Controllability of switched time-delay systems under constrained switching

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Abstract

Different from the existing mathematical models for switched systems, where the switching from one subsystem to another subsystem is finished instantly, in this paper it is assumed that the switching is a transfer process. Moreover, there exists a basic transfer subsystem such that in the transfer process, the transfer subsystem is active. Based on the model of switched systems under constrained switching, this paper studies the controllability of such systems with time delay in the control function. A necessary and sufficient condition for controllability of such systems is established. Finally, an example is given to illustrate the utility of our results.

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1. Introduction

Switched systems arise in varied contexts in manufacturing, communication networks, auto pilot design, automotive engine control, computer synchronization, traffic control, chemical processes, and so on. Switched systems are a special class of hybrid dynamical systems which consist of a family of subsystems and a switching law specifying the
switching between the subsystems. In recent years, there has been increasing interest in
the control of switched systems due to their significance both in theory and applications
[1–11].

In the analysis and synthesis of dynamic systems, controllability and observability
are the two most fundamental concepts in modern control theory [12–14]. Ezzine and
Haddad [15] first studied the controllability, observability and stability for periodic type
switched systems, and some necessary and sufficient conditions were established. Then Xie
and Zheng [16] introduced the multiple-period controllability and observability concepts
and derived some necessary and sufficient criteria. It also pointed out that the controlla-
bility can be realized in \( n \) periods at least, where \( n \) is the state dimension. A sufficient
condition and a necessary condition for controllability of general switched linear systems
were presented in [17] and it was proved that the necessary condition is also sufficient
for 3-dimensional systems with only two subsystems. Following this work, Xie et al. [18]
extended the results to 3-dimensional systems with arbitrary number of subsystems. Nec-
essary and sufficient geometric type criteria for controllability and observability of general
switched systems were established in [19]. Then Ge et al. [20] extended the results to the
discrete-time case. Meanwhile, it was proved that the controllability can be realized by a
single switching sequence [21,22]. For the discrete-time case, a corresponding result was
also given in [23]. Different from the above work, Xu and Antsaklis [24] investigated the
reachability of a class of 2-dimensional switched systems. Yang [25] studied the controlla-
bility of linear switched systems and presented a sufficient condition and a necessary
condition by an algebraic approach.

A common characteristic of the models in the above work is that the switching between
two subsystems is finished instantly. However, in many real systems, this is not true. For
example, in an automobile power train, switching from one gear to another gear involves
two steps: first get out of gear, then put into another gear. Thus, from the former gear
being disconnected to the latter gear being connected, there is a transfer process. Other
examples include autopilot systems, car driving systems, etc. [26]. In order to describe this
phenomenon, a transfer subsystem is introduced into the model. We will investigate the
controllability of this new class of switched systems.

Meanwhile, time-delay phenomena are very common in practical systems, for instance,
economic, biological and physiological systems and so on. The controllability for general
time-delay systems was studied in [27–29]. Necessary and sufficient conditions for the
controllability of switched linear continuous-time systems with time-delay in the control
function were presented in [30].

In this paper, based on the model of switched linear systems under constrained switch-
ing, the controllability problem for such systems with time delay in control is formulated
and studied. A necessary and sufficient condition for controllability of such systems is
derived.

This paper is organized as follows. Section 2 formulates the problem and gives some
preliminaries. In Section 3, the controllable state set and the geometric criterion for
switched linear systems with time delay are discussed in detail. An example is given in
Section 4. Finally, Section 5 concludes the whole paper.
2. Preliminaries

Consider a switched linear system with time delay in control given by
\[
\dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t) + D_{r(t)}u(t-d),
\]
where \(x(t) \in \mathbb{R}^n\) is the state, \(u(t) \in \mathbb{R}^p\) is the input, the left continuous piecewise constant function \(r(t) : \mathbb{R}^+ \to \{0, 1, \ldots, N\}\) is the switching signal to be designed, where the integer \(N < \infty\), and \(d > 0\) is the fixed time delay in control.

**Assumption 1.** There exist \(N\) subsystems \((A_i, B_i, D_i), i \in \{1, \ldots, N\}\). Moreover, \(r(t) = i\) implies that the subsystem \((A_i, B_i, D_i)\) is activated.

**Assumption 2.** There exists a transfer subsystem \((A_0, B_0, D_0)\). For any \(i, j \in \{1, \ldots, N\}, i \neq j\), subsystem \((A_i, B_i, D_i)\) cannot be switched to subsystem \((A_j, B_j, D_j)\), but must pass through the transfer subsystem \((A_0, B_0, D_0)\) for some period of time (see Fig. 1).

Now, we introduce the concept of switching sequence to describe the switching signal.

**Definition 1 (Switching sequence).** A switching sequence is defined as
\[
\pi \overset{\text{def}}{=} \{(i_1, h_1), (i_2, h_2), \ldots; (i_m, h_m), \tau_m; \ldots; (i_{M-1}, h_{M-1}), \tau_{M-1}; (i_M, h_M)\},
\]
where \(M < \infty\) is the length of \(\pi\), \(i_m \in \{1, \ldots, N\}\) is the index of the \(m\)th subsystem, and \(h_m > 0\) is the dwell time of the \(m\)th subsystem, for \(m = 1, \ldots, M\). \(\tau_m \in (0, \tau_{\text{max}}]\) is the dwell time of the transfer subsystem \((A_0, 0, 0)\) between the subsystem \((A_{i_m}, B_{i_m}, D_{i_m})\) and the subsystem \((A_{i_{m+1}}, B_{i_{m+1}}, D_{i_{m+1}})\), for \(m = 1, \ldots, M - 1\). \(\tau_{\text{max}} < \infty\) is a positive scalar. Denote \(T[\pi] = h_1 + \tau_1 + h_2 + \tau_2 + \cdots + \tau_{M-1} + h_M\), and we call \(T[\pi]\) the period of the switching sequence \(\pi\).

**Remark 1.** In order to avoid unnecessary complexity, we assume that the dwell time \(h_m\) of the \(m\)th subsystem satisfies \(h_m > d\), where \(m \in \{1, \ldots, M\}\).
Given a switching sequence \( \pi \) defined by (2), an associated switching signal \( r(t), t \in [t_0, t_0 + T_\pi] \) can be determined as

\[
\begin{align*}
  r(t) &= \begin{cases} 
    i_m, & \text{if } t \in [t_m-1, t_m-1 + h_m), \\
    0, & \text{if } t \in [t_m-1 + h_m, t_m),
  \end{cases} 
\end{align*}
\]

(3)

where \( t_m = t_0 + \sum_{l=1}^m (h_l + \tau_l) \), for \( m = 1, \ldots, M - 1 \). If \( t \in [t_{M-1}, t_0 + T_\pi] \), \( r(t) = i_M \) (see Fig. 2).

Now, we introduce some mathematical preliminaries as the basic tools for the discussion in the remaining parts of the paper. Given a matrix \( B \in \mathbb{R}^{n \times p} \), let \( I_m(B) \) be the range of \( B \). Given a matrix \( A \in \mathbb{R}^{n \times n} \) and a linear subspace \( W \subseteq \mathbb{R}^n \), let \( \langle A \vert W \rangle \) be the minimal invariant subspace, i.e.,

\[
\langle A \vert W \rangle = \bigoplus_{i=1}^n A_i^{-1} W.
\]

(4)

It is easy to see that the minimal invariant subspace is a linear subspace and satisfies [21, 30, 31]

- \( A^m W \subseteq \langle A \vert W \rangle \),
- \( \exp(At)W \subseteq \langle A \vert W \rangle \),
- \( \exp(At)\langle A \vert W \rangle = \langle A \vert W \rangle \),
- \( \langle A \vert \exp(At)W \rangle = \langle A \vert W \rangle \),
- \( \langle A \vert W + V \rangle = \langle A \vert W \rangle + \langle A \vert V \rangle \).

For clarity, denote \( \langle A \vert B \rangle = \langle A \vert \text{Im}(B) \rangle \).

The following lemma is very basic, but it is the starting point to investigate the controllability problems.

**Lemma 1** [31]. Given matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times p} \), for any \( 0 \leq t_0 < t_f < +\infty \), we have

\[
\begin{cases}
  x = \int_{t_0}^{t_f} \exp[A(t_f - s)]Bu(s) \, ds, \forall \text{ piecewise continuous } u
\end{cases} \in \langle A \vert B \rangle.
\]

(5)

**Lemma 2.** Given matrices \( A_1 \in \mathbb{R}^{n \times n}, B_1, B_2 \in \mathbb{R}^{n \times p} \), we have

\[
\langle A_1 \vert B_1 + B_2 \rangle + \langle A_1 \vert B_2 \rangle = \langle A_1 \vert B_1 \rangle + \langle A_1 \vert B_2 \rangle,
\]

(6)

where \( \langle A_1 \vert B_1 + B_2 \rangle = \langle A_1 \vert \text{Im}(B_1 + B_2) \rangle \).
Proof. For any positive integer \( m \), we have
\[
\mathcal{I}m\left( A_1^m (B_1 + B_2) \right) + \mathcal{I}m\left( A_1^m B_2 \right) = \mathcal{I}m\left( \begin{bmatrix} A_1^m B_1 & A_1^m B_2, A_1^m B_2 \end{bmatrix} \right)
\]
\[
= \mathcal{I}m\left( \begin{bmatrix} I & 0 \\ I & 1 \end{bmatrix} \right)
\]
\[
= \mathcal{I}m\left( \begin{bmatrix} A_1^m B_1, A_1^m B_2 \end{bmatrix} \right).
\]
Therefore,
\[
\langle A_1 | B_1 + B_2 \rangle + \langle A_1 | B_2 \rangle = \sum_{m=0}^{n-1} \mathcal{I}m\left( A_1^m (B_1 + B_2) \right) + \mathcal{I}m\left( A_1^m B_2 \right)
\]
\[
= \sum_{m=0}^{n-1} \mathcal{I}m\left( \begin{bmatrix} A_1^m B_1, A_1^m B_2 \end{bmatrix} \right)
\]
\[
= \langle A_1 | B_1 \rangle + \langle A_1 | B_2 \rangle. \quad \square
\]

Lemma 3. Given matrices \( E, A_1, A_2 \in \mathbb{R}^{n \times n}, B_1, B_2 \in \mathbb{R}^{n \times p}, \) we have
\[
\left\{ x \mid x = \int_{t_0}^{t_f} \left( E \exp\left[ A_1 (t_f - s) \right] B_1 + \exp\left[ A_2 (t_f - s) \right] B_2 \right) u(s) ds, \forall \text{ piecewise continuous } u \right\}
\]
\[
\subseteq E \langle A_1 | B_1 \rangle + \langle A_2 | B_2 \rangle. \quad (7)
\]

Proof. It is easy to see that
\[
\left\{ x \mid x = \int_{t_0}^{t_f} \left( E \exp\left[ A_1 (t_f - s) \right] B_1 + \exp\left[ A_2 (t_f - s) \right] B_2 \right) u(s) ds, \forall \text{ piecewise continuous } u \right\}
\]
\[
\subseteq \left\{ x \mid x = \int_{t_0}^{t_f} \exp\left[ A_1 (t_f - s) \right] B_1 u_1(s) ds, \forall \text{ piecewise continuous } u_1 \right\}
\]
\[
+ \left\{ x \mid x = \int_{t_0}^{t_f} \exp\left[ A_2 (t_f - s) \right] B_2 u_2(s) ds, \forall \text{ piecewise continuous } u_2 \right\}
\]
\[
= E \langle A_1 | B_1 \rangle + \langle A_2 | B_2 \rangle. \quad \square \quad (8)
\]
Lemma 4 [21,22,30]. Given a matrix $A \in \mathbb{R}^{n \times n}$, for almost all $T \in \mathbb{R}$, we have
$$\langle A|\mathcal{W} \rangle = \langle \exp(AT)|\mathcal{W} \rangle, \forall \text{ linear subspace } \mathcal{W} \subseteq \mathbb{R}^n.$$

3. Controllability

In this section, we study the controllability of system (1).

Definition 2 (State controllability). For system (1), given initial state $x_0$ and initial input $u_0(t)$, $t \in [t_0-d,t_0)$, the state $x_f$ is said to be $(x_0,u_0)$-controllable, if there exist a switching sequence $\pi$ and a piecewise continuous function $u(t)$, $t \in [t_0,t_0+T_{\pi}]$, such that $x(t_0) = x_0$ and $x(t_0+T_{\pi}) = x_f$.

Definition 3 (System controllability). System (1) is said to be completely controllable, if for any initial state $x_0$ and initial input $u_0(t)$, $t \in [t_0-d,t_0)$, any state $x_f$ is $(x_0,u_0)$-controllable.

In order to establish the criterion for controllability, we first introduce the controllable state set.

3.1. Controllable state set

Definition 4 (Controllable state set). For system (1), given initial state $x_0$, initial input $u_0(t)$, $t \in [t_0-d,t_0)$, and a switching sequence $\pi$, the set of all the controllable states starting from $x_0$ and $u_0(t)$, evolving through the switching sequence $\pi$ is defined as the $(x_0,u_0)$-controllable state set of the switching sequence $\pi$, denoted by $\mathcal{C}(x_0,u_0,\pi)$.

In particular, when $x_0 = 0$ and $u_0 = 0$, $\mathcal{C}(x_0,u_0,\pi)$ is denoted by $\mathcal{C}(\pi)$ for clarity.

Theorem 1. For system (1), given a switching sequence
$$\pi = \{(i_1,h_1),\tau_1; (i_2,h_2),\tau_2; \ldots; (i_{M-1},h_{M-1}),\tau_{M-1}; (i_M,h_M)\},$$
its controllable state set is
$$\mathcal{C}(x_0,u_0,\pi) = \mathcal{I}n(x_0,u_0,\pi) + \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij}h_j) \exp(A_0\tau_{j-1}) \langle A_{im}|[B_{im},D_{im}] \rangle + \langle A_{i_M}[B_{i_M},D_{i_M}] \rangle;$$
where
$$\mathcal{I}n(x_0,u_0,\pi) = \prod_{m=M}^{2} \exp(A_{im}h_m) \exp(A_0\tau_{m-1}) \exp(A_{i_1}h_1) \times \left\{ x_0 + \int_{t_0-d}^{t_0} \exp[A_{i_1}(t_0-s)] \exp(-A_{i_1}d) D_{i_1}u_0(s) \, ds \right\}. \quad (9)$$

In particular, when $x_0 = 0$ and $u_0 = 0$,

$$
C(0, 0, \pi) = C(\pi) = \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \left[A_{i_m} \left[B_{i_m}, D_{i_m}\right]\right] \\
+ \left[A_{i_M} \left[B_{i_M}, D_{i_M}\right]\right].
$$

(11)

**Proof.** See Appendix A. □

Let $C$ denote the set of all the controllable states of system (1). It is easy to see that

$$
C = \bigcup_{\pi} C(\pi).
$$

(12)

And it is obvious that system (1) is completely controllable if and only if its controllable state set $C$ is the full space, i.e.,

$$
C = \mathbb{R}^n.
$$

(13)

In the following, we define two operations of the switching sequences and discuss the associated controllable state sets.

**Definition 5** (Concatenation of switching sequences). Given two switching sequences

$$
\pi_1 = \{(i_1, h_1), \mu_1; (i_2, h_2), \mu_2; \ldots; (i_{M-1}, h_{M-1}), \mu_{M-1}; (i_M, h_M)\},
$$

$$
\pi_2 = \{(j_1, g_1), v_1; (j_2, g_2), v_2; \ldots; (j_{L-1}, g_{L-1}), v_{L-1}; (j_L, g_L)\},
$$

and a scalar $\tau \in (0, \tau_{\text{max}}]$. The concatenation of $\pi_1$ and $\pi_2$ with $\tau$ is defined as

$$
\pi_1 \wedge \tau \pi_2 \text{ def } = \{(i_1, h_1), \mu_1; (i_2, h_2), \mu_2; \ldots; (i_{M-1}, h_{M-1}), \mu_{M-1}; (i_M, h_M), \tau; \\
(j_1, g_1), v_1; (j_2, g_2), v_2; \ldots; (j_{L-1}, g_{L-1}), v_{L-1}; (j_L, g_L)\}. 
$$

(14)

Since it is easy to verify that $(\pi_1 \wedge \mu \pi_2) \wedge \nu \pi_3 = \pi_1 \wedge \mu (\pi_2 \wedge \nu \pi_3)$, we just denote it by $\pi_1 \wedge \mu \pi_2 \wedge \nu \pi_3$.

**Definition 6** (Power of a switching sequences). Given a switching sequence $\pi$ and a scalar $\tau \in (0, \tau_{\text{max}}]$, the $n$th power of the switching sequence $\pi$ with $\tau$ is defined as

$$
\pi \wedge \tau^n \text{ def } = \pi \wedge \tau \cdots \wedge \tau \pi.
$$

(15)

**Definition 7** (Exponential matrix). Given a switching sequence

$$
\pi = \{(i_1, h_1), \tau_1; (i_2, h_2), \tau_2; \ldots; (i_{M-1}, h_{M-1}), \tau_{M-1}; (i_M, h_M)\},
$$

and suppose $T_{[\pi]} < \infty$, the exponential matrix is defined as

$$
\exp(\pi) \text{ def } = \prod_{j=M}^{2} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \exp(A_i h_1).
$$

(16)
Theorem 2. Given two switching sequences $\pi_1$, $\pi_2$ and a scalar $\tau \in (0, \tau_{\text{max}}]$, we have
\begin{equation}
\mathcal{C}(\pi_1 \land \tau \pi_2) = \exp(\pi_2) \exp(A_0 \tau) \mathcal{C}(\pi_1) + \mathcal{C}(\pi_2).
\end{equation}

Proof. It is straightforward by Theorem 1. \hfill \Box

Theorem 3. Given a switching sequence $\pi$ and a scalar $\tau \in (0, \tau_{\text{max}}]$, we have
\begin{equation}
\mathcal{C}(\pi \land \tau^n) = \langle \exp(\pi) \exp(A_0 \tau) \mathcal{C}(\pi) \rangle.
\end{equation}

Proof.
\begin{align*}
\mathcal{C}(\pi \land \tau^n) & = \mathcal{C}(\pi) + \exp(\pi) \exp(A_0 \tau) \mathcal{C}(\pi \land \tau^{n-1}) \\
& = \mathcal{C}(\pi) + \exp(\pi) \exp(A_0 \tau) \mathcal{C}(\pi) + \left[\exp(\pi) \exp(A_0 \tau)\right]^2 \mathcal{C}(\pi \land \tau^{n-2}) \\
& \quad \vdots \\
& = \sum_{i=1}^{n} \left[\exp(\pi) \exp(A_0 \tau)\right]^{(i-1)} \mathcal{C}(\pi) \\
& = \langle \exp(\pi) \exp(A_0 \tau) \mathcal{C}(\pi) \rangle. \hfill \Box
\end{align*}

By (18) and by the property of minimal invariant subspace, it is easy to see that
\begin{equation}
\exp(\pi \land \tau^n) \mathcal{C}(\pi \land \tau^n) = \mathcal{C}(\pi \land \tau^n).
\end{equation}

3.2. Geometric criterion

For system (1), we define a subspace sequence as follows:
\begin{align*}
\mathcal{W}_1 & = \sum_{i=0}^{N} \langle A_i | [B_i, D_i] \rangle, \\
\mathcal{W}_2 & = \sum_{i=0}^{N} \langle A_i | \mathcal{W}_1 \rangle, \quad \ldots, \\
\mathcal{W}_n & = \sum_{i=0}^{N} \langle A_i | \mathcal{W}_{n-1} \rangle.
\end{align*}

(20)

where $B_0 = 0$ and $D_0 = 0$.

If there exists a positive integer $m$ such that $\mathcal{W}_m = \mathcal{W}_{m+1}$, by the above definition, it is easy to prove that $\mathcal{W}_{m+1} = \mathcal{W}_{m+2} = \cdots$. Since $0 \leq \dim(\mathcal{W}_1) \leq \cdots \leq \dim(\mathcal{W}_n) \leq n$, we have $\mathcal{W}_n = \mathcal{W}_{n+1}$. This implies that $\mathcal{W}_{n+l} \subseteq \mathcal{W}_n$, for $l = 1, 2, \ldots$.

Theorem 4. System (1) is controllable if and only if
\begin{equation}
\mathcal{W}_n = \mathbb{R}^n.
\end{equation}

(21)

We will complete the proof of Theorem 4 in three steps: (i) first, we will prove that $\mathcal{C} \subseteq \mathcal{W}_n$; (ii) next, we will define a new subspace sequence $\mathcal{V}_1, \ldots, \mathcal{V}_n$ and prove that $\mathcal{V}_n = \mathcal{W}_n$; (iii) finally, we will construct a basic switching sequence $\pi_b$ such that $\mathcal{C}(\pi_b) = \mathcal{V}_n$. 
Step (i).

**Lemma 5.** For system (1), for any switching sequence \( \pi \), we have \( \mathcal{C}(\pi) \subseteq \mathcal{W}_n \).

**Proof.** For any switching sequence \( \pi \) given by (2), by Theorem 1, we have

\[
\mathcal{C}(\pi) = \exp(A_i h_M) \exp(A_0 \tau_{M-1}) \cdots \exp(A_i h_2) \exp(A_0 \tau_1) [A_i [B_i, D_i]] \\
+ \exp(A_i h_M) \exp(A_0 \tau_{M-1}) \cdots \exp(A_i h_3) \exp(A_0 \tau_2) [A_i [B_i, D_i]] \\
+ \cdots + \exp(A_i h_M) \exp(A_0 \tau_{M-1}) \exp(A_{iM-1} [B_{iM-1}, D_{iM-1}]) \\
+ [A_i [B_i, D_i]].
\]

(22)

By the property of minimal invariant subspace, it follows that

\[
[A_i [B_i, D_i]] \subseteq \mathcal{W}_1,
\]

\[
\exp(A_i h_M) \exp(A_0 \tau_{M-1}) \exp(A_{iM-1} [B_{iM-1}, D_{iM-1}]) \subseteq \mathcal{W}_3,
\]

\[
\exp(A_i h_M) \exp(A_0 \tau_{M-1}) \exp(A_{iM-1} [B_{iM-1}, D_{iM-1}]) \exp(A_0 \tau_{M-2}) \exp(A_{iM-2} [B_{iM-2}, D_{iM-2}]) \subseteq \mathcal{W}_5,
\]

\[
\vdots
\]

\[
\exp(A_i h_M) \exp(A_0 \tau_{M-1}) \cdots \exp(A_i h_2) \exp(A_0 \tau_1) [A_i [B_i, D_i]] \subseteq \mathcal{W}_{2M-1}.
\]

(23)

On the other hand,

\[
\mathcal{W}_1 \subseteq \mathcal{W}_3 \cdots \subseteq \mathcal{W}_{2M-1} \subseteq \mathcal{W}_n.
\]

(24)

Thus, we get \( \mathcal{C}(\pi) \subseteq \mathcal{W}_n \). □

By Lemma 5, it is easy to see that \( \mathcal{C} \subseteq \mathcal{W}_n \).

Step (ii). We define another subspace sequence as follows:

\[
\mathcal{V}_1 = \sum_{i=1}^{N} [A_i [B_i, D_i]],
\]

\[
\mathcal{V}_2 = \sum_{i=1}^{N} [A_i [A_0 \mathcal{V}_1]],
\]

\[
\vdots
\]

\[
\mathcal{V}_n = \sum_{i=1}^{N} [A_i [A_0 \mathcal{V}_{n-1}]].
\]

(25)

**Lemma 6.** For system (1), we have \( \mathcal{W}_n = \mathcal{V}_n \).
Proof. First, we will prove that $W_n \subseteq V_n$. In fact, we have

$$W_1 = V_1,$$

$$W_2 = \sum_{i=1}^{N} \langle A_i | W_1 \rangle + \langle A_0 | W_1 \rangle = \sum_{i=1}^{N} \langle A_i | V_1 \rangle + \langle A_0 | V_1 \rangle \subseteq \sum_{i=1}^{N} \langle A_i | \langle A_0 | V_1 \rangle \rangle = V_2,$$

$$
\vdots
$$

$$W_n = \sum_{i=1}^{N} \langle A_i | W_{n-1} \rangle + \langle A_0 | W_{n-1} \rangle \subseteq \sum_{i=1}^{N} \langle A_i | V_{n-1} \rangle + \langle A_0 | V_{n-1} \rangle \subseteq \sum_{i=1}^{N} \langle A_i | \langle A_0 | V_{n-1} \rangle \rangle = V_n.$$

Next, we will prove that $V_n \subseteq W_n$. In fact, we have

$$V_1 \subseteq W_1,$$

$$V_2 = \sum_{i=1}^{N} \langle A_i | \langle A_0 | V_1 \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | \langle A_0 | V_1 \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | W_2 \rangle = W_3,$$

$$V_3 = \sum_{i=1}^{N} \langle A_i | \langle A_0 | V_2 \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | \langle A_0 | V_3 \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | W_4 \rangle = W_5,$$

$$
\vdots
$$

$$V_n = \sum_{i=1}^{N} \langle A_i | \langle A_0 | V_{n-1} \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | \langle A_0 | V_{2n-3} \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | W_{2n-2} \rangle = W_{2n-1} \subseteq W_n. \quad (26)$$

Hence, $W_n = V_n$ holds. □

Step (iii). We construct a basic switching sequence $\pi_b$ such that $C(\pi_b) = V_n$.

Lemma 7. For system (1), there exists a switching sequence $\pi_b$, such that $C(\pi_b) = V_n$.

Proof. First, by Lemma 4, there must exist $\tau_b > 0$ satisfying $n \tau_b < \tau_{\text{max}}$ and $h_1, \ldots, h_N > 0$, such that we can redefine $V_1, \ldots, V_n$ as follows:

$$V_1 = \sum_{i=1}^{N} \langle A_i | [B_i, D_i] \rangle,$$

$$V_2 = \sum_{i=1}^{N} \langle \exp(A_i h_i) | \exp(A_0 \tau_b) | V_1 \rangle \rangle.$$

$$\vdots$$

$$V_n = \sum_{i=1}^{N} \langle A_i | \langle A_0 | V_{n-1} \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | \langle A_0 | V_{2n-3} \rangle \rangle \subseteq \sum_{i=0}^{N} \langle A_i | W_{2n-2} \rangle = W_{2n-1} \subseteq W_n.$$
\[ V_n = \sum_{i=1}^{N} \langle \exp(A_i h_i) | \exp(A_0 \tau b) | V_{n-1} \rangle \].

(27)

Moreover, by the property of minimal invariant subspace, we can rewrite (27) as follows:

\[ V_1 = \sum_{i=1}^{N} \langle A_i | [B_i, D_i] \rangle, \]

\[ V_2 = \sum_{i=1}^{N} \exp(A_i h_i) \langle \exp(A_i h_i) \rangle \langle \exp(A_0 \tau b) \rangle \langle \exp(A_0 \tau b) \rangle \langle V_1 \rangle \],

\[ \vdots \]

\[ V_n = \sum_{i=1}^{N} \exp(A_i h_i) \langle \exp(A_i h_i) \rangle \langle \exp(A_0 \tau b) \rangle \langle \exp(A_0 \tau b) \rangle \langle V_{n-1} \rangle \].

(28)

By (28), we can rewrite \( V_n \) in the following form:

\[ V_n = \sum_{i_1, \ldots, i_n, j_1, \ldots, j_n \in \{1, \ldots, n\}} \langle \exp(A_{i_1, h_{i_1}} \rangle \langle \exp(A_0 \tau b) \rangle \langle \exp(A_0 \tau b) \rangle \langle A_j | [B_j, D_j] \rangle \].

(29)

Suppose \( \dim(V_n) = d \), then there exist linear subspaces \( X_1, \ldots, X_d \) such that

\[ V_n = \sum_{m=1}^{d} X_m \]

and each \( X_m, m = 1, \ldots, d \), has the form

\[ \langle \exp(A_{i_{m,1}, h_{i_{m,1}}}) \rangle \langle \exp(A_0 \tau b) \rangle \langle \exp(A_0 \tau b) \rangle \langle A_{j_m} | [B_{j_m}, D_{j_m}] \rangle. \]

(30)

where \( i_{m,1}, \ldots, i_{m,n}, j_m \in \{1, \ldots, N\}, l_{m,1}, \ldots, l_{m,n}, s_{m,1}, \ldots, s_{m,n} \in \{1, \ldots, n\} \).

Consider the subspace \( X_m \) which has the form (30), we can select the switching sequence as

\[ \pi_m = \{(j_{m,1}, s_{m,n} \tau b; (l_{m,n}, l_{m,n} h_{i_{m,n}}), s_{m,n-1} \tau b; \ldots; (i_{m,2}, l_{m,2} h_{i_{m,2}}), s_{m,1} \tau b; (i_{m,1}, l_{m,1} h_{i_{m,1}}) \}. \]

(31)

It is easy to verify that the last term of \( C(\pi_m) \) is just (30). Thus, we can select switching sequences \( \pi_1, \ldots, \pi_d \) such that \( X_m \subseteq C(\pi_m), m = 1, \ldots, d \). Then we have

\[ V_n = \sum_{m=1}^{d} C(\pi_m). \]

(32)
Now we can construct the switching sequence $\pi_b$ as follows. First, if $C(\wedge \tau_b \pi_1) = W_n$, we can take $\pi_b = \pi_1 \wedge \tau_b$; if not, there must exist $k \in \{2, \ldots, d\}$ (without loss of generality, let $k = 2$) such that $C(\pi_2) \not\subseteq C(\pi_1 \wedge \tau_b)$. By Theorem 2, it follows that $C(\pi_2 \wedge \tau_b \pi_1) = \exp(\pi_1 \wedge \tau_b) \exp(A_0 \tau_b) C(\pi_2) + C(\pi_1 \wedge \tau_b)$. By (19), we have $C(\pi_2 \wedge \tau_b \pi_1) = \exp(\pi_1 \wedge \tau_b) \exp(A_0 \tau_b) (C(\pi_2) + C(\pi_1 \wedge \tau_b))$. This implies that $\dim(C(\pi_2 \wedge \tau_b \pi_1)) = \dim(C(\pi_2) + C(\pi_1 \wedge \tau_b)) \geq 1 + \dim(C(\pi_1 \wedge \tau_b)) = 2$. Thus, we can construct switching sequences

$$\begin{align*}
\pi_1 &= \pi_1, \\
\pi_2 &= \pi_2 \wedge \tau_b \pi_1, \\
&\vdots \\
\pi_d &= \pi_d \wedge \tau_b (\pi_{d-1} \wedge \tau_b), \\
\pi_b &= \pi_d.
\end{align*}$$

It is easy to verify that $\dim(C(\pi_b)) \geq d$. Thus, $C(\pi_b) = V_n$. $\square$

Now, we can complete the proof of Theorem 4 easily.

**Proof of Theorem 4.** For system (1), by Lemma 5, we get $C \subseteq W_n$. Then, by Lemma 6 and by Lemma 7, there exists a basic switching sequence $\pi_b$ such that $C(\pi_b) = V_n = W_n$. Thus, we have $W_n = C(\pi_b) \subseteq C \subseteq W_n$. This implies $W_n = C$. Hence, system (1) is controllable if and only if $W_n = \mathbb{R}^n$. $\square$

**Remark 2.** By the proof of Lemma 7, it is obvious that $\pi_b$ is not unique. One reason is that $\tau_b$ and $h_i$, $i \in \{1, \ldots, N\}$, are not unique, and another reason is that $V_1, \ldots, V_d$ are not unique. For system (1), we can use only one basic switching sequence $\pi_b$ to realize the controllability. The proof of Lemma 7 provides a method to construct $\pi_b$.

**Remark 3.** For the existing relevant results [19,21,22,30], it is easy to see that they are special cases of Theorem 4.
4. Example

In this section, we present a numerical example.

Example 1. Consider the 4-dimensional switched linear system

\[
A_1 = 0_{4 \times 4}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

\[
A_2 = 0_{4 \times 4}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (33)
\]

with \( d = 2, h_1 = 3, h_2 = 4 \) and \( \tau_{\text{max}} = 1 \).

By simple calculation, we have

\[
W_1 = \text{span}\{B_1, B_2, D_2\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad (34)
\]

\[
W_2 = \text{span}\{B_1, B_2, D_2, A_0D_2\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}. \quad (35)
\]

By Theorem 4, the system is controllable. In fact, if we choose a basic switching sequence as \( \pi_b = \{(2, 4), (1, 2, 4), (2, 4), (1, 3)\} \), and let \( x(0) = x_0 \) and

\[
u(t) = \begin{cases} 
c_1, & t \in [0, 4), 
c_2, & t \in [5, 9), 
c_3, & t \in [10, 14), 
c_4, & t \in [15, 18]. 
\end{cases}
\]

then it is easy to calculate that

\[
x(18) = x_0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \exp(3) & 0 & 0 \\ 0 & 0 & \exp(6) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \int_0^1 u_0(s) \, ds + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}. \quad (36)
\]

Obviously, \( x(18) \) can be driven to anywhere by selecting appropriate \( c_1, c_2, c_3 \) and \( c_4 \). Thus, the system is completely controllable.
5. Conclusion

This paper has studied the controllability of a class of switched systems with time delay in the control function, where the switching between two subsystems is not an instant event, but a transfer process. A necessary and sufficient condition for the controllability of such systems has been presented. A basic switching sequence has been constructed such that the controllability can be realized by this basic switching sequence. Finally, an example has been given to illustrate the results.

Appendix A

Proof of Theorem 1. Let $x_0$ be the initial state, then after evolving through the switching sequence $\pi$, the terminal state $x_f$ can be expressed as

$$x_f = \prod_{m=M}^{2} \exp(A_{i_m}h_m) \exp(A_0\tau_{m-1}) \exp(A_{i_1}h_1)x_0$$

$$+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{i_j}h_j) \exp(A_0\tau_{j-1})$$

$$\times \int_{t_{m-1}+h_m}^{t_{m-1}+h_m+h_M} \exp[A_{i_m}(t_{m-1}+h_m-s)]B_{i_m}u(s) + D_{i_m}u(s-d)ds$$

$$+ \int_{t_{M-1}+h_M}^{t_{M-1}+h_M+h_m} \exp[A_{i_{M-1}}(t_{M-1}+h_M-s)]B_{i_{M-1}}u(s) + D_{i_{M-1}}u(s-d)ds.$$  \hspace{1cm} (A.1)

Since

$$\int_{t_{m-1}+h_m}^{t_{m-1}+h_m+h_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]B_{i_m}u(s) + D_{i_m}u(s-d)ds$$

$$= \int_{t_{m-1}}^{t_{m-1}+h_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]B_{i_m}u(s)ds$$

$$+ \int_{t_{m-1}}^{t_{m-1}+h_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]D_{i_m}u(s-d)ds$$

$$= \int_{t_{m-1}}^{t_{m-1}+h_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]B_{i_m}u(s)ds$$
\[
\begin{align*}
&= \int_{t_{m-1}+d}^{t_m} \exp[A_{i_m}(t_{m-1}+d-s)]D_{i_m}u(s) \, ds \\
&\quad + \int_{t_{m-1}+h_m}^{t_m-h_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]D_{i_m}u(s) \, ds \\
&\quad + \int_{t_{m-1}}^{t_{m-1}+d} \exp[A_{i_m}(t_{m-1}+d-s)]D_{i_m}u(s) \, ds \\
&\quad + \int_{t_{m-1}+h_m}^{t_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]D_{i_m}u(s) \, ds \\
&\quad + \int_{t_{m-1}+h_m}^{t_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]B_{i_m}u(s) \, ds \\
&\quad + \int_{t_{m-1}}^{t_{m-1}+d} \exp[A_{i_m}(t_{m-1}+d-s)]B_{i_m}u(s) \, ds \\
&\quad + \int_{t_{m-1}+h_m}^{t_m} \exp[A_{i_m}(t_{m-1}+h_m-s)]B_{i_m}u(s) \, ds. \\
\end{align*}
\]

In the following, we will discuss two cases: \( \tau_m \geq d \) and \( \tau_m < d \), respectively. First, when \( \tau_m \geq d \) (see Fig. 3), Eq. (A.1) can be rewritten as

\[
x_f = \prod_{m=M}^2 \exp(A_{i_m}h_m) \exp(A_0\tau_{m-1}) \exp(A_{i_1}h_1) \\
\times \left\{ x_0 + \int_{t_0}^{t_0-d} \exp[A_{i_1}(t_0-s)] \exp(-A_{i_1}d)D_{i_1}u(t_0) \, ds \right\} \\
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{i_j}h_j) \exp(A_0\tau_{j-1})
\]
By (A.3), we have
\[
\mathcal{C}(x_0, u_0, \pi) = I_n(x_0, u_0, \pi) + \left\{ x \mid x = \sum_{m=1}^{M-1} \prod_{j=M}^m \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \right\}
\]
\[
\times \int_{t_{m-1}+h_m-d}^{t_{m-1}} \exp[A_{im}(t_{m-1} + h_m - s)] \left[ B_{im} + \exp(-A_{im} d) D_{im} \right] u(s) \, ds
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^m \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \cdot \left[ B_{im} + \exp(-A_{im} d) D_{im} \right] u(s) \, ds
\]
\[
+ \int_{t_{m-1}+h_m-d}^{t_{M-1}+h_{M-d}} \exp[A_{im}(t_{m-1} + h_m - s)] \left[ B_{im} + \exp(-A_{im} d) D_{im} \right] u(s) \, ds
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^m \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \cdot \left[ B_{im} + \exp(-A_{im} d) D_{im} \right] u(s) \, ds.
\]
\[
\times \int_{t_m}^{t_m+1} \exp[A_{m+1}(t_m - s)] \exp(-A_{m+1}d)D_{m+1}u(s) \, ds \\
+ \int_{t_{M-1}+h_{M-d}}^{t_{M-1}+h_{M-d}} \exp[A_{M-1}(t_{M-1} + h_{M} - s)][B_{M-1} + \exp(-A_{M}d)D_{M}u(s) \, ds \\
+ \int_{t_{M-1}+h_{M-d}}^{t_{M-1}+h_{M-d}} \exp[A_{M}(t_{M-1} + h_{M} - s)]B_{M}u(s) \, ds. \\
\forall \text{ piecewise continuous } u \}
\]

\[
= \mathcal{I}(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \\
\times \left\{ x \mid x = \int_{t_{m-1}}^{t_{m-1}+h_m-d} \exp[A_{m}(t_{m-1} + h_{m} - s)] \right\} \\
\times \left\{ x \mid x = \int_{t_{m-1}+h_m-d}^{t_{m-1}+h_m} \exp[A_{m}(t_{m-1} + h_{m} - s)]B_{m}u(s) \, ds, \forall \text{ piecewise continuous } u \right\} \\
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \\
\times \left\{ x \mid x = \int_{t_{m-1}+h_m-d}^{t_{m-1}} \exp[A_{m}(t_{m-1} + h_{m} - s)]B_{m}u(s) \, ds, \forall \text{ piecewise continuous } u \right\} \\
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \exp(A_{m+1} h_{m+1}) \\
\times \left\{ x \mid x = \int_{t_{m-1}+h_m-d}^{t_{m}} \exp[A_{m+1}(t_{m} - s)] \exp(-A_{m+1}d)D_{m+1}u(s) \, ds, \forall \text{ piecewise continuous } u \right\}
\]
\[ I_n(x_0, u_0, \pi) = \prod_{m=1}^{2} \exp(A_{i_m} h_m) \exp(A_0 \tau_{m-1}) \exp(A_{i_1} h_1) \times \left\{ x_0 + \int_{t_0-d}^{t_0} \exp[A_{i_1}(t_0 - s)] \exp(-A_{i_1} d) D_{i_1} u_0(s) \, ds \right\}. \]  

(A.5)

Then, by Lemma 1, it follows that

\[
C(x_0, u_0, \pi) = I_n(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{j} h_j) \exp(A_0 \tau_{j-1}) \times \left\{ (A_{i_m} | B_{i_m}) + \exp(-A_{i_m} d) D_{n+m} \right\} + \sum_{m=1}^{M-1} \prod_{j=M}^{m+2} \exp(A_{j} h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1}) \times \left\{ (A_{i_{m+1}} | \exp(-A_{i_{m+1}} d) D_{m+1}) + \exp(-A_{i_{m+1}} d) D_{i_{m+1}} \right\} + \left( A_{i_{m+1}} | B_{i_{m+1}} \right) + \left( A_{i_{m+1}} | B_{i_{m+1}} \right) \times \left( A_{i_{m+1}} | \exp(-A_{i_{m+1}} d) D_{i_{m+1}} \right) + \left( A_{i_{m+1}} | B_{i_{m+1}} \right). \]  

(A.6)

In particular, let \( x_0 = 0 \) and \( u_0 = 0 \), then by Lemma 2, we have

\[
C(0, 0, \pi) = C(\pi) = \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{j} h_j) \exp(A_0 \tau_{j-1}) \times \left\{ (A_{i_m} | \exp(-A_{i_m} d) D_{i_m}) + \left( A_{i_m} | B_{i_m} \right) \right\} + \sum_{m=1}^{M-1} \prod_{j=M}^{m+2} \exp(A_{j} h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1}) \times \left\{ (A_{i_{m+1}} | \exp(-A_{i_{m+1}} d) D_{m+1}) + \left( A_{i_{m+1}} | B_{i_{m+1}} \right) \right\} + \left( A_{i_{m+1}} | \exp(-A_{i_{m+1}} d) D_{i_{m+1}} \right) + \left( A_{i_{m+1}} | B_{i_{m+1}} \right). \]  

(A.7)
By the property of minimal invariant subspace, it is obvious that

\[
C(\pi) = \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij}) \exp(A_0 \tau_{j-1}) \left( [A_{im} \mid \exp(-A_{im} d)D_{im}] + [A_{im} \mid B_{im}] \right) \\
+ \sum_{m=2}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij}) \exp(A_0 \tau_{j-1}) [A_{im} \mid \exp(-A_{im} d)D_{im}] \\
+ [A_{im} \mid \exp(-A_{im} d)D_{im}] + [A_{im} \mid B_{im}] \\
= \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij}) \exp(A_0 \tau_{j-1}) \left( [A_{im} \mid D_{im}] + [A_{im} \mid B_{im}] \right) \\
+ [A_{im} \mid D_{im}] + [A_{im} \mid B_{im}] \\
= \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij}) \exp(A_0 \tau_{j-1}) \left( [A_{im} \mid [B_{im}, D_{im}]] \right) \\
+ [A_{im} \mid [B_{im}, D_{im}]]. \tag{A.8}
\]

Second, when \( \tau_m < d \) (see Fig. 4), Eq. (A.1) can be rewritten as

\[
x_f = \prod_{m=M}^{m=2} \exp(A_{im} h_m) \exp(A_0 \tau_{m-1}) \exp(A_{i1} h_1) \\
\times \left\{ x_0 + \int_{t_0}^{t_0 - d} \exp[A_{i1}(t_0 - s)] \exp(-A_{i1} d)D_{i1} u_0(s) \, ds \right\} \\
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij}) \exp(A_0 \tau_{j-1}) \\
\times \left\{ \int_{t_{m-1} + h_{m-1} - d}^{t_{m-1} + h_{m-1}} \exp[A_{im}(t_{m-1} + h_m - s)] \left[ B_{im} \exp(-A_{im} d)D_{im} \right] u(s) \, ds \right\}
\]
\[ + \int_{t_{m-1} + h_m - d}^{t_m - d} \exp[A_{i_m}(t_m - 1 + h_m - s)] B_{i_m} u(s) \, ds \]
\[ + \sum_{m=1}^{M-1} \prod_{j=M} \exp(A_{j} h_{j}) \exp(A_{0} \tau_{j-1}) \exp(A_{j+1} h_{j+1}) \times \left\{ \int_{t_{m-1} + h_m - d}^{t_m - d} \exp(A_{0} \tau_{m}) \exp[A_{i_m}(t_m - 1 + h_m - s)] B_{i_m} \exp(-A_{i_m} d) \, ds \right\} \]
\[ + \exp[A_{i_{m+1}}(t_m - s)] \exp(-A_{i_{m+1}} d) D_{i_{m+1}} u(s) \, ds \]
\[ + \int_{t_{M-1} + h_{M-1} - d}^{t_{M-1} + h_{M-1}} \exp[A_{i_M}(t_{M-1} + h_{M-1} - s)] \left[ B_{i_M} + \exp(-A_{i_M} d) D_{i_M} \right] u(s) \, ds \]
\[ = \prod_{m=M} \exp(A_{i_m} h_m) \exp(A_{0} \tau_{m-1}) \exp(A_{i_1} h_1) \times \left\{ x_0 + \int_{t_0 - d}^{t_0} \exp[A_{i_1}(t_0 - s)] \exp(-A_{i_1} d) D_{i_1} u_0(s) \, ds \right\} \]
\[ + \sum_{m=1}^{M-1} \prod_{j=M} \exp(A_{j} h_{j}) \exp(A_{0} \tau_{j-1}) \times \left\{ \int_{t_{m-1} + h_m - d}^{t_m - d} \exp[A_{i_m}(t_m - 1 + h_m - s)] \left[ B_{i_m} + \exp(-A_{i_m} d) D_{i_m} \right] u(s) \, ds \right\} \]
\[ + \exp[A_{i_{m+1}}(t_m - s)] \exp(-A_{i_{m+1}} d) B_{i_{m+1}} u(s) \, ds \]
\[ + \sum_{m=1}^{M-1} \prod_{j=M} \exp(A_{j} h_{j}) \exp(A_{0} \tau_{j-1}) \exp(A_{j+1} h_{j+1}) \]
\[
\times \left\{ \int_{t_m - d}^{t_m - 1 + h_m} \left( \exp(A_0 \tau_m) \exp[A_{i_m}(t_m - 1 + h_m - s)] \right) B_{i_m} \\
+ \exp[A_{i_m + 1}(t_m - 1 + h_m - s)] \exp(A_{i_m + 1} \tau_m) \exp(-A_{i_m + 1} d) D_{i_m + 1} u(s) \, ds \\
+ \int_{t_m}^{t_m - 1 + h_m} \exp[A_{i_m + 1}(t_m - s)] \exp(-A_{i_m + 1} d) D_{i_m + 1} u(s) \, ds \right\} \\
+ \int_{t_{M-1} + h_M - d}^{t_{M-1} + h_M} \exp[A_{i_M}(t_M - s)] [B_{i_M} + \exp(-A_{i_M} d) D_{i_M}] u(s) \, ds \\
+ \int_{t_{M-1} + h_M}^{t_{M-1} + h_M - d} \exp[A_{i_M}(t_M - s)] B_{i_M} u(s) \, ds. \quad (A.9)
\]

By (A.9), we have

\[
\mathcal{C}(x_0, u_0, \pi) = \mathcal{I}n(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \\
\times \left\{ x \mid x = \int_{t_m - 1 + h_m - d}^{t_m - d} \exp[A_{i_m}(t_m - 1 + h_m - s)] \right\} \\
\times [B_{i_m} + \exp(-A_{i_m} d) D_{i_m}] u(s) \, ds, \forall \text{ piecewise continuous } u \\
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \\
\times \left\{ x \mid x = \int_{t_m - 1 + h_m - d}^{t_m - d} \exp[A_{i_m}(t_m - 1 + h_m - s)] \exp(-A_{i_m} \tau_m) B_{i_m} u(s) \, ds, \forall \text{ piecewise continuous } u \right\} \\
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+2} \exp(A_j h_j) \exp(A_0 \tau_{j-1}) \exp(A_{im+1} h_{im+1}) \\
\times \left\{ x \mid x = \int_{t_m - 1 + h_m - d}^{t_m - d} \exp(A_0 \tau_m) \exp[A_{i_m}(t_m - 1 + h_m - s)] B_{i_m} \right\}
\]
\[
+ \exp\left[A_{i_{m+1}}(t_{m-1} + h_m - s)\right] \exp(A_{i_{m+1}} \tau_m) \exp(-A_{i_{m+1}} d) D_{i_{m+1}} u(s) ds,
\]
\[
\forall \text{ piecewise continuous } u \}
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+2} \exp(A_i h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1})
\times \left\{ x \mid x = \int_{t_{m-1} + h_m}^{t_m} \exp\left[A_{i_{m+1}}(t_m - s)\right] \exp(-A_{i_{m+1}} d) D_{i_{m+1}} u(s) ds, \forall \text{ piecewise continuous } u \right\}
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+2} \exp(A_i h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1})
\times \left\{ x \mid x = \int_{t_{m-1}}^{t_m} \exp\left[A_{i_M}(t_{M-1} + h_M - s)\right] B_{i_M} u(s) ds, \forall \text{ piecewise continuous } u \right\}
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+2} \exp(A_i h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1})
\times \left\{ x \mid x = \int_{t_{m-1} + h_M}^{t_{M-1} + h_M - d} \exp\left[A_{i_M}(t_{M-1} + h_M - s)\right] B_{i_M} u(s) ds, \forall \text{ piecewise continuous } u \right\}.
\]
(A.10)

Then, by Lemma 1, it follows that

\[
C(x_0, u_0, \pi) = \mathcal{I} n(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_i h_j) \exp(A_0 \tau_{j-1})
\times \left\{ x \mid x = \int_{t_{m-1} + h_m}^{t_m} \exp\left[A_{i_{m+1}}(t_m - s)\right] \exp(-A_{i_{m+1}} d) D_{i_{m+1}} u(s) ds, \forall \text{ piecewise continuous } u \right\}
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_i h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1})
\times \left\{ x \mid x = \int_{t_{m-1} + h_M}^{t_{M-1} + h_M - d} \exp\left[A_{i_M}(t_{M-1} + h_M - s)\right] B_{i_M} u(s) ds, \forall \text{ piecewise continuous } u \right\}
\]
\[
+ \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_i h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1})
\times \left\{ x \mid x = \int_{t_{m-1} + h_M}^{t_{M-1} + h_M - d} \exp\left[A_{i_M}(t_{M-1} + h_M - s)\right] B_{i_M} u(s) ds, \forall \text{ piecewise continuous } u \right\}.
\]
(A.11)

where

\[
C_\infty = \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_i h_j) \exp(A_0 \tau_{j-1}) \exp(A_{i_{m+1}} h_{m+1})
\]
\begin{align*}
  & \times \left\{ x \mid x = \int_{t_{m-1}}^{t_m} \left( \exp(A_0 \tau_m) \exp\left[ A_{in} (t_{m-1} + h_m - s) \right] B_{im} \\
  & + \exp[A_{in+1} (t_{m-1} + h_m - s)] \exp(A_{in+1} \tau_m) \exp(-A_{in+1} d) D_{im+1} \right) u(s) \, ds, \right\}, \quad \forall \text{ piecewise continuous } u \right) \}. \quad (A.12) \\
  \text{By the property of minimal invariant subspace and by Lemma 2, it is obvious that} \\
  & C(x_0, u_0, \pi) \\
  & = \mathcal{I}(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \\
  & \times \left( [A_{im} | B_{im} + \exp(-A_{im} d) D_{im}] + \langle A_{im} | B_{im} \rangle \right) \\
  & + C_\infty + \sum_{m=2}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \langle A_{im} | D_{im} \rangle \\
  & + \langle A_{im} | B_{im} \rangle + \exp(-A_{im} d) D_{im} \rangle + \langle A_{im} | B_{im} \rangle \\
  & = \mathcal{I}(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \langle A_{im} | B_{im} \rangle + \langle A_{im} | D_{im} \rangle \rangle \\
  & + C_\infty + \langle A_{im} | [B_{im}, D_{im}] \rangle. \quad (A.13) \\
  \text{By Lemma 3 and by the property of minimal invariant subspace, we have} \\
  & C_\infty \leq \sum_{m=1}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \exp(A_{im+1} h_{m+1}) \\
  & \times \left( \exp(A_0 \tau_m) \langle A_{im} | B_{im} \rangle + \langle A_{im+1} | \exp(A_{im+1} \tau_m) \exp(-A_{im+1} d) D_{im+1} \rangle \right) \\
  & = \sum_{m=1}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \langle A_{im} | B_{im} \rangle \\
  & + \sum_{m=2}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \langle A_{im} | D_{im} \rangle. \quad (A.14) \\
  \text{By Eq. (A.14), it is easy to see that} \\
  & C_\infty \leq \sum_{m=1}^{M-1} \prod_{j=1}^{M} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \langle A_{im} | [B_{im}, D_{im}] \rangle. \quad (A.15)
\end{align*}
Thus, Eq. (A.13) can be written as
\[
C(x_0, u_0, \pi) = I_n(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1}) \langle A_{im} | [B_{im}, D_{im}] \rangle
+ \langle A_{im} | [B_{im}, D_{im}] \rangle.
\] (A.16)

In particular, let \(x_0 = 0\) and \(u_0 = 0\), then
\[
C(0, 0, \pi) = C(\pi) = \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{ij} h_j) \exp(A_0 \tau_{j-1})
\times (\langle A_{im} | [B_{im}, D_{im}] \rangle + \langle A_{im} | [B_{im}, D_{im}] \rangle).
\] (A.17)

This completes the whole proof.

References