View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Topology and its Applications 156 (2009) 911–914

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Reasonable non-Radon–Nikodym ideals

Vladimir Kanovei *,[1], Vassily Lyubetsky [2]

*Russian Academy of Sciences, Institute for Information Transmission Problems, Moscow, Russian Federation*

A R T I C L E  I N F O

*Article history:*
Received 8 August 2008
Received in revised form 29 September 2008
Accepted 11 November 2008

*Keywords:*
Non-Radon–Nikodym ideals

A B S T R A C T

Following a research line suggested by Ilijas Farah, we prove that for any abelian Polish $\sigma$-compact group $\mathbb{H}$ there exists an $\mathbf{F}_\sigma$ Radon–Nikodym ideal, that is, an ideal $\mathscr{Z} \subseteq \mathscr{P}(\mathbb{N})$ together with a Borel $\mathscr{Z}$-approximate homomorphism $f:\mathbb{H} \to \mathbb{H}^{\mathbb{N}}$ which is not $\mathscr{Z}$-approximable by a continuous true homomorphism $g:\mathbb{H} \to \mathbb{H}^{\mathbb{N}}$.

© 2008 Elsevier B.V. All rights reserved.

## 0. Introduction

The main goal of this article is to generalize the important concept of Radon–Nikodym ideal introduced by Ilijas Farah and prove that for uncountable abelian Polish groups there always exists an analytic non-Radon–Nikodym ideal; in case of a $\sigma$-compact group the ideal is even $\mathbf{F}_\sigma$.

Let $G$, $H$ be abelian Polish groups, and $\mathscr{Z}$ be an ideal over a countable set $A$. We consider $H^A$ as a product group. For $s, t \in H^A$ put

$$\Delta_{s,t} = \{a \in A:\ s(a) \neq t(a)\}.$$

Suppose that $\mathscr{Z}$ is an ideal over $A$. A map $f:G \to H^A$ is a *$\mathscr{Z}$-approximate homomorphism* iff $\Delta_{f(x)+f(y),f(x+y)} \in \mathscr{Z}$ for all $x, y \in G$. Thus it is required that the set of all $a \in A$ such that $f_a(x) + f_a(y) \neq f_a(x+y)$ belongs to $\mathscr{Z}$. Here $f_a:G \to H$ is the $a$th co-ordinate map of the map $f:G \to H^A$.

And $\mathscr{Z}$ is a *Radon–Nikodym* ideal (for this pair of groups) iff for any measurable $\mathscr{Z}$-approximate homomorphism $f:G \to H^{\mathbb{N}}$ there is a continuous exact homomorphism $g:G \to H^{\mathbb{N}}$ which $\mathscr{Z}$-approximates $f$ in the sense that $\Delta_{f(x),g(x)} \in \mathscr{Z}$ for all $x \in G$. Here the measurability condition can be understood as Baire measurability, or, if $G$ is equipped with a $\sigma$-additive Borel measure, as measurability with respect to that measure.

The idea of this (somewhat loose) concept is quite clear: the Radon–Nikodym ideals are those which allow us to approximate non-exact homomorphisms by true ones. This type of problems appears in different domains of mathematics. Closer to the context of this note, Veličković [7] proved that any Baire-measurable FIN-approximate Boolean-algebra automorphism $f$ of $\mathscr{P}(\mathbb{N})$ (so that the symmetric differences between $f(x) \cup f(y)$ and $f(x \cup y)$ and between $f(\mathbb{N} \smallsetminus x)$ and $\mathbb{N} \smallsetminus f(x)$ are finite for all $x, y \subseteq \mathbb{N}$) is FIN-approximable by a true automorphism $g$ induced by a bijection between two cofinite subsets of $\mathbb{N}$. Kanovei and Reeken proved in [3] that any Baire-measurable $\mathbb{Q}$-approximate homomorphism $f:\mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$-approximable by a homomorphism of the form $f(x) = cx$, $c$ being a real constant. ($\mathbb{Q}$ is the additive group of rational numbers.) See also some results in [1,4,5].

* Corresponding author.
*E-mail address:* kanovei@rambler.ru (V. Kanovei).

[1] Partially supported by RFFI grants 06-01-00608 and 07-01-00445.
[2] Partially supported by RFFI grant 07-01-00445.

0166-8641/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2008.11.008

The term "Radon–Nikodym ideal" was introduced by Farah [1,2] in the context of Baire-measurable Boolean algebra homomorphisms of $\mathscr{P}(\mathbb{N})$. Many known Borel ideals were demonstrated to be Radon–Nikodym, see [1,2,4,5]. Suitable counterexamples, again in the context of Boolean algebra homomorphisms, were defined by Farah on the base of so-called pathological submeasures. A different and, perhaps, more transparent counterexample, related to homomorphisms $\mathbb{T} \to \mathbb{T}^{\mathbb{N}}$ (where $\mathbb{T} = \mathbb{R}/\mathbb{N}$), is defined in [5] as a modification of an ideal introduced in [6]. The next theorem generalizes this result.

**Theorem 1.** *Suppose that $\mathbb{H}$ is an uncountable abelian Polish group. Then there is an analytic ideal $\mathscr{Z}$ over $\mathbb{N}$ that is not a Radon–Nikodym ideal for maps $\mathbb{H} \to \mathbb{H}^{\mathbb{N}}$ in the sense that there is a Borel and $\mathscr{Z}$-approximate homomorphism $f : \mathbb{H} \to \mathbb{H}^{\mathbb{N}}$ not $\mathscr{Z}$-approximable by a continuous homomorphism $g : \mathbb{H} \to \mathbb{H}^{\mathbb{N}}$. If moreover $\mathbb{H}$ is $\sigma$-compact then $\mathscr{Z}$ can be chosen to be $\mathbf{F}_\sigma$.*

Note that the theorem will not become stronger if we require $g$ to be only Baire-measurable, or just measurable with respect to a certain Borel measure on $\mathbb{H}$—because by the Pettis theorem any such a measurable group homomorphism must be continuous.

The remainder of the note contains the proof of Theorem 1. It would be interesting to prove the theorem for non-abelian Polish groups. (The assumption that $\mathbb{H}$ is abelian is used in the proof of Lemma 7.) And it will be interesting to find non-Radon–Nikodym ideals for homomorphisms $G \to H^{\mathbb{N}}$ in the case when the Polish groups $G$ and $H$ are not necessarily equal.

## 1. Countable subgroup

Let us fix a group $\mathbb{H}$ as in the theorem, that is, an uncountable abelian Polish group. By $\mathbb{0}$ we denote the neutral element, by $\oplus$ the group operation, by $d$ a compatible complete separable distance (and we do not assume it to be invariant). The first step is to choose a certain countable subgroup $D \subseteq \mathbb{H}$ of "rational elements".

It is quite clear that there exists a countable dense subgroup $D \subseteq \mathbb{H}$ satisfying the following requirement of elementary equivalence type.

(∗) Suppose that $n \geqslant 1$, $c_1, \dots, c_n \in D$, $\varepsilon$ is a positive rational, $U_i = \{x \in \mathbb{H}: d(x, c_i) \leqslant \varepsilon\}$, and $P(x_1, \dots, x_n)$ is a finite system of linear equations with integer coefficients, unknowns $x_1, \dots, x_n$, and constants in $D$, of the form:

$$b_1 x_1 \oplus \cdots \oplus b_n x_n = r, \quad \text{where } b_i \in \mathbb{Z} \text{ and } r \in D.$$

Suppose also that this system $P$ has a solution $\langle x_1, \dots, x_n\rangle$ in $\mathbb{H}$ such that $x_i \in U_i$ for all $i$. Then $P$ has a solution in $D$ as well. (That is, all $x_i$ belong to $D \cap U_i$.)

Let us fix such a subgroup $D$.

## 2. The index set

Let *rational ball* mean any subset of $\mathbb{H}$ of the form $\{x \in \mathbb{H}: d(c, x) < \varepsilon\}$, where $c \in D$ (the center), and $\varepsilon$ is a positive rational number.

**Definition 2.** Let $A$, the index set, consist of all objects $a$ of the following kind. Each $a \in A$ consists of:

- an open non-empty set $U^a \subsetneqq \mathbb{H}$,
- a partition $U^a = U_1^a \cup \cdots \cup U_n^a$ of $U^a$ onto a finite number $n = n^a$ of pairwise disjoint non-empty rational balls $U_i^a \subseteq \mathbb{H}$, and
- a set of points $r_i^a \in U_i^a \cap D$ such that, for all $i, j = 1, 2, \dots, n$:
  (1) *either* $r_i^a \oplus r_j^a$ is $r_k^a$ for some $k$, and $(U_i^a \oplus U_j^a) \cap U^a \subseteq U_k^a$,
  (2) *or* $(U_i^a \oplus U_j^a) \cap U^a = \emptyset$.

Under the conditions of Definition 2, if $\mathbb{0} \in U_i^a$ then $s_i = \mathbb{0}$: for take $j = i$.

**Lemma 3.** *$A$ is an infinite* (*countable*) *set.*

**Proof.** For any $\varepsilon > 0$ there is $a \in A$ such that $U^a$ a set of diameter $\leqslant \varepsilon$: just take $n^a = 1$, $r_1^a = \mathbb{0}$, and let $U^a = U_1^a$ be the $\frac{\varepsilon}{2}$-nbhd of $\mathbb{0}$ in $\mathbb{H}$. □

The next lemma will be used below.

**Lemma 4.** *If $y_1, \dots, y_n \in \mathbb{H}$ are pairwise distinct then there exists $a \in A$ such that $n^a = n$ and $y_i \in U_i^a$ for all $i = 1, \dots, n$.*

**Proof.** As the operation is continuous, we can pick pairwise disjoint rational balls $B_1, \dots, B_n$ such that $y_i \in B_i$ for all $i$ and the following holds: if $1 \leqslant i, j \leqslant n$ then either there exists $k$ such that $(B_i \oplus B_j) \cap B \subseteq B_k$, where $B = B_1 \cup \cdots \cup B_n$, or just $(B_i \oplus B_j) \cap B = \emptyset$. Put $U_i^a = B_i$.

To obtain a system of points $r_i^a$ required, let $P(x_1, \dots, x_n)$ be the system of all equations of the form $x_i + x_j = x_k$ with unknowns $x_i$, $x_j$, $x_k$, where $1 \leqslant i, j, k \leqslant n$ and in reality $y_i + y_j = y_k$. It follows from the choice of $D$ that this system has a solution $\langle r_1, \dots, r_n \rangle$ such that $r_i \in U_i^a \cap D$ for all $i$. In other words we have: $r_i + r_j = r_k$ whenever $y_i + y_j = y_k$. Let $r_i^a = r_i$. This ends the definition of $a \in A$ as required. (An extra care to guarantee that $U^a = \bigcup_{1 \leqslant i \leqslant n} U_i^a$ is a proper subset of $\mathbb{H}$ is left to the reader.) □

## 3. The ideal

Let $\mathscr{X}$ be the set of all sets $X \subseteq A$ such that there is a finite set $u \subseteq \mathbb{H}$ satisfying the following: for any $a \in X$ we have $u \nsubseteq U^a$.

The idea of this ideal goes back to Solecki [6], where a certain ideal over the set $\Omega$ of all clopen sets $U \subseteq 2^{\mathbb{N}}$ of measure $\frac{1}{2}$ (also a countable set) is considered. In our case the index set $A$ is somewhat more complicated.

**Lemma 5.** *$\mathscr{X}$ is an ideal containing all finite sets $X \subseteq A$, but $A \notin \mathscr{X}$.*

**Proof.** If $a \in A$ then the singleton $\{a\}$ belongs to $\mathscr{X}$. Indeed by definition $U^a$ is a non-empty subset of $\mathbb{H}$. Therefore there is a point $x \in \mathbb{H} \smallsetminus U^a$. Then $u = \{x\}$ witnesses $A \in \mathscr{X}$. To see that $\mathscr{X}$ is closed under finite unions, suppose that finite sets $u, v \subseteq \mathbb{H}$ witness that respectively $X$, $Y$ belong to $\mathscr{X}$. Then $w = u \cup v$ obviously witnesses that $Z = X \cup Y \in \mathscr{X}$. Finally by Lemma 4 for any finite $u = \{x_1, \dots, x_n\} \subseteq \mathbb{H}$ there is an element $a \in A$ such that $u \subseteq U^a$. This implies that $A$ itself does not belong to $\mathscr{X}$. □

**Proposition 6.** *$\mathscr{X}$ is an analytic ideal. If $\mathbb{H}$ is $\sigma$-compact then $\mathscr{X}$ is $\mathbf{F}_\sigma$.*

**Proof.** We claim that $X \in \mathscr{X}$ iff there are a natural $n$ and a partition $X = \bigcup_{1 \leqslant k \leqslant n} X_k$ such that for any $k$ the set $X_k \subseteq A$ satisfies $\bigcup_{a \in X_k} U^a \neq \mathbb{H}$. Indeed suppose that $X \in \mathscr{X}$ and this is witnessed by a finite set $u = \{x_1, \dots, x_n\} \subseteq \mathbb{H}$, that is, $u \nsubseteq U^a$ for all $a \in X$. It follows that $X = \bigcup_{1 \leqslant k \leqslant n} X_k$, where $X_k = \{a \in X\colon\ x_k \notin U^a\}$. Clearly $x_k \notin \bigcup_{a \in X_k} U^a$. To prove the converse suppose that $X = \bigcup_{1 \leqslant k \leqslant n} X_k \subseteq A$ and $\bigcup_{a \in X_k} U^a \neq \mathbb{H}$ for all $k$. Let us pick arbitrary points $x_k \in \mathbb{H} \smallsetminus \bigcup_{a \in X_k} U^a$ for all $k$. Then $u = \{x_1, \dots, x_n\}$ witnesses $X \in \mathscr{X}$, as required.

It easily follows that $\mathscr{X}$ is analytic.

Now suppose that $\mathbb{H} = \bigcup_{\ell \in \mathbb{N}} H_\ell$, where all sets $H_\ell$ are compact. Then the inequality $\bigcup_{a \in X_k} U^a \neq \mathbb{H}$ is equivalent to $\exists \ell\ (H_\ell \nsubseteq \bigcup_{a \in X_k} U^a)$. And by the compactness, the non-inclusion $H_\ell \nsubseteq \bigcup_{a \in X_k} U^a$ is equivalent to the following statement: $H_\ell \nsubseteq \bigcup_{a \in X'} U^a$ for every finite $X' \subseteq X_k$. Fix an enumeration $A = \{a_n\}_{n \in \mathbb{N}}$. Put $A \restriction m = \{a_j\colon\ j < m\}$. Using König's lemma, we conclude that $X \in \mathscr{X}$ iff there exist natural $\ell$, $n$ such that for any $m$ there exists a partition $X \cap (A \restriction m) = \bigcup_{k<n} X_k$, where for every $k$ we have $H_\ell \nsubseteq \bigcup_{a \in X_k} U^a$. And this is a $\mathbf{F}_\sigma$ definition for $\mathscr{X}$. □

## 4. The main result

Here we prove Theorem 1. Define a Borel map $f : \mathbb{H} \to \mathbb{H}^A$ as follows. Suppose that $x \in \mathbb{H}$ and $a \in A$, $n^a = n$. If $x \in U_i^a$, $1 \leqslant i \leqslant n$, then put $f_a(x) = x \ominus r_i^a$. ($\ominus$ in the sense of the group $\mathbb{H}$.) If $x \notin U^a$ then put simply $f_a(x) = \mathbb{0}$.

Finally define $f(x) = \{f_a(x)\}_{a \in A}$. Clearly $f$ is a Borel map.

The maps $f_a$ do not look like homomorphisms $\mathbb{H} \to \mathbb{H}$. Nevertheless their combination surprisingly turns out to be an approximate homomorphism!

**Lemma 7.** *$f : \mathbb{H} \to \mathbb{H}^A$ is a Borel and $\mathscr{X}$-approximate homomorphism.*

**Proof.** Let $x, y \in \mathbb{H}$ and $z = x \oplus y$. Prove that the set

$$C_{xy} = \{a\colon\ f_a(x) \oplus f_a(y) \neq f_a(z)\}$$

belongs to $\mathscr{X}$. We assert that this is witnessed by the set $u = \{x, y, z\}$, that is, if $a \in C_{xy}$ then at least one of the points $x$, $y$, $z$ is not a point in $U^a$. Or, equivalently, if $a \in A$ and $x$, $y$, $z$ belong to $U^a$ then $f_a(x) \oplus f_a(y) = f_a(z)$.

To prove this fact suppose that $a \in A$ and $x, y, z \in U^a$. By definition, $U^a = U_1^a \cup \cdots \cup U_n^a$, where $n = n^a$ and $U_i^a$ are disjoint rational balls in $\mathbb{H}$. We have $x \in U_i^a$, $y \in U_j^a$, $z \in U_k^a$, where $1 \leqslant i, j, k \leqslant n$. Then by definition

$$f_a(x) = x \ominus r_i^a, \qquad f_a(y) = y \ominus r_j^a, \qquad f_a(z) = z \ominus r_k^a.$$

Therefore $f_a(x) \oplus f_a(y) = x \oplus y \ominus (s_i \oplus s_j)$. (Here we clearly use the assumption that the group is abelian.) We assert that $r_i^a \oplus r_j^a = r_k^a$—then obviously $f_a(x) \oplus f_a(y) = f_a(z)$ by the above, and we are done.

Note that $z = x \oplus y \in U^a$, hence $(U_i^a \oplus U_j^a) \cap U^a \neq \emptyset$. We conclude that (2) of Definition 2 fails. Therefore (1) holds, $r_i^a \oplus r_j^a = r_{k'}^a$ for some $k'$ and $(U_i^a \oplus U_j^a) \cap U^a \subseteq U_{k'}^a$. But the set $(U_i^a \oplus U_j^a) \cap U^a$ obviously contains $z$, and $z \in U_k^a$. It follows that $k' = k$, $r_{k'}^a = r_k^a$, $r_i^a \oplus r_j^a = r_k^a$, as required. □

**Lemma 8.** *The approximate homomorphism $f$ is not $\mathscr{Z}$-approximable by a continuous homomorphism $g : \mathbb{H} \to \mathbb{H}^A$.*

**Proof.** Assume towards the contrary that $g : \mathbb{H} \to \mathbb{H}^A$ is a continuous homomorphism which $\mathscr{Z}$-approximates $f$. Thus if $x \in \mathbb{H}$ then the set $\Delta_x = \{a : f_a(x) \neq g_a(x)\}$ belongs to $\mathscr{Z}$, where, as usual, $g_a(x) = g(x)(a)$. Note that all of these projection maps $g_a : \mathbb{H} \to \mathbb{H}$ are continuous group homomorphisms since such is $g$ itself.

Thus if $x \in \mathbb{H}$ then $\Delta_x \in \mathbb{Z}$, and hence there is a finite set $u_x \subseteq D$ satisfying the following: if $a \in A$ and $u_x \subseteq U^a$ then $a \notin \Delta_x$, that is, $f_a(x) = g_a(x)$. Put

$$X_u = \bigl\{x \in \mathbb{H}:\ \forall a \in A\ \bigl(u \subseteq U^a \Rightarrow f_a(x) = g_a(x)\bigr)\bigr\}$$

for every finite $u \subseteq D$. These sets are Borel since so are maps $f$, $g$ (and $g$ even continuous). Moreover $\mathbb{H} = \bigcup_{u \subseteq D \text{ finite}} X_u$ since every $x \in \mathbb{H}$ belongs to $X_{u_x}$. Thus at least one of the sets $X_u$ is not meager, therefore, is comeager on a certain rational ball $B \subseteq \mathbb{H}$. Fix $u$ and $B$. By definition for comeager-many $x \in B$ and all $a \in A$ satisfying $u \subseteq U^a$ we have $f_a(x) = g_a(x)$.

Arguing as in the proof of Lemma 4, we obtain an element $a \in A$ satisfying the following properties: $u \subseteq U^a$, $U^a \cap B \neq \emptyset$, but the set $B \smallsetminus U^a$ is non-empty and moreover is not dense in $B$. Fix such $a$. Thus there exists a non-empty rational ball $B' \subseteq B$ that does not intersect $U^a$. By definition $f_a(x) = \mathbb{0}$ for all $x \in B'$, and hence $g_a(x) = \mathbb{0}$ for comeager-many $x \in B'$ by the choice of $B$. We conclude that $g_a(x) = \mathbb{0}$ for all $x \in B$ in general, because $g$ is continuous.

Now, let $n^a = n$. Then $U^a = U_1^a \cup \cdots \cup U_n^a$. Recall that the intersection $B \cap U^a$ of two open sets is non-empty by the choice of $a$. It follows that there exists an index $i$, $1 \leqslant i \leqslant n$, and a non-empty rational ball $B'' \subseteq B \cap U_i^a$. Then by definition $f_a(x) = x \ominus r$ for all $x \in B''$, where $r = r_i^a$. Therefore $g_a(x) = x \ominus r$ for comeager-many $x \in B''$, and then $g_a(x) = x \ominus r$ for all $x \in B''$ since $g$ is continuous.

To conclude, $g_a$, a continuous group homomorphism, is constant $\mathbb{0}$ on a non-empty open set $B'$, and is bijective on another non-empty open set $B''$. But this cannot be the case. □

Lemmas 7 and 8 complete the proof of Theorem 1.

## References

[1] I. Farah, Approximate homomorphisms. II: Group homomorphisms, Combinatorica 20 (1) (2000) 47–60.
[2] I. Farah, Analytic quotients. Theory of liftings for quotients over analytic ideals on the integers, Mem. Amer. Math. Soc. 702 (2000) 171.
[3] V. Kanovei, M. Reeken, On Baire measurable homomorphisms of quotients of the additive group of the reals, MLQ Math. Log. Q. 46 (3) (2000) 377–384.
[4] V. Kanovei, M. Reeken, On Ulam stability of the real line, in: Jair Minoro Abe, et al. (Eds.), Unsolved Problems on Mathematics for the 21st Century. A Tribute to Kiyoshi Iséki's 80th Birthday, IOS Press, Amsterdam, 2001, pp. 169–181.
[5] V. Kanovei, M. Reeken, On Ulam's problem of stability of non-exact homomorphisms, in: R.I. Grigorchuk (Ed.), Dynamical Systems, Automata, and Infinite Groups, in: Proc. Steklov Inst. Math., vol. 231, 2000, pp. 238–270.
[6] S. Solecki, Filters and sequences, Fund. Math. 163 (3) (2000) 215–228.
[7] B. Veličković, Definable automorphisms of $\mathscr{P}(\omega)/\mathtt{fin}$, Proc. Amer. Math. Soc. 96 (1986) 130–135.