Classification of nonorientable regular embeddings of complete bipartite graphs

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\begin{abstract}
A 2-cell embedding of a graph $G$ into a closed (orientable or nonorientable) surface is called regular if its automorphism group acts regularly on the flags – mutually incident vertex–edge–face triples. In this paper, we classify the regular embeddings of complete bipartite graphs $K_{n,n}$ into nonorientable surfaces. Such a regular embedding of $K_{n,n}$ exists only when $n$ is of the form $n = 2p_1^{a_1}p_2^{b_2} \cdots p_k^{a_k}$ where the $p_i$ are primes congruent to $\pm 1 \mod 8$. In this case, up to isomorphism the number of those regular embeddings of $K_{n,n}$ is $2^k$.
\end{abstract}

1. Introduction

A 2-cell embedding of a connected graph into a connected closed surface is called a topological map. An automorphism of a map is an automorphism of the underlying graph which can be extended to a self-homeomorphism of the supporting surface in the embedding. The automorphisms of a map $\mathcal{M}$ act semi-regularly on its flags – mutually incident vertex–edge–face triples. If the automorphism group $\text{Aut}(\mathcal{M})$ of $\mathcal{M}$ acts regularly on the flags, then the map $\mathcal{M}$ as well as the corresponding embedding are also called regular. For a given map $\mathcal{M}$ whose supporting surface is orientable, the set $\text{Aut}^+(\mathcal{M})$ of orientation-preserving automorphisms of the map $\mathcal{M}$ acts semi-regularly on its arcs – mutually incident vertex–edge pairs. If $|\text{Aut}^+(\mathcal{M})|$ acts regularly on its arcs, then the map $\mathcal{M}$ is called orientably regular. Therefore if a supporting surface is orientable, a regular map means an orientably regular map having an orientation-reversing automorphism.

One of the standard problems in topological graph theory is the classification of orientably regular embeddings or regular embeddings of a given class of graphs. In recent years, there has been
particular interest in the orientably regular embeddings and regular embeddings of complete bipartite graphs $K_{n,n}$ exhibited in papers by several authors [4,5,8,10,9,12,11,14]. The regular (or reflexible regular) embeddings and self-Petrie dual regular embeddings of $K_{n,n}$ into orientable surfaces were classified by the authors in [12]. Recently, G.A. Jones [8] classified the orientably regular embeddings of complete graphs, while their nonorientable regular embeddings were classified by S.E. Wilson [15]. Recently, D.A. Catalano et al. [3] classified the orientably regular embeddings of complete graphs, while their nonorientable regular embeddings were classified by N.L. Biggs [2] and of L.D. James and G.A. Jones [7] yields a classification of their nonorientable regular embeddings. The following theorem is the main result in this paper.

For other classes of graphs, the classification of their regular embeddings has been done: for instance, the work of N.L. Biggs [2] and of L.D. James and G.A. Jones [7] yields a classification of the orientably regular embeddings of complete graphs, while their nonorientable regular embeddings were classified by S.E. Wilson [15]. Recently, D.A. Catalano et al. [3] classified the orientably regular embeddings of $n$-dimensional cubes $Q_n$, and R. Nedela and the second author [13] classified their nonorientable regular embeddings. The following theorem is the main result in this paper.

**Theorem 1.1.** For any integer $n$ congruent to 0, 1 or 3 mod 4, no nonorientable regular embedding of $K_{n,n}$ exists. For $n = 2p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$ (the prime decomposition of $n$), up to isomorphism the number of nonorientable regular embeddings of $K_{n,n}$ is $2^k$ if every $p_i$ is congruent to $\pm 1$ mod 8; and 0 otherwise.

This paper is organized as follows. In the next section, we introduce a triple of three graph automorphisms of $G$, called an *admissible triple* for $G$, which corresponds to a regular embedding of $G$. In Section 3, we construct some nonorientable regular embeddings of $K_{n,n}$ by forming admissible triples for $K_{n,n}$. In the last two sections, Theorem 1.1 is proved by showing that up to isomorphism no other nonorientable regular embedding of $K_{n,n}$ exists beyond those constructed in Section 3.

**2. An admissible triple of graph automorphisms**

For a given simple graph $G$, the automorphism group $\text{Aut}(G)$ acts faithfully on both the vertex set $V(G)$ and the arc set $D(G)$. Moreover, in an embedding $\mathcal{M}$ of $G$ whose valency is greater than two, $\text{Aut}(\mathcal{M}) \subseteq \text{Aut}(G)$ and $\text{Aut}(\mathcal{M})$ acts faithfully on the flag set $F(\mathcal{M})$. So we consider a graph automorphism as a permutation of $V(G)$, $D(G)$ or $F(\mathcal{M})$ according to the context.

Let $G$ be a graph and let $\mathcal{M}$ be a regular embedding of $G$. Then for some incident vertex-edge pair $(v,e)$, there exist three involutory graph automorphisms $\ell$, $r$ and $t$ in $\text{Aut}(G)$ which generate $\text{Aut}(\mathcal{M})$ and satisfy the following properties:

(i) $\Gamma = \langle \ell, r, t \rangle$ acts transitively on the arc set $D(G)$.

(ii) The stabilizer $\Gamma_v$ of the vertex $v$ is $\Gamma_v = \langle r, t \rangle$ and is isomorphic to the dihedral group $D_n$, and its cyclic subgroup $\langle rt \rangle$ acts regularly on the arcs emanating from $v$, where $n$ is the valency of $G$.

(iii) The stabilizer $\Gamma_e$ of the edge $e$ is $\Gamma_e = \langle \ell, t \rangle$ and is isomorphic to the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We call such automorphisms $\ell$, $r$ and $t$ *basic generators for the regular map* $\mathcal{M}$ with respect to the incident vertex-edge pair $(v,e)$. Also we call a triple $(\ell, r, t)$ of involutory automorphisms of $G$ satisfying properties (i), (ii) and (iii) an *admissible triple* for a regular embedding of $G$, or simply an *admissible triple* for $G$. The vertex $v$ and the edge $e$ are called a root vertex and a root edge, respectively.

Conversely, for a given admissible triple $(\ell, r, t)$ for $G$ with a root vertex $v$, the group $\Gamma = \langle \ell, r, t \rangle$ is the automorphism group of a regular map $\mathcal{M} = \mathcal{M}(\ell, r, t)$, called the *derived map* of the admissible triple $(\ell, r, t)$. Flags of $\mathcal{M}$ are elements of $\Gamma$, and vertices, edges and faces of $\mathcal{M}$ are left cosets of the subgroups $(r, t), (\ell, t)$ and $(\ell, r)$, respectively. Mutual incidence of map elements is given by nonempty intersection. In fact, $\Gamma$ acts on the map $\mathcal{M}$ by the left multiplication. Let $\mathcal{G}$ be the underlying graph of the map $\mathcal{M}$. If we label each vertex $g(r, t)$ in $\mathcal{G}$ by $g(v)$, then the graph $\mathcal{G}$ is exactly $G$ and the action of $\Gamma$ on $\mathcal{G}$ corresponds to a subgroup of automorphisms of $G$. Hence we consider $\mathcal{M}(\ell, r, t)$ as a regular embedding of $G$ from now on. In [6, Theorem 3], Gardiner et al. showed how to construct a regular embedding of $G$ by an admissible triple. The method of [6] looks different from ours, but they are essentially the same.
For a regular embedding $\mathcal{M}$ of $G$ with basic generators $\ell, r$ and $t$ with respect to an incident vertex–edge pair $(v, e)$, one can see that the derived map $\mathcal{M}(\ell, r, t)$ is isomorphic to $\mathcal{M}$. Hence it suffices to consider derived maps $\mathcal{M}(\ell, r, t)$ of admissible triples $(\ell, r, t)$ for $G$ in order to classify regular embeddings of $G$.

Let $(\ell_1, r_1, t_1)$ and $(\ell_2, r_2, t_2)$ be two admissible triples for $G$. If there exists a graph automorphism $\phi$ of $G$ such that $\phi \ell_1 \phi^{-1} = \ell_2$, $\phi r_1 \phi^{-1} = r_2$ and $\phi t_1 \phi^{-1} = t_2$, then $\phi$ induces a map isomorphism from $\mathcal{M}(\ell_1, r_1, t_1)$ to $\mathcal{M}(\ell_2, r_2, t_2)$, and hence $\mathcal{M}(\ell_1, r_1, t_1)$ and $\mathcal{M}(\ell_2, r_2, t_2)$ are isomorphic. Conversely, assume that two derived regular maps $\mathcal{M}(\ell_1, r_1, t_1)$ and $\mathcal{M}(\ell_2, r_2, t_2)$ are isomorphic. Since two underlying graphs of the two maps $\mathcal{M}(\ell_1, r_1, t_1)$ and $\mathcal{M}(\ell_2, r_2, t_2)$ are the same graph $G$, a map isomorphism is a graph automorphism of $G$, which implies that there exists a graph automorphism $\phi$ of $G$ such that $\phi \ell_1 \phi^{-1} = \ell_2$, $\phi r_1 \phi^{-1} = r_2$ and $\phi t_1 \phi^{-1} = t_2$. Therefore we have the following obvious but important consequence of the above considerations.

**Proposition 2.1.** Let $G$ be a graph. Then every regular embedding $\mathcal{M}$ of $G$ into an orientable or nonorientable surface is isomorphic to a derived regular map $\mathcal{M}(\ell, r, t)$ of an admissible triple $(\ell, r, t)$ for $G$ and its isomorphism class corresponds to the conjugacy class of the triple $(\ell, r, t)$ in $\text{Aut}(G)$.

It follows from Proposition 2.1 that for a given graph $G$, up to isomorphism the number of nonorientable regular embeddings of $G$ equals the number of orbits of admissible triples $(\ell, r, t)$ for $G$ satisfying $(\ell t, r t) = (\ell, r, t)$ under the conjugate action by $\text{Aut}(G)$.

### 3. Constructions of nonorientable embeddings

The complete bipartite graph $K_{2,2}$ is just the 4-cycle, and there is only one nonorientable regular embedding of the 4-cycle with the projective plane as the supporting surface. So from now on, we assume that $n \geq 3$. For a complete bipartite graph $K_{n,n}$, let $[n] = \{0, 1, \ldots, n-1\}$ and $[n]^\prime = \{0^\prime, 1^\prime, \ldots, (n-1)^\prime\}$ be the vertex sets of $K_{n,n}$ as the partite sets, and let $D = \{(i,j), (j^\prime, i) \mid 0 \leq i, j \leq n \}$ be the arc set. We denote the symmetric group on $[n] = \{0, 1, \ldots, n-1\}$ by $S_n$, and the stabilizer of $i$ as a subgroup of $S_n$ by $\text{Stab}(i)$. We identify the integers $0, 1, \ldots, n-1$ with their residue classes modulo $n$ according to the context.

Since $\text{Aut}(K_{n,n}) \cong S_n \rtimes \mathbb{Z}_2$ contains all permutations of vertices of each partite set and the interchanging of two partite sets $[n]$ and $[n]^\prime$, one can assume that up to isomorphism every orientable or nonorientable regular embedding of $K_{n,n}$ is derived from an admissible triple $(\ell, r_{\delta}, t)$ for $K_{n,n}$ of the following type:

$$
\ell = (0, 0^\prime)(1, (n-1)^\prime)(2, (n-2)^\prime) \cdots (n-1, 1^\prime),
$$

$$
r_{\delta} = \delta(0, 1^\prime)(0^\prime, 1)(n-1, 2^\prime)(n-2, 3^\prime) \cdots \left( \left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil \right),
$$

$$
t = (0)(1, n-1)(2, n-2) \cdots \left( \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right) (0^\prime)(1^\prime, (n-1)^\prime)(2^\prime, (n-2)^\prime) \cdots \left( \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right)
$$

for some $\delta \in \text{Stab}(0)$; see Fig. 1. Note that the root vertex and the root edge of the above admissible triple are 0 and $[0, 0^\prime]$, respectively. In fact, the admissibility of the triple $(\ell, r_{\delta}, t)$ depends only on the permutation $\delta \in \text{Stab}(0)$. Clearly, $\ell t = t \ell$ and so $(\ell, t) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover

$$
r_{\delta} t = \delta(0)(1, n-1)(2, n-2) \cdots \left( \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right) (0^\prime, 1^\prime, 2^\prime, \ldots, (n-1)^\prime)
$$

generates the cyclic subgroup which acts regularly on the arcs emanating from the root vertex 0.

For an even integer $n$, let us write $\bar{n} = n/2$ for notational convenience.

**Lemma 3.1.** Suppose the derived regular maps $\mathcal{M}(\ell, r_{\delta_1}, t)$ and $\mathcal{M}(\ell, r_{\delta_2}, t)$ are isomorphic, where $(\ell, r_{\delta_1}, t)$ and $(\ell, r_{\delta_2}, t)$ are admissible triples with $\delta_1, \delta_2 \in \text{Stab}(0)$. Then either (i) $\delta_1 = \delta_2$ or (ii) $n$ is even and $\delta_2(k) = \delta_1(k + \bar{n}) + \bar{n}$ for all $k \in [n]$. 


the vertex 0 and the vertex \( \bar{\delta} \) direction-reversing automorphism of the root edge vertex 0 and the \( \bar{\delta} \) involution. For an admissible triple conditions. Let \( \psi \in \text{Aut}(K_{n,n}) \) such that \( \psi \ell \psi^{-1} = \ell \), \( \psi \bar{\delta}_1 \psi^{-1} = \bar{\delta}_2 \) and \( \psi t \psi^{-1} = t \). Note that the vertex 0 and the vertex \( \bar{n} \) when \( n \) is even are the only vertices which can be fixed by both \( t \) and \( r_\delta \). The relations \( \psi t = t \psi \) and \( \psi r_\delta = r_\delta \psi \) imply that \( \psi \) permutes these vertices. Hence our discussion can be divided into the following two cases.

**Case 1.** Suppose \( \psi(0) = 0 \). Since \( \psi r_\delta = r_\delta \psi \) and \( \psi \) commutes with \( \ell \) and \( t \), \( \psi \) should be the identity. This means that \( \delta_1 = \delta_2 \).

**Case 2.** Suppose \( n \) is even and \( \psi(0) = \bar{n} \). Since \( \psi r_\delta = r_\delta \psi \) and \( \psi \) commutes with \( \ell \) and \( t \), one can show that \( \psi(k) = k + \bar{n} \) and \( \psi(k') = (k + \bar{n})' \) for all \( k \in [n] \). So for every \( k \in [n] \), \( \delta_2(k) = r_\delta(k) = \psi r_\delta \psi^{-1}(k) = \psi r_\delta(k + \bar{n}) = \delta_1(k + \bar{n}) + \bar{n} \).

It will be shown later that the case (ii) in Lemma 3.1 cannot occur.

For any involution \( \delta \in \text{Stab}(0) \), set

\[
\bar{\delta} = r_\delta \cdot t |_{[n]} = \delta \cdot (0, 1, -1, 2, -2) \cdots \in S[n].
\]

Then \( \bar{\delta} \) also belongs to \( \text{Stab}(0) \) and satisfies the equation \( \bar{\delta}^{-1}(-k) = -\bar{\delta}(k) \) for all \( k \in [n] \), since \( \delta \) is an involution. For an admissible triple \((\ell, r_\delta, t)\) for \( K_{n,n} \) with \( \delta \in \text{Stab}(0) \), let \( R_{\bar{\delta}} = r_\delta t \) and \( L = \ell t \), namely,

\[
R_{\bar{\delta}} = r_\delta t = \bar{\delta}(0', 1', \ldots, (n - 1)')
\]

and

\[
L = t \ell = (0, 0')(1, 1') \cdots ((n - 1), (n - 1)').
\]
as permutations on the vertex set \([n] \cup [n]'\). These are the local rotation automorphism at the root vertex 0 and the direction-reversing automorphism of the root edge \((0 0')\), respectively. In fact, \( L \) is an automorphism which interchange partite sets. Note that any one of \( \delta, \bar{\delta}, R_{\bar{\delta}} \) determines the automorphism which interchange partite sets. Recall that the regular map \( \mathcal{M}(\ell, r_\delta, t) \) is nonorientable if and only if \((\ell, r_\delta, t) = (R_{\bar{\delta}}, L)\). Hence if \((\ell, r_\delta, t)\) is an admissible triple for \( K_{n,n} \) and \( \mathcal{M}(\ell, r_\delta, t) \) is nonorientable, then \( \bar{\delta}(0) = 0 \), \( \bar{\delta}^{-1}(-k) = -\bar{\delta}(k) \) for all \( k \in [n] \), \(|(R_{\bar{\delta}}, L)| = 4 |E(K_{n,n})| = 4n^2 \) and \( t \in (R_{\bar{\delta}}, L) \). So in order to construct all nonorientable regular embeddings of \( K_{n,n} \), we need to examine \( \delta \) satisfying the aforementioned conditions. Let

\[
\mathcal{M}_n^{\text{non}} = \{ \delta \in S[n] \mid \delta(0) = 0, \bar{\delta}^{-1}(-k) = -\bar{\delta}(k) \text{ for all } k \in [n], |(R_{\bar{\delta}}, L)| = 4n^2 \text{ and } (R_{\bar{\delta}}, L) \text{ contains } t \}.
\]

We shall show that there is a one-to-one correspondence between the nonorientable regular embeddings of \( K_{n,n} \) for \( n \geq 3 \) up to isomorphism and the elements in \( \mathcal{M}_n^{\text{non}} \). From now on, we shall deal with \( \delta \) instead of \( \bar{\delta} \), as we construct and classify the nonorientable regular embeddings of \( K_{n,n} \).
Lemma 3.2. For every involution \( \delta \in S_{[n]} \) with \( \delta(0) = 0 \), the following statements are equivalent:

1. The triple \((\ell, r_{\delta}, t)\) is admissible and the derived regular map \( \mathcal{M}(\ell, r_{\delta}, t) \) is nonorientable.
2. \( \delta \in \mathcal{M}_{n}^{\text{non}} \), where \( \delta = r_{\delta} \cdot t_{[n]} \).

Proof. For \( \delta = r_{\delta} \cdot t_{[n]} \), we already know \( \delta(0) = 0 \) and \( \delta^{-1}(-k) = -\delta(k) \) for all \( k \in [n] \).

(1) \( \Rightarrow \) (2) Let \((\ell, r_{\delta}, t)\) be admissible, and let the map \( \mathcal{M}(\ell, r_{\delta}, t) \) be nonorientable. Then \((\ell, r_{\delta}, t) = (R_{\delta}, L) \) and \(|(\ell, r_{\delta}, t)| = |(R_{\delta}, L)| = 4E(K_{n,n})| = 4n^2 \), so \( \delta \in \mathcal{M}_{n}^{\text{non}} \).

(2) \( \Rightarrow \) (1) Let \( \delta \in \mathcal{M}_{n}^{\text{non}} \). Since \( t \in (R_{\delta}, L) \), it follows that \( R_{\delta} t = r_{\delta} \in (R_{\delta}, L) \) and \( tL = \ell \in (R_{\delta}, L) \). Hence, \((\ell, r_{\delta}, t) = (R_{\delta}, L) \).

For any \( i, j \in [n] \), we have

\[
R_{\delta}^i L R_{\delta}^j t(0, 0') = R_{\delta}^i L R_{\delta}^j(0, 0') = R_{\delta}^i L(0, j') = R_{\delta}^i(0', j) = (i', \delta^i(j))
\]

and by taking \( L \) on both sides, we have

\[
LR_{\delta}^i L R_{\delta}^j t(0, 0') = LR_{\delta}^i L R_{\delta}^j(0, 0') = (i, \delta^i(j')).
\]

This shows that the arc \((0, 0')\) can be mapped to any other arc by the action of the group \((R_{\delta}, L)\). This means that \((\ell, r_{\delta}, t) = (R_{\delta}, L) \) acts transitively on both the arc set \( D(K_{n,n}) \) and the vertex set \( V(K_{n,n}) \). For \( 0 \in V(K_{n,n}) \), \((R_{\delta}, L) \leq (R_{\delta}, L)_{0} \). Since \(|(R_{\delta}, t)| = |(R_{\delta}, L)| = 2n \), one can see that \((R_{\delta}, t) = (R_{\delta}, L)_{0} \). Since \((\ell, r_{\delta}, t) = (R_{\delta}, L) \), the derived regular map \( \mathcal{M}(\ell, r_{\delta}, t) \) is nonorientable.

In order to determine \( \tilde{\delta} \) satisfying \(|(R_{\tilde{\delta}}, L)| = 4n^2 \), we need to consider certain elements in the group \((R_{\tilde{\delta}}, L)\). For later use, we define two properties, called \((P_{1})\) and \((P_{2})\), as follows

\[
\tilde{\delta}(k + i) = \tilde{\delta}^b(i)(k) + a(i) \quad \text{and} \quad \tilde{\delta}^i(k + 1) = \tilde{\delta}^a(i)(k + b(i)) \quad \text{for all} \ k \in [n], \quad (P_{1})
\]

and

\[
\tilde{\delta}(k + i) = \tilde{\delta}^b(i)(-k) + a(i) \quad \text{and} \quad \tilde{\delta}^i(k + 1) = \tilde{\delta}^a(i)(-k + b(i)) \quad \text{for all} \ k \in [n]. \quad (P_{2})
\]

In fact, \((P_{1})\) and \((P_{2})\) are equivalent to the equalities \( R_{\tilde{\delta}} L R_{\tilde{\delta}}^i L \) and \( R_{\tilde{\delta}} L R_{\tilde{\delta}}^i L = L R_{\tilde{\delta}}^i L R_{\tilde{\delta}}^b(i), L \), respectively.

Lemma 3.3. Let \( \delta \in S_{[n]} \) be an involution such that \( \delta(0) = 0 \), or equivalently, \( \delta^{-1}(-k) = -\delta(k) \) for all \( k \in [n] \), and \( \delta(0) = 0 \). Then the following statements are equivalent:

1. \( \tilde{\delta} \in \mathcal{M}_{n}^{\text{non}} \).
2. The subgroup \((R_{\tilde{\delta}}, L)\) of \( S_{[n]} \) is a disjoint union of the four sets

\[
B := \{ R_{\tilde{\delta}}^i L R_{\tilde{\delta}}^j | i, j \in [n] \}, \\
LB := \{ L R_{\tilde{\delta}}^i L R_{\tilde{\delta}}^j | i, j \in [n] \}, \\
Bt := \{ R_{\tilde{\delta}}^i L R_{\tilde{\delta}}^j t | i, j \in [n] \}, \\
LBt := \{ L R_{\tilde{\delta}}^i L R_{\tilde{\delta}}^j t | i, j \in [n] \}.
\]

3. For each \( i \in [n] \), there exist \( a(i), b(i) \in [n] \) such that either \((P_{1})\) or \((P_{2})\) holds. In addition, \((P_{2})\) holds for at least one \( i \in [n] \).
Proof. (1) $\Rightarrow$ (2) For any $\tilde{\delta} \in \mathcal{M}^{\text{non}}_n$ and for any $i, j \in [n]$, we have
\[ R^{i}_\delta LR^{j}_\delta t(0,0') = R^{i}_\delta LR^{j}_\delta (0,0') = R^{i}_\delta L(0, j') = R^{i}_\delta (O', j) = (i', \tilde{\delta}^i(j)) \]
and by taking $L$ on both sides we obtain
\[ LR^{i}_\delta LR^{j}_\delta t(0,0') = LR^{i}_\delta LR^{j}_\delta (0,0') = (i, \tilde{\delta}^i(j')). \]
By comparing images of the arc $(0,0')$, one can see that
\[ (B \cup Bt) \cap (LB \cup LBt) = \emptyset, \]
and for any $(i, j) \neq (k, \ell)$, also \{\(R^{i}_\delta LR^{j}_\delta \), \(R^{k}_\delta LR^{j}_\delta \)\} \(\cap\) \{\(R^{i}_\delta LR^{j}_\delta\), \(LR^{k}_\delta LR^{j}_\delta \)\} = \emptyset and \{\(LR^{i}_\delta LR^{j}_\delta \), \(LR^{k}_\delta LR^{j}_\delta \)\} \(\cap\) \{\(LR^{k}_\delta LR^{j}_\delta \), \(LR^{i}_\delta LR^{j}_\delta \)\} = \emptyset. Now, it suffices to show that for all $(i, j) \in [n] \times [n]$ in order to show that $B \cap Bt \neq \emptyset$ and $LB \cap LBt \neq \emptyset$. In fact, for any $(i, j) \in [n] \times [n]$, \[ R^{i}_\delta LR^{j}_\delta t(0,1') = R^{i}_\delta LR^{j}_\delta (0, -1') \]
and \[ LR^{i}_\delta LR^{j}_\delta t(0,1') = LR^{i}_\delta LR^{j}_\delta (0, -1'). \]
These imply that $R^{i}_\delta LR^{j}_\delta \neq R^{i}_\delta LR^{j}_\delta t$ and $R^{i}_\delta LR^{j}_\delta \neq LR^{i}_\delta LR^{j}_\delta t$. Hence the four sets $B, LB, Bt$ and $LBt$ are mutually disjoint, and the cardinality of their union is $4n^2$, which equals $|\langle R_\tilde{\delta}, L \rangle|$. (2) $\Rightarrow$ (3) Since the group $\langle R_\tilde{\delta}, L \rangle$ is the union of the four sets, for each $i \in [n]$, there exist $a(i), b(i) \in [n]$ such that $R^{i}_\delta L = LR^{a(i)}_\delta LR^{b(i)}_\delta$ or $R^{i}_\delta L = LR^{a(i)}_\delta LR^{b(i)}_\delta t$. By comparing the associated values of $k$ and $k'$, we have
\[ \tilde{\delta}(k + i) = \tilde{\delta}^i(k) + a(i) \quad \text{and} \quad \tilde{\delta}^i(k) + 1 = \tilde{\delta}^{a(i)}(k + b(i)) \quad \text{for all} \quad k \in [n] \]
or
\[ \tilde{\delta}(k + i) = \tilde{\delta}^i(-k) + a(i) \quad \text{and} \quad \tilde{\delta}^i(-k) + 1 = \tilde{\delta}^{a(i)}(-k + b(i)) \quad \text{for all} \quad k \in [n]. \]
In other words, either $(P_1)$ or $(P_2)$ holds. Suppose that $(P_1)$ holds for all $i \in [n]$, and hence that $R^{i}_\delta L = LR^{a(i)}_\delta LR^{b(i)}_\delta$. Then one can easily check that
\[ \langle R_\tilde{\delta}, L \rangle = \{ LR^{i}_\delta LR^{j}_\delta | i, j \in [n] \} \cup \{ R^{i}_\delta LR^{j}_\delta | i, j \in [n] \} \quad \text{(disjoint union),} \]
which contradicts the assumption. Hence there exists at least one $i \in [n]$ for which $(P_2)$ holds.
(3) $\Rightarrow$ (1) Since $(P_1)$ and $(P_2)$ are equivalent to the equalities $R^{i}_\delta L = LR^{a(i)}_\delta LR^{b(i)}_\delta$ and $R^{i}_\delta L = LR^{a(i)}_\delta LR^{b(i)}_\delta t$, the group $\langle R_\tilde{\delta}, L \rangle$ contains $t$, and $\langle R_\tilde{\delta}, L \rangle$ is the same as the union of the four sets in (2). The disjointness of the union can be shown in a similar way to show (1) $\Rightarrow$ (2). This means that $|\langle R_\tilde{\delta}, L \rangle| = 4n^2$, so $\tilde{\delta} \in \mathcal{M}^{\text{non}}_n$. □

In fact, two numbers $a(i)$ and $b(i)$ in Lemma 3.3(3) are completely determined by the values of $\tilde{\delta}$ in the following sense.

Lemma 3.4. Let $\tilde{\delta} \in \mathcal{M}^{\text{non}}_n$ and let $a(i)$ and $b(i)$ be the numbers given in $(P_1)$ and $(P_2)$. If they satisfy $(P_1)$, then
\[ a(i) = \tilde{\delta}(i) \quad \text{and} \quad b(i) = \tilde{\delta}^i(1) = \tilde{\delta}^{a(i)}(1), \]
and if they satisfy $(P_2)$, then
\[ a(i) = \tilde{\delta}(i) \quad \text{and} \quad b(i) = \tilde{\delta}^{a(i)}(1) = \tilde{\delta}^{\tilde{\delta}(i)}(1). \]
Proof. Let \( a(i) \) and \( b(i) \) satisfy (P1). By taking \( k = 0 \) in the equation \( \tilde{\delta}(k + i) = \tilde{\delta} b(i)(k) + a(i) \), we have \( a(i) = \tilde{\delta}(i) \). Also, by taking \( k = 0 \) and \( k = -b(i) \) in the equation \( \tilde{\delta}(k + 1) = \tilde{\delta} a(i)(k + b(i)) \), we have \( b(i) = \tilde{\delta} a(i)(1) = \tilde{\delta} \tilde{\delta}^{-1}(1) \) and \( b(i) = -\tilde{\delta}^{-1}(-1) \). Since \( \tilde{\delta}^{-1}(-k) = -\tilde{\delta}(k) \) for all \( k \in [n] \), it follows that

\[
 b(i) = -\tilde{\delta}^{-i}(-1) = -\tilde{\delta}^{-i+1}(-\tilde{\delta}^{-1}(-1)) = -\tilde{\delta}^{-i+1}(-\tilde{\delta}(1)) = -\tilde{\delta}^{-i+2}(-\tilde{\delta}^2(1)) = \cdots = -\tilde{\delta}^{-i-1}(1) = -\tilde{\delta}^i(1).
\]

Let \( a(i) \) and \( b(i) \) satisfy (P2). By taking \( k = 0 \) in the equation \( \tilde{\delta}(k + i) = \tilde{\delta} b(i)(k) + a(i) \), we have \( a(i) = \tilde{\delta}(i) \). Also, by taking \( k = 0 \) and \( k = b(i) \) in the equation \( \tilde{\delta}(k + 1) = \tilde{\delta} a(i)(-k + b(i)) \), we have \( b(i) = \tilde{\delta} a(i)(1) = \tilde{\delta} \tilde{\delta}^{-1}(1) \) and \( b(i) = -\tilde{\delta}^{-1}(-1) \). Since \( \tilde{\delta}^{-1}(-k) = -\tilde{\delta}(k) \) for all \( k \in [n] \), we find \( b(i) = -\tilde{\delta}^{-i-1}(1) = -\tilde{\delta}^i(1) \).

Now, let us consider the even numbers \( n \) as the first case in which we construct nonorientable regular embeddings of \( K_{n,n} \). As a candidate of \( \tilde{\delta} \in \mathcal{M}_{n}^{\text{non}} \), we define a permutation \( \tilde{\delta}_{n,x} \in S[n] \) by

\[
\tilde{\delta}_{n,x}(i) = (0)(2, -2)(4, -4) \cdots (1, 1 + x, 1 + 2x, 1 + 3x, \ldots)
\]

for any positive even integer \( x \) such that the greatest common divisor of \( n \) and \( x \) is 2.

Suppose that \( \tilde{\delta}_{n,x} \) belongs to \( \mathcal{M}_{n}^{\text{non}} \) for some even \( x \). For each \( i \in [n] \), there exist \( a(i), b(i) \in [n] \) satisfying (P1) or (P2) by Lemma 3.3. For every even \( i \in [n] \), it follows that

\[
b(i) = \tilde{\delta}_{n,x}^{-b(i)}(i) = \tilde{\delta}_{n,x}^{-1}(1),
\]

which implies that \( a(i), b(i) \) satisfy (P1) by Lemma 3.4. So there exists an odd integer \( j \in [n] \) such that \( a(j), b(j) \in [n] \) satisfying (P2) by Lemma 3.3. For such odd \( j \in [n] \), we have \( b(j) = -\tilde{\delta}_{n,x}^j(1) = -(1 + jx) \) by Lemma 3.4. Since

\[
b(j) = \tilde{\delta}_{n,x}^{-b(j)}(1) = \tilde{\delta}_{n,x}^{1-j}(1) = 1 - (j + x)x = (1 + jx) + (2 - x^2),
\]

we have \((-1 + jx) = -(1 + jx) + (2 - x^2)\), and hence \( x^2 \equiv 2 \mod n \). Therefore the condition \( x^2 \equiv 2 \mod n \) is necessary for \( \tilde{\delta}_{n,x} \) to belong to \( \mathcal{M}_{n}^{\text{non}} \). The following two lemmas show that the condition is also sufficient. Furthermore it will be shown that up to isomorphism there is a one-to-one correspondence between the set of nonorientable regular embeddings of \( K_{n,n} \) and the set of solutions of \( x^2 \equiv 2 \mod n \) in \( \mathbb{Z}_n \). So square roots of 2 mod \( n \) are very important in this paper.

**Lemma 3.5.** Let \( n \) and \( x \) be even integers such that \( n > x > 3 \) and \( x^2 \equiv 2 \mod n \):

1. For an even integer \( 2i \in [n] \), if we define \( a(2i) = -2i \) and \( b(2i) = 2ix + 1 \), then \( a(2i) \) and \( b(2i) \) satisfy (P1) with \( \tilde{\delta} = \tilde{\delta}_{n,x} \).
2. For an odd integer \( 2i + 1 \in [n] \), if we define \( a(2i + 1) = 2i + x + 1 \) and \( b(2i + 1) = -2ix - x - 1 \) then \( a(2i + 1) \) and \( b(2i + 1) \) satisfy (P2) with \( \tilde{\delta} = \tilde{\delta}_{n,x} \).

**Proof.** We prove (1), and note that a proof of (2) is similar. For an even integer \( 2i \), let \( a(2i) = -2i \) and \( b(2i) = 2ix + 1 \). Then for any even integer \( 2k \in [n] \), we have

\[
\tilde{\delta}_{n,x}(2k + 2i) = -2k - 2i,
\]

\[
\tilde{\delta}_{n,x}^{b(2i)}(2k) + a(2i) = \tilde{\delta}_{n,x}^{2ix + 1}(2k) - 2i = -2k - 2i,
\]

\[
\tilde{\delta}_{n,x}^{2i}(2k) + 1 = 2k + 1,
\]

and

\[
\tilde{\delta}_{n,x}^{a(2i)}(2k + b(2i)) = \tilde{\delta}_{n,x}^{-2i}(2k + 2ix + 1) = 2k + 2ix + 1 - 2ix = 2k + 1.
\]

For any odd integer \( 2k + 1 \in [n] \), we have
Proof. Since the sufficiency is clear, we prove only the necessity. Recall that if for a given \( n \) \( \tilde{\delta}_{n,x} \) of \( H \) subgroup of \( M \), 
\[ \tilde{\delta}_{n,x}(2k + 1 + 2i) = 2k + 2i + x + 1, \]
and 
\[ \tilde{\delta}_{n,x}(2k + 1) + a(2i) = \tilde{\delta}_{n,x}^2(2k + 1) - 2i = 2k + 1 + (2ix + 1)x - 2i = 2k + 4i + x - 2i = 2k + 2i + x + 1 \] 
for any two different \( n \) and \( x \) such that \( n > x > 3 \) and \( x^2 \equiv 2 \pmod{n} \) because \( x^2 \equiv 2 \pmod{n} \) becomes \( x^2 \equiv 2 \pmod{4} \) and hence there is no such \( x \) satisfying \( x^2 \equiv 2 \pmod{n} \). For every \( n \equiv 0 \pmod{4} \), \( N_n^\text{non} = \emptyset \). The smallest integer \( n \) such that \( N_n^\text{non} \neq \emptyset \) is 14. This will show that \( K_{14,14} \) is the smallest complete bipartite graph (other than \( K_{2,2} \)) which can be regularly embedded into a nonorientable surface.

\[ N_n^\text{non} = \begin{cases} \{ \tilde{\delta}_{n,x} \mid n > x > 3 \text{ and } x^2 \equiv 2 \pmod{n} \} & \text{if } n \text{ is even,} \\ \emptyset & \text{if } n \text{ is odd.} \end{cases} \]

Note that for any even integers \( n, x \) such that \( n \equiv 0 \pmod{4} \) and \( n > x > 1 \), \( x^2 \) is a multiple of 4 and \( x \) is even. Hence \( x^2 \equiv 2 \pmod{n} \) becomes \( x^2 \equiv 2 \pmod{4} \) and there is no such \( x \) satisfying \( x^2 \equiv 2 \pmod{n} \). For every \( n \equiv 0 \pmod{4} \), \( N_n^\text{non} = \emptyset \). The smallest integer \( n \) such that \( N_n^\text{non} \neq \emptyset \) is 14. This will show that \( K_{14,14} \) is the smallest complete bipartite graph (other than \( K_{2,2} \)) which can be regularly embedded into a nonorientable surface.

Remark. For any \( \tilde{\delta}_{n,x} \in N_n^\text{non} \) with \( \delta = \tilde{\delta}_{n,x} \cdot (0)(1 - 1)(2 - 2)\cdots \), the automorphism group \( \langle \ell, r, t \rangle = (R_{\tilde{\delta}_{n,x}}, L) \) of the derived map \( M(\ell, r, t) \) is a split extension (semidirect product) of an abelian normal subgroup \( K = \langle R_{\tilde{\delta}_{n,x}}, L \rangle \) of rank 2, order \( n^2 \) and exponent \( n \) by a Sylow 2-subgroup \( H = \langle R_{\tilde{\delta}_{n,x}}^2, L \rangle \) of order 16. The two cyclic summands of the subgroup \( K \) act as the stabilizers of vertices in the two different parts of the bipartition of \( K_{n,n} \), and these are interchanged by the arc-reversing automorphism \( L \). Note that \( H \) is isomorphic to the automorphism group of the nonorientable regular embedding \( M_1 \) of \( K_{2,2} \) into the projective plane, and so there is a regular homomorphism \( M(\ell, r, t) \to M_1 \) with the fibre transformation group isomorphic to \( K \). The covalency (face size) of \( M(\ell, r, t) \) is the order of \( LR_{\tilde{\delta}_{n,x}}^2 \), which is in fact 8. Hence the number of faces of the map \( M(\ell, r, t) \) is \( n^2/4 \). By the Euler formula, the supporting surface of \( M(\ell, r, t) \) is nonorientable surface with \( (3n^2 - 8n + 8)/4 \) crosscaps.

In the final two sections, it will be shown that \( M_n^\text{non} = N_n^\text{non} \) for every \( n \), which means that \( M_n^\text{non} = \emptyset \) if \( n \) is odd or \( n \equiv 0 \pmod{4} \). In the remaining part of this section, we shall show that for any two different \( \tilde{\delta}_{n,x_1}, \tilde{\delta}_{n,x_2} \in N_n^\text{non} \), their derived regular embeddings of \( K_{n,n} \) are not isomorphic. Also for a given \( n \equiv 2 \pmod{4} \), we will estimate the cardinality \( |N_n^\text{non}| \), that is, the number of solutions of \( x^2 = 2 \) in \( \mathbb{Z}_n \).

\[ \text{Lemma 3.7. For any two } \tilde{\delta}_{n,x_1}, \tilde{\delta}_{n,x_2} \in N_n^\text{non} \text{ with } n > 3, \text{ let } \delta_i = \tilde{\delta}_{n,x_i} \cdot (0)(1 - 1)(2 - 2)\cdots \text{ for } i = 1, 2. \text{ Then the derived regular maps } M(\ell, r_{\delta_1}, t) \text{ and } M(\ell, r_{\delta_2}, t) \text{ are isomorphic if and only if } x_1 = x_2. \]

Proof. Since the sufficiency is clear, we prove only the necessity. Recall that if \( n \equiv 0 \pmod{4} \), then \( N_n^\text{non} = \emptyset \). So let \( n \equiv 2 \pmod{4} \).
Suppose the two regular maps $\mathcal{M}(\ell, r_{\delta_1}, t)$ and $\mathcal{M}(\ell, r_{\delta_2}, t)$ are isomorphic. By Lemma 3.1, $\delta_1 = \delta_2$ or $\delta_2(k) = \delta_1(k + \bar{n}) + \bar{n}$ for any $k \in [n]$. If $\delta_1 = \delta_2$ then $x_1 = x_2$. Assume instead that $\delta_2(k) = \delta_1(k + \bar{n}) + \bar{n}$ for any $k \in [n]$. By taking $k = 0$ in the equation $\delta_2(k) = \delta_1(k + \bar{n}) + \bar{n}$, we obtain

$$0 = \delta_2(0) = \delta_1(\bar{n}) + \bar{n} = \bar{n}, x_1(\bar{n}) + \bar{n} = \bar{n} + x_1 + \bar{n} = x_1.$$

Since $x_1^2 \equiv 2 \pmod{n}$, this is impossible. □

The following two lemmas are well known in number theory, so we state them without proof. (See p. 112 and p. 77 of the book [1].)

Lemma 3.8 (Gauss’ lemma). Let $p$ be an odd prime and let $a$ be an integer such that $p \nmid a$. Consider a sequence of integers $a, 2a, 3a, \ldots, (\frac{p-1}{2})a$. Replace each integer in the sequence by the one congruent to it modulo $p$ which lies between $-\frac{p-1}{2}$ and $\frac{p-1}{2}$. Let $v$ be the number of negative integers in the resulting sequence. Then $x^2 \equiv a \pmod{p}$ has a solution if and only if $p \equiv \pm 1 \pmod{8}$.

Corollary 3.9. For any odd prime $p$, $x^2 \equiv 2 \pmod{p}$ has a solution if and only if $p \equiv \pm 1 \pmod{8}$.

Lemma 3.10. Let $p$ be an odd prime and let $a$ be an integer such that $p \nmid a$. Then, for any positive integer $m$, $x^2 \equiv a \pmod{p^m}$ has a solution in $\mathbb{Z}_p$ if and only if $x^2 \equiv a \pmod{p^n}$ has a solution in $\mathbb{Z}_{p^n}$. Moreover they have the same number of solutions, which is $0$ or $2$.

Since $\mathcal{N}_n^{\text{non}} = \emptyset$ for $n \equiv 0 \pmod{4}$, we need to estimate $|\mathcal{N}_n^{\text{non}}|$ only for $n \equiv 2 \pmod{4}$.

Lemma 3.11. For $n = 2p_1^{\nu_1}p_2^{\nu_2} \ldots p_k^{\nu_k}$ (a prime decomposition), the number $|\mathcal{N}_n^{\text{non}}|$ of solutions of $x^2 = 2$ in $\mathbb{Z}_n$ is $2^k$ if $p_i \equiv \pm 1 \pmod{8}$ for all $i = 1, 2, \ldots, k$; $0$ otherwise.

Proof. For $x \in \mathbb{Z}_n$, $x^2 \equiv 2 \pmod{n}$ if and only if $x$ is even and $x^2 \equiv 2 \pmod{p_i^{\nu_i}}$ for all $i = 1, 2, \ldots, k$. Hence, by Corollary 3.9 and Lemma 3.10, if $p_i \equiv \pm 1 \pmod{8}$ for some $i \geq 1$, the cardinality $|\mathcal{N}_n^{\text{non}}|$ is zero. If $p_i \equiv \pm 1 \pmod{8}$ for all $i = 1, 2, \ldots, k$, then $|\mathcal{N}_n^{\text{non}}| = 2^k$ by Corollary 3.9, Lemma 3.10 and the Chinese remainder theorem. □

4. Reduction

In this section, we show that if there exists some $\bar{d} \in \mathcal{M}_n^{\text{non}} - \mathcal{N}_n^{\text{non}}$, then $d = |\langle \bar{d} \rangle| < n$ and there is a proper divisor $\delta_1$ of $\mathcal{N}_n^{\text{non}}$, which we call the reduction of $\bar{d}$. If such $\delta_1$ is also contained in $\mathcal{M}_d^{\text{non}} - \mathcal{N}_d^{\text{non}}$ then one can choose the next reduction $\delta_2$ of $\delta_1$. By continuing such reduction, one can have a non-negative integer $j$ such that $\bar{d}_{i+j} \in \mathcal{M}_{d_j}^{\text{non}} - \mathcal{N}_{d_j}^{\text{non}}$ but its reduction $\delta_j$ is the identity or belongs to $\mathcal{N}_{d_j}^{\text{non}}$. In the next section, we prove that $\mathcal{M}_n^{\text{non}} = \mathcal{N}_n^{\text{non}}$ for any $n$ by showing that such an element $\bar{d}_{i+j}$ does not exist.

Lemma 4.1. Suppose that $\bar{d} \in \mathcal{M}_n^{\text{non}} - \mathcal{N}_n^{\text{non}}$ exists. Then the order of the cyclic group $\langle \bar{d} \rangle$ equals the size of the orbit of 1 under $\langle \bar{d} \rangle$, namely, $|\langle \bar{d} \rangle(1) | = |\{ i | i \in [n] \}|$. Furthermore, this is a proper divisor of $n$.

Proof. For any $k \in [n]$, let $O(k) = \{ \bar{d}_{i+j}(k) | i \in [n] \}$ be the orbit of $k$ under $\langle \bar{d} \rangle$. Let $|O(1)| = d$. Then $d$ is a proper divisor of $n$ because $0 \notin O(1)$. Moreover, we have

$$\bar{d}^d(1) = 1 \quad \text{and} \quad (LR_{\bar{d}}L)^{-1}R_{\bar{d}}(LR_{\bar{d}}L)(0) = 0.$$

Hence the conjugate $(LR_{\bar{d}}L)^{-1}R_{\bar{d}}(LR_{\bar{d}}L)$ of $R_{\bar{d}}$ belongs to the vertex stabilizer $\langle R_{\bar{d}}, L \rangle_{0} = \langle R_{\bar{d}}, t \rangle$, which is isomorphic to a dihedral group $D_n$ of order $2n$. 
Assume that \((LR_n^{-1} L R_n^{-1}) = R_n^m\) for some \(m \in [n]\). Because \(R_n^m\) and \(R_n^d\) are conjugate in \(\langle R_n \rangle\), we have \(R_n^m = R_n^d\) as subgroups of the cyclic group \(\langle R_n \rangle\). Since \(d\) is a divisor of \(n\), there exists \(\ell \in [\frac{n}{d}]\) such that \(m = d\ell\) and \((\ell, \frac{n}{d}) = 1\). Suppose that \(\ell \neq d\). Then there exists \(k \in [n]\) such that \(\delta^d(k) \neq k\). Let \(q\) be the largest such \(k\). Then \(\delta^d(q) \neq q\). On the other hand,

\[
\delta^d(q) = R_n^d(q) = R_n^m(q) = (LR_n^{-1} L R_n^{-1} R_n^d)(LR_n^{-1} L R_n^{-1} R_n^d)(q + 1) = q,
\]

a contradiction. Therefore \(\ell = |O(1)| = d\) is a proper divisor of \(n\).

Next, suppose that \((LR_n^{-1} L R_n^{-1})(LR_n^{-1} L R_n^{-1})(k) \neq R_n^m t\) for some \(m \in [n]\). Since the order of \(R_n^m t\) is 2 and \(d < n\), we find that \(n\) is even and \(d = \bar{n}\). If \(|O(k)|\) divides \(\bar{n}\) for all \(k \in [n]\), then \(\delta\) is a cyclic group of order \(\bar{n}\) and the result follows. Hence we may assume that there is some \(i \in [n]\) such that \(|O(i)|\) does not divide \(\bar{n}\). By comparing two values \((LR_n^{-1} L R_n^{-1})(LR_n^{-1} L R_n^{-1})(k)\) and \(R_n^m t(k)\), we have

\[
\delta^m(k + 1) - 1 = \delta^m(-k), \quad \text{or equivalently,} \quad \delta^m(k + 1) = \delta^m(-k) + 1,
\]

for all \(k \in [n]\). If \(\delta^m(k + 1) = k + 1\) for some \(k \in [n]\), then \(\delta^m(-k) = k\). Since \(\delta^m(j) = j\) for any \(j \in O(1) \cup \{0\}\), there are at least \(\bar{n} + 1\) elements \(k \in [n]\) satisfying \(\delta^m(-k) = k\). Note that there exist at most two \(k\) satisfying \(\delta^m(-k) = k\) in any orbit \(\delta\) when \(|O|\) is even, and at most one when \(|O|\) is odd because \(\delta^{-1}(-k) = -\delta(k)\) for all \(k \in [n]\). So there exist at most 2 elements \(k \in O(1)\) satisfying \(\delta^m(-k) = k\). If there exists an orbit \(\delta\) which is not \(O(1)\) and whose size is greater than or equal to 3, then there exist at least two \(k \in O(1)\) such that \(\delta^m(-k) \neq k\). This implies that there exist at most \(\bar{n}\) elements \(k \in [n]\) satisfying \(\delta^m(-k) = k\), a contradiction. Hence apart from \(O(1)\), the size of every orbit under \(\delta\) is either 1 or 2. By our assumption that there is an orbit under \(\delta\) whose size doesn’t divide \(\bar{n}\), the value \(\bar{n}\) should be odd. This means that there is only one element \(k \in O(1)\) satisfying \(\delta^m(-k) = k\), and implies that for all \(k \in [n] \setminus O(1)\), \(\delta^m(-k) = k\). Hence each orbit under \(\delta\) containing neither 0 nor 1 is \(\{i, -i\}\) for some \(i \in [n]\), and \(m\) is odd. Since \(\delta^{-1}(-1) = -\delta(k)\) for all \(k \in [n]\) and there exists one \(k \in O(1)\) such that \(\delta^m(-k) = k\), it follows that \(-O(1) = \{k \mid k \in O(1)\} = O(1)\). Recall that for any \(k \in [n]\), \(\delta^m(k) = k\) if and only if \(k \in O(1) \cup \{0\}\). For any orbit \(\{i, -i\}\) under \(\delta\), \(\delta^m(i + 1) = \delta^m(-i + 1) = i + 1\) and \(\delta^m(i - 1) = \delta^m(i) + 1 = -i + 1\). This implies that \(i - 1, i + 1, -i - 1, -i + 1 \in O(1)\) because \(i \neq \pm 1\) and \(O(1) = -O(1)\). Hence there exist no two consecutive elements \(i, i + 1 \in [n]\) satisfying \(|O(i)| = |O(i + 1)| = 2\). Note that \(|O(0)| = 1\) and \(|O(1)| = |O(-1)| = \bar{n}\). Since there are \(\bar{n} - 1\) elements \(j \in [n]\) such that \(|O(j)| = 2\), for any even \(2k \in [n]\), we have \(O(2k) = \{2k, -2k\}\) or equivalently, \(\delta(2k) = -2k\). Moreover, \(O(1)\) is composed of all odd numbers. Hence for any even \(2k \in [n]\),

\[
\delta^m(2k + 1) = \delta^m((-2k - 1)) = \delta^m(-2k) = 2k - 1 = 2k - 1.
\]

This implies that \(m\) and \(\bar{n}\) are relative prime. Moreover, \(m\) and \(n\) are relative prime because \(m\) is odd, so there exists \(s \in [n]\) such that \(sm \equiv 1 \pmod{n}\). For any even \(2k \in [n]\),

\[
\delta(2k + 1) = \delta^m(2k + 1) = \delta^{(s-1)m}(2k - 1) = \cdots = 2k + 1 - 2s.
\]

Let \(x = -2s\). Then, \(\delta = (0)(2 - 2)(4 - 4) \cdots (1 + x + 2x + 3x \cdots)\). By Lemma 3.3(3), there exist \(a(\bar{n})\) and \(b(\bar{n})\) satisfying \((P1)\) or \((P2)\) with \(\delta\). Suppose that \(a(\bar{n})\) and \(b(\bar{n})\) satisfy \((P1)\). Then, by Lemma 3.4,

\[
b(\bar{n}) = \delta(1) = 1 \quad \text{and} \quad b(\bar{n}) = \delta^{-\delta}(1) = \delta^{-\delta(1)}(1) \neq 1,
\]

which is a contradiction. Hence, we can assume that \(a(\bar{n})\) and \(b(\bar{n})\) satisfy \((P2)\). By Lemma 3.4,

\[
b(\bar{n}) = -\delta(1) = -1 \quad \text{and} \quad b(\bar{n}) = \delta^{-\delta}(1) = \delta^{-\delta(1)}(1) = 1 - (\bar{n} + x)x = 1 - x^2
\]

because \(x\) is even. This implies that \(x^2 \equiv 2 \pmod{n}\). So, \(\delta \in \mathcal{L}^\text{non}_{n}\), which contradicts the assumption. \(\square\)
Remark. As one can see in the proof of Lemma 4.1, any element in $\mathcal{M}^\text{non}_n$ which does not satisfy the condition in Lemma 4.1 is $\delta_{n,x}$ for some even $n$ and some $x$ such that $x^2 \equiv 2 \pmod{n}$. This is a reason why we define $\delta_{n,x}$ in Section 3.

Proposition 4.2. (See [11].) If $\delta$ is the identity permutation of $[n]$ then $|\langle R_3, L \rangle| = 2n^2$. Furthermore, if we define $\delta : [n] \rightarrow [n]$ by $\delta(k) = k + rd$ for all $k \in [n]$, where $n \geq 3$, $d$ is a divisor of $n$ and $r$ is a positive integer such that the order of $1 + rd$ in the multiplicative group $\mathbb{Z}_n^*$ of units is $d$, then $|\langle R_3, L \rangle| = 2n^2$.

Lemma 4.3. If $n \geq 3$, $|\langle \delta \rangle| \neq 2$ for every $\delta \in \mathcal{M}^\text{non}_n$.

Proof. Suppose that there exists $\delta \in \mathcal{M}^\text{non}_n$ satisfying $|\langle \delta \rangle| = 2$. Then $n$ is even. By Lemma 3.3(3), there exist $a(1), b(1) \in [n]$ satisfying $(P_1)$ or $(P_2)$ with $\delta$. In both cases, $a(1) = \delta(1)$ and $b(1) = \delta^{-1}(1)$ by Lemma 4.4. Suppose $a(1) = \delta(1)$ is even and let $\delta(1) = 2r$. Then $b(1) = 1$ and

$$\delta(k + 1) = \delta^b(1)(k) + a(1) = \delta(k) + 2r \quad \text{for all } k \in [n]$$

or

$$\delta(k + 1) = \delta^b(1)(-k) + a(1) = \delta(-k) + 2r = -\delta(k) + 2r \quad \text{for all } k \in [n].$$

In both cases, one can show inductively that $\delta(k)$ is even for all $k \in [n]$. This is impossible, since $\delta \in \mathcal{S}[n]$. Therefore we can assume that $a(1) = \delta(1)$ is odd. Let $\delta(1) = 1 + 2r$. By Lemma 4.3, $b(1) = \delta^{-1}(1) = \delta(1) = 1 + 2r$.

Suppose that $(P_1)$ holds. Then

$$\delta(k + 1) = \delta^b(1)(k) + a(1) = \delta(k) + 1 + 2r = \delta(k - 1) + 2(1 + 2r) = \cdots = (k + 1)(1 + 2r).$$

Moreover 2 is the smallest positive integer $d$ satisfying $\delta^d(1) = (1 + 2r)^d = 1$. By Proposition 4.2, $|\langle R_3, L \rangle| = 2n^2$, and so $\delta \notin \mathcal{M}^\text{non}_n$, a contradiction.

Now suppose that $(P_2)$ holds. Then $b(1) = -\delta(1)$ and hence $b(1) = \delta(1) = -\delta(1)$. Since $-\delta(1) = -\delta^d(-1) = -\delta(-1)$, we have $\delta(1) = \delta(-1)$. This implies that $n = 2$, a contradiction. $\square$

From Proposition 4.2 and Lemma 4.3, one can see that for any $n \geq 3$ and for every $\delta \in \mathcal{M}^\text{non}_n$, $\delta$ is neither the identity nor an involution.

Lemma 4.4. Suppose that there exists $a \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$ with $n \geq 3$, and let $|\langle \delta \rangle| = d$. If $k_1 \equiv k_2 \pmod{d}$ for some $k_1, k_2 \in [n]$, then $\delta(k_1) \equiv \delta(k_2) \pmod{d}$.

Proof. By Lemma 4.1, $d$ is a proper divisor of $n$. By Lemma 3.3(3), there exist $a(d)$ and $b(d)$ satisfying $(P_1)$ or $(P_2)$. Assume that $(P_1)$ holds. By Lemma 3.4, $b(d) = \delta(d)(1) = 1$, and so $k + 1 = \delta^d(k) + 1 = \delta^a(d)(k + b(d)) = \delta^a(d)(k + 1)$. This implies that $a(d)$ is a multiple of $d$, say $a(d) = rd$. So the first equation in $(P_1)$ is $\delta(k + d) = \delta^b(d)(k) + a(d) = \delta(k + rd)$. Hence if $k_1 \equiv k_2 \pmod{d}$ for some $k_1, k_2 \in [n]$, then $\delta(k_1) \equiv \delta(k_2) \pmod{d}$.

Next, suppose that $(P_2)$ holds. By Lemma 3.4, $b(d) = -\delta(d)(1) = -1$, and so $k + 1 = \delta^d(k) + 1 = \delta^a(d)(-k + b(d)) = \delta^a(d)(-k + 1).$ By taking $k = -2$ and $k = -\delta(1) - 1$ in the equation $\delta^a(d)(-k + 1) = k + 1$, we obtain $\delta^a(d)(1) = -1$ and $\delta^a(d)(1) = -\delta(1)$. Since $\delta^a(d)(1) = \delta^a(d)(1) = -\delta^d(-1) = -\delta^d(-1)$, we have $\delta^d(-1) = \delta(d)(1).$ By Lemma 4.1, $\delta^d = \delta$, or equivalently, $d = 1$ or 2. This is impossible by Proposition 4.2 and Lemma 4.3. $\square$

Suppose that $\delta \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$ with $|\langle \delta \rangle| = d$. By Lemma 4.4, the function $\delta_{(1)} : [d] \rightarrow [d]$ defined by $\delta_{(1)}(k) \equiv \delta(k) \pmod{d}$ for any $k \in [d]$ is well defined. Furthermore $\delta_{(1)}$ is a bijection, namely, a permutation of $[d]$. We call the permutation $\delta_{(1)}$ the (mod $d$)-reduction of $\delta$. In fact, $\delta_{(1)}$ belongs to $\mathcal{M}^\text{non}_d$ as the following lemma shows in a general setting.
Lemma 4.5. Suppose that $\bar{d} \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$ with $|\langle \bar{d} \rangle| = d \geq 3$. Let $m$ be a divisor of $n$ such that

1. $m$ is a multiple of $d$, and
2. if $k_1 \equiv k_2 \pmod{m}$ for some $k_1, k_2 \in [n]$, then $\bar{d}(k_1) \equiv \bar{d}(k_2) \pmod{m}$.

Define $\bar{d}' : [m] \to [m]$ by $\bar{d}'(k) \equiv \bar{d}(k) \pmod{m}$ for any $k \in [m]$. Then $\bar{d}'$ is a well-defined bijection and it belongs to $\mathcal{M}^\text{non}_m$.

Proof. By the assumption that $\bar{d}(k_1) \equiv \bar{d}(k_2) \pmod{m}$ for any $k_1, k_2 \in [n]$ satisfying $k_1 \equiv k_2 \pmod{m}$, $\bar{d}' : [m] \to [m]$ is well-defined. Since $\bar{d}$ is a bijection, so is $\bar{d}'$. By the fact $\bar{d}(0) = 0$, we have $\bar{d}'(0) = 0$.

Now, we show that $\bar{d}' \in \mathcal{M}^\text{non}_n$ using Lemma 3.3(3). For any $k \in [n]$, let $k'$ denote the remainder of $k$ on division by $m$. By Lemma 3.3(3), for any $i \in [n]$ there exist $a(i)$ and $b(i)$ satisfying $(P_1)$ or $(P_2)$ with $\bar{d}$. If we define $a(i') = a(i)'$ and $b(i') = b(i)'$ then $a(i')$ and $b(i')$ also satisfy $(P_1)$ or $(P_2)$ depending on whether $a(i)$ and $b(i)$ satisfy $(P_1)$ or $(P_2)$. Since $\bar{d} \in \mathcal{M}^\text{non}_n$, there exists at least one $j \in [n]$ such that $\bar{d}(k + j) = \bar{d}^b(j)(-k) + a(j)$ and $\bar{d}^j(k) + 1 = \bar{d}^a(j)(-k + b(j))$ for all $k \in [n]$, by Lemma 3.3(3). This implies that $\bar{d}'(k' + j') = \bar{d}^b(j')(\bar{k}' - k') + a(j')$ (mod $m$) and $\bar{d}^j(k') + 1 \equiv \bar{d}^a(j')(\bar{k}' - k' + b(j'))$ (mod $m$) for all $k' \in [m]$. So by Lemma 3.3, $\bar{d}' \in \mathcal{M}^\text{non}_m$. $
$
Corollary 4.6. Suppose that $\bar{d} \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$ with $|\langle \bar{d} \rangle| = d \geq 3$. Then $\bar{d}(1)$ belongs to $\mathcal{M}^\text{non}_d$.

Proof. By Lemmas 4.4 and 4.5, the (mod $d$)-reduction $\bar{d}(1)$ of $\bar{d}$ belongs to $\mathcal{M}^\text{non}_d$. $
$
5. Proof of Theorem 1.1

To prove Theorem 1.1, we need to show that for any integer $n \equiv 0, 1$ or $3 \pmod{4}$, no nonorientable regular embedding of $K_{n,n}$ exists, while for $n \equiv 2 \pmod{4}$, $\mathcal{M}^\text{non}_n = \mathcal{N}^\text{non}_n$.

For a nonnegative integer $k$, we define $\bar{d}(0) = \bar{d} \in S_n$ and $\bar{d}(k+1) = (\bar{d}(k))_{(1)}$ by taking inductive reduction.

Lemma 5.1. Suppose that $\bar{d} \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$ with $n \geq 3$. Then

1. $\bar{d}(1)$ is not the identity, and
2. $|\langle \bar{d} \rangle|$ is even.

Proof. Suppose that there exists a $\bar{d} \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$. Let $|\langle \bar{d} \rangle| = d$. By Proposition 4.2 and Lemma 4.3, we find that $d \geq 3$. This implies that $\bar{d}(1)$ belongs to $\mathcal{M}^\text{non}_d$ by Corollary 4.6. Hence $\bar{d}(1)$ is not the identity by Proposition 4.2.

Suppose that $|\langle \bar{d} \rangle| = d$ is odd. By Lemma 4.1, $d$ is less than $n$. Since $\mathcal{N}^\text{non}_n = \emptyset$ with any odd $n$, $\bar{d}(1)$ is an element in $\mathcal{M}^\text{non}_d - \mathcal{N}^\text{non}_d$ by Corollary 4.6, and the order of $\bar{d}(1)$ is also odd. By continuing the same process, we obtain $j \geq 1$ and $d_j \geq 3$ such that $\bar{d}(j) \in \mathcal{M}^\text{non}_{d_j} - \mathcal{N}^\text{non}_{d_j}$, and $\bar{d}(j+1)$ is the identity permutation on $[d_{j+1}]$, where $d_{j+1} = |\langle \bar{d}(j) \rangle|$ and $d_j = |\langle \bar{d}(j-1) \rangle| \geq 3$. But such $\bar{d}(j)$ cannot exist by (1). $
$
Corollary 5.2. If $n$ is odd, then $\mathcal{M}^\text{non}_n = \emptyset$, or equivalently, there is no nonorientable regular embedding of $K_{n,n}$.

Proof. Suppose that $\bar{d} \in \mathcal{M}^\text{non}_n$ exists. Since $\mathcal{N}^\text{non}_n = \emptyset$ for odd $n$, $\bar{d}$ belongs to $\mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$. By Lemma 4.1, the order $|\langle \bar{d} \rangle|$ is a divisor of $n$. Hence $|\langle \bar{d} \rangle|$ is odd, which is a contradiction by Lemma 5.1. $
$
Lemma 5.3. There is no $\bar{d} \in \mathcal{M}^\text{non}_n - \mathcal{N}^\text{non}_n$ with $|\langle \bar{d} \rangle| = d \geq 3$ such that $\bar{d}(1) \in \mathcal{M}^\text{non}_d$. 

Proof. Suppose that there exists an element \( \tilde{\delta} \in \mathcal{M}_d^{\text{non}} - \mathcal{N}_d^{\text{non}} \) of order \( d \geq 3 \) such that \( \tilde{\delta}(1) \in \mathcal{N}_d^{\text{non}} \).

Note that \( d \equiv 2 \pmod{4} \). We consider two cases separately.

**Case 1.** \( n \equiv 2 \pmod{4} \).

Here \( \tilde{n} \) is an odd integer. Let \( O \) be the orbit of \( \tilde{n} \) under \( \langle \tilde{\delta} \rangle \). Then the size \( |O| \) is a divisor of \( d = |\langle \tilde{\delta} \rangle| \). Furthermore, \( |O| \) is a multiple of \( d/2 \) because all odd numbers in \([d]\) are in the same orbit under \( \langle \tilde{\delta} \rangle \), whose size is \( d/2 \). So \( |O| \) is \( d/2 \) or \( d \). Since \( -\tilde{n} = \tilde{n} \) and \( \tilde{\delta}^{-1}(-k) = -\tilde{\delta}(k) \) for any \( k \in [n] \), one can see that \( -O = O \) and the size \( |O| \) is odd, which implies \( |O| = d/2 \). Since all odd numbers in \([d]\) are in the same orbit under \( \langle \tilde{\delta}(1) \rangle \), there exists a number \( 1 + jd \in [n] \) such that \( 1 + jd \in O \). This implies that \( \tilde{\delta}^{d/2}(1 + jd) = 1 + jd \) and hence \( (LR_{\tilde{\delta}}^{1+jd}L)^{-1}R_{\tilde{\delta}}^{d/2}(LR_{\tilde{\delta}}^{1+jd}L)(0) = 0 \).

So \( (LR_{\tilde{\delta}}^{1+jd}L)^{-1}R_{\tilde{\delta}}^{d/2}(LR_{\tilde{\delta}}^{1+jd}L) \) belongs to the vertex stabilizer \( \langle L_{\tilde{\delta}}R, L \rangle_0 = \langle L_{\tilde{\delta}}, L \rangle \), which is isomorphic to the dihedral group \( D_n \) of order \( 2n \). Since the order of \( (LR_{\tilde{\delta}}^{1+jd}L)^{-1}R_{\tilde{\delta}}^{d/2}(LR_{\tilde{\delta}}^{1+jd}L) \) is not 2, \( (LR_{\tilde{\delta}}^{1+jd}L)^{-1}R_{\tilde{\delta}}^{d/2}(LR_{\tilde{\delta}}^{1+jd}L) = R_{\tilde{\delta}}^m \) for some \( m \in [n] \). Because \( R_{\tilde{\delta}}^m \) and \( R_{\tilde{\delta}}^{d/2} \) are conjugate in \( \langle L_{\tilde{\delta}}, L \rangle \), they have the same order, and consequently, \( \langle R_{\tilde{\delta}}^m \rangle = \langle R_{\tilde{\delta}}^{d/2} \rangle \) as subgroups of the cyclic group \( \langle L_{\tilde{\delta}} \rangle \). Since \( d/2 \) is a divisor of \( n \), there exists \( \ell \in [n, d/2] \) such that \( m = \ell d/2 \) and \( (\ell, n, d/2) = 1 \). By considering two images of \( 1 + jd \) under the permutations \( (LR_{\tilde{\delta}}^{1+jd}L)^{-1}R_{\tilde{\delta}}^{d/2}(LR_{\tilde{\delta}}^{1+jd}L) \) and \( R_{\tilde{\delta}}^{d/2} \), we have

\[
\tilde{\delta}^{d/2}(2 + 2jd) - 1 - jd = (LR_{\tilde{\delta}}^{1+jd}L)^{-1}R_{\tilde{\delta}}^{d/2}(LR_{\tilde{\delta}}^{1+jd}L)(1 + jd) = R_{\tilde{\delta}}^{d/2}(1 + jd) = 1 + jd.
\]

This implies that \( \tilde{\delta}^{d/2}(2 + 2jd) = 2 + 2jd \). Since \( d/2 \) is odd and for any even \( k \in [d] \) with \( k \neq 0 \), the orbit of \( k \) under \( \langle \tilde{\delta}(1) \rangle \) is \([k, d - k]\), the even number \( 2 + 2jd \) should be a multiple of \( d \). This means that \( d \) is 1 or 2, a contradiction. Therefore for any \( n \equiv 2 \pmod{4} \), \( \delta \in \mathcal{M}_n^{\text{non}} - \mathcal{N}_n^{\text{non}} \) exists with \( |\langle \delta \rangle| = d \geq 3 \) such that \( \tilde{\delta}(1) \in \mathcal{N}_d^{\text{non}} \).

**Case 2.** \( n \equiv 0 \pmod{4} \).

Let \( n = 2sd \) for some even integer \( 2s \). Here \( \tilde{n} = sd \) is even. By Lemma 3.3(3), there exist \( a(\tilde{n}) \) and \( b(\tilde{n}) \) satisfying \((P_1)\) or \((P_2)\). In both cases, \( a(\tilde{n}) = \tilde{\delta}(\tilde{n}) \) by Lemma 3.4. Since \( \tilde{n} \) is a multiple of \( d \), \( a(\tilde{n}) \) is also a multiple of \( d \) by Lemma 4.4.

Suppose that \((P_2)\) holds. Then

\[
k + 1 = \tilde{\delta}(k + 1) = \tilde{\delta}(a(\tilde{n}))(k + b(\tilde{n})) = -k + b(\tilde{n})
\]

for all \( k \in [n] \). This means that \( b(\tilde{n}) = 2k + 1 \) for all \( k \in [n] \). Since \( b(\tilde{n}) \) is a constant, \( n \leq 2 \), a contradiction. So \((P_1)\) holds, that is, \( \delta(k + \tilde{n}) = \delta(b(\tilde{n})(k) + a(\tilde{n})) \) and \( k + 1 = \tilde{\delta}(k + 1) = \tilde{\delta}(a(\tilde{n})(k + b(\tilde{n}))) = k + b(\tilde{n}) \) for all \( k \in [n] \). This means that \( b(\tilde{n}) = 1 \), and hence \( \delta(k + \tilde{n}) = \delta(b(\tilde{n})(k) + a(\tilde{n})) = \delta(k) + \tilde{\delta}(\tilde{n}) \).

By taking \( k = \tilde{n} \) in the above equation, we have \( 2\tilde{\delta}(\tilde{n}) = 0 \). Since \( \tilde{\delta}(\tilde{n}) \neq 0 \), \( \tilde{\delta}(\tilde{n}) = \tilde{n} \) and

\[
\tilde{\delta}(k + \tilde{n}) = \tilde{\delta}(k) + \tilde{\delta}(\tilde{n}) = \tilde{\delta}(k) + \tilde{n}.
\]

This implies that if \( k_1 \equiv k_2 \pmod{\tilde{n}} \), then \( \tilde{\delta}(k_1) \equiv \tilde{\delta}(k_2) \pmod{\tilde{n}} \). Let \( \tilde{\delta}' : [\tilde{n}] \to [\tilde{n}] \) be defined by \( \tilde{\delta}'(k) \equiv \tilde{\delta}(k) \pmod{\tilde{n}} \) for any \( k \in [\tilde{n}] \). By Lemma 4.5, \( \tilde{\delta}' \) is well defined and belongs to \( \mathcal{M}_{\tilde{n}}^{\text{non}} \) because \( \tilde{n} \) is a multiple of \( d \). Note that the size of the orbit of 1 under \( \langle \tilde{\delta}' \rangle \) is \( d/2 \) or \( d \).

**Subcase 2.1.** The size of the orbit of 1 under \( \langle \tilde{\delta}' \rangle \) is \( d/2 \).

Let \( d' = |\langle \tilde{\delta}' \rangle| \). Then \( d' = d/2 \) or \( d \). Since \( d \) is a divisor of \( \tilde{n} \) and the orbit of 2 under \( \langle \tilde{\delta}(1) \rangle \) is \([2, d - 2]\), the size of the orbit of 2 under \( \langle \tilde{\delta}' \rangle \) is even. Hence \( d' \) is even and consequently equals \( d \). Since the order of \( \tilde{\delta}' \) is not equal to the size of the orbit of 1 under \( \langle \tilde{\delta}' \rangle \), \( \tilde{\delta}' \in \mathcal{N}_{\tilde{n}}^{\text{non}} \) by Lemma 4.1. Hence, \( \tilde{\delta}' = \tilde{\delta}'_{x, x} \) for some \( x \in [\tilde{n}] \) satisfying \( x^2 \equiv 2 \pmod{\tilde{n}} \). Moreover, \( d' = d = \tilde{n} \). This implies that
\[\tilde{d}' = \tilde{d}(1)\] and all odd numbers in \([n]\) belong to the same orbit under \(\langle \tilde{d} \rangle\). Furthermore, for any even number \(2k \in [n] \setminus \{0, \bar{n}\}\), the size of the orbit of \(2k\) under \(\langle \tilde{d} \rangle\) is 2 or 4. Since this is a divisor of \(d\) and \(d \equiv 2 \pmod{4}\), it is 2. Note that the orbit of \(2k\) under \(\langle \tilde{d} \rangle\) is \(\{2k, -2k\}\) or \(\{2k, \bar{n} - 2k\}\).

First, we want to show that \(\tilde{d}(2k) = -2k\) for all even \(2k \in [n]\). By Lemma 3.3(3), there exist \(a(2)\) and \(b(2)\) satisfying \((P_1)\) or \((P_2)\). In both cases, \(a(2) = \tilde{d}(2)\) and \(b(2) = \tilde{d}^{-a(2)}(1) = \tilde{d}^{-\tilde{d}(2)(1)} = \tilde{d}^2(1)\) by Lemma 3.4.

Suppose that \((P_2)\) holds. By Lemma 3.4, \(b(2) = -\tilde{d}^2(1)\). Hence \(b(2) = -\tilde{d}^2(1) = \tilde{d}^2(1)\), which implies \(2\tilde{d}^2(1) = 0\). Since \(\tilde{d}^2(1)\) is not \(0\), \(\tilde{d}^2(1) = \bar{n}\). This contradicts the fact that the orbit of 1 under \(\langle \tilde{d} \rangle\) is composed of all odd numbers in \([n]\). So \((P_1)\) holds. By Lemma 3.4, \(b(2) = \tilde{d}^2(1) \equiv 1 + 2x \pmod{\bar{n}}\), and so the first equation in \((P_1)\) can be written as \(\tilde{d}(k + 2) = \tilde{d}(b(2)) + a(2) = \tilde{d}^{1+2x}(k) + \tilde{d}(2)\). Suppose that \(\tilde{d}(2) = \bar{n} - 2\). Then

\[\tilde{d}(k + 2) = \tilde{d}^{1+2x}(k) + \bar{n} - 2.\]

Taking \(k = 2\) in the equation \(\tilde{d}(k + 2) = \tilde{d}^{1+2x}(k) + \bar{n} - 2\), we have \(\tilde{d}(4) = \tilde{d}^{1+2x}(2) + \bar{n} - 2 = \tilde{d}(2) + \bar{n} - 2 = -4\). Taking \(k = 4\), we have \(\tilde{d}(6) = \tilde{d}(4) + \bar{n} - 2 = \bar{n} - 6\). By continuing the same process, one can see that \(\tilde{d}(4k) = -4k\) and \(\tilde{d}(4k + 2) = \bar{n} - 4k - 2\). Since \(\bar{n} \equiv 2 \pmod{4}\), we have \(\tilde{d}(\bar{n}) = \bar{n} - \bar{n} = 0\), which is a contradiction. Hence \(\tilde{d}(2) = -2\), so it follows that

\[\tilde{d}(k + 2) = \tilde{d}^{1+2x}(k) - 2.\]

Taking \(k = 2\) in the equation \(\tilde{d}(k + 2) = \tilde{d}^{1+2x}(k) - 2\), we have \(\tilde{d}(4) = \tilde{d}^{1+2x}(2) - 2 = \tilde{d}(2) - 2 = -4\). Taking \(k = 4\), we have \(\tilde{d}(6) = \tilde{d}(4) - 2 = -6\). By continuing the same process, one can see that \(\tilde{d}(2k) = -2k\) for all even \(2k \in [n]\).

Now we aim to apply Lemma 3.3(3) once more to show that Subcase 2.1 cannot occur. There exist \(a(1)\) and \(b(1)\) satisfying \((P_1)\) or \((P_2)\). In both cases, \(a(1) = \tilde{d}(1) \equiv 1 + x \pmod{\bar{n}}\). For our convenience, let \(\tilde{d}(1) = 1 + x_1\).

Suppose that \((P_1)\) holds. Then \(b(1) = \tilde{d}(1) = 1 + x_1\), and so

\[\tilde{d}(k + 1) = \tilde{d}^{b(1)}(k) + a(1) = \tilde{d}^{b(1)}(k) + \tilde{d}(1) = \tilde{d}^{1+x_1}(k) + 1 + x_1.\]

By taking \(k = 2\), we have \(\tilde{d}(3) = \tilde{d}^{1+x_1}(2) + 1 + x_1 = -2 + 1 + x_1 \equiv x - 1 \pmod{\bar{n}}\). Since \(\tilde{d}(3) \equiv 3 + x \pmod{\bar{n}}\), we have \(4 \equiv 0 \pmod{\bar{n}}\). By the assumption that \(\bar{n} \equiv 2 \pmod{4}\), we have \(\bar{n} = d = 2\), which contradicts the assumption that \(d \geq 3\). So \((P_2)\) holds. Hence \(b(1) = -\tilde{d}(1) = -1 - x_1\) and it follows that

\[\tilde{d}(k) + 1 = \tilde{d}^{a(1)}(-k + b(1)) = \tilde{d}^{1+x_1}(-k - 1 - x_1)\]

for all \(k \in [n]\). By taking odd \(2k + 1 \in [n]\), we have \(\tilde{d}(2k + 1) + 1 = \tilde{d}^{1+x_1}(-2k + 2 - x_1) = 2k + 2 + x_1\). Hence \(\tilde{d}(2k + 1) = 2k + 1 + x_1\). By taking \(k = 2\) in the equation \(\tilde{d}(k) + 1 = \tilde{d}^{1+x_1}(-k - 1 - x_1)\), we obtain

\[-1 = \tilde{d}(2) + 1 = \tilde{d}^{1+x_1}(-3 - x_1) = -3 - x_1 + (1 + x_1)x_1 = -3 + x_1^2.\]

So \(x_1^2 = 2 \pmod{n}\). This is impossible because \(n \equiv 0 \pmod{4}\). Hence Subcase 2.1 cannot occur.

**Subcase 2.2.** The size of the orbit of 1 under \(\langle \tilde{d}' \rangle\) is \(d\).

Since the order of \(\tilde{d}'\) divides that of \(\tilde{d}\), the order of \(\tilde{d}'\) is \(d\), which equals the size of the orbit of 1 under \(\langle \tilde{d}' \rangle\). This implies that \(\tilde{d}' \in \mathcal{M}_n^{\text{non}} - \mathcal{N}_n^{\text{non}}\). Moreover, since \(d\) divides \(\bar{n}\), it follows that \(\tilde{d}'(1) = \tilde{d}(1) \in \mathcal{N}_n^{\text{non}}\). Since Subcase 2.1 cannot occur, by repeating the same process continually, we obtain \(n_1 \equiv 2 \pmod{4}\) and \(\tilde{d} \in \mathcal{M}_{n_1}^{\text{non}} - \mathcal{N}_{n_1}^{\text{non}}\) with \(|\langle \tilde{d} \rangle| = d = |\langle \tilde{d}' \rangle|\) such that \(\tilde{d}(1) = \tilde{d}'(1) \in \mathcal{N}_{n_1}^{\text{non}}\). But this brings us back to Case 1. \(\square\)

Now we can prove Theorem 1.1. We know that there exists only one nonorientable regular embedding of \(K_{2,2}\) into the projective plane, so let \(n \geq 3\).
Suppose that $\mathcal{M}_n^{\text{non}} \supseteq \mathcal{N}_n^{\text{non}}$ and let $\tilde{\delta} \in \mathcal{M}_n^{\text{non}} - \mathcal{N}_n^{\text{non}}$ and $|\langle \tilde{\delta} \rangle| = d$. Note that $d < n$. By Lemma 5.1 and Lemma 4.3, $\tilde{\delta}(1)$ is not the identity, and $d \geq 3$ is even. By Lemmas 4.6 and 5.3, $\tilde{\delta}(1) \in \mathcal{M}_d^{\text{non}} - \mathcal{N}_d^{\text{non}}$. By continuing the same process, we obtain $j \geq 1$ and $d_j \geq 3$ such that $\tilde{\delta}(j) \in \mathcal{M}_{d_j}^{\text{non}} - \mathcal{N}_{d_j}^{\text{non}}$, and $\tilde{\delta}(j+1)$ is the identity permutation on $[d_{j+1}]$, where $d_{j+1} = |\langle \tilde{\delta}(j) \rangle|$ and $d_j = |\langle \tilde{\delta}(j-1) \rangle| \geq 3$. But this is impossible by Lemma 5.1. Hence $\mathcal{M}_n^{\text{non}} = \mathcal{N}_n^{\text{non}}$ for every $n \geq 3$. This means there is no nonorientable regular embedding of $K_{n,n}$ for any $n$ congruent to 0, 1 or 3 mod 4. Finally, for $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ (the prime decomposition of $n$), up to isomorphism the number of nonorientable regular embeddings of $K_{n,n}$ is $2^k$ if every $p_i$ is congruent to $\pm 1$ mod 8; 0 otherwise by Lemma 3.11.

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References