Algebraic identities for the Nijenhuis tensors

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Abstract
The general algebraic identities are discovered for the Nijenhuis and Haantjes tensors on an arbitrary manifold $M^n$. For $n = 3$, the special algebraic identities involving the symmetric bilinear form $(u, v)_H$ are derived.

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1. Introduction

I. In his well-known papers [18,19], Nijenhuis formulated the problem on the relations between the tensors $N_{B(A)}(u, v)$ and $N_A(u, v)$ where the (1, 1) tensor $B(A)$ is an arbitrary polynomial of $A$, a (1, 1) tensor on some $n$-dimensional manifold $M^n$. In the present paper we give a complete solution of the Nijenhuis problem and a solution of a more general problem on the relations between the Nijenhuis tensors $N_{B,C}(u, v)$ and $N_A(u, v)$ where $B$ and $C$ are arbitrary polynomials of $A$. In Section 3 we obtain the explicit formulae connecting the tensors $N_{B(A)}(u, v)$, $N_{B,C}(u, v)$ and $N_A(u, v)$. To derive these formulae we introduce in Section 2 a representation of the commutative ring $R[z, \lambda, \mu]$ of polynomials of the three independent variables in the space of (1, 2) tensors on the manifold $M^n$.

We derive the explicit formulae that relate the Haantjes tensors $H_{B(A)}(u, v)$ and $H_A(u, v)$. One of these identities shows that the Haantjes tensor $H_A(u, v)$ [8] is a gauge invariant with respect to the gauge transformations of the (1, 1) tensor $A^i_j(x)$, in the contrast with the Nijenhuis tensor $N_A(u, v)$.

In Section 4 we prove a general algebraic identity for the Nijenhuis tensor $N_A(u, v)$ that is connected with the characteristic polynomial for the (1, 1) tensor $A^i_j(x)$. We derive also a general algebraic identity for the Haantjes tensor $H_A(u, v)$.

The alternating (1, 2) Nijenhuis tensor $N_A(u, v)$ and the Haantjes tensor $H_A(u, v)$ define deformations of the structures of non-associative and alternating algebras in the tangent bundle $T(M^n)$. In papers [1–3], we introduce the (1, 3) tensors $B^i_{Njk\ell}$ and $B^i_{Hjk\ell}$ that characterize the deviation of the algebraic structures from the Lie algebraic structures. A general theory of deformations of Lie algebraic structures is developed in [12,20].
In Section 5 we show that an arbitrary (1, 1) tensor \( A^i_j(x) \) on a 3-dimensional manifold \( M^3 \) is symmetric with respect to the bilinear form \((u, v)_H\) connected with the Haantjes tensor \( H_A(u, v) \). We prove that the Haantjes tensor \( H_A(u, v) \) defines a deformation of the Lie algebraic structures in the tangent bundle \( T(M^3) \), and derive the algebraic identities involving the Cartan–Killing form \((u, v)_H\).

II. The Nijenhuis tensor is defined by the formula \[ N_A(u, v) = A^2[u, v] + [A\overline{u}, A\overline{v}] - A([A\overline{u}, \overline{v}] + [\overline{u}, A\overline{v}]), \] (1.1)

where \( u \) and \( v \) are tangent vectors at a point \( x \in M^n \), and \( \overline{u} \) and \( \overline{v} \) are arbitrary vector fields extending the vectors \( u \) and \( v \) (expression (1.1) is independent of the extensions \( \overline{u} \) and \( \overline{v} \)).

The Nijenhuis tensor appears in many problems of differential geometry and mathematical physics; however until now the main applications of the Nijenhuis and Haantjes [8] tensors were connected with the vanishing conditions \( N_A(u, v) = 0 \) or \( H_A(u, v) = 0 \): The Newlander–Nirenberg theorem [17] states that a quasi-complex structure \( A(x) \), \( A^2(x) = -1 \), is complex if and only if the Nijenhuis tensor \( N_A(u, v) \) vanishes. The Nijenhuis–Haantjes theorem [8,18] states that for a (1, 1) tensor \( A^i_j(x) \) with real and distinct eigenvalues on a smooth manifold \( M^n \) the linear combinations of any \( n - 1 \) eigenvectors generate an integrable distribution if and only if the Haantjes tensor \( H_A(u, v) \) vanishes.

The condition \( N_A(u, v) = 0 \) is used in [5,11] as a sufficient condition for the existence of the special Lie algebraic structures on the space of vector fields on a manifold \( M^n \). It is applied in [10] as the condition of integrability of \( G \)-structures on \( M^n \). The Gelfand–Dorfman–Magri–Morosi theorem [7,13] states that the two Poisson structures \( P_1 \) and \( P_2 \) are compatible in Magri’s sense [14] if and only if \( N_A(u, v) = 0 \) where \( A = P_1 P_2^{-1} \). The condition \( N_A(u, v) = 0 \) is used as the definition of the Poisson–Nijenhuis structures in [9,13,15] and as the definition of the Nijenhuis \( G \)-manifolds in [16]. The condition \( N_A(u, v) = 0 \) is applied in [21,22] as a sufficient condition for the existence of conservation laws for the systems of partial differential equations

\[ u^i_j = \sum_{j=1}^{n} A^i_j(u^1, \ldots, u^n)u^j_s, \quad i = 1, \ldots, n. \] (1.2)

In papers [1–3], we applied the non-zero Nijenhuis and Haantjes tensors to study the necessary criteria for the existence of the Hamiltonian and bi-Hamiltonian structures [6] for systems (1.2). In paper [4] we used the non-zero Nijenhuis and Haantjes tensors to formulate the necessary criteria for the existence of conservation laws for systems (1.2).

2. The (1, 2)-tensors representation of the ring of polynomials \( R[z, \lambda, \mu] \)

I. Let us consider the commutative ring \( R_3 = R[z, \lambda, \mu] \) of polynomials of three independent variables \( z, \lambda, \mu \) with coefficients depending on a point \( x \) of the manifold \( M^n \). Elements of the ring \( R_3 \) are polynomials

\[ S(z, \lambda, \mu) = \sum_{i,j,k}^{N} a_{ijk}(x)z^i\lambda^j\mu^k, \quad x \in M^n, \] (2.1)

where coefficients \( a_{ijk}(x) \) are arbitrary smooth functions on \( M^n \).

We introduce a representation \( T \) of the ring \( R_3 \) in the linear space of (1, 2) tensors \( V(u, v) \). The representation depends on an arbitrary (1, 1) tensor \( A(x) \) and is defined for an arbitrary polynomial \( S(z, \lambda, \mu) \) (2.1) by the formula

\[ (T_S V)(u, v) = \sum_{i,j,k}^{N} a_{ijk}(x)A^i V(A^j u, A^k v). \] (2.2)

Here the action of \( \lambda \) and \( \mu \) is associated respectively with the first and the second entries of \( V(u, v) \); the action of \( z \) is associated with the value of \( V \). Representation (2.2) possess the standard properties

\[ T_{S_1 + S_2} = T_{S_1} + T_{S_2}, \quad T_{S_1 \cdot S_2} = T_{S_1} \cdot T_{S_2}. \] (2.3)

The first identity (2.3) is obvious. The second identity (2.3) is evidently true for any monomials \( S_1 = a(x)z^i\lambda^j\mu^k \) and \( S_2 = b(x)z^p\lambda^q\mu^r \); hence the general case follows by the bilinearity.
II. For any two commuting $(1, 1)$ tensors $B^i_j(x)$ and $C^k_\ell(x)$, Nijenhuis defined in paper [18] a $(1, 2)$ tensor

$$N_{B,C}(u,v) = BC[\tilde{u}, \tilde{v}] + [B\tilde{u}, C\tilde{v}] - B[\tilde{u}, C\tilde{v}] - C[B\tilde{u}, \tilde{v}],$$

(2.4)

that is not alternating in general. The alternating Nijenhuis tensor (1.1) is a special case of tensor (2.4) for $B = C = A$. Tensor $N_{B,C}$ (2.4) has the following entries

$$(N_{B,C})^j_i = B^i_j C^k_\ell - C^k_\ell B^i_j, \quad \ell = k,$$

where $B^i_j = \partial B^i_j(x)/\partial x^\alpha$, etc.

The Haantjes tensor [8] is defined by the formula

$$H_A(u,v) = A^2 N_A(u,v) + N_A(Au, Av) - A(N_A(Au, v) + N_A(u, Av)).$$

(2.5)

In terms of the representation $T_S$ (2.2), formula (2.5) takes the form

$$H_A(u,v) = T_D N_A(u,v),$$

(2.6)

where $D$ is the polynomial

$$D(z, \lambda, \mu) = (z - \lambda)(z - \mu).$$

(2.7)

It is evident that any $(1, 2)$ tensor $W(u,v)$ of the form

$$W(u,v) = \sum_{m=0}^N (u(g_m)A^m v + v(h_m)A^m u)$$

belongs to the kernel of the operator $T_D$:

$$T_D W(u,v) = 0.$$  

(2.8)

III. Let $B(z)$ be a polynomial, $B(z) = \sum_{m=0}^k b_m(x)z^m$. We will use repeatedly the known Bezout identity

$$B(z) - B(\lambda) = (z - \lambda)Q_B(z, \lambda),$$

(2.9)

where

$$Q_B(z, \lambda) = \sum_{m=0}^k b_m(x) \sum_{p+q=m-1} z^p \lambda^q,$$

(2.10)

and its consequence

$$Q_B(z, \lambda) - Q_B(z, \mu) = (\lambda - \mu)R_B(z, \lambda, \mu),$$

(2.11)

where

$$R_B(z, \lambda, \mu) = \sum_{m=2}^k b_m(x) \sum_{p+q+r=m-2} z^p \lambda^q \mu^r.$$  

(2.12)

Identity (2.11) has an equivalent symmetric form

$$(\lambda - \mu)B(z) + (\mu - z)B(\lambda) + (z - \lambda)B(\mu) = (\lambda - \mu)(z - \lambda)(z - \mu)R_B(z, \lambda, \mu).$$

(2.13)

It is evident that the polynomials $Q_B(z, \lambda)$ and $R_B(z, \lambda, \mu)$ are symmetric with respect to their variables.

IV. Let us introduce the following $(1, 2)$ tensors

$$\tilde{X}_B(u,v) = \sum_{m=1}^k b_m(x) \sum_{p+q=m-1} A^p H_A(A^q u, v),$$

$$\tilde{Y}_B(u,v) = \sum_{m=1}^k b_m(x) \sum_{p+q=m-1} H_A(A^p u, A^q v),$$

where $b_m(x) = \partial b_m(x)/\partial x^\alpha$, etc.
\[
\tilde{Z}_B(u,v) = \sum_{m=2}^{k} b_m(x) \sum_{p+q+r=m-2} A^p H_A(A^q u, A^r v).
\]  
(2.14)

Formulae (2.2), (2.10) and (2.12) lead to the expressions
\[
\tilde{X}_B = T_{Q_B(z,\lambda)} H_A, \quad \tilde{Y}_B = T_{Q_B(\lambda,\mu)} H_A, \quad \tilde{Z}_B = T_{R_B(z,\lambda,\mu)} H_A.
\]  
(2.15)

Identity (2.11) yields
\[
T_{Q_B(z,\lambda)} H_A - T_{Q_B(z,\mu)} H_A = T_{\lambda - \mu} T_{R_B(z,\lambda,\mu)} H_A,
\]
that in view of (2.15) means
\[
\tilde{X}_B(u,v) + \tilde{X}_B(v,u) = \tilde{Z}_B(Au,v) - \tilde{Z}_B(u,Av).
\]  
(2.16)

The symmetricity of the polynomials \( Q_B(z, \lambda) \) and \( R_B(z, \lambda, \mu) \) and the identity
\[
Q_B(\lambda, z) - Q_B(\lambda, \mu) = (z - \mu) R_B(z, \lambda, \mu)
\]
lead analogously to the tensor identity
\[
\tilde{X}_B(u,v) - \tilde{Y}_B(u,v) = A \tilde{Z}_B(u,v) - \tilde{Z}_B(u,Av).
\]  
(2.17)

**Proposition 1.** Let an operator \( A(x_1) \) satisfies at a point \( x_1 \) an algebraic equation
\[
B(A,x_1) = \sum_{m=0}^{k} b_m(x_1) A^m(x_1) = 0.
\]  
(2.18)

Then the following tensor equations hold at the point \( x_1 \):
\[
\tilde{X}_B(u,v) = 0, \quad \tilde{Y}_B(Au,v) = \tilde{Y}_B(u,Av), \quad \tilde{Z}_B(Au,v) = \tilde{Z}_B(u,Av).
\]  
(2.19-2.21)

**Proof.** Formulae (2.15), (2.6), (2.7) and the Bezout identity (2.9) yield
\[
\tilde{X}_B = T_{Q_B(z,\lambda)} T_D N_A = T_{z - \mu} T_{B(z) - B(\lambda)} N_A.
\]
Hence we obtain
\[
\tilde{X}_B(u,v) = T_{z - \mu} \left( B(A) N_A(u,v) - N_A(B(A)u,v) \right) = 0,
\]
in view of Eq. (2.18). Eq. (2.21) follows from (2.16) and (2.19). Eq. (2.20) follows from (2.17), (2.19) and (2.21). \( \square \)

**Remark 1.** Let a \((k,l)\) tensor \( U(A) \) analytically depends on the entries of the tensor \( A^i_j(x) \) and their partial derivatives up to a finite order \( N \). If tensor \( U(A) \) is equal to zero for all \((1,1)\) tensors \( A^i_j(x) \) with distinct (complex) eigenvalues then \( U(A) \equiv 0 \) for any \((1,1)\) tensor \( A^i_j(x) \). This evidently follows by continuation from the non-degenerate case \( A^i_j(x) \) with distinct eigenvalues.

**Remark 2.** All tensors \( U(A) \) considered in this paper have the form
\[
U \left( A^i_j, \frac{\partial^k A^i_j(x)}{\partial x_{q_1} \cdots \partial x_{q_k}} \right)
\]
and are polynomials with respect to their arguments. Therefore the tensors \( U(A) \) can be continued on the complexifications of the tangent bundle \( T(M^n) \) and the cotangent bundle \( T^*(M^n) \). We will mean this continuation when the \((1,1)\) tensor \( A^i_j(x) \) has complex eigenvalues and eigenvectors.
3. Algebraic relations between the Nijenhuis tensors

In this section we derive a formula that connects the Nijenhuis tensor \(N_{B,C}(u,v)\) with the Nijenhuis tensor \(N_A(u,v)\). The formula implies a complete solution of the problem on the interconnections between the tensors \(N_{B(A)}(u,v)\) and \(N_A(u,v)\) raised by Nijenhuis in [18,19].

Let \(B(A,x)\) and \(C(A,x)\) be any polynomials with the variable coefficients

\[
B(A,x) = \sum_{m=0}^{k} b_m(x) A^m(x), \quad C(A,x) = \sum_{m=0}^{k} c_m(x) A^m(x). \tag{3.1}
\]

Let \(\lambda_1(x), \ldots, \lambda_n(x)\) be the eigenvalues of the operator \(A(x)\), corresponding to the eigenvectors \(e_1(x), \ldots, e_n(x)\). The operators \(B(A,x)\) and \(C(A,x)\) have the same eigenvectors with the eigenvalues \(B(\lambda_i(x),x)\) and \(C(\lambda_i(x),x)\).

**Lemma 1.** For the \((1,2)\) tensor \(N_{B,C}(u,v)\), the formula holds

\[
N_{B,C}(e_i,e_j) = (B(A) - B(\lambda_i))[C(A) - C(\lambda_j)] [e_i,e_j] + (B(\lambda_i) - B(\lambda_j)) e_i (C(\lambda_j)) e_j + (C(\lambda_i) - C(\lambda_j)) e_j (B(\lambda_i)) e_i. \tag{3.2}
\]

Indeed, formula (3.2) follows from definition (2.4) by a direct calculation.

**Theorem 1.** Tensor \(N_{B,C}(u,v)\) is connected with the Nijenhuis tensor \(N_A(u,v)\) by the formula

\[
N_{B,C}(u,v) = \sum_{m,l=1}^{k} b_m c_l \sum_{p<\ell, q<l} A^{m+l-p-q-2} N_A(A^p u, A^q v) + \sum_{m=0}^{k} [B(A)u(c_m)A^m v - C(A)v(b_m)A^m u - u(c_m)B(A)A^m v + v(b_m)C(A)A^m u]. \tag{3.3}
\]

**Proof.** We first assume that the operator \(A(x)\) has distinct eigenvalues \(\lambda_1(x), \ldots, \lambda_n(x)\). Using Bezout identities

\[
B(z) - B(\lambda) = (z - \lambda) Q_B(z,\lambda), \quad C(z) - C(\lambda) = (z - \lambda) Q_C(z,\lambda), \tag{3.4}
\]

we find

\[
Q_B(\lambda,\lambda) = \frac{\partial B(\lambda)}{\partial \lambda}, \quad Q_C(\lambda,\lambda) = \frac{\partial C(\lambda)}{\partial \lambda},
\]

\[
e(B(\lambda)) = \sum_{m=0}^{k} \left( b_m e(\lambda_m) + e(b_m) \lambda_m^{m} \right) = Q_B(\lambda,\lambda) e(\lambda) + \sum_{m=0}^{k} e(b_m) \lambda_m^{m},
\]

\[
e(C(\lambda)) = \sum_{m=0}^{k} \left( c_m e(\lambda_m) + e(c_m) \lambda_m^{m} \right) = Q_C(\lambda,\lambda) e(\lambda) + \sum_{m=0}^{k} e(c_m) \lambda_m^{m}. \tag{3.5}
\]

where \(e\) is an arbitrary tangent vector, \(e \in T_e(M^n)\). In view of identities (3.4) and (3.5), formula (3.2) takes the form:

\[
N_{B,C}(e_i,e_j) = Q_B(A,\lambda_i) Q_C(A,\lambda_j)(A - \lambda_i)(A - \lambda_j)[e_i,e_j]
+ Q_B(\lambda_i,\lambda_j) Q_C(\lambda_j,\lambda_i)(\lambda_i - \lambda_j) e_i(\lambda_j) e_j + Q_B(\lambda_i,\lambda_i) Q_C(\lambda_i,\lambda_i)(\lambda_i - \lambda_j) e_j(\lambda_i) e_i
+ \sum_{m=0}^{k} [(B(\lambda_i) - B(\lambda_j)) e_i(c_m) \lambda_j^{m} e_j + (C(\lambda_i) - C(\lambda_j)) e_j(b_m) \lambda_i^{m} e_i].
\]
This expression has the form

\[
N_{B,C}(e_i, e_j) = Q_B(A, \lambda_i) Q_C(A, \lambda_j) N_A(e_i, e_j) \\
+ \sum_{m=0}^{k} \left[ B(A)e_i(c_m)A^m e_j - C(A)e_j(b_m)A^m e_i - e_i(c_m)B(A)A^m e_j + e_j(b_m)C(A)A^m e_i \right],
\]  

(3.6)

where we use the Nijenhuis formula [18]

\[
N_A(e_i, e_j) = (A - \lambda_i)(A - \lambda_j)[e_i, e_j] + (\lambda_i - \lambda_j)(e_i(\lambda_j)e_j + e_j(\lambda_i)e_i).
\]

In view of (2.4), the first term in (3.6) has the form

\[
Q_B(A, \lambda_i) Q_C(A, \lambda_j) N_A(e_i, e_j) = T_{Q_B(z, \lambda)Q_C(z, \mu)} N_A(e_i, e_j).
\]

Substituting this expression into (3.6), we obtain

\[
N_{B,C}(u, v) = T_{Q_B(z, \lambda)Q_C(z, \mu)} N_A(u, v) \\
+ \sum_{m=0}^{k} \left[ B(A)u(c_m)A^m v - C(A)v(b_m)A^m u - u(c_m)B(A)A^m v + v(b_m)C(A)A^m u \right],
\]  

(3.7)

where \(u = e_i\) and \(v = e_j\). Hence for the general vectors \(u\) and \(v\), formula (3.7) follows by the bilinearity. Formula (3.3) coincides with (3.7) in view of definitions (2.2) and (2.10). Therefore formula (3.3) is proven for any \((1, 1)\) tensor \(A^j_i(x)\) having distinct eigenvalues. Applying Remark 1, we obtain that formula (3.3) holds for an arbitrary \((1, 1)\) tensor \(A^j_i(x)\).

\[\square\]

**Corollary 1.** For an arbitrary polynomial \(B(A, x)\), the Nijenhuis tensor \(N_{B,A}(u, v)\) satisfies the identity

\[
N_{B,A}(u, v) = \sum_{m=1}^{k} b_m \sum_{p+q=m-1} A^p N_A(A^q u, v) + \sum_{m=0}^{k} \left[ v(b_m)A^{m+1} u - A v(b_m)A^m u \right].
\]  

(3.8)

Indeed, formula (3.3) for \(C(A) = A\) reduces to formula (3.8) because \(c_1 = 1\) and \(c_l = 0\) for \(l \neq 1\).

**Corollary 2.** For any polynomial \(B(A, x)\) (3.1), the Nijenhuis tensor \(N_B(u, v)\) is connected with \(N_A(u, v)\) by the formula

\[
N_B(u, v) = T_{Q_B(z, \lambda)Q_B(z, \mu)} N_A(u, v) \\
+ \sum_{m=0}^{k} \left[ B(A)u(b_m)A^m v - B(A)v(b_m)A^m u - u(b_m)B(A)A^m v + v(b_m)B(A)A^m u \right].
\]  

(3.9)

Indeed, the Nijenhuis tensor \(N_B(u, v)\) by definition (1.1) is equal to the tensor \(N_{B,B}(u, v)\) (2.4). Substituting \(C(A) = B(A)\) into (3.7), we arrive at formula (3.9).

**Remark 3.** An equivalent form of (3.9) follows from (3.3) after substituting \(C(A) = B(A)\):

\[
N_B(u, v) = \sum_{m,l=1}^{k} b_mb_l \sum_{p<m, q<l} A^{m+l-p-q-2} N_A(A^p u, A^q v) \\
+ \sum_{m=0}^{k} \left[ B(A)u(b_m)A^m v - B(A)v(b_m)A^m u - u(b_m)B(A)A^m v + v(b_m)B(A)A^m u \right].
\]  

(3.10)

Formulae (3.9) and (3.10) give a complete solution of the Nijenhuis problem [18,19] on the interconnections between the tensors \(N_{B(A)}(u, v)\) and \(N_A(u, v)\). For any polynomial \(B(A)\) (3.1) with constant coefficients, formula (3.9) takes the simple form \(N_B(u, v) = T_{Q_B(z, \lambda)Q_B(z, \mu)} N_A(u, v)\).
Corollary 3. For any polynomial $B(A, x)$ with variable coefficients, the Haantjes tensor $H_B(u, v)$ is connected with $H_A(u, v)$ by the formula

$$H_B(u, v) = T_{Q_B(z, \lambda)Q_B(z, \mu)}^2 H_A(u, v).$$

(3.11)

Proof. The Haantjes tensor $H_B(u, v)$ is connected with the Nijenhuis tensor $N_B(u, v)$ by the relation

$$H_B(u, v) = B^2 N_B(u, v) + N_B(Bu, Bv) - B(N_B(Bu, v) + N_B(u, Bv)),$$

that is equivalent to the expression

$$H_B(u, v) = T_{B(z) - B(\lambda)B(z) - B(\mu)} N_B(u, v).$$

Using here identity (3.4), we obtain

$$H_B(u, v) = T^{Q_B(z, \lambda)Q_B(z, \mu)D} N_B(u, v),$$

(3.12)

where $D = (z - \lambda)(z - \mu)$. Formula (3.11) follows from (3.12) after substituting (3.9) and using Eqs. (2.6) and (2.8). □

Corollary 4. For arbitrary smooth functions $f(x)$ and $g(x)$, the following identities hold: for the Haantjes tensor

$$H_{f(x)A + g(x)}(u, v) = f^4(x) H_A(u, v),$$

(3.13)

where $I$ is the unit $(1, 1)$ tensor, $I^k_j = \delta^k_j$, and for the Nijenhuis tensor

$$N_{f(x)A + g(x)}(u, v) = f^2(x) N_A(u, v) + f\left[(Au(f) - u(g))Av - (Av(f) - v(g))Au - u(f)A^2v + v(f)A^2u + Au(g)v - Av(g)u\right].$$

(3.14)

Indeed, for the operator

$$B(A, x) = f(x)A(x) + g(x)I,$$

(3.15)

we have

$$b_1(x) = f(x), \quad b_0(x) = g(x), \quad Q_B(z, \lambda) = f(x).$$

(3.16)

Hence $Q_B(z, \lambda)Q_B(z, \mu) = f^2(x)$ and formula (3.11) implies identity (3.13). Identity (3.14) follows from formula (3.9) after substituting expressions (3.15) and (3.16).

Remark 4. Formula (3.13) means that the Haantjes tensor $H_A(u, v)$ is a gauge invariant with respect to the gauge transformations (3.15). Formula (3.14) shows that the Nijenhuis tensor $N_A(u, v)$ is not a gauge invariant.

4. The general algebraic identities for the Nijenhuis and Haantjes tensors

I. For an arbitrary polynomial $B(A, x)$ (3.1), we introduce the following alternating $(1, 2)$ tensor:

$$Z_B(u, v) = \sum_{m=2}^{k} b_m(x) \sum_{p+q+r=m-2} A^p N_A(A^q u, A^r v) + \sum_{m=0}^{k} \left[v(b_m) A^m u - u(b_m) A^m v\right].$$

(4.1)

Lemma 2. The tensor identity holds:

$$N_B, A(u, v) + N_B, A(v, u) = Z_B(Au, v) - Z_B(u, Av).$$

(4.2)

Indeed, this identity follows after a direct substitution of formulae (3.8) and (4.1).
Lemma 3. If a (1, 1) tensor $A^j_i(x)$ satisfies an algebraic equation $B(A, x) = 0$ then the tensor equality holds:

$$Z_B(Au, v) = Z_B(u, Av).$$

(4.3)

Indeed, equation $B(A, x) = 0$ evidently yields $N_{B,A}(u, v) = 0$. Hence identity (4.2) implies equality (4.3).

Theorem 2. The Nijenhuis tensor $N_A(u, v)$ satisfies the tensor identity

$$\sum_{m=2}^{n} a_m(x) \sum_{p+q+r=m-2} A^p N_A(A^q u, A^r v) = \sum_{m=0}^{n} \left( u(a_m) A^m v - v(a_m) A^m u \right),$$

(4.4)

where $a_m(x)$ are coefficients of the characteristic polynomial

$$P(\lambda, x) = \det(A(x) - \lambda I) = \sum_{m=0}^{n} a_m(x) \lambda^m.$$

(4.5)

Proof. Note that Eq. (4.4) has the form $Z_P(u, v) = 0$, where tensor $Z_P(u, v)$ has form (4.1) for $B(\lambda, x) = P(\lambda, x)$. We first assume that the (1, 1) tensor $A^j_i(x)$ has distinct eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$ with the eigenvectors $e_1(x), \ldots, e_n(x)$. The Cayley–Hamilton theorem states that $P(A, x) = 0$. Hence Lemma 3 and Eq. (4.3) give

$$(\lambda_i - \lambda_j) Z_P(e_i, e_j) = 0.$$ Hence due to the bilinearity and skew-symmetricity of the tensor $Z_P(u, v)$ it follows $Z_P(u, v) = 0$. Using Remark 1, we obtain that equation $Z_P(u, v) = 0$ and the equivalent identity (4.4) hold for any (1, 1) tensor $A^j_i(x)$. 

Remark 5. Theorem 2 is based on the universal properties of the characteristic polynomial $P(\lambda, x)$ of an operator $A(x)$. If an operator $A(x)$ has coinciding eigenvalues and $B(\lambda, x)$ is its minimal polynomial, so $B(A(x), x) = 0$, then the corresponding tensor $Z_B(u, v)$ (4.1) does satisfy equality (4.3). However $Z_B(u, v) \neq 0$ in general and there is no identity of a smaller degree analogous to (4.4).

Corollary 5. The Haantjes tensor $H_A(u, v)$ satisfies the tensor identity

$$\sum_{m=2}^{n} a_m(x) \sum_{p+q+r=m-2} A^p H_A(A^q u, A^r v) = 0,$$

(4.6)

that has the form

$$Z_P(u, v) = T_{RP(\zeta, \lambda, \mu)} H_A(u, v) = 0,$$

(4.7)

where $P = P(\lambda, x)$ is the characteristic polynomial (4.5).

Proof. Applying transformation $T_{D(\zeta, \lambda, \mu)}$ (2.7) to identity (4.4) and using relations (2.6) and (2.8) we obtain identity (4.6). This identity has form (4.7) in view of formulae (2.14) and (2.15). 

II. For $n = 2$, the characteristic polynomial (4.5) has the form $P(\lambda) = \lambda^2 - (\text{Tr } A) \lambda + \det A$; hence identity (4.4) implies

$$N(u, v) = u(\det A)v - v(\det A)u - u(\text{Tr } A)A^2 v + v(\text{Tr } A)A^2 u.$$

For $n = 3$, the characteristic polynomial (4.5) is $P(\lambda) = \lambda^3 - (\text{Tr } A)\lambda^2 + a_1 \lambda - \det A$, where $2a_1 = (\text{Tr } A)^2 - 2(\text{Tr } A^2)$. Hence identity (4.4) yields

$$(A - \text{Tr } A) N_A(Au, v) + N_A(Au, Av) + N_A(Au, u)\]

$$= v(\det A)u - u(\det A)v - v(a_1)Av + u(a_1)Av + v(\text{Tr } A)A^2 v - u(\text{Tr } A)A^2 u.$$

For $n = 3$, identity (4.6) takes the form

$$(A - \text{Tr } A) H_A(u, v) + H_A(Au, v) + H_A(Au, Av) = 0.$$ (4.8)

Identity (4.8) implies that the Haantjes tensor defines a deformation of the Lie algebraic structures in the tangent bundle $T(M^3)$; see Section 5.
5. The special algebraic identities for \( n = 3 \)

We denote \( N(u, v) = N_A(u, v) \), \( H(u, v) = H_A(u, v) \), \( M(u, v) = N(Au, v) - AN(u, v) \) and define for a fixed tangent vector \( u \) the linear operators \( N_A(v) = N(u, v) \), \( H_A(v) = H(u, v) \), \( M_A(v) = M(u, v) \). Evidently we have

\[
M_u = N_{Au} - AN_u, \quad H_u = [M_u, A].
\] (5.1)

Eqs. (5.1) imply

\[
\text{Tr} \, H_u = 0, \quad \text{Tr}(A^k H_u) = 0, \quad k = 1, 2, \ldots.
\] (5.2)

Let us consider the symmetric bilinear form \((u, v)_H = \text{Tr}(H_u H_v)\) and the alternating (1, 3) tensor \([1–3]\)

\[
B_H(u, v, w) = H(H(u, v), w) + H(H(v, w), u) + H(H(w, u), v).
\] (5.3)

Hence for the linear operator \( B_{uv}(w) = B_H(u, v, w) \) we find

\[
B_{uv} = H_{H(u,v)} - [H_u, H_v].
\] (5.4)

**Theorem 3.** For \( n = 3 \), the Nijenhuis and Haantjes tensors have the following properties:

1. The Haantjes tensor \( H(u, v) \) defines a deformation of Lie algebra structures in the tangent bundle \( T(M^3) \).
2. The operators \( H_u \) are skew-symmetric with respect to the form \((u, v)_H\):

\[
(H_u u, v)_H = -(u, H_u v)_H.
\] (5.5)

3. The operators \( A^j_i(x) \) are symmetric with respect to the bilinear form \((u, v)_H\):

\[
(A u, v)_H = (u, A v)_H.
\] (5.6)

4. The Nijenhuis tensor \( N(u, v) \) satisfies the following identity:

\[
(\langle B(A) N(u, w), v \rangle)_H + \langle B(A) N(v, w), u \rangle_H + \langle B(A) N(w, u), v \rangle_H = R_B(\lambda_1, \lambda_2, \lambda_3)(H(u, v), w)_H,
\] (5.7)

where \( B(A, x) \) is an arbitrary polynomial (3.1), \( R_B(\lambda_1, \lambda_2, \lambda_3) \) is the corresponding symmetric polynomial (2.12), \( \lambda_1(x), \lambda_2(x) \) and \( \lambda_3(x) \) are the eigenvalues of the \( (1, 1) \) tensor \( A^j_i(x) \) and \( u, v, w \in T_x(M^3) \) are arbitrary tangent vectors.

**Proof.** 1. For \( n = 3 \), the alternating tensor \( B^{ij}_k \) has non-zero components only for \( i \neq j \neq k \). Let \( s = k \); then we have

\[
B^{ij}_s = B^{ij}_s + B^{ij}_s = \text{Tr}(B_{e_i e_j}).
\] Eqs. (5.2) and (5.4) yield \( \text{Tr}(B_{uv}) = 0 \). Hence all components \( B^{ij}_s = 0 \), or tensor \( B_H(u, v, w) \equiv 0 \). Therefore the Haantjes tensor \( H(u, v) \) defines a Lie algebra structure in the tangent space \( T_x(M^3) \) for any point \( x \in M^3 \).

2. Eq. (5.5) follows from the general theory of Lie algebras since for \( n = 3 \) the bilinear form \((u, v)_H\) coincides with the Cartan–Killing form for the corresponding Lie algebra.

3. We first assume that the operator \( A^j_i(x) \) has distinct (complex) eigenvalues \( \lambda_1(x) \), \( \lambda_2(x) \) and \( \lambda_3(x) \) and \( e_1(x), e_2(x), e_3(x) \) are the corresponding eigenvectors, \( A e_i = \lambda_i e_i \), belonging to the complexification of the tangent bundle \( T(M^3) \).

Formula (2.5) yields

\[
H(e_i, e_j) = (A - \lambda_i)(A - \lambda_j)N(e_i, e_j).
\] (5.8)

Applying operator \((A - \lambda_k)\) to (5.8) and using the Cayley–Hamilton theorem for \( n = 3 \) we obtain \((A - \lambda_k)H(e_i, e_j) = P(A)N(e_i, e_j) = 0\). Hence we find \( H(e_i, e_j) = H^k_{ij} e_k \) for \( i \neq j \neq k \). These relations imply \( H_{e_i} H_{e_j} e_i = H^k_{ij} H^j_{ik} e_j, H_{e_i} H_{e_j} e_j = 0, H_{e_i} H_{e_j} e_k = 0 \). Hence

\[
(e_i, e_j)_H = \text{Tr}(H_{e_i} H_{e_j}) = 0, \quad i \neq j.
\] (5.9)

Eq. (5.6) is an evident consequence of the equations \( A e_i = \lambda_i e_i \) and (5.9). Using Remark 1, we obtain that identity (5.6) is true for an arbitrary \((1, 1)\) tensor \( A^j_i(x) \).
4. First we assume that tensor $A^j_i(x)$ has distinct eigenvalues $\lambda_k(x)$. The left-hand side of equality (5.7) is alternating because the Nijenhuis tensor is alternating. The right-hand side of (5.7) is alternating in view of identity (5.5). Therefore it is sufficient to prove identity (5.7) only for the eigenvectors $u = e_i(x)$. Let us apply an operator $B(A,x)$ (3.1) to equality (5.8) and then consider the scalar product $(u,v)_H$ of the resulting vector with eigenvector $e_k$. Using identity (5.6) and equations $Ae_i = \lambda_i e_i$ we obtain the equality

$$
(B(A)N(e_i,e_j),e_k)_H = B(\lambda_k)(\lambda_k - \lambda_i)^{-1}(\lambda_k - \lambda_j)^{-1}(H(e_i,e_j),e_k)_H.
$$

In view of (5.5) we find

$$
(B(A)N(e_i,e_j),e_k)_H + (B(A)N(e_j,e_k),e_i)_H + (B(A)N(e_k,e_i),e_j)_H = R_B(\lambda_i,\lambda_j,\lambda_k)(H(e_i,e_j),e_k)_H,
$$

where

$$
R_B(\lambda_i,\lambda_j,\lambda_k) = B(\lambda_i)(\lambda_i - \lambda_j)^{-1}(\lambda_i - \lambda_j)^{-1} + B(\lambda_j)(\lambda_j - \lambda_i)^{-1}(\lambda_j - \lambda_k)^{-1} + B(\lambda_k)(\lambda_k - \lambda_j)^{-1}(\lambda_k - \lambda_i)^{-1}.
$$

Identity (2.13) proves that function (5.10) coincides with the symmetric polynomial $R_B(z,\lambda,\mu)$ (2.12). Therefore identity (5.7) is proven by the bilinearity when $\lambda_1(x) \neq \lambda_2(x) \neq \lambda_3(x)$. Applying Remark 1, we obtain that identity (5.7) is true for an arbitrary $(1,1)$ tensor $A^j_i(x)$.

Remark 6. For any polynomial of second degree $B(A,x) = b_2(x)A^2(x) + b_1(x)A(x) + b_0(x)$, formula (2.12) implies $R_B(z,\lambda,\mu) = b_2(z)$. Therefore identity (5.7) for $B(A) = 1, A$ and $A^2$ yields:

$$
\begin{align*}
(N(u,v),w)_H + (N(v,w),u)_H + (N(w,u),v)_H &= 0, \\
(A^2N(u,v),w)_H + (A^2N(v,w),u)_H &= 0.
\end{align*}
$$

These identities have generalizations [2,3] for the $n$-dimensional case for the special $(1,1)$ tensors $A^j_i(x)$ corresponding to the Hamiltonian systems (1.2).

References


