Available online at www.sciencedirect.com

ScienceDirect

J. Math. Pures Appl. 89 (2008) 248–277

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

The periodic unfolding method for perforated domains and Neumann sieve models

D. Cioranescu [a], A. Damlamian [b,*], G. Griso [a], D. Onofrei [c]

[a] *Laboratoire Jacques-Louis Lions, Boite courier 187, Universite Pierre et Marie Curie, 4 place Jussieu, France*

[b] *Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, CMC, 94010 Créteil Cedex, France*

[c] *Department of Mathematical Sciences, Worcester Polytechnic Institute, MA, USA*

Received 11 June 2007

Available online 8 December 2007

Abstract

The periodic unfolding method, introduced in [D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 99–104], was developed to study the limit behavior of periodic problems depending on a small parameter ε. The same philosophy applies to a range of periodic problems with small parameters and with a specific period (as well as to almost any combinations thereof). One example is the so-called Neumann sieve.

In this work, we present these extensions and show how they apply to known results and allow for generalizations (some in dimension $N \geqslant 3$ only). The case of the Neumann sieve is treated in details. This approach is significantly simpler than the original ones, both in spirit and in practice.

Résumé

La méthode de l'éclatement périodique, introduite dans [D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 99–104], a pour but l'étude du comportement asymptotique de problèmes périodiques avec période tendant vers zéro. La même approche permet de traiter toute une famille de problèmes caractérisés par des périodicités de tailles tendant vers zéro. Un exemple est donné par le problème connu sous le nom de la passoire de Neumann.

Nous présentons ici divers prolongements et généralisations de l'éclatement périodique (certains nécéssitant que la dimension N soit supérieure à 3) et nous l'appliquons à la passoire de Neumann. Pour ce type de problèmes, cette approche apparaît comme élémentaire, directe et plus efficace que les méthodes classiques.

Keywords: Periodic unfolding; Neumann sieve; Strange term; Homogenization

1. Introduction

The periodic unfolding method (see [8]), as a simpler alternative to the two-scale convergence, was developed to study the limit behavior of periodic problems depending on a small parameter ε. As it turns out, the same philosophy

* Corresponding author.
E-mail address: damlamian@univ-paris12.fr (A. Damlamian).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2007.12.008

applies to a whole range of periodic problems with small parameters, provided they have a specific period. The method is flexible enough to apply as well to almost any combinations of the preceding cases.

In this work, we present these various extensions and show how they apply to known results and allow for generalizations. This approach is significantly simpler than the original ones, both in spirit and in practice.

The plan of the paper is as follows.

Section 2 is devoted to the presentation of various unfolding operators and their main properties for domains in $\mathbb{R}^N$, $N \in \mathbb{N}^*$. More precisely, in Section 2.1, we recall the definition of the unfolding operator $\mathcal{T}_\varepsilon$ for the periodic case in fixed domains ([8] and [12]). In Section 2.2, we present the unfolding operator adapted to the case of holes of size ε (with Neumann boundary condition) with period of same size (see [9] for details and applications). Section 2.3 introduces the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending of two small parameters ε and δ (corresponding to the scales ε and $\varepsilon\delta$) and which was first introduced in a similar form in [6] and [7]. The following subsections deal again with an unfolding operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$ depending on the scales ε and $\varepsilon\delta$ when the latter occurs only on a layer. This approach never assumes the existence of an extension operator in the cells but is based on the Poincaré–Wirtinger inequality (Section 2.1) and Sobolev–Poincaré–Wirtinger inequality (Sections 2.2 and 2.3). The latter requires that the dimension N be larger than 2.

The remainder of the paper is devoted to the application to various linear problems in perforated domains and with oscillating coefficients. For simplicity, we assume a homogeneous Dirichlet boundary condition on the outer boundary of the domain, but more general boundary conditions can be handled provided the outer boundary is Lipschitz and the perforations do not intersect it. In each case, we obtain both the unfolded and the classical (standard) form for the limit problem. The operator $\mathcal{T}_\varepsilon$ allows to homogenize the coefficients of the differential operators, whereas the operators $\mathcal{T}_{\varepsilon,\delta}$ (or $\mathcal{T}^{bl}_{\varepsilon,\delta}, \ldots$) generates the "strange terms" in the limit.

Section 3 concerns the homogenization of elliptic problems with oscillating coefficients, for volume ε-periodically distributed small holes of size $\varepsilon\delta$ with Dirichlet condition. These results are well known for the Laplace operator, with the appearance of the "strange term" (see [10] and references therein). For the case of oscillating coefficients, we refer to [11] where H-convergence is used. It should be noted that for technical reasons, our method fails to apply in dimension $N = 2$. See also [2] for the nonlinear case.

Section 4 considers small perforations of size $\varepsilon\delta$ which are distributed ε-periodically in a layer of thickness ε. It generalizes the results of [21,17] and [10] to the case of oscillating coefficients.

Section 5 deals with the Neumann sieve problem with zero thickness and oscillating coefficients. For the case of constant coefficients, we refer the reader to [4,12,16,20,1] and [19]. We also refer to the recent paper [3] for a different approach. In Section 6, the thick sieve is treated (for which we refer to [15] for the case of constant coefficients). The unfolding method was applied for the first time for sieve problems in [18], also in the case of constant coefficients.

To conclude this section, we would like to point out that using the various unfolding operators introduced in this paper, one can treat any combination of the previous problems, for instance, a medium with ε-size Neumann perforations and $\varepsilon\delta$-size Dirichlet holes in the bulk (see Fig. 10), or even a thick sieve in such a medium. This will be presented in a forthcoming paper which will also include the proof of convergence for the energies.

2. The periodic unfolding operator

In this section we recall the general properties of the periodic unfolding operator introduced in [8] and include variants and generalizations, all based on the technique of unfolding. In particular, we introduce the notion of *unfolding criterion for integrals* (in short u.c.i.), in order to simplify the proofs where unfolding is used.

For N in $\mathbb{N}^*$, let Y be the unit cube of $\mathbb{R}^N$ centered in the origin, $Y \doteq]-\frac{1}{2}, \frac{1}{2}[^N$ (more general sets Y having the paving property in $\mathbb{R}^N$ can also be used, cf. [14]). We consider the periodical net on $\mathbb{R}^N$ (i.e. the subgroup $\mathbb{Z}^N$) and all the corresponding translates of Y. By analogy with the one-dimensional case, to each $x \in \mathbb{R}^N$ we can associate its integer part, $[x]_Y$ belonging to the net, such that $x - [x]_Y \in Y$, the latter being its fractional part, respectively, i.e, $\{x\}_Y = x - [x]_Y$ (see Fig. 1). These definitions are ambiguous, but only on a set of measure zero, which is enough for our purpose.

Therefore we have:

$$x = \varepsilon\left\{\frac{x}{\varepsilon}\right\}_Y + \varepsilon\left[\frac{x}{\varepsilon}\right]_Y \quad \text{for every } x \in \mathbb{R}^N.$$

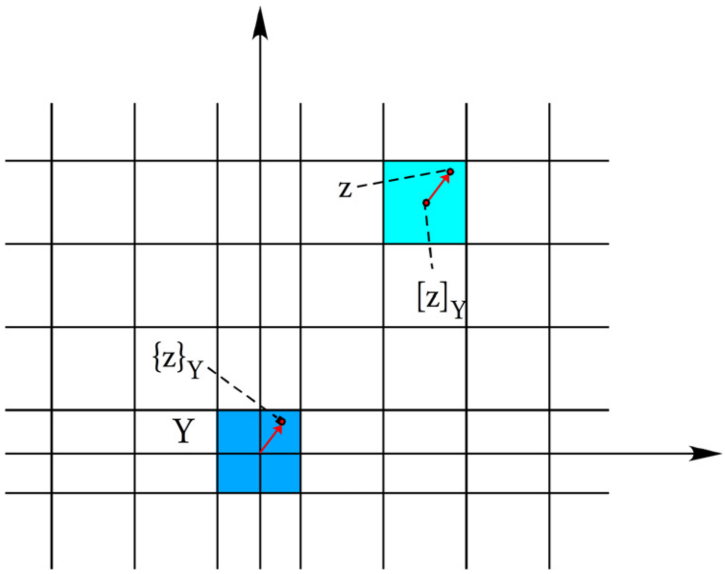

Fig. 1. The basic decomposition.

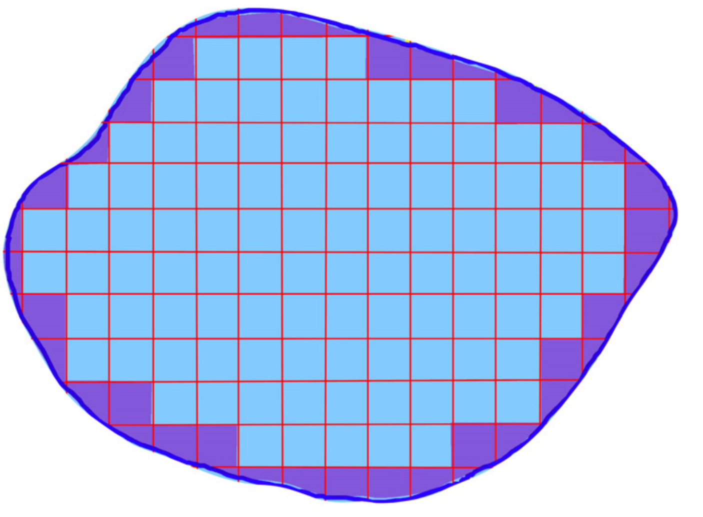

Fig. 2. The sets Ω, $\widehat{\Omega}_\varepsilon$ and Λ_ε.

Let Ω be open and bounded in $\mathbb{R}^N$. We use the following notations:

$$\widehat{\Omega}_\varepsilon = \left\{ x \in \Omega, \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon Y \right) \subset \Omega \right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon. \tag{2.1}$$

The set $\widehat{\Omega}_\varepsilon$ is the largest union of εY cells contained in Ω, while Λ_ε is the subset of Ω containing the parts from εY cells intersecting the boundary $\partial\Omega$ (see Fig. 2).

2.1. The case of fixed domains: the operator $\mathcal{T}_\varepsilon$

We recall here the definition of the unfolding operator and its main properties (for details and proofs we refer the reader to [8] and [13]).

Definition 2.1. For $\phi \in L^p(\Omega)$, the unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$ is defined as follows:

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y) & \text{if } (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{if } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Theorem 2.2 *(Properties of the operator $\mathcal{T}_\varepsilon$).*

1. *For any $v, w \in L^p(\Omega)$, $\mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\mathcal{T}_\varepsilon(w)$.*
2. *For any $w \in L^p(\Omega)$, one has the following "exact integration" formula*:
$$\int_{\Omega \times Y} \mathcal{T}_\varepsilon(w)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega w(x)\,\mathrm{d}x - \int_{\Lambda_\varepsilon} w(x)\,\mathrm{d}x = \int_{\widehat{\Omega}_\varepsilon} w(x)\,\mathrm{d}x.$$
3. *For any $u \in L^1(\Omega)$,*
$$\int_{\Omega \times Y} \left|\mathcal{T}_\varepsilon(u)\right|\mathrm{d}x\,\mathrm{d}y \leqslant \int_\Omega |u|\,\mathrm{d}x.$$
4. *For any $u \in L^1(\Omega)$,*
$$\left|\int_\Omega u\,\mathrm{d}x - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(u)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \int_{\Lambda_\varepsilon} |u|\,\mathrm{d}x. \tag{2.2}$$
5. *Let $\{w_\varepsilon\} \subset L^2(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^2(\Omega)$. Then*
$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w \quad \text{strongly in } L^2(\Omega \times Y).$$
6. *Let $w_\varepsilon \rightharpoonup w$ weakly in $H^1(\Omega)$. Then, there exists a subsequence and $\widehat{w} \in L^2(\Omega; H^1_{per}(Y))$ such that*
$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla_x w + \nabla_y \widehat{w} \quad \text{weakly in } L^2(\Omega \times Y).$$

Property 4 shows that any integral of a function w on Ω, is "almost equivalent" to the integral of its unfolded on $\Omega \times Y$, the "integration defect" arises only from the cells intersecting the boundary $\partial\Omega$ and is controlled by the right-hand side integral in (2.2).

The next proposition, which we call unfolding criterion for integrals (u.c.i.), is a very useful tool when treating homogenization problems.

Proposition 2.3 *(u.c.i.). If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |w_\varepsilon|\,\mathrm{d}x \to 0,$$

then

$$\int_\Omega w_\varepsilon\,\mathrm{d}x - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y \to 0.$$

Based on this result, in order to simplify the proofs in the sequel, we introduce the following notation:

Notation 2.4. *If* $\{w_\varepsilon\}$ *is a sequence satisfying u.c.i., we write*:

$$\int_\Omega w_\varepsilon \, dx \overset{\mathcal{T}_\varepsilon}{\cong} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(w_\varepsilon)\, dx\, dy.$$

Corollary 2.5. *Let* $\{u_\varepsilon\}$ *be bounded in* $L^2(\Omega)$ *and* $\{v_\varepsilon\}$ *be bounded in* $L^p(\Omega)$ *with* $p > 2$. *Then we have*:

$$\int_\Omega u_\varepsilon v_\varepsilon \, dx \overset{\mathcal{T}_\varepsilon}{\cong} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(v_\varepsilon)\, dx\, dy.$$

We end this subsection with the notion of local average of a function.

Definition 2.6. The local average $M_Y^\varepsilon : L^p(\Omega) \mapsto L^p(\Omega)$, is defined for any ϕ in $L^p(\Omega)$, $1 \leqslant p < \infty$, by

$$M_Y^\varepsilon(\phi)(x) \doteq \int_Y \mathcal{T}_\varepsilon(\phi)(x, y)\, dy.$$

Remark 2.7. The function $M_Y^\varepsilon(\phi)$ is indeed a local average, since

$$M_Y^\varepsilon(\phi)(x) = \int_Y \mathcal{T}_\varepsilon(\phi)(x, y)\, dy = \begin{cases} \frac{1}{\varepsilon^N}\int_{\varepsilon[\frac{x}{\varepsilon}]+\varepsilon Y} \phi(\zeta)\, d\zeta & \text{if } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{if } x \in \Lambda_\varepsilon. \end{cases}$$

Remark 2.8. Note that $\mathcal{T}_\varepsilon(M_Y^\varepsilon(\phi)) = M_Y^\varepsilon(\phi)$ on the set $\Omega \times Y$.

The next proposition, which will be frequently used as well, is classical:

Proposition 2.9. *Let* $\{w_\varepsilon\}$ *be a sequence such that* $w_\varepsilon \to w$ *strongly in* $L^p(\Omega)$ *where* $1 \leqslant p < \infty$. *Then we have*:

$$M_Y^\varepsilon(w_\varepsilon) \to w \quad \textit{strongly in } L^p(\Omega).$$

2.2. Unfolding in domains with volume-distributed "small" holes: the operator $\mathcal{T}_{\varepsilon,\delta}$

In Section 4 below, we will consider domains with εY-periodically distributed holes of size $\varepsilon\delta$ ($\delta \to 0$ with ε). More precisely (see Fig. 3), for a given open $B \Subset Y$ we denote $Y_\delta^* = Y \setminus \delta\overline{B}$ and define the perforated domain $\Omega_{\varepsilon,\delta}^*$ as

$$\Omega_{\varepsilon,\delta}^* = \left\{ x \in \Omega, \text{ such that} \left\{\frac{x}{\varepsilon}\right\} \in Y_\delta^* \right\}. \tag{2.3}$$

This geometry of domains with "small" holes requires another unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending on both parameters ε and δ. In the next sections, we will consider functions $v_{\varepsilon,\delta}$ which vanish on the whole boundary of the perforated domain $\Omega_{\varepsilon,\delta}^*$, namely belonging to the space $H_0^1(\Omega_{\varepsilon,\delta}^*)$. These functions are naturally extended by zero to the whole of Ω and these extensions belong to $H_0^1(\Omega)$. *Consequently, from now on, we will not distinguish elements of* $H_0^1(\Omega_{\varepsilon,\delta}^*)$ *and their extensions in* $H_0^1(\Omega)$. This justifies the introduction of $\mathcal{T}_{\varepsilon,\delta}$ on the fix domain Ω, while it may (and, in Section 4, will) be applied to elements of $H_0^1(\Omega_{\varepsilon,\delta}^*)$.

Definition 2.10. For $\phi \in L^p(\Omega)$, $p \in [1, \infty[$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ is defined by:

$$\mathcal{T}_{\varepsilon,\delta}(\phi)(x, z) = \begin{cases} \mathcal{T}_\varepsilon(x, \delta z) & \text{if } (x, z) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases}$$

For $N \geqslant 3$, the Sobolev exponent $\frac{2N}{N-2}$ associated to 2 is denoted 2^*.

The next results follow from Theorem 2.2 by using the change of variable $z = (1/\delta)y$.

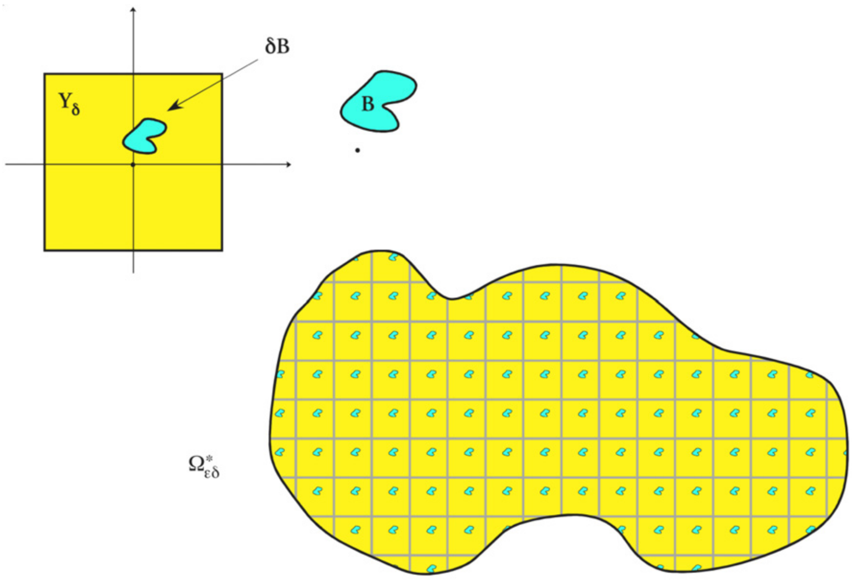

Fig. 3. The sets B and Y^*_δ and the corresponding $\Omega^*_{\varepsilon,\delta}$.

Theorem 2.11 *(Properties of the operator $\mathcal{T}_{\varepsilon,\delta}$).*

1. *For any $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$.*
2. *For any $u \in L^1(\Omega)$, one has*

$$\delta^N \int_{\Omega\times\mathbb{R}^N} \left|\mathcal{T}_{\varepsilon,\delta}(u)\right| \mathrm{d}x\,\mathrm{d}z \leqslant \int_\Omega |u|\,\mathrm{d}x.$$

3. *For any $u \in L^2(\Omega)$,*

$$\left\|\mathcal{T}_{\varepsilon,\delta}(u)\right\|^2_{L^2(\Omega\times\mathbb{R}^N)} \leqslant \frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}.$$

4. *For any $u \in L^1(\Omega)$,*

$$\left|\int_\Omega u\,\mathrm{d}x - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u)\,\mathrm{d}x\,\mathrm{d}z\right| \leqslant \int_{\Lambda_\varepsilon} |u|\,\mathrm{d}x.$$

5. *Let $u \in H^1(\Omega)$. Then*

$$\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u)\big) \quad \text{in } \Omega\times\frac{1}{\delta}Y.$$

6. *Suppose $N \geqslant 3$ and let ω be open and bounded in $\mathbb{R}^N$. Then the following estimates hold*:

$$\left\|\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u)\big)\right\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leqslant \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.4}$$

$$\left\|\mathcal{T}_{\varepsilon,\delta}\big(u - M^\varepsilon_Y(u)\big)\right\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \leqslant \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.5}$$

and

$$\left\|\mathcal{T}_{\varepsilon,\delta}(u)\right\|^2_{L^2(\Omega\times\omega)} \leqslant \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{2/N}\|\nabla u\|^2_{L^2(\Omega)} + 2|\omega|\|u\|^2_{L^2(\Omega)}, \tag{2.6}$$

where C denotes the Sobolev–Poincaré–Wirtinger constant for $H^1(Y)$.

7. *Suppose $N \geqslant 3$. Let $\{w_{\varepsilon,\delta}\}$ be a sequence in $H^1(\Omega)$ which is uniformly bounded when both ε and δ go to zero. Then, up to a subsequence, there is W in $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Omega \times \mathbb{R}^N)$ such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M_Y^\varepsilon(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2\big(\Omega; L^{2^*}(\mathbb{R}^N)\big),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big)\, 1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in } L^2\big(\Omega \times \mathbb{R}^N\big).$$

Assuming furthermore that $\limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{N/2-1}}{\varepsilon} < +\infty$, *one can choose the subsequence above and some U in $L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^N))$ with*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2\big(\Omega; L^2_{\text{loc}}(\mathbb{R}^N)\big).$$

Remark 2.12. In order to establish (2.5)–(2.6) from (2.4), the Sobolev–Poincaré–Wirtinger inequality is used (because of its scale-invariance). The use of the standard Poincaré–Wirtinger inequality would give,

$$\big\|\mathcal{T}_{\varepsilon,\delta}\big(u - M_Y^\varepsilon(u)\big)\big\|^2_{L^2(\Omega'\times\mathbb{R}^N)} \leqslant \frac{1}{\delta^2}\frac{C'\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$

where C' is the Poincaré–Wirtinger constant of Y. This estimate is not compatible with (2.4).

Concerning the integral formulas, we have the following results, similar to those of the previous subsection.

Proposition 2.13 *(u.c.i.). If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |w_\varepsilon|\,\mathrm{d}x \to 0,$$

then

$$\int_\Omega w_\varepsilon\,\mathrm{d}x \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}z.$$

Corollary 2.14. *Let $\{u_\varepsilon\}$ be bounded in $L^2(\Omega)$ and $\{v_\varepsilon\}$ be bounded in $L^p(\Omega)$ with $p > 2$. Then*

$$\int_\Omega u_\varepsilon v_\varepsilon\,\mathrm{d}x \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u_\varepsilon)\mathcal{T}_{\varepsilon,\delta}(v_\varepsilon)\,\mathrm{d}x\,\mathrm{d}z.$$

2.3. The boundary-layer unfolding operator: the operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$

For sieve-type problems (Sections 4 and 5 below), we consider the case of holes of size $\varepsilon\delta$, distributed in Σ'_ε, a layer of thickness ε parallel to a hyperplane in the open domain Ω in $\mathbb{R}^N$. We denote $x' \doteq (x_1, \dots, x_{N-1})$, $\Pi \doteq \{x_N = 0\}$ and set $\Sigma = \Pi \cap \Omega$.

The layer Σ'_ε is defined as

$$\Sigma'_\varepsilon = \Omega \cap \left\{x;\, |x_N| < \frac{\varepsilon}{2}\right\},$$

and by analogy with (2.1), we introduce the corresponding sets,

$$\widehat{\Sigma}'_\varepsilon = \left\{x \in \Sigma'_\varepsilon,\, \left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon Y\right) \subset \Sigma'_\varepsilon\right\}, \qquad \Lambda'_\varepsilon = \Sigma'_\varepsilon \setminus \widehat{\Sigma}'_\varepsilon,$$

and denote $\widehat{\Sigma}_\varepsilon = \widehat{\Sigma}'_\varepsilon \cap \Pi$.

The set $\widehat{\Sigma}'_\varepsilon$ is the largest union of εY cells contained in Σ'_ε (see Fig. 4).

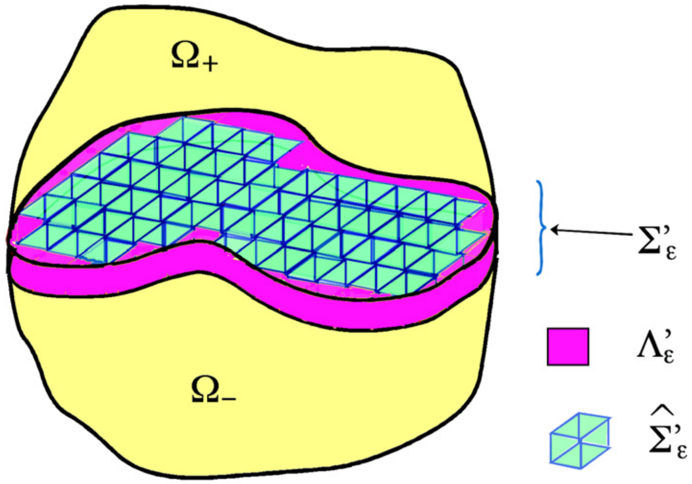

Fig. 4. The sets Σ'_ε, $\widehat{\Sigma}_\varepsilon$ and Λ'_ε.

Definition 2.15. For $\phi \in L^p(\Sigma'_\varepsilon)$, $p \in [1, \infty[$ the unfolding operator $\mathcal{T}^{bl}_{\varepsilon,\delta} : L^p(\Sigma'_\varepsilon) \to L^p(\Sigma \times \mathbb{R}^N)$ is defined by:

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)(x', z) = \begin{cases} \phi(\varepsilon[\frac{(x',0)}{\varepsilon}]_Y + \varepsilon\delta z) & \text{if } (x', z) \in \widehat{\Sigma}_\varepsilon \times \frac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases}$$

This operation, designed to capture the contribution of the barriers in the limit process, was originally used in [18]. We also introduce the notion of local average related to the hyperplane Σ.

Definition 2.16. The local average $M^{\varepsilon,bl}_Y : L^p(\Sigma'_\varepsilon) \mapsto L^p(\Sigma)$, is defined for every ϕ in $L^p(\Sigma'_\varepsilon)$, $1 \leqslant p < \infty$, by

$$M^{\varepsilon,bl}_Y(\phi)(x') = \delta^N \int_{\frac{1}{\delta}Y} \mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)(x', z)\,\mathrm{d}z = \begin{cases} \frac{1}{\varepsilon^N} \int_{\varepsilon[\frac{x'}{\varepsilon}]+\varepsilon Y} \phi(\zeta)\,\mathrm{d}\zeta & \text{if } x' \in \widehat{\Sigma}_\varepsilon, \\ 0 & \text{if } x' \in \Sigma \setminus \widehat{\Sigma}_\varepsilon. \end{cases}$$

Remark 2.17. Since elements of $L^p(\Sigma)$ can be considered as functions of $L^p(\Sigma'_\varepsilon)$, $M^{\varepsilon,bl}_Y$ can be applied to them. With this convention, $\mathcal{T}^{bl}_{\varepsilon,\delta}(M^{\varepsilon,bl}_Y(\phi)) = M^{\varepsilon,bl}_Y(\phi)$ on the set Σ.

We also have an equivalent of Proposition 2.9.

Proposition 2.18. *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \rightharpoonup w$ weakly in $H^1(\Omega)$. Then*

$$M^{\varepsilon,bl}_Y(w_\varepsilon) \to w_{|\Sigma} \quad \text{strongly in } L^2(\Sigma).$$

It is easy to check that most of the results stated in the previous subsection extend to $\mathcal{T}^{bl}_{\varepsilon,\delta}$.

Theorem 2.19 *(Properties of the operator $\mathcal{T}^{bl}_{\varepsilon,\delta}$).*

1. *For any $v, w \in L^p(\Sigma'_\varepsilon)$,*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(vw) = \mathcal{T}^{bl}_{\varepsilon,\delta}(v)\mathcal{T}^{bl}_{\varepsilon,\delta}(w).$$

2. *For any $u \in L^1(\Sigma'_\varepsilon)$,*

$$\varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(u)\,\mathrm{d}x'\,\mathrm{d}z = \int_{\widehat{\Sigma}'_\varepsilon} u\,\mathrm{d}x, \quad \textit{and} \quad \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \left|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\right| \mathrm{d}x'\,\mathrm{d}z \leqslant \int_{\Sigma'_\varepsilon} |u|\,\mathrm{d}x.$$

3. *For any* $u \in L^2(\Sigma'_\varepsilon)$,

$$\left\| \mathcal{T}^{bl}_{\varepsilon,\delta}(u) \right\|^2_{L^2(\Sigma\times\mathbb{R}^N)} \leqslant \frac{1}{\varepsilon\delta^N} \|u\|^2_{L^2(\Sigma'_\varepsilon)}.$$

4. *For any* $u \in L^1(\Sigma'_\varepsilon)$, *one has*

$$\left| \int_{\Sigma'_\varepsilon} u \, \mathrm{d}x - \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(u) \, \mathrm{d}x' \, \mathrm{d}z \right| \leqslant \int_{\Lambda'_\varepsilon} |u| \, \mathrm{d}x.$$

5. *Let* u *be in* $H^1(\Sigma'_\varepsilon)$. *Then,*

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta} \nabla_z \big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u) \big) \quad \text{in } \Sigma \times \frac{1}{\delta} Y.$$

6. *Suppose* $N \geqslant 3$ *and let* ω *be open and bounded in* $\mathbb{R}^N$. *Then the following estimates hold*:

$$\left\| \nabla_z \big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u) \big) \right\|^2_{L^2(\Sigma\times\frac{1}{\delta}Y)} \leqslant \frac{\varepsilon}{\delta^{N-2}} \|\nabla u\|^2_{L^2(\Sigma'_\varepsilon)},$$

$$\left\| \mathcal{T}^{bl}_{\varepsilon,\delta}\big(u - M^{\varepsilon,bl}_Y(u)\big) \right\|^2_{L^2(\Sigma;L^{2^*}(\mathbb{R}^N))} \leqslant \frac{C\varepsilon}{\delta^{N-2}} \|\nabla u\|^2_{L^2(\Sigma'_\varepsilon)},$$

and

$$\left\| \mathcal{T}^{bl}_{\varepsilon,\delta}(u) \right\|^2_{L^2(\Sigma\times\omega)} \leqslant 2\frac{C\varepsilon}{\delta^{N-2}} |\omega|^{2/N} \|\nabla u\|^2_{L^2(\Sigma'_\varepsilon)} + 2|\omega| \|u\|^2_{L^2(\Sigma'_\varepsilon)},$$

where C *denotes the Sobolev–Poincaré–Wirtinger constant for* $H^1(Y)$.

7. *Suppose* $N \geqslant 3$. *Let* $\{w_{\varepsilon,\delta}\}$ *be a sequence in* $H^1(\Sigma'_\varepsilon)$ *such that* $\|\nabla w_{\varepsilon,\delta}\|_{L^2(\Sigma'_\varepsilon)}$ *is bounded. Then, up to a subsequence, there exist* W *in* $L^2(\Sigma; L^{2^*}(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^2(\Sigma \times \mathbb{R}^N)$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M^\varepsilon_Y(w_{\varepsilon,\delta}) 1_{\frac{1}{\delta}Y} \big) \rightharpoonup W \quad \text{weakly in } L^2\big(\Sigma; L^{2^*}(\mathbb{R}^N)\big),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \nabla_z \big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \big) 1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in } L^2\big(\Sigma \times \mathbb{R}^N\big).$$

Assuming furthermore that $\limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{N/2-1}}{\sqrt{\varepsilon}} < +\infty$, *one can choose the subsequence above and some* U *in* $L^2(\Sigma; L^2_{\mathrm{loc}}(\mathbb{R}^N))$ *with*

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2\big(\Sigma; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big).$$

Proposition 2.20 *(u.c.i.). If* $\{w_\varepsilon\}$ *is a sequence in* $L^1(\Sigma'_\varepsilon)$ *satisfying,*

$$\int_{\Lambda'_\varepsilon} |w_\varepsilon| \, \mathrm{d}x \to 0,$$

then

$$\int_{\Sigma'_\varepsilon} w_\varepsilon \, \mathrm{d}x \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\cong} \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(w_\varepsilon) \, \mathrm{d}x' \, \mathrm{d}z.$$

Corollary 2.21. *Let* $\{u_\varepsilon\} \subset L^2(\Sigma'_\varepsilon)$ *and* $\{v_\varepsilon\} \subset L^p(\Sigma'_\varepsilon)$ *with* $p > 2$, *such that* $\|u_\varepsilon\|_{L^2(\Sigma'_\varepsilon)}$ *and* $\|v_\varepsilon\|_{L^p(\Sigma'_\varepsilon)}$ *are bounded independently of* ε. *Then*

$$\int_{\Sigma'_\varepsilon} u_\varepsilon v_\varepsilon \, \mathrm{d}x \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\cong} \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(u_\varepsilon) \mathcal{T}^{bl}_{\varepsilon,\delta}(v_\varepsilon) \, \mathrm{d}x' \, \mathrm{d}z.$$

For sieve problems, there is a need to distinguish between the subdomains above and below Σ. Set

$$\Omega_+ = \mathbb{R}^N_+ \cap \Omega, \qquad \Omega_- = \mathbb{R}^N_- \cap \Omega, \qquad Y_+ = \mathbb{R}^N_+ \cap Y, \qquad Y_- = \mathbb{R}^N_- \cap Y.$$

We suppose that the two domains Ω_+ and Ω_- have a Lipschitz boundary.

For simplicity, we will make the convention that all the results stated for Ω_+, are true also for Ω_- unless specified otherwise. For any function u defined in Ω, we denote by u^+ its restriction to the domain Ω_+, i.e., $u^+ \equiv u|_{\Omega_+}$. Analogously, $u^- \equiv u|_{\Omega_-}$.

The corresponding definitions and propositions are the following:

Definition 2.22. The local average $M^{\varepsilon,bl}_{Y_\pm} : L^p(\Sigma'_{\varepsilon\pm}) \mapsto L^p(\Sigma)$, is defined for every ϕ in $L^p(\Sigma'_{\varepsilon\pm})$, $1 \leqslant p < \infty$, by

$$M^{\varepsilon,bl}_{Y_\pm}(\phi)(x') \doteq \frac{\delta^N}{|Y_\pm|} \int\limits_{\frac{1}{\delta}Y_\pm} \mathcal{T}^{bl}_{\varepsilon,\delta}(\phi)(x',z)\,\mathrm{d}z.$$

Proposition 2.23. *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \rightharpoonup w^\pm$ weakly in $H^1(\Omega_\pm)$. Then*

$$M^{\varepsilon,bl}_{Y_\pm}(w_\varepsilon) \to w^\pm|_\Sigma \quad \textit{strongly in } L^2(\Sigma).$$

Theorem 2.24. 1. *For all $\phi \in L^2(\Omega_\pm)$,*

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big\|^2_{L^2(\Sigma\times\mathbb{R}^N_\pm)} \leqslant \frac{1}{\varepsilon\delta^N}\|u\|^2_{L^2(\Sigma'_{\varepsilon\pm})}.$$

2. *Suppose $N \geqslant 3$ and let u belong to $H^1(\Omega_\pm)$. For every ω open and bounded in $\mathbb{R}^N_+$ the following estimates hold*:

$$\big\|\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big)\big\|^2_{L^2(\Sigma\times\frac{1}{\delta}Y_\pm)} \leqslant \frac{\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_{\varepsilon\pm})},$$

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}\big(u - M^{\varepsilon,bl}_Y(u)\big)\big\|^2_{L^2(\Sigma;L^{2^*}(\mathbb{R}^N_\pm))} \leqslant \frac{C\varepsilon}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Sigma'_{\varepsilon\pm})},$$

and

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}(u)\big\|^2_{L^2(\Sigma\times\omega)} \leqslant 2\frac{C\varepsilon}{\delta^{N-2}}|\omega|^{2/N}\|\nabla u\|^2_{L^2(\Sigma'_{\varepsilon+})} + 2|\omega|\|u\|^2_{L^2(\Sigma'_{\varepsilon+})},$$

where C denotes the Sobolev–Poincaré–Wirtinger constant for $H^1(Y_\pm)$.

A similar inequality is true for bounded open subsets of $\mathbb{R}^N_-$.

3. *Suppose $N \geqslant 3$. Let $\{w_{\varepsilon,\delta}\}$ be a sequence in $H^1(\Sigma'_{\varepsilon+})$ such that $\|\nabla w_{\varepsilon,\delta}\|_{L^2(\Sigma'_{\varepsilon+})}$ is bounded. Then, up to a subsequence there exists W^+ in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N_\pm))$ with $\nabla_z W^+$ in $L^2(\Sigma\times\mathbb{R}^N_+)$ such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M^\varepsilon_{Y_+}(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y_+}\big) \rightharpoonup W^+ \quad \textit{weakly in } L^2\big(\Sigma; L^{2^*}(\mathbb{R}^N_+)\big),$$

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\big)1_{\frac{1}{\delta}Y_+} \rightharpoonup \nabla_z W^+ \quad \textit{weakly in } L^2\big(\Sigma\times\mathbb{R}^N_+\big).$$

Assuming furthermore that $\limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{N/2-1}}{\sqrt{\varepsilon}} < +\infty$, one can choose the subsequence above and some U^+ in $L^2(\Sigma; L^2_{\mathrm{loc}}(\mathbb{R}^N_+))$ with

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\mathcal{T}^{bl}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U^+ \quad \textit{weakly in } L^2\big(\Sigma; L^2_{\mathrm{loc}}(\mathbb{R}^N_+)\big).$$

The same result holds true for sequences in $H^1(\Sigma'_{\varepsilon-})$.

The equivalent of Proposition 2.20 (u.c.i.) also holds true in $\Omega_\pm$.

3. Homogenization in domains with small holes which are periodically distributed in volume

3.1. Functional setting

Let α and β be two real numbers such that $0 < \alpha < \beta$. For any open set $\mathcal{O}$ in $\mathbb{R}^N$, denote by $M(\alpha, \beta, \mathcal{O})$ the set of the $N \times N$ matrix-fields $A = (a_{ij})_{1\leqslant i,j\leqslant N} \in (L^\infty(\mathcal{O}))^{N\times N}$ such that

$$\alpha|\lambda|^2 \leqslant \big(A(x)\lambda, \lambda\big) \quad \text{and} \quad \big|A(x)\lambda\big|^2 \leqslant \beta\big(A(x)\lambda, \lambda\big),$$

for any $\lambda \in \mathbb{R}^N$ and a.e. x in $\mathcal{O}$.

The perforated domain $\Omega^*_{\varepsilon,\delta}$ is defined by (2.3). Assume that the matrix field $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\leqslant i,j\leqslant N}$ belongs to $M(\alpha, \beta, \Omega)$. For $f \in L^2(\Omega)$, consider the following problem:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in H^1_0(\Omega^*_{\varepsilon,\delta}) \text{ satisfying} \\ \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\phi = \int_{\Omega^*_{\varepsilon,\delta}} f\phi, \\ \forall\phi \in H^1_0(\Omega^*_{\varepsilon,\delta}). \end{cases} \tag{$\mathcal{P}_{\varepsilon,\delta}$}$$

In this section we suppose that $N \geqslant 3$ and study the asymptotic behavior of problem $(\mathcal{P}_{\varepsilon,\delta})$ as ε and $\delta = \delta(\varepsilon)$ are such that there exists a positive constant k_1 satisfying,

$$k_1 = \lim_{\varepsilon\to 0} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon}, \quad \text{with } 0 \leqslant k_1 < \infty. \tag{3.1}$$

3.2. Unfolded homogenization result

We now derive the unfolded formulation of the limit problem for $\mathcal{P}_{\varepsilon,\delta}$. In the limit we will observe the contribution of the periodic oscillations as well as the contribution of the perforations.

In order to state the result, we introduce the functional space K_B defined as follows:

$$K_B = \big\{\Phi \in L^{2^*}(\mathbb{R}^N);\ \nabla\Phi \in L^2(\mathbb{R}^N),\ \Phi \text{ constant on } B\big\}. \tag{3.2}$$

Theorem 3.1. *Let A^ε belong to $M(\alpha, \beta, \Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x, y) \to A(x, y) \quad \text{a.e. in } \Omega \times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that as ε and $\delta \to 0$,

$$\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(x, z) \to A_0(x, z) \quad \text{a.e. in } \Omega \times (\mathbb{R}^N \setminus B). \tag{3.3}$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega), \tag{3.4}$$

and there exists $\hat{u}$ in $L^2(\Omega; H^1_{per}(Y))$, and U vanishing on $\Omega \times B$ with $U - k_1u_0$ belonging to $L^2(\Omega; K_B)$, such that the triplet $(u_0, \hat{u}, U)$ satisfies the following three conditions:

$$\int_Y A(x, y)\big(\nabla_x u_0(x) + \nabla_y \hat{u}(x, y)\big)\nabla_y\phi(y)\,\mathrm{d}y = 0, \tag{3.5}$$

for a.e. x in Ω and all $\phi \in H^1_{per}(Y)$;

$$\int_{\mathbb{R}^N\setminus B} A_0(x, z)\nabla_z U(x, z)\,\nabla_z v(z)\,\mathrm{d}z = 0, \tag{3.6}$$

for a.e. x in Ω and all $v \in K_B$ with $v(B) = 0$;

$$\int_{\Omega\times Y} A(\nabla_x u_0 + \nabla_y\hat{u})\nabla\psi\,\mathrm{d}x\,\mathrm{d}y - k_1 \int_{\Omega\times\partial B} A_0\nabla_z U\nu_B\psi\,\mathrm{d}\sigma_z = \int_\Omega f\psi\,\mathrm{d}x, \tag{3.7}$$

for all $\psi \in H^1_0(\Omega)$, where ν_B is the inward normal on ∂B and $\mathrm{d}\sigma_z$ the surface measure.

The proof of this theorem, makes use of the next two elementary results.

Lemma 3.2. *Let $N \geqslant 3$. Then, for every $\delta_0 > 0$, the set*

$$\bigcup_{0<\delta<\delta_0} \{\phi \in H^1_{per}(Y);\ \phi = 0 \text{ on } \delta B\}$$

is dense in $H^1_{per}(Y)$.

Proof. Let $\psi \in C^\infty_{per}(\overline{Y})$ be fixed. For $\delta_k \xrightarrow{k\to\infty} 0$ consider $\phi_k \in H^1_{per}(Y)$ smooth with

$$\phi_k = \begin{cases} 0 & \text{on } \delta_k B, \\ 1 & \text{on } Y \setminus 2\delta_k B, \end{cases}$$

and such that $|\nabla \phi_k| \leqslant \frac{C}{\delta_k}$. Define $\Phi_k = \phi_k \psi$. We claim that Φ_k converges to ψ strongly in $H^1_{per}(Y)$. To do so, observe that

$$\|\Phi_k - \psi\|_{L^2(Y)} + \|\nabla\Phi_k - \nabla\psi\|_{L^2(Y)} \leqslant \int_{2\delta_k B} |\psi|^2\,\mathrm{d}y + \int_{2\delta_k B} |\nabla\psi|^2\,\mathrm{d}y + \int_{2\delta_k B} |\nabla\phi_k|^2|\psi|^2\,\mathrm{d}y.$$

For the last integral, using the definition of ϕ_k, one gets:

$$\int_{2\delta_k B} |\nabla\phi_k|^2|\psi|^2\,\mathrm{d}y \leqslant C^2\delta_k^{N-2}\|\psi\|^2_{L^\infty(Y)}.$$

Hence,

$$\Phi_k \to \psi \quad \text{strongly in } H^1_{per}(Y).$$

Since $H^1_{per}(Y)$ is the closure of $C^\infty_{per}(\overline{Y})$ in the H^1-norm, a density argument completes the proof. □

Lemma 3.3. *Let v in $\mathcal{D}(\mathbb{R}^N) \cap K_B$ (i.e. $v = const. = v(B)$ on B), and set*

$$w_{\varepsilon,\delta}(x) = v(B) - v\left(\frac{1}{\delta}\left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{for } x \in \mathbb{R}^N.$$

Then,

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \quad \text{weakly in } H^1(\Omega). \tag{3.8}$$

Proof. For δ small enough, the support of v is compact in $\frac{1}{\delta}Y$ and consequently,

$$\int_{\frac{1}{\delta}Y} |v(z)|^2\,\mathrm{d}z = \|v\|^2_{L^2(\mathbb{R}^n)}.$$

Clearly, $w_{\varepsilon,\delta}$ is uniformly bounded on $\mathbb{R}^N$. Observe that the set where $w_{\varepsilon,\delta}$ differs from $v(B)$ is $\bigcup_{\xi\in\mathbb{Z}^N}(\varepsilon\xi + \varepsilon\delta\{\text{Support}(v)\})$, so that the measure of its intersection with Ω, is at most of order δ^N. Thus, $w_{\varepsilon,\delta}$ converges to $v(B)$ in every $L^q(\Omega)$ for finite q.

Since $\mathcal{T}_{\varepsilon,\delta}(w_\varepsilon)(x,z) = v(B) - v(z)$, property (5) from Theorem 2.11 gives:

$$\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = -\frac{1}{\varepsilon\delta}\nabla_z v \quad \text{in } \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \tag{3.9}$$

hence (see Theorem 2.2(2)),

$$\|\nabla w_{\varepsilon,\delta}\|^2_{L^2(\hat{\Omega}_\varepsilon)} = \delta^N \big\|T_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\big\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \leqslant \frac{\delta^{N-2}}{\varepsilon^2}|\Omega|\|\nabla_z v\|^2_{L^2(\mathbb{R}^N)}.$$

Due to (3.1), $\nabla w_{\varepsilon,\delta}$ is bounded in $L^2_{\mathrm{loc}}(\Omega)$ which concludes the proof, since $w_{\varepsilon,\delta}$ is εY-periodic in $\mathbb{R}^N$. □

Proof of Theorem 3.1 (for the case $k_1 > 0$)**.** Observe first that by the Lax–Milgram theorem, there exists a unique solution $u_{\varepsilon,\delta}$ of $(\mathcal{P}_{\varepsilon,\delta})$ and it satisfies

$$\|u_{\varepsilon,\delta}\|_{H^1_0(\Omega^*_{\varepsilon,\delta})} \leqslant C\|f\|_{L^2(\Omega)}, \tag{3.10}$$

which implies convergence (3.4), up to a subsequence. Next, by Theorem 2.2(6), there exists $\hat{u} \in L^2(\Omega; H^1_{per}(Y))$ such that

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \hat{u} \quad \text{weakly in } L^2(\Omega \times Y). \tag{3.11}$$

By Theorem 2.11(7), there exists some U in $L^2(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^N))$ such that, up to a subsequence,

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2\big(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big). \tag{3.12}$$

By Proposition 2.9, one has

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}M^\varepsilon_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to k_1 u_0 \quad \text{strongly in } L^2\big(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big). \tag{3.13}$$

On the other hand, by Theorem 2.11(7) there exists a W in $L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Omega \times \mathbb{R}^N)$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\big(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) - M^\varepsilon_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2\big(\Omega; L^{2^*}(\mathbb{R}^N)\big). \tag{3.14}$$

From (3.12), (3.13) and (3.14), one concludes:

$$U = W + k_1 u_0, \quad \text{and} \quad \nabla_z U = \nabla_z W,$$

and, by Theorem 2.11(5) and (7) again

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\,1_{\frac{1}{\delta}Y} = \delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_z U \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^N). \tag{3.15}$$

From Definition 2.10, $\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Omega \times B$, so that by (3.12),

$$U = 0 \quad \text{on } \Omega \times B. \tag{3.16}$$

Due to (3.16), $W = U - k_1 u_0$ actually belongs to $L^2(\Omega; K_B)$.

Using $\Phi(\cdot) = \varepsilon\psi(\cdot)\phi(\frac{\cdot}{\varepsilon})$ as a test function in $(\mathcal{P}_{\varepsilon,\delta})$, with $\psi \in \mathcal{D}(\Omega)$ and $\phi \in C^1_{per}(Y)$ vanishing in a neighborhood of the origin, we have:

$$\varepsilon\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\nabla\psi\phi\left(\frac{\cdot}{\varepsilon}\right) + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\,\psi\,\nabla\phi\left(\frac{\cdot}{\varepsilon}\right) = \varepsilon\int_{\Omega^*_{\varepsilon,\delta}} f\psi\phi\left(\frac{\cdot}{\varepsilon}\right).$$

It is easy to see that the first integral as well as the right-hand side of the above equality converge to zero. The second integral above is unfolded with $\mathcal{T}_\varepsilon$ noting that $\mathcal{T}_\varepsilon(\nabla\phi(\cdot/\varepsilon))(x,y) = \nabla\phi(y)$. Applying Theorem 2.2(1) and (4), then Corollary 2.5, one gets:

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\psi\nabla\phi\left(\frac{\cdot}{\varepsilon}\right) \overset{\mathcal{T}_\varepsilon}{\cong} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)(x,y)\mathcal{T}_\varepsilon(\nabla_x u_{\varepsilon,\delta})(x,y)\nabla\phi(y)\mathcal{T}_\varepsilon(\psi)(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.17}$$

(the unfolding criterion of integrals (u.c.i.) is trivially satisfied since ψ is compactly supported in Ω). From (3.11), we can pass to the limit with respect to ε in (3.17). Then, by Lemma 3.2, we obtain (3.5), the first equation of the unfolded formulation for the limit problem. This equation describes the effect of the periodic oscillations of the coefficients in $(\mathcal{P}_{\varepsilon,\delta})$.

In order to describe the contribution of the perforations, we use the function $w_{\varepsilon,\delta}$ introduced in Lemma 3.3. For ψ in $\mathcal{D}(\Omega)$, use $w_{\varepsilon,\delta}\,\psi$ as a test function in $(\mathcal{P}_{\varepsilon,\delta})$. By the definition of $w_{\varepsilon,\delta}$ this function vanishes on the holes and by the choice of ψ, it vanishes near the boundary of Ω. Thus, we obtain,

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\nabla w_{\varepsilon,\delta}\,\psi + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta}\nabla\psi\, w_{\varepsilon,\delta} = \int_{\Omega^*_{\varepsilon,\delta}} f w_{\varepsilon,\delta}\psi. \tag{3.18}$$

The first term in (3.18) is unfolded with $\mathcal{T}_{\varepsilon,\delta}$. Again, the choice of the test function implies that the u.c.i. is satisfied, so by Theorem 2.11 and Corollary 2.14, we can write,

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta} \psi \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\psi). \tag{3.19}$$

Therefore (3.19), together with (3.9), yields

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta} \psi \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi). \tag{3.20}$$

From the following obvious inequality,

$$\left\|\mathcal{T}_{\varepsilon,\delta}(\psi)-\psi\right\|_{L^\infty(\widehat{\Omega}_\varepsilon\times\frac{1}{\delta}Y)} \leqslant C\varepsilon\|\nabla\psi\|_{L^\infty(\Omega)},$$

we obtain:

$$\mathcal{T}_{\varepsilon,\delta}(\psi)\nabla_z v \to \psi\nabla_z v \quad \text{strongly in } L^2(\Omega\times\mathbb{R}^N). \tag{3.21}$$

Convergences (3.15), (3.21), as well as hypothesis (3.3), allows us to pass to the limit in (3.20) to obtain:

$$\lim_{\varepsilon\to 0} \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta} \psi \,\mathrm{d}x = -k_1 \int_{\Omega\times(\mathbb{R}^N\setminus B)} A_0(x,z)\nabla_z U(x,z)\nabla_z v(z)\psi(x)\,\mathrm{d}x\,\mathrm{d}z, \tag{3.22}$$

which by density, is true for every $v\in K_B$.

The second term in (3.18) is unfolded with $\mathcal{T}_\varepsilon$ and we have,

$$\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} w_{\varepsilon,\delta} \nabla\psi \overset{\mathcal{T}_\varepsilon}{\cong} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi).$$

Using Theorem 2.2(5) and convergences (3.8) and (3.11), we can pass to the limit with respect to ε in the above equality to get:

$$\lim_{\varepsilon\to 0} \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} w_{\varepsilon,\delta} \nabla\psi = v(B)\int_{\Omega\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x\psi, \tag{3.23}$$

where we also used the fact that $\mathcal{T}_\varepsilon(\nabla\psi)$ converges uniformly to $\nabla\psi$ (hence strongly in every $L^q(\Omega\times Y)$ for $1\leqslant q\leqslant\infty$).

Passing to the limit with respect to ε in (3.18) and using (3.22) and (3.23), we obtain,

$$v(B)\int_{\Omega\times Y} A(\nabla_x u_0 + \nabla_y\hat{u})\nabla\psi - k_1 \int_{\Omega\times(\mathbb{R}^N\setminus B)} A_0\nabla_z U\nabla v\psi = v(B)\int_\Omega f\psi, \tag{3.24}$$

which, by density, holds true for all $\psi\in H^1_0(\Omega)$ and $v\in K_B$. Choosing $v(B)=0$ in (3.24) yields Eq. (3.6), whereupon the Stokes formula transforms (3.24) into (3.7). This concludes the proof of the theorem. □

3.3. Standard form for the limit problem

Here we show that the unfolded problem is well-posed and we give the formulation in terms of the macroscopic solution u_0 alone.

First, consider the classical correctors $\widehat{\chi}_j$, $j=1,\dots,N$, defined by the cell problems (see [5]),

$$\begin{cases} \widehat{\chi}_j \in L^\infty(\Omega; H^1_{per}(Y)), \\ \int_Y A(x,y)\nabla(\widehat{\chi}_j - y_j)\nabla\phi\,\mathrm{d}y = 0 \quad \text{a.e. } x\in\Omega, \\ \forall\phi\in H^1_{per}(Y). \end{cases} \tag{3.25}$$

Assuming u_0 is known and solving Eq. (3.5) for $\hat{u}$ as a function of u_0, gives:

$$\hat{u}(x,y) = -\sum_{j=1}^{N} \frac{\partial u_0}{\partial x_j}(x)\widehat{\chi}_j(x,y),$$

which used in Eq. (3.7) from Theorem 3.1 yields

$$\int_{\Omega} \mathcal{A}^{\text{hom}} \nabla u_0 \nabla \psi \, \mathrm{d}x - k_1 \int_{\Omega\times\partial B} A_0 \nabla_z U \nu_B \psi \, \mathrm{d}\sigma_z = \int_{\Omega} f\psi \, \mathrm{d}x, \tag{3.26}$$

where, for a.e. x in Ω, $\mathcal{A}^{\text{hom}}(x)$ is the homogenized matrix defined as

$$\mathcal{A}_{ij}^{\text{hom}}(x) = \int_{Y} \left(a_{ij}(x,y) - \sum_{k=1}^{N} a_{ik}(x,y) \frac{\partial \widehat{\chi}_j}{\partial y_k}(x,y) \right) \mathrm{d}y. \tag{3.27}$$

Eq. (3.26) is the variational formulation for

$$-\operatorname{div}\big(\mathcal{A}^{\text{hom}} \nabla u_0\big) - k_1 \int_{\partial B} A_0 \nabla_z U \nu_B \, \mathrm{d}\sigma_z = f. \tag{3.28}$$

It remains to clarify the connection between the second term in (3.28) and u_0. In order to do so, let θ be the solution of the corresponding "cell problem":

$$\begin{cases} \theta \in L^\infty(\Omega; K_B), \quad \theta(x,B) \equiv 1, \\ \int_{\mathbb{R}^N\setminus B} {}^t A_0(x,z) \nabla_z \theta(x,z) \nabla_z \Psi(z) \, \mathrm{d}z = 0 \quad \text{a.e. for } x \in \Omega, \\ \forall \Psi \in K_B \text{ with } \Psi(B) = 0. \end{cases} \tag{3.29}$$

From (3.29), (3.16) and Green's formula together with Eq. (3.6), we get:

$$\int_{\partial B} A_0 \nabla_z U \nu_B \, \mathrm{d}\sigma_z = \int_{\partial B} A_0 \nabla_z (U - k_1 u_0) \nu_B \, \mathrm{d}\sigma_z = -k_1 u_0 \left(\int_{\partial B} {}^t A_0 \nabla_z \theta \nu_B \, \mathrm{d}\sigma_z \right),$$

so that Eq. (3.28) becomes

$$-\operatorname{div}\big(\mathcal{A}^{\text{hom}} \nabla u_0\big) + k_1^2 \boldsymbol{\Theta} u_0 = f,$$

where

$$\boldsymbol{\Theta}(x) \doteq \int_{\partial B} {}^t A_0(x,z) \nabla_z \theta(x,z) \nu_B \, \mathrm{d}\sigma_z. \tag{3.30}$$

Remark 3.4. From definition (3.30) the function $\boldsymbol{\Theta}(x)$ equals,

$$\boldsymbol{\Theta}(x) = \int_{\mathbb{R}^N\setminus B} A_0(x,z) \nabla_z \theta(x,z) \nabla_z \theta(x,z) \, \mathrm{d}z,$$

which is non-negative and can be interpreted as the local capacity of the set B.

In conclusion, by Lax–Milgram's theorem, we have:

Theorem 3.5. *The limit function u_0 given by Theorem* 3.1 *is the unique solution of the homogenized equation*:

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ \int_\Omega \mathcal{A}^{\text{hom}} \nabla u_0 \nabla \psi + k_1^2 \int_\Omega \boldsymbol{\Theta} u_0 \psi = \int_\Omega f\psi, \\ \forall \psi \in H_0^1(\Omega). \end{cases} \tag{3.31}$$

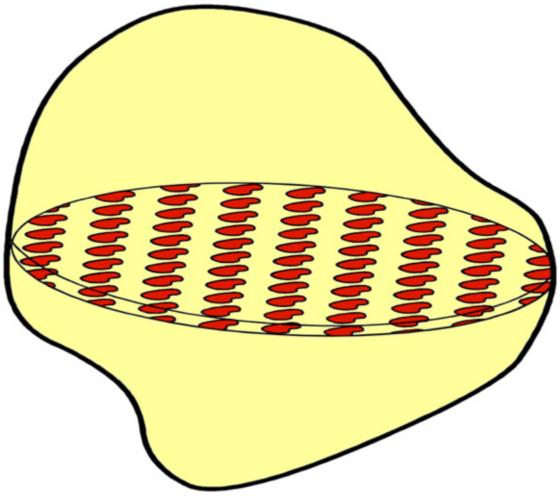

Fig. 5. An example of set $\Omega'_{\varepsilon,\delta}$: an electrostatic screen.

Remark 3.6. The contribution of the oscillations of the matrix A^ε in the homogenized problem are reflected by the first term of the left-hand side in (3.31). The contribution of the perforations is the zero order "strange term" $k_1^2\boldsymbol{\Theta}(x)u_0$.

Remark 3.7.

1. The proof is actually simpler for the case $k_1 = 0$ and the statement is included in Theorem 3.5: the small holes have no influence at the limit.
2. The case of $\lim \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} = \infty$ is easy to analyze: from Theorem 2.11(6),
$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0 \quad \text{weakly in } L^2\big(\Omega; L^2_{\text{loc}}(\mathbb{R}^N)\big).$$
On the other hand, since $\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Omega \times B$, this implies that $u_0 = 0$.

4. Homogenization in domains with small holes which are periodically distributed in a layer

4.1. Functional setting

As in the preceding section, we suppose that $N \geqslant 3$. We use the notations introduced in Section 2.3 for domains with small holes contained in the layer Σ'_ε. The corresponding perforated layer $\Sigma'_{\varepsilon,\delta}$ is given by:

$$\Sigma'_{\varepsilon,\delta} = \left\{ x \in \Sigma'_\varepsilon \text{ such that } \left\{\frac{x}{\varepsilon}\right\}_Y \in Y^*_\delta \right\}.$$

The perforated domain is now (see Fig. 5 for an example),

$$\Omega'_{\varepsilon,\delta} = \Omega \setminus \left\{ x \in \Sigma'_\varepsilon \text{ such that } \left\{\frac{x}{\varepsilon}\right\}_Y \in \delta B \right\}.$$

The small perforations are of size $\varepsilon\delta$ with $\delta = \delta(\varepsilon)$ satisfying,

$$k_2 = \lim_{\varepsilon\to 0} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}, \quad \text{where } 0 \leqslant k_2 < \infty. \tag{4.1}$$

We consider the asymptotic behavior for the following problem:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in H^1_0(\Omega'_{\varepsilon,\delta}) \text{ satisfying} \\ \int_{\Omega'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega'_{\varepsilon,\delta}} f\phi, \quad f \in L^2(\Omega), \\ \forall \phi \in H^1_0(\Omega'_{\varepsilon,\delta}). \end{cases} \tag{$\mathcal{P}'_{\varepsilon,\delta}$}$$

4.2. Unfolded homogenization result

Theorem 4.1. *Let A^ε belong to $M(\alpha,\beta,\Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,y)\to A(x,y)\quad \text{a.e. in } \Omega\times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that, as ε and $\delta\to 0$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)(x',z)\to A_0(x',z)\quad \text{a.e. in } \Sigma\times(\mathbb{R}^N\setminus B). \tag{4.2}$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta}\rightharpoonup u_0\quad \text{weakly in } H^1_0(\Omega),$$

and there exists $\hat u\in L^2(\Omega;H^1_{per}(Y))$, and U satisfying (4.11) with $U-k_2u_0$ in $L^2(\Sigma;K_B)$, such that $(u_0,\hat u,U)$ solves the equations

$$\int_Y A(x,y)\big(\nabla_x u_0(x)+\nabla_y\hat u(x,y)\big)\nabla_y\phi(y)\,\mathrm{d}y=0, \tag{4.3}$$

for a.e. x in Ω and all $\phi\in H^1_{per}(Y)$;

$$\int_{\mathbb{R}^N\setminus B} A_0(x',z)\nabla_z U(x',z)\nabla_z v(z)\,\mathrm{d}z=0, \tag{4.4}$$

for a.e. x' in Σ and all $v\in K_B$ with $v(B)=0$;

$$\int_{\Omega\times Y} A(\nabla_x u_0+\nabla_y\hat u)\nabla\psi-k_2\int_{\Sigma\times\partial B} A_0\nabla_z U\,\nu_B\psi\,\mathrm{d}\sigma_z=\int_\Omega f\psi, \tag{4.5}$$

for all $\psi\in H^1_0(\Omega)$, where ν_B and $\mathrm{d}\sigma_z$ are the inward normal and the surface measure on ∂B.

For the proof of this theorem, we need the equivalent of Lemma 3.3 with a similar proof (where $\mathcal{T}_{\varepsilon,\delta}$ is replaced by $\mathcal{T}^{bl}_{\varepsilon,\delta}$).

Lemma 4.2. *Let v in $\mathcal{D}(\mathbb{R}^N)\cap K_B$ and, for δ small enough, set*

$$w^{bl}_{\varepsilon,\delta}(x)=v(B)-v\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}_Y,\frac{x_N}{\varepsilon\delta}\right)\quad \text{for } x\in\mathbb{R}^N.$$

Then,

$$w^{bl}_{\varepsilon,\delta}\rightharpoonup v(B)\quad \text{weakly in } H^1(\Omega). \tag{4.6}$$

Proof of Theorem 4.1 (for the case $k_2>0$)**.** We denote $u_{\varepsilon,\delta}$ the extension by zero to the whole of Ω of the solution of $(\mathcal{P}'_{\varepsilon,\delta})$. The reasoning is similar to that of the previous section. The following estimate is straightforward from $(\mathcal{P}'_{\varepsilon,\delta})$:

$$\|u_{\varepsilon,\delta}\|_{H^1_0(\Omega)}\leqslant C\|f\|_{L^2(\Omega)},$$

so that, up to a subsequence,

$$u_{\varepsilon,\delta}\rightharpoonup u_0\quad \text{weakly in } H^1_0(\Omega).$$

Eq. (4.3) is obtained exactly as in the proof of Theorem 3.1.

By Theorem 2.19(7), there exists some U in $L^2(\Sigma;L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\rightharpoonup U\quad \text{weakly in } L^2\big(\Sigma;L^2_{loc}(\mathbb{R}^N)\big). \tag{4.7}$$

Since $\mathcal{T}^{bl}_{\varepsilon,\delta}\big(M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})\big) = M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}$, Proposition 2.18 implies:

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to k_2 u_{0|\Sigma} \quad \text{strongly in } L^2\big(\Sigma; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big). \tag{4.8}$$

On the other hand, Theorem 2.19(7) gives a W in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W$ in $L^2(\Sigma\times\mathbb{R}^N)$, such that

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) - M^{\varepsilon,bl}_Y(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\big) \rightharpoonup W \quad \text{weakly in } L^2\big(\Sigma; L^{2^*}(\mathbb{R}^N)\big). \tag{4.9}$$

From (4.7), (4.8) and (4.9), one concludes:

$$U = W + k_2 u_0, \quad \text{and} \quad \nabla_z U = \nabla_z W,$$

and, by Theorem 2.19(5) and (7) again,

$$\sqrt{\varepsilon}\delta^{\frac{N}{2}}\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) = \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad \text{weakly in } L^2(\Sigma\times\mathbb{R}^N). \tag{4.10}$$

From Definition 2.15, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Sigma\times B$, so (4.7) implies:

$$U = 0 \quad \text{on } \Sigma\times B. \tag{4.11}$$

Therefore, $W = U - k_2 u_0$ belongs to $L^2(\Sigma; K_B)$.

In order to capture the contribution of the perforations to the limit problem, we adapt the proof of Theorem 3.1 and use Lemma 4.2. For $\psi\in\mathcal{D}(\Omega)$, let $\Phi \doteq \psi w^{bl}_{\varepsilon,\delta}$, be a test function in problem $(\mathcal{P}'_{\varepsilon,\delta})$. Since $w^{bl}_{\varepsilon,\delta}$ is constant outside Σ'_ε for δ small enough, one obtains:

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi + \int_{\Omega'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla\psi w^{bl}_{\varepsilon,\delta} = \int_{\Omega'_{\varepsilon,\delta}} f w^{bl}_{\varepsilon,\delta}\psi. \tag{4.12}$$

Observe that since $w^{bl}_{\varepsilon,\delta}$ vanishes in the holes, one actually has

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi = \int_{\Sigma'_\varepsilon} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi,$$

which unfolded with $\mathcal{T}^{bl}_{\varepsilon,\delta}$ gives:

$$\int_{\Sigma'_\varepsilon} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\cong} \varepsilon\delta^N \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}\big(\nabla w^{bl}_{\varepsilon,\delta}\big)\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \tag{4.13}$$

Properties (5) of Theorem 2.19 implies:

$$\mathcal{T}^{bl}_{\varepsilon,\delta}\big(\nabla w^{bl}_{\varepsilon,\delta}\big) = -\frac{1}{\varepsilon\delta}\nabla_z v,$$

so that (4.10) and (4.13) yield,

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\cong} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \int_{\Sigma\times\mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}\nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)(-\nabla_z v)\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \tag{4.14}$$

From the compactness of the support of v and the straightforward inequality,

$$\big\|\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi) - \psi\big\|_{L^\infty(\widehat{\Sigma}_\varepsilon\times\frac{1}{\delta}Y)} \leqslant c\varepsilon\|\nabla_x\psi\|_{L^\infty(\Omega)^N},$$

we obtain:

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi)\nabla_z v \to \psi\nabla_z v \quad \text{strongly in } L^2(\Sigma\times\mathbb{R}^N). \tag{4.15}$$

This, together with convergences (4.1) and (4.10), as well as hypothesis (4.2), allows us to pass to the limit in (4.14) which now reads

$$\lim_{\varepsilon\to 0}\int_{\Sigma'_{\varepsilon,\delta}} A^{\varepsilon}\nabla u_{\varepsilon,\delta}\nabla w^{bl}_{\varepsilon,\delta}\psi\,\mathrm{d}x = -k_2\int_{\Sigma\times\mathbb{R}^N} A_0(x',z)\nabla_z U(x',z)\nabla_z v\psi\,\mathrm{d}x'\,\mathrm{d}z. \tag{4.16}$$

By a density argument, (4.15) is true for every v in K_B.

The second term in (4.12) is unfolded with $\mathcal{T}_\varepsilon$ and using Theorem 2.2, we get at the limit

$$\lim_{\varepsilon\to 0}\int_{\Omega'_{\varepsilon,\delta}} A^{\varepsilon}\nabla u_{\varepsilon,\delta} w^{bl}_{\varepsilon,\delta}\nabla\psi\,\mathrm{d}x = v(B)\int_{\Omega\times Y} A(x,y)(\nabla_x u_0+\nabla_y\hat{u})\nabla_x\psi\,\mathrm{d}x\,\mathrm{d}y,$$

which, with (4.16) gives Eq. (4.4). Eq. (4.5) is obtained similarly. □

4.3. Standard form of the homogenized equation

Like in Section 3.4, one can rewrite system (4.3)–(4.5) in the standard form. The result is stated in the next theorem, the proof of which follows the same lines as that of Theorem 3.5.

Theorem 4.3. *The limit function u_0 given by Theorem* 4.1 *is the solution of the homogenized equation*:

$$\begin{cases} u_0\in H^1_0(\Omega),\\ \int_\Omega \mathcal{A}^{\mathrm{hom}}\nabla u_0\nabla\psi + k_2^2\int_\Sigma \boldsymbol{\Theta}' u_0\psi = \int_\Omega f\psi,\\ \forall\psi\in H^1_0(\Omega),\end{cases} \tag{4.17}$$

where $\boldsymbol{\Theta}'$ is defined by (3.30) *with x' in place of x.*

Remark 4.4. The strong formulation for (4.17) is the following:

$$\begin{cases} -\operatorname{div}\mathcal{A}^{\mathrm{hom}}\nabla u_0 = f & \text{in } \Omega\setminus\Sigma,\\ -[\mathcal{A}^{\mathrm{hom}}\nabla u_0] = (k_2)^2\boldsymbol{\Theta}' u_0 & \text{on } \Sigma,\\ u_0 = 0 & \text{on } \partial\Omega,\end{cases}$$

where $[\mathcal{A}^{\mathrm{hom}}\nabla u_0]$ denotes the jump across Σ,

$$\big[\mathcal{A}^{\mathrm{hom}}\nabla u_0\big] \doteq \mathcal{A}^{\mathrm{hom}}\nabla u_0^- n^- + \mathcal{A}^{\mathrm{hom}}\nabla u_0^+ n^+ \quad \text{on } \Sigma,$$

n^+ and n^- denoting the respective exterior unit normal to Ω_+ and Ω_- on Σ.

Remark 4.5. 1. The proof for the case $k_2=0$ is actually simpler, and the statement is included in Theorem 4.3: the small holes have no influence at the limit, i.e. the equation $-\operatorname{div}\mathcal{A}^{\mathrm{hom}}\nabla u_0 = f$ is satisfied in the whole of Ω.

2. As in Remark 3.7, for the case of $\lim \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} = \infty$, Theorem 2.19(6) implies:

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0|_\Sigma \quad \text{weakly in } L^2\big(\Sigma; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big).$$

On the other hand, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Sigma\times B$ implies that $u_0|_\Sigma = 0$. Therefore, the limit problem splits into two separate homogeneous Dirichlet problems in Ω_+ and Ω_-,

$$\begin{cases} -\operatorname{div}\mathcal{A}^{\mathrm{hom}}\nabla u_0 = f & \text{in } \Omega_\pm,\\ u_0 = 0 & \text{on } \partial\Omega_\pm.\end{cases}$$

5. The thin Neumann sieve with variable coefficients

5.1. Functional setting

We use the same notations as in Sections 2 and 4. For an open subset S of $Y\cap\Pi$ such that $\overline{S}\subset(Y\cap\Pi)$, set

$$Y_\delta = Y_+\cup Y_-\cup\delta S,$$

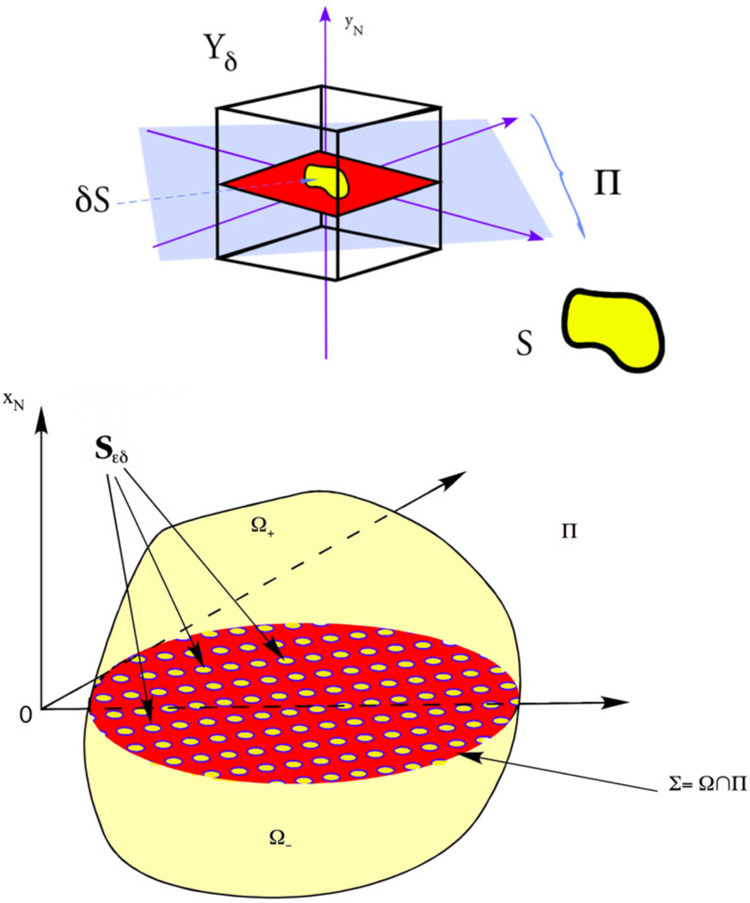

Fig. 6. The set Y_δ and the thin sieve $\Omega^{bl}_{\varepsilon\delta}$.

and

$$S_{\varepsilon,\delta} = \left\{ x \in \Sigma \text{ such that } \left\{ \frac{x}{\varepsilon} \right\}_Y \in \delta S \right\}.$$

For Ω open and bounded in $\mathbb{R}^N$ $(N \geqslant 3)$, define:

$$\Omega^{bl}_{\varepsilon\delta} = \Omega_+ \cup \Omega_- \cup S_{\varepsilon,\delta} \quad \text{and} \quad \Sigma'_{\varepsilon,\delta} \doteq \Sigma'_\varepsilon \cap \Omega^{bl}_{\varepsilon\delta}.$$

The connection between Ω_+ and Ω_- occurs through the "sieve" consisting of the set $S_{\varepsilon,\delta}$ (see Fig. 6). We assume that ε and δ satisfy assumption (4.1) of Section 4:

$$k_2 = \lim_{\varepsilon \to 0} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}}, \quad \text{where } 0 \leqslant k_2 < \infty.$$

Consider the space

$$V = \left\{ v \in H^1(\Omega_+ \cup \Omega_-);\ v = 0 \text{ on } \partial\Omega \right\},$$

which is a Hilbert space for the scalar product,

$$\langle u, v \rangle_V = \int\limits_{\Omega_+ \cup \Omega_-} \nabla u \nabla v \quad \text{for all } u, v \in V.$$

For simplicity, when v belongs to V, we denote ∇v the $L^2(\Omega)-$function which equals the gradient of v in $\Omega_+ \cup \Omega_-$ (this is the restriction to $\Omega_+ \cup \Omega_-$ of the distributional gradient of v). We also denote by $[v]$ the jump of v across Σ, $[v] \doteq v^+|_\Sigma - v^-|_\Sigma$, which belongs to $H^{1/2}(\Sigma)$. Finally set

$$V_{\varepsilon,\delta} = \{v \in V, [v] = 0 \text{ on } S_{\varepsilon,\delta}\}.$$

The thin Neumann sieve model is:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in V_{\varepsilon,\delta} \text{ satisfying,} \\ \int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega^{bl}_{\varepsilon,\delta}} f\phi, \quad f \in L^2(\Omega), \\ \forall \phi \in V_{\varepsilon,\delta}. \end{cases} \tag{$\mathcal{P}^{bl}_{\varepsilon,\delta}$}$$

5.2. *Unfolded homogenization result*

In this problem, the equivalent of the space K_B of Section 3 (see (3.2)), is

$$\widehat{K_S} = \{\Phi \in H^1_{\text{loc}}(\mathbb{R}^N_+ \cup \mathbb{R}^N_-);\ \nabla\Phi \in L^2(\mathbb{R}^N_+ \cup \mathbb{R}^N_-),\ [\Phi] = 0 \text{ on } S\}. \tag{5.1}$$

Proposition 5.1. *There exist two linear forms $l^\pm$ on $\widehat{K_S}$ such that for every Φ in $\widehat{K_S}$, the functions $\Phi^\pm - l^\pm(\Phi)$ belong to $L^{2^*}(\mathbb{R}^N_\pm)$.*

The space $\widehat{K_S}$ is Hilbert space for the norm,

$$\|\Phi\|^2_{\widehat{K_S}} \doteq \|\nabla\Phi\|^2_{L^2(\mathbb{R}^N_+ \cup \mathbb{R}^N_-)} + \left(\frac{l^+(\Phi) + l^-(\Phi)}{2}\right)^2. \tag{5.2}$$

Furthermore,

$$\widehat{K_S}^\infty \doteq \{\Phi \in \widehat{K_S}, \Phi^\pm \in C^\infty(\mathbb{R}^N_\pm),\ \text{supp}(\nabla\Phi^\pm) \text{ bounded in } \mathbb{R}^N_\pm\},$$

is dense in $\widehat{K_S}$ for this norm.

Proof. Due to the Sobolev–Poincaré–Wirtinger inequality (applied in the sets $\frac{1}{\delta}Y_\pm$ with $\delta \to 0$), for every Φ in $\widehat{K_S}$, there exist two constants $l^\pm(\Phi)$ such that $(\Phi^\pm - l^\pm(\Phi))$ belong to $L^{2^*}(\mathbb{R}^N_\pm)$.

It is well known that the first term in (5.2) is a Hilbert semi-norm on the space $\widehat{K_S}$, so that, with the second term, it defines a norm. The density of $\widehat{K_S}^\infty$ in $\widehat{K_S}$ follows by a standard argument of truncation and regularization. □

Theorem 5.2. *Let A^ε belong to $M(\alpha, \beta, \Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x, y) \to A(x, y) \quad \text{a.e. in } \Omega \times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that, as ε and $\delta \to 0$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)(x', z) \to A_0(x', z) \quad \text{a.e. in } \Sigma \times \mathbb{R}^N. \tag{5.3}$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}^{bl}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } V,$$

and there exists $\hat{u} \in L^2(\Omega; H^1_{per}(Y))$, $U \in L^2(\Sigma; \widehat{K_S})$ satisfying,

$$l^\pm(U) = k_2 u^\pm_{0\,|\Sigma} \quad \text{for a.e. } x' \in \Sigma, \tag{5.4}$$

and such that $(u_0, \hat{u}, U)$ solves the following three equations:

$$\int_Y A(x, y)\big(\nabla_x u_0(x) + \nabla_y \hat{u}(x, y)\big)\nabla_y \phi(y)\, \mathrm{d}y = 0, \tag{5.5}$$

for a.e. x in Ω and all $\phi \in H^1_{per}(Y)$,

$$\int_{\mathbb{R}^N} A_0(x', z)\nabla_z U(x', z)\nabla_z v(z)\, \mathrm{d}z = 0, \tag{5.6}$$

for a.e. x' *in* Σ *and all* $v \in \widehat{K}_S$ *with* $l^{\pm}(v) = 0$*, and*

$$\int_{\Omega \times Y} A(\nabla_x u_0 + \nabla_y \hat{u}) \nabla \phi - k_2 \int_{\Sigma \times S} A_0 \, \nabla_z U n^+ [\phi]_\Sigma = \int_\Omega f \phi, \tag{5.7}$$

for all $\phi \in V$.

Proof (for the case $k_2 > 0$)**.** Let $u_{\varepsilon,\delta}$ be a test function in $(\mathcal{P}^{bl}_{\varepsilon,\delta})$. Using the Poincaré inequality on Ω_+ and Ω_-, there is a constant C (independent of ε, δ) such that,

$$\|u_{\varepsilon,\delta}\|_V \leqslant C \|f\|_{L^2(\Omega)}.$$

Consequently, up to a subsequence, there exists $u_0 \in V$ such that

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } V.$$

By Theorem 2.2, one can also assume that there exists $\hat{u} \in L^2(\Omega; H^1_{per}(Y))$ with,

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \hat{u} \quad \text{weakly in } L^2(\Omega \times Y).$$

Using $\psi \in \mathcal{D}(\Omega)$ as a test function in $(\mathcal{P}^{bl}_{\varepsilon,\delta})$, and unfolding with operator $\mathcal{T}_\varepsilon$, we get:

$$\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \psi \, dx \overset{\mathcal{T}_\varepsilon}{\cong} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(A^\varepsilon) \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \mathcal{T}_\varepsilon(\nabla \psi) \, dx \, dy.$$

Applying properties (5) and (6) of Theorem 2.2 we can pass to the limit to obtain,

$$\int_{\Omega \times Y} A(x,y) \big[\nabla_x u_0 + \nabla_y \hat{u}\big] \nabla_x \psi \, dx \, dy = \int_\Omega f \psi \, dx.$$

Next, consider $\phi \in H^1_{per}(Y)$ and $\psi \in \mathcal{D}(\Omega_+) \cup \mathcal{D}(\Omega_-)$. Using $\Phi(x) = \varepsilon \psi(x) \phi(\frac{x}{\varepsilon})$ as a test function in $(\mathcal{P}^{bl}_{\varepsilon,\delta})$ yields,

$$\varepsilon \int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla \psi \phi\left(\frac{\cdot}{\varepsilon}\right) + \int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \psi \nabla \phi\left(\frac{\cdot}{\varepsilon}\right) = \varepsilon \int_{\Omega^{bl}_{\varepsilon,\delta}} f \psi \phi\left(\frac{\cdot}{\varepsilon}\right).$$

As in Section 3.3, passing to the limit gives (5.5).

By Theorem 2.24(3), there exists $U \in L^2(\Sigma; L^2_{\text{loc}}(\mathbb{R}^N_\pm))$ such that (up to a subsequence),

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \mathcal{T}^{bl}_{\varepsilon,\delta}\big(u^\pm_{\varepsilon,\delta}\big) \rightharpoonup U^\pm \quad \text{weakly in } L^2\big(\Sigma; L^2_{\text{loc}}\big(\mathbb{R}^N_\pm\big)\big). \tag{5.8}$$

By construction $\mathcal{T}^{bl}_{\varepsilon,\delta}(M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta})) = M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta}) 1_{\frac{1}{\delta}Y_\pm}$. By Proposition 2.23, one has:

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} M^{\varepsilon,bl}_{Y_\pm}\big(u^\pm_{\varepsilon,\delta}\big) 1_{\frac{1}{\delta}Y} \to k_2 u^\pm_{0\,|\Sigma} \quad \text{strongly in } L^2\big(\Sigma; L^2_{\text{loc}}(\mathbb{R}^N_\pm)\big). \tag{5.9}$$

By Theorem 2.24(3) there exists a W in $L^2(\Sigma; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W^\pm$ in $L^2(\Sigma \times \mathbb{R}^N_\pm)$ such that

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \big(\mathcal{T}^{bl}_{\varepsilon,\delta}(u^\pm_{\varepsilon,\delta}) - M^{\varepsilon,bl}_{Y_\pm}(u^\pm_{\varepsilon,\delta}) 1_{\frac{1}{\delta}Y}\big) \rightharpoonup W^\pm \quad \text{weakly in } L^2\big(\Sigma; L^{2^*}(\mathbb{R}^N_\pm)\big). \tag{5.10}$$

From (5.8), (5.9) and (5.10), one concludes:

$$U^\pm = W^\pm + k_2 u^\pm_{0\,|\Sigma}, \quad \text{and} \quad \nabla_z U^\pm = \nabla_z W^\pm. \tag{5.11}$$

Again by Theorem 2.24(3), one has the convergence:

$$\frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \nabla_z\big(\mathcal{T}^{bl}_{\varepsilon,\delta}\big(u^\pm_{\varepsilon,\delta}\big)\big) = \sqrt{\varepsilon} \delta^{\frac{N}{2}} \mathcal{T}^{bl}_{\varepsilon,\delta}\big(\nabla u^\pm_{\varepsilon,\delta}\big) \rightharpoonup \nabla_z U^\pm \quad \text{weakly in } L^2\big(\Sigma \times \mathbb{R}^N_\pm\big). \tag{5.12}$$

From Definition 2.15, $\mathcal{T}^{bl}_{\varepsilon,\delta}(u^+_{\varepsilon,\delta}) = \mathcal{T}^{bl}_{\varepsilon,\delta}(u^-_{\varepsilon,\delta})$ on $\Sigma \times S$, so that by convergences (5.8), (5.12) one has:

$$[U(x', \cdot)] = 0 \quad \text{on } S \text{ for a.e. } x' \in \Sigma.$$

Therefore, $U \in L^2(\Sigma; \widehat{K_S})$, and (5.11) implies (5.4).

In order to obtain equations (5.6) and (5.7), choose a function v in $\widehat{K_S}^{\infty}$ and set:

$$w_{\varepsilon,\delta}(x', x_N) = v\left(\frac{1}{\delta}\left\{\frac{x'}{\varepsilon}\right\}_Y, \frac{x_N}{\varepsilon\delta}\right).$$

Clearly, $[w_{\varepsilon,\delta}] = 0$on $S_{\varepsilon,\delta}$ and $\nabla w^{\pm}_{\varepsilon,\delta}$ vanishes outside $\Sigma'_{\varepsilon,\delta}$ for δ small enough. One easily shows (as in Lemma 4.2) that

$$\begin{cases} \mathcal{T}_\varepsilon(w^{\pm}_{\varepsilon,\delta}) \to l^{\pm}(v) & \text{strongly in } L^2(\Omega_\pm), \\ w^{\pm}_{\varepsilon,\delta} \rightharpoonup l^{\pm}(v) & \text{weakly in } H^1(\Omega_\pm). \end{cases} \tag{5.13}$$

For $\psi \in \mathcal{D}(\Omega)$, using $\psi w_{\varepsilon,\delta}$ as a test function in problem $(\mathcal{P}^{bl}_{\varepsilon,\delta})$ gives:

$$\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\psi\, w_{\varepsilon,\delta} + \int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\psi = \int_{\Omega^{bl}_{\varepsilon,\delta}} f w_{\varepsilon,\delta}\psi. \tag{5.14}$$

The first term in (5.14) is unfolded with $\mathcal{T}_\varepsilon$ as usual. This yields

$$\int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\psi\, w_{\varepsilon,\delta} \overset{\mathcal{T}_\varepsilon}{\cong} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\, \mathrm{d}x\, \mathrm{d}y.$$

Applying (5.13) and properties (5) and (6) of Theorem 2.2, one obtains:

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega^{bl}_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla\psi\, w_{\varepsilon,\delta} = {} & l^+(v) \int_{\Omega_+ \times Y} A(x, y)(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x\psi\, \mathrm{d}x\, \mathrm{d}y \\ & + l^-(v) \int_{\Omega_- \times Y} A(x, y)(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x\psi\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

The second term in (5.14) is unfolded with $\mathcal{T}^{bl}_{\varepsilon,\delta}$. The choice of the test function implies that u.c.i. is satisfied, so

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\cong} \varepsilon\delta^N \int_{\Sigma \times \mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \tag{5.15}$$

Property (5) from Theorem 2.19 gives:

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = \frac{1}{\varepsilon\delta}\nabla_z v,$$

which, together with (5.15), yields

$$\int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\psi \overset{\mathcal{T}^{bl}_{\varepsilon,\delta}}{\cong} \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} \int_{\Sigma \times \mathbb{R}^N} \mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)\sqrt{\varepsilon}\delta^{\frac{N}{2}}\mathcal{T}^{bl}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\nabla_z v\,\mathcal{T}^{bl}_{\varepsilon,\delta}(\psi). \tag{5.16}$$

Convergences (5.3), (5.12), allow to pass to the limit in (5.16) to obtain:

$$\lim_{\varepsilon\to 0} \int_{\Sigma'_{\varepsilon,\delta}} A^\varepsilon \nabla u_{\varepsilon,\delta} \nabla w_{\varepsilon,\delta}\psi = k_2 \int_{\Sigma \times \mathbb{R}^N} A_0\, \nabla_z U(x', z)\nabla_z v\psi\, \mathrm{d}x'\, \mathrm{d}z.$$

Now, the limit in (5.14) becomes:

$$
\begin{aligned}
& l^+(v) \int\limits_{\Omega_+\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x \psi + l^-(v) \int\limits_{\Omega_-\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x \psi \\
& \quad + k_2 \int\limits_{\Sigma\times\mathbb{R}^N} A_0(x', z)\nabla_z U(x', z)\nabla_z v\psi \,\mathrm{d}x'\,\mathrm{d}z \\
& = l^+(v) \int\limits_{\Omega_+} f\psi + l^-(v) \int\limits_{\Omega_-} f\psi,
\end{aligned} \tag{5.17}
$$

which, by density, holds for every $v \in \widehat{K_S}$. Eq. (5.6) is then simply obtained by choosing $l^+(v) = l^-(v) = 0$ in (5.17).

Using (5.6) with an arbitrary v in $\widehat{K_S}^\infty$ one deduces by Green's formula that

$$
\int\limits_{\mathbb{R}^N_\pm} A_0 \nabla_z U \nabla_z v \,\mathrm{d}z = \int\limits_S A_0 \nabla_z U n^\pm \big(v(z') - l^\pm(v)\big)\,\mathrm{d}z', \tag{5.18}
$$

which still holds for every $v \in \widehat{K_S}$. Then, (5.18) together with (5.17) leads to,

$$
\begin{aligned}
& l^+(v)\Bigg(\int\limits_{\Omega_+\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x \psi - k_2 \int\limits_{\Sigma\times S} A_0 \nabla_z U n^+ \psi - \int\limits_{\Omega_+} f\psi \Bigg) \\
& \quad + l^-(v)\Bigg(\int\limits_{\Omega_-\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x \psi - k_2 \int\limits_{\Sigma\times S} A_0 \nabla_z U n^- \psi - \int\limits_{\Omega_-} f\psi \Bigg) \\
& \quad + k_2 \int\limits_{\Sigma\times S} \big(A_0 \nabla_z U n^+ + A_0 \nabla_z U n^-\big) v\psi = 0.
\end{aligned} \tag{5.19}
$$

Taking $l^+(v) = l^-(v) = 0$ in (5.19), implies that

$$
A_0 \nabla_z U n^+ + A_0 \nabla_z U n^- \doteq [A_0 \nabla_z U]_S = 0 \quad \text{a.e. on } \Sigma \times S. \tag{5.20}
$$

Since $l^+(v)$ and $l^-(v)$ are independent, (5.19) now gives the following two formulas:

$$
\begin{aligned}
& \int\limits_{\Omega_+\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x \psi - k_2 \int\limits_{\Sigma\times S} A_0 \nabla_z U n^+ \psi = \int\limits_{\Omega_+} f\psi, \\
& \int\limits_{\Omega_-\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla_x \psi - k_2 \int\limits_{\Sigma\times S} A_0 \nabla_z U n^- \psi = \int\limits_{\Omega_-} f\psi,
\end{aligned} \tag{5.21}
$$

which, by density, hold for every ψ in $H^1_0(\Omega)$. Let ϕ be arbitrary in V. Eq. (5.7) is obtained by choosing $\psi = \phi^+$, respectively $\psi = \phi^-$ in (5.21), and adding the two corresponding equations. □

5.3. Standard form of the homogenized equation

As in Section 4.4, one can write system (5.4)–(5.7) in a standard form, with only u_0 as unknown.

First, from (5.6), the first term in the left-hand side of (5.7), can be written in terms of the standard homogenized operator:

$$
\int\limits_{\Omega\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla\phi = \int\limits_{\Omega} \mathcal{A}^{\mathrm{hom}} \nabla u_0 \nabla\phi,
$$

for every ϕ in the space V, using the same cell-problems (3.25) and the same $\mathcal{A}^{\mathrm{hom}}$ given by (3.27).

Next, observe that for a given u_0, problem (5.4)–(5.6) for U, has a unique solution by the Lax–Milgram theorem (applied on a closed affine subspace of $\widehat{K_S}$).

Now, we show how Eq. (5.7) can be brought to the standard form. More precisely, it remains to clarify the connection between the term $-k_2 \int_S A_0 \nabla_z U n^+$ and $[u_0]_\Sigma$. In order to do so, let θ be the solution of the following "cell problem":

$$\begin{cases} \theta \in L^\infty(\Sigma; \widehat{K_S}), \quad l^\pm(\theta) \equiv \pm 1, \\ \int_{\mathbb{R}^N} {}^t A_0(x', z) \nabla_z \theta(x', z) \nabla_z \Psi(z)\, \mathrm{d}z = 0 \quad \text{for a.e. } x' \in \Sigma, \\ \forall \Psi \in \widehat{K_S} \text{ with } l^\pm(\Psi) = 0. \end{cases} \tag{5.22}$$

From (5.18) follows:

$$\int_{\mathbb{R}^N_+ \cup \mathbb{R}^N_-} A_0 \nabla_z U \nabla_z v\, \mathrm{d}z = \big(l^+(v) - l^-(v)\big) \int_S A_0 \nabla_z U n^- \, \mathrm{d}z'. \tag{5.23}$$

Similarly, the solution of (5.22) is unique and satisfies for a.e. x' in Σ,

$$\begin{gathered} {}^t A_0 \nabla_z \theta n^+ + {}^t A A_0 \nabla_z \theta n^- \doteq \big[{}^t A_0 \nabla_z \theta\big]_S = 0, \\ \int_{\mathbb{R}^N_+ \cup \mathbb{R}^N_-} {}^t A_0 \nabla_z \theta \nabla_z v\, \mathrm{d}z = \big(l^+(v) - l^-(v)\big) \int_S {}^t A_0 \nabla_z \theta n^- \, \mathrm{d}z'. \end{gathered} \tag{5.24}$$

Formula (5.23) holds for $v = \theta$, whereas (5.24) does for $v = U$, so that combining the two yields,

$$\int_S A_0 \nabla_z U n^- \, \mathrm{d}z' = \frac{l^+(\theta) - l^-(\theta)}{2} \int_S A_0 \nabla_z U n^- \, \mathrm{d}z' = \frac{l^+(U) - l^-(U)}{2} \int_S {}^t A_0 \nabla_z \theta n^- \, \mathrm{d}z'.$$

Consequently, by (5.4),

$$k_2 \int_S A_0(x', z) \nabla_z U(x', z) n^- \, \mathrm{d}z' = \frac{k_2^2}{2} \boldsymbol{\Theta}(x') [u_0]_\Sigma(x'),$$

where

$$\boldsymbol{\Theta}(x') \doteq \int_S {}^t A_0 \nabla_z \theta n^- \, \mathrm{d}z' = -\int_S {}^t A_0 \nabla_z \theta n^+ \, \mathrm{d}z',$$

the latter equality deriving from (5.23). Thus, Eq. (5.7) becomes:

$$\int_\Omega \mathcal{A}^{\mathrm{hom}} \nabla u_0 \nabla \phi \, \mathrm{d}x + \frac{k_2^2}{2} \int_\Sigma \boldsymbol{\Theta}(x') [u_0]_\Sigma(x') [\phi]_\Sigma(x')\, \mathrm{d}x' = \int_\Omega f \phi \, \mathrm{d}x.$$

We have proved the following theorem:

Theorem 5.3. *The limit function u_0 given by Theorem* 5.2 *is the solution of the homogenized equation*:

$$\begin{cases} u_0 \in V, \\ \int_\Omega \mathcal{A}^{\mathrm{hom}} \nabla u_0 \nabla \phi + \frac{k_2^2}{2} \int_\Sigma \boldsymbol{\Theta} [u_0]_\Sigma [\phi]_\Sigma = \int_\Omega f \phi, \\ \forall \phi \in V. \end{cases} \tag{5.25}$$

Remark 5.4. Taking $v = \theta$ in (5.24) shows that

$$\boldsymbol{\Theta}(x') = \frac{1}{2} \int_{\mathbb{R}^N_+ \cup \mathbb{R}^N_-} A_0(x', z) \nabla_z \theta(x', z) \nabla_z \theta(x', z)\, \mathrm{d}z$$

is non-negative. This implies existence and uniqueness of the solution u_0 of (5.25).

Remark 5.5. The strong formulation for the solution u_0 of the limit problem is:

$$\begin{cases} -\operatorname{div} \mathcal{A}^{\text{hom}} \nabla u_0 = f \quad \text{in } \Omega \setminus \Sigma, \\ \mathcal{A}^{\text{hom}} \nabla u_0 n^-|_\Sigma = -\mathcal{A}^{\text{hom}} \nabla u_0 n^+|_\Sigma = \frac{k_2^2}{2} \boldsymbol{\Theta}[u_0]_\Sigma, \\ u_0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Remark 5.6. In the case where A_0 is even with respect to z_N, θ vanishes on S. Then, $\boldsymbol{\Theta}(x')$ can be interpreted as the local capacity of the set S, the capacitary potential being $(1 \mp \theta^\pm)$.

Remark 5.7. 1. The proof for the case $k_2 = 0$ is actually simpler and the statement is included in Theorem 5.3: the holes are too small to keep any connection between Ω_+ and Ω_-. The limit problem is split into two independent problems in each of these sets with mixed homogeneous boundary conditions,

$$\begin{cases} -\operatorname{div} \mathcal{A}^{\text{hom}} \nabla u_0 = f & \text{in } \Omega_\pm, \\ \mathcal{A}^{\text{hom}} \nabla u_0 n^\pm|_\Sigma = 0 & \text{on } \Sigma, \\ u_0 = 0 & \text{on } \partial\Omega_\pm \setminus \Sigma. \end{cases}$$

2. For the case of $\lim \frac{\delta^{\frac{N}{2}-1}}{\sqrt{\varepsilon}} = \infty$, Theorem 2.24(2) implies:

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0^\pm|_\Sigma \quad \text{weakly in } L^2\big(\Sigma; L^2_{\text{loc}}\big(\mathbb{R}^N_\pm\big)\big).$$

On the other hand, $[\mathcal{T}^{bl}_{\varepsilon,\delta}(u_{\varepsilon,\delta})]_S = 0$ on $\Sigma \times S$ implies that $[u_0]|_\Sigma = 0$. Therefore, u_0 belongs to $H^1_0(\Omega)$ so that the limit problem is satisfied in the whole of Ω.

6. The thick Neumann sieve with variable coefficients

In this section we extend the results of Section 5 to the case of a thick Neumann sieve of thickness of order $\varepsilon > 0$. We will use the same notations, unless specified otherwise, and we only sketch the main modifications of setting and of the proof.

For an open subset S of $Y \cap \Pi$ such that $S \Subset (Y \cap \Pi)$, we introduce the class $\mathcal{F}_\mathcal{S}$ of admissible sets, which we use to describe a thick sieve with holes shaped according to S.

Definition 6.1. The subset set F of $\mathbb{R}^N$ is in $\mathcal{F}_\mathcal{S}$, if

(i) F is closed with connected complement in $\mathbb{R}^N$,
(ii) F is symmetric with respect to all the hyperplanes of equations $\{z_j = 0,\ j \in 1, \dots, N-1\}$ and $F = F_+ \cup F_- \cup \{\Pi \setminus S\}$,
(iii) F is such that $F \cap \frac{1}{\delta}\overline{Y} \subset \{|z_N| \leqslant \frac{1}{2\delta}\}$ for every $0 < \delta \ll 1$,
(iv) F_+ and F_- are unbounded with Lipschitz boundary,
(v) there exists some positive R such that the boundaries ∂F_+ and ∂F_- outside the ball of radius R, are Lipschitz graphs over $\mathbb{R}^{N-1}$.

For $F \in \mathcal{F}_\mathcal{S}$, set

$$F_\delta = \delta F \cap Y, \quad \text{and} \quad F_{\varepsilon,\delta} = \left\{ x \in \Sigma'_\varepsilon \text{ such that } \left\{ \frac{x}{\varepsilon} \right\}_Y \in F_\delta \right\}.$$

Define:

$$\Omega^{ns}_{\varepsilon\delta} = \Omega \setminus F_{\varepsilon\delta} \quad \text{and} \quad S_{\varepsilon,\delta} = \Omega^{ns}_{\varepsilon\delta} \cap \Pi.$$

Fig. 7 present an example of admissible set F in dimension 3. Fig. 8 is the corresponding sieve. Fig. 9 is a two dimensional cross-section.

We use the same space V as in Section 5, while the $V_{\varepsilon,\delta}$ is now:

$$V_{\varepsilon,\delta} = \big\{ v \in H^1\big(\Omega^{ns}_{\varepsilon\delta\,+} \cup \Omega^{ns}_{\varepsilon\delta\,-}\big),\ v|_{\partial\Omega} = 0,\ [v]_{S_{\varepsilon,\delta}} = 0 \big\}.$$

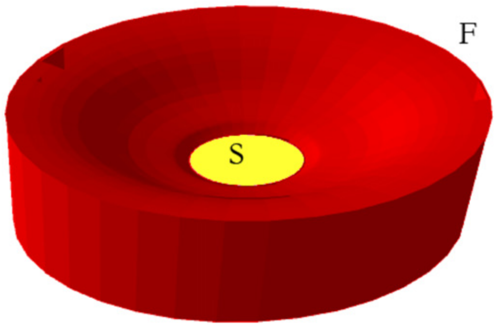

Fig. 7. An example of set F: the hole in the sieve.

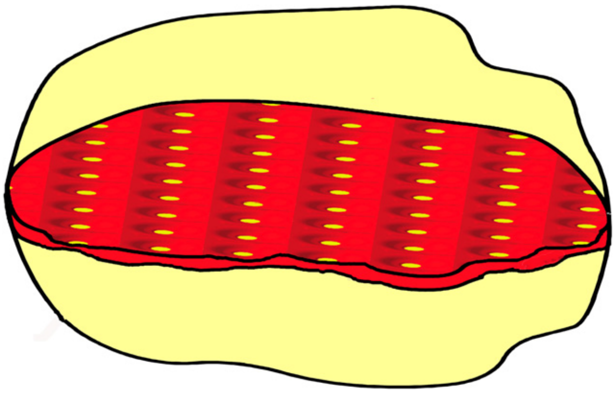

Fig. 8. The 3D geometry of the thick Neumann sieve.

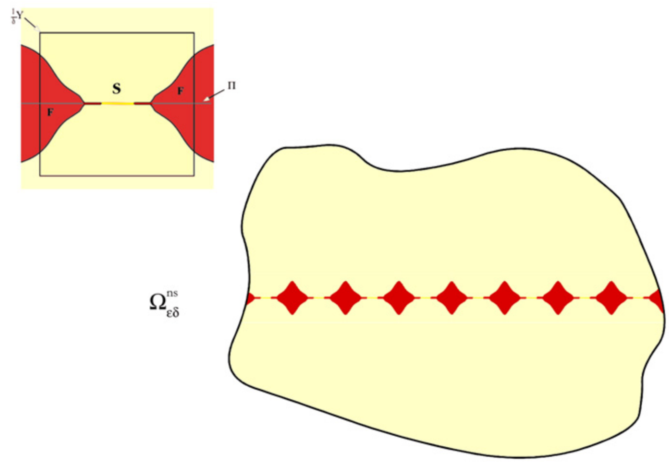

Fig. 9. A 2D cross-section of the set F and the domain $\Omega^{ns}_{\varepsilon,\delta}$.

The thick Neumann sieve problem can be stated as follows:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in V_{\varepsilon,\delta} \text{ satisfying,} \\ \int_{\Omega^{ns}_{\varepsilon,\delta}} A^{\varepsilon} \nabla u_{\varepsilon,\delta} \nabla \phi = \int_{\Omega^{ns}_{\varepsilon,\delta}} f\phi, \quad f \in L^2(\Omega), \\ \forall \phi \in V_{\varepsilon,\delta}. \end{cases} \qquad (\mathcal{P}^{ns}_{\varepsilon,\delta})$$

The equivalent of the space $\widehat{K_S}$ (see (5.1)) is the following, where G denotes the complement of F:

$$\widetilde{K_G} = \{\Phi \in H^1_{\mathrm{loc}}(G);\ \nabla\Phi \in L^2(G)\}. \tag{6.1}$$

Proposition 6.2. *There exist two linear forms $l^\pm$ on $\widetilde{K_G}$ such that for every Φ in $\widetilde{K_G}$, the functions $\Phi^\pm - l^\pm(\Phi)$ belong to $L^{2^*}((\mathbb{R}^N \setminus F)_\pm)$.*

The space $\widetilde{K_G}$ is a Hilbert space for the norm given by:

$$\|\Phi\|^2_{\widetilde{K_G}} \doteq \|\nabla\Phi\|^2_{L^2((\mathbb{R}^N_+ \cup \mathbb{R}^N_-)\setminus F)} + l^+(\Phi)^2 + l^-(\Phi)^2.$$

Furthermore, for this norm, l^+ and l^- are continuous on $\widetilde{K_G}$, and

$$\widetilde{K_G}^\infty \doteq \{\Phi \in \widetilde{K_G}, \Phi \in C^\infty(G),\ \mathrm{supp}(\nabla\Phi) \text{ bounded in } G\},$$

is dense in $\widetilde{K_G}$.

Proof. The proof is the same as that of Proposition 5.1. The only modification concerns the sequence of sets on which the Sobolev–Poincaré–Wirtinger inequality (with a uniform constant) is applied. In view of Definition 6.1(iv), this can be achieved on the sets $\frac{1}{\delta}Y_\pm \cap \{\pm z_N > R\} \cap G$ (making use of [22]). □

The unfolded limit problem and the standard homogenized equation are given in the next two theorems. Up to the modifications of notations indicated above, theirs proofs are the same as in Section 5.

Theorem 6.3. *Let Ω be open and bounded in $\mathbb{R}^N$, $N \geqslant 3$, and A^ε belong to $M(\alpha, \beta, \Omega)$. Suppose that, as ε goes to 0, there exists a matrix A such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x, y) \to A(x, y) \quad \text{a.e. in } \Omega \times Y.$$

Furthermore, suppose that there exists a matrix field A_0 such that, as ε and $\delta \to 0$,

$$\mathcal{T}^{bl}_{\varepsilon,\delta}(A^\varepsilon)(x', z) \to A_0(x', z) \quad \text{a.e. in } \Sigma \times (\mathbb{R}^N \setminus F).$$

Let $u_{\varepsilon,\delta}$ be the solution of the problem $(\mathcal{P}^{ns}_{\varepsilon,\delta})$. Then

$$u_{\varepsilon,\delta} \rightharpoonup u_0 \quad \text{weakly in } H^1_{\mathrm{loc}}(\Omega \setminus \Sigma),$$

and there exist $\hat{u} \in L^2(\Omega; H^1_{per}(Y))$, $U \in L^2(\Sigma; \widetilde{K}_G)$ satisfying,

$$l^\pm(U) = k_2(u_0^\pm)_{|\Sigma} \quad \text{for a.e. } x' \in \Sigma,$$

and such that $(u_0, \hat{u}, U)$ solves the equations,

$$\int_Y A(x, y)\big(\nabla_x u_0(x) + \nabla_y \hat{u}(x, y)\big)\nabla_y \phi(y)\, \mathrm{d}y = 0,$$

for a.e. x in Ω and all $\phi \in H^1_{per}(Y)$;

$$\int_G A_0(x', z)\nabla_z U(x', z)\nabla_z v(z)\, \mathrm{d}z = 0,$$

for a.e. x' in Σ and all $v \in \widetilde{K}_G$ with $l^\pm(v) = 0$,

$$\int_{\Omega\times Y} A(\nabla_x u_0 + \nabla_y \hat{u})\nabla\phi - k_2 \int_{\Sigma\times S} A_0 \nabla_z U n^+ [\phi]_\Sigma = \int_\Omega f\phi,$$

for all $\phi \in V$.

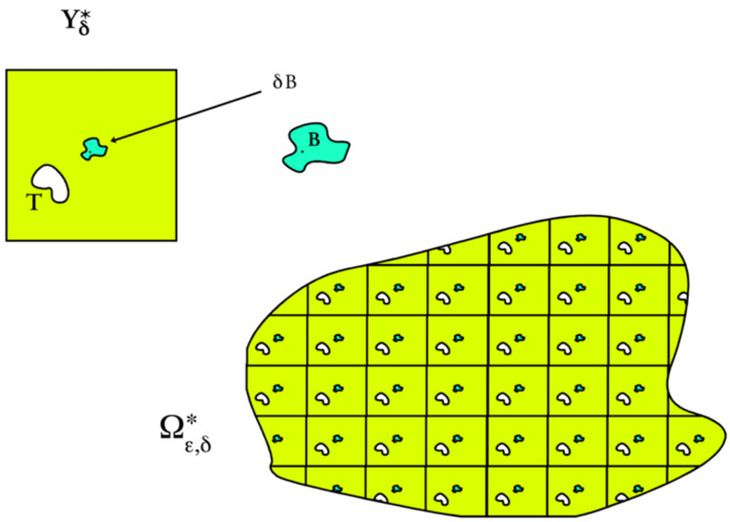

Fig. 10. The combination of a Neumann hole T and a Dirichlet hole δB.

Theorem 6.4. *The limit function u_0 given by Theorem* 6.3 *is the solution of the homogenized equation*:

$$\begin{cases} u_0 \in V, \\ \int_\Omega \mathcal{A}^{\text{hom}} \nabla u_0 \nabla \phi + \frac{k_2^2}{2} \int_\Sigma \boldsymbol{\Theta}[u_0]_\Sigma [\phi]_\Sigma = \int_\Omega f\phi, \\ \forall \phi \in V, \end{cases}$$

where

$$\boldsymbol{\Theta}(x') = \frac{1}{2} \int_G A_0(x', z) \nabla_z \theta(x', z) \nabla_z \theta(x', z)\, \mathrm{d}z,$$

and θ is the solution of the cell-problem,

$$\begin{cases} \theta \in L^\infty(\Sigma; \widetilde{K_G}), \quad l^\pm(\theta(x', \cdot)) \equiv \pm 1, \\ \int_G {}^t A_0(x', z) \nabla_z \theta(x', z) \nabla_z \Psi(z)\, \mathrm{d}z = 0, \quad \textit{a.e. for } x' \in \Sigma, \\ \forall \Psi \in \widetilde{K_G} \textit{ with } l^\pm(\Psi) = 0. \end{cases}$$

Remark 6.5. The function $\boldsymbol{\Theta}(x')$ can be interpreted as the local relative capacity (in G) of the set $\mathcal{C}(x')$ defined as the set where $\theta(x', \cdot)$ vanishes, the capacitary potential being $(1 - \theta(x', \cdot))$ "above $\mathcal{C}(x')$" and $(1 + \theta(x', \cdot))$ "below $\mathcal{C}(x')$".

Acknowledgements

We would like to thank the anonymous referee for her/his very constructive and thorough report. (S)he pointed out several typing errors and also made editing suggestions which significantly improved the readability of our original manuscript.

References

[1] N. Ansini, The nonlinear sieve problem and applications to thin films, Asymptotic Anal. 39 (2) (2004) 113–145.
[2] N. Ansini, A. Braides, Asymptotic analysis of periodically-perforated nonlinear media, J. Math. Pures Appl. 81 (5) (2002) 439–451; Erratum: J. Math. Pures Appl. 84 (5) (2005) 147–148.
[3] N. Ansini, J.-F. Babadjian, C.I. Zeppieri, The Neumann sieve problem and dimensional reduction: A multiscale approach, Math. Models Methods Appl. Sci. 17 (5) (2007) 681–735.
[4] H. Attouch, Variational Convergence for Functions and Operators, Pitman, Boston, 1984.
[5] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[6] J. Casado-Díaz, Two-scale convergence for nonlinear Dirichlet problems in perforated domains, Proc. Royal Soc. Edinburgh 130 A (2000) 249–276.
[7] J. Casado-Díaz, M. Luna-Laynez, J.D. Martín, An adaptation of the multi-scale methods for the analysis of very thin reticulated structures, C. R. Acad. Sci. Paris Sér. 1 332 (2001) 223–228.
[8] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Acad. Sci. Paris Sér. I 335 (2002) 99–104.
[9] D. Cioranescu, P. Donato, R. Zaki, Periodic unfolding and Robin problems in perforated domains, C. R. Acad. Sci. Paris Sér. I 342 (2006) 469–474.
[10] D. Cioranescu, F. Murat, Un terme étrange venu d'ailleurs, in: H. Brezis, J.L. Lions (Eds.), Nonlinear Partial Differential Equations and their Applications, College de France Seminar, II & III, in: Research Notes in Math., vol. 60–70, Pitman, Boston, 1982, pp. 98–138, 154–178.
[11] G. Dal Maso, F. Murat, Asymptotic behavior and correctors for linear Dirichlet problems with simultaneously varying operators and domains, Ann. Inst. H. Poincaré, Analyse nonlinéaire 21 (1982) 445–486.
[12] A. Damlamian, Le problème de la passoire de Neumann, Rend. Sem. Mat. Univ. Politecn. Torino 43 (3) (1985) 427–450.
[13] A. Damlamian, An elementary introduction to periodic unfolding, in: A. Damlamian, D. Lukkassen, A. Meidell, A. Piatnitski (Eds.), Proc. of the Narvik Conference 2004, in: Gakuto Int. Series, Math. Sci. Appl., vol. 24, Gakkokotosho, 2006, pp. 119–136.
[14] A. Damlamian, P. Donato, Which sequences of holes are admissible for periodic homogenization with Neumann boundary condition? ESAIM: COCV 8 (2006) 555–585.
[15] T. Del Vecchio, The thick Neumann's sieve, Ann. Mat. Pura Appl. 4 (147) (1987) 363–402.
[16] F. Murat, The Neuman sieve, in: Nonlinear Variational Problems, Isola d'Elba, 1983, in: Research Notes in Math., vol. 127, Pitman, Boston, 1985, pp. 24–32.
[17] G. Nguetseng, Problemes d'ecrans perforés pour l'équation de Laplace, RAIRO Model. Math. Anal. Numer. 19 (1) (1985) 33–63.
[18] D. Onofrei, The unfolding operator near a hyperplane and its application to the Neumann Sieve model, AMSA 16 (2006) 239–258.
[19] D. Onofrei, B. Vernescu, Asymptotics of a spectral problem associated with the Neumann Sieve, Anal. Appl. 3 (1) (2005) 69–87.
[20] C. Picard, Analyse limite d'équations variationelles dans un domain contenant une grille, RAIRO Model. Math. Anal. Numer. 21 (2) (1987) 293–326.
[21] E. Sanchez-Palencia, Boundary value problems in domains containing perforated walls, in: Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, III (Paris, 1980/1981), in: Res. Notes in Math., vol. 70, Pitman, Boston, 1982, pp. 309–325.
[22] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1971.