Journal of Computational and Applied Mathematics 235 (2011) 2662–2669

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On a class of inverse quadratic eigenvalue problem

Yongxin Yuan [a,*], Hua Dai [b]

[a] School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, PR China

[b] Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China

ARTICLE INFO

Article history:
Received 25 September 2009
Received in revised form 3 December 2009

MSC:
65F18
15A24

Keywords:
Quadratic eigenvalue problem
Inverse eigenvalue problem
Monic quadratic pencil
Partially prescribed spectral information
Best approximation

ABSTRACT

In this paper, we first give the representation of the general solution of the following inverse monic quadratic eigenvalue problem (IMQEP): given matrices $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_p\} \in \mathbf{C}^{p\times p}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, $i, j = 1, \ldots, p$, $X = [x_1, \ldots, x_p] \in \mathbf{C}^{n\times p}$, $\mathrm{rank}(X) = p$, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \ldots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l + 1, \ldots, p$, find real-valued symmetric matrices D and K such that $X\Lambda^2 + DX\Lambda + KX = 0$. Then we consider a best approximation problem: given $\tilde{D}, \tilde{K} \in \mathbf{R}^{n\times n}$, find $(\hat{D}, \hat{K}) \in \mathcal{S}_{DK}$ such that $\|(\hat{D}, \hat{K}) - (\tilde{D}, \tilde{K})\|_W = \min_{(D,K)\in\mathcal{S}_{DK}} \|(D, K) - (\tilde{D}, \tilde{K})\|_W$, where $\|\cdot\|_W$ is a weighted Frobenius norm and $\mathcal{S}_{DK}$ is the solution set of IMQEP. We show that the best approximation solution $(\hat{D}, \hat{K})$ is unique and derive an explicit formula for it.

1. Introduction

In a recent treatise, Tisseur and Meerbergen [1] have surveyed many applications, mathematical properties, and a variety of numerical techniques for the so-called quadratic eigenvalue problem (QEP). The problem concerns, given $n \times n$ complex matrices M, D and K, find scalars λ and nonzero vectors x such that

$$Q(\lambda)x = 0, \tag{1.1}$$

where

$$Q(\lambda) := Q(\lambda; M, D, K) = \lambda^2 M + \lambda D + K \tag{1.2}$$

is called a quadratic pencil defined by M, D and K. The scalars λ and the corresponding nonzero vectors x are called, respectively, eigenvalues and eigenvectors of the pencil. The QEP is currently receiving much attention because of its extensive applications in areas such as applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, gyroscopic systems, and signal processing. It is known that the QEP has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M is nonsingular. The "direct" problem is, of course, to find the eigenvalues and eigenvectors when the coefficient matrices M, D and K are given, and an inverse QEP is to determine the matrix coefficients M, D and K from a prescribed set of eigenvalues and eigenvectors.

In most of the applications involving (1.1), specifications of the underlying physical system are embedded in the matrix coefficients M, D and K while the resulting bearing of the system usually can be interpreted via its eigenvalues and eigenvectors. A direct QEP, therefore, is meant generally to induce the dynamical behavior from given physical parameters.

* Corresponding author. Tel.: +86 0511 4401748.
E-mail address: yuanyx_703@163.com (Y. Yuan).

0377-0427/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2010.11.017

The inverse QEP reverse the line of induction. It is meant to construct the physical parameters from a given or desired behavior. The inverse problem is equally important in practice as the direct problem.

Matrix coefficients in QEPs arising in practice often maintain some additional properties. A typical case is that each of the matrix coefficients is of certain symmetry and that M is symmetric positive definite. Thus, for practicality, inverse QEPs should be solved with these additional constraints on the matrix coefficients in mind. An inverse QEP of interest here is that when the leading matrix coefficient M is known and fixed, only D and K are to be determined. Our study in this paper stems from the speculation that the notion of the inverse QEP has the potential of leading to an important modification tool for model updating, model tuning, and model correction [2–5], when compared with an analytical model.

Consider the fact that if $M = LL^T$ denotes the Cholesky decomposition of M, then

$$Q(\lambda)x = 0 \Leftrightarrow \tilde{Q}(\lambda)(L^T x) = 0, \tag{1.3}$$

where

$$\tilde{Q}(\lambda) := \lambda^2 I_n + \lambda L^{-1} D L^{-T} + L^{-1} K L^{-T}. \tag{1.4}$$

Thus, without loss of generality, we may assume that the given matrix M in our inverse problem is the $n \times n$ identity matrix I_n. For this reason, the inverse problem we are dealing with will be called an inverse monic quadratic eigenvalue problem (IMQEP). This paper is mainly concerned with solving the following two problems.

Problem I (*IMQEP*)**.** Given a pair of matrices (Λ, X) in the form

$$\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_p\} \in \mathbf{C}^{p\times p} \tag{1.5}$$

and

$$X = [x_1, \ldots, x_p] \in \mathbf{C}^{n\times p}, \tag{1.6}$$

where diagonal elements of Λ are all distinct, X is of full column rank p, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \ldots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l + 1, \ldots, p$, find real-valued and symmetric matrices D and K that satisfy the equation

$$X\Lambda^2 + DX\Lambda + KX = 0. \tag{1.7}$$

In other words, each pair (λ_t, x_t), $t = 1, \ldots, p$, is an eigenpair of the monic quadratic pencil

$$Q(\lambda) = \lambda^2 I_n + \lambda D + K.$$

Problem II (*Approximation Problem*)**.** Given $\tilde{D}, \tilde{K} \in \mathbf{R}^{n\times n}$, find $(\hat{D}, \hat{K}) \in \mathcal{S}_{DK}$ such that

$$\|(\hat{D}, \hat{K}) - (\tilde{D}, \tilde{K})\|_W = \min_{(D,K)\in\mathcal{S}_{DK}} \|(D, K) - (\tilde{D}, \tilde{K})\|_W, \tag{1.8}$$

where $\|\cdot\|_W$ is the weighted Frobenius norm, and $\mathcal{S}_{DK}$ is the solution set of Problem I.

In Section 2, we show that Problem I is always solvable, and the representation of the solution set of Problem I, denoted by $\mathcal{S}_{DK}$, is presented. In Section 3, we prove that Problem II is uniquely solvable, and the expression of the unique solution $(\hat{D}, \hat{K})$ is given. In Section 4, a numerical algorithm to acquire the best approximation solution under a weighted Frobenius norm sense is described and two numerical examples are provided. Some concluding remarks are given in Section 5.

In this paper, we shall adopt the following notation. Let $\mathbf{C}^{m\times n}$, $\mathbf{R}^{m\times n}$ denote the set of all $m \times n$ complex and real matrices, respectively. $\mathbf{OR}^{n\times n}$ denotes the set of all orthogonal matrices in $\mathbf{R}^{n\times n}$. Capital letters $A, B, C, \ldots$, denote matrices, lower case letters denote column vectors, Greek letters denote scalars, $\bar{\alpha}$ denotes the conjugate of the complex number α, A^T denotes the transpose of the matrix A, and $\|\cdot\|$ stands for the matrix Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbf{R}^{m\times n}$, $A * B$ represents the Hadamard product of the matrices A and B, that is, $A * B = [a_{ij}b_{ij}] \in \mathbf{R}^{m\times n}$. We write $A > 0$ ($A \geq 0$) if A is real symmetric positive definite (positive semi-definite).

2. Solving Problem I

Let $\alpha_i = \mathrm{Re}(\lambda_i)$ (the real part of the complex number λ_i), $0 < \beta_i = \mathrm{Im}\,(\lambda_i)$ (the imaginary part of the complex number λ_i), $y_i = \mathrm{Re}(x_i)$, $z_i = \mathrm{Im}\,(x_i)$ for $i = 1, 3, \ldots, 2l - 1$ and

$$\tilde{\Lambda} = \mathrm{diag}\left\{\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{2l-1} & \beta_{2l-1} \\ -\beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \ldots, \lambda_p\right\} \in \mathbf{R}^{p\times p}, \tag{2.1}$$

$$\tilde{X} = \left[y_1, z_1, \ldots, y_{2l-1}, z_{2l-1}, x_{2l+1}, \ldots, x_p\right] \in \mathbf{R}^{n\times p}. \tag{2.2}$$

Then, the equation (1.7) is equivalent to the equation

$$\tilde{X}\tilde{\Lambda}^2 + D\tilde{X}\tilde{\Lambda} + K\tilde{X} = 0. \tag{2.3}$$

Since $\operatorname{rank}(X) = \operatorname{rank}(\tilde{X}) = p$, the singular value decomposition (SVD) of $\tilde{X}$ is of the form

$$\tilde{X} = U\begin{bmatrix}\Sigma\\0\end{bmatrix}Q^T = U_1\Sigma Q^T, \tag{2.4}$$

where $U = [U_1, U_2] \in \mathbf{OR}^{n\times n}$ with $U_1 \in \mathbf{R}^{n\times p}$, $Q \in \mathbf{OR}^{p\times p}$, and $\Sigma = \operatorname{diag}\{\sigma_1, \ldots, \sigma_p\} > 0$.

Now, based on the SVD of $\tilde{X}$, for an arbitrary matrix $A \in \mathbf{R}^{n\times n}$, we can define a norm termed weighted Frobenius norm that

$$\|A\|_W := \|WAW\|, \tag{2.5}$$

where $W = U\begin{bmatrix}\Sigma & 0\\0 & I\end{bmatrix}U^T > 0$. It is easy to check that $\|\cdot\|_W$ is a norm induced by the inner product $\langle A, B\rangle := \operatorname{trace}(W^2B^TW^2A)$ for all matrices $A, B \in \mathbf{R}^{n\times n}$. So, $\mathbf{R}^{n\times n}$ with $\|\cdot\|_W$ is a Hilbert space.

Plugging (2.4) into (2.3), we have

$$\begin{bmatrix}\Sigma\\0\end{bmatrix}Q^T\tilde{\Lambda}^2Q + U^TDU\begin{bmatrix}\Sigma\\0\end{bmatrix}Q^T\tilde{\Lambda}Q + U^TKU\begin{bmatrix}\Sigma\\0\end{bmatrix} = 0. \tag{2.6}$$

Let

$$U^TDU = \begin{bmatrix}D_{11} & D_{12}\\D_{12}^T & D_{22}\end{bmatrix}, \qquad U^TKU = \begin{bmatrix}K_{11} & K_{12}\\K_{12}^T & K_{22}\end{bmatrix}. \tag{2.7}$$

Thus, the equation (2.6) is equivalent to the following matrix equations:

$$\Sigma Q^T\tilde{\Lambda}^2Q + D_{11}\Sigma Q^T\tilde{\Lambda}Q + K_{11}\Sigma = 0, \tag{2.8}$$

$$D_{12}^T\Sigma Q^T\tilde{\Lambda}Q + K_{12}^T\Sigma = 0. \tag{2.9}$$

For convenience, we shall denote

$$G = Q\Sigma^2Q^T, \qquad Q\Sigma D_{11}\Sigma Q^T = A, \qquad Q\Sigma K_{11}\Sigma Q^T = B, \tag{2.10}$$

then the equation (2.8) can be written as

$$G\tilde{\Lambda}^2 + A\tilde{\Lambda} + B = 0, \tag{2.11}$$

where A, B are yet to be determined. It is easy to see that the equation (2.11) with unknown matrix B has a symmetric solution if and only if

$$G\tilde{\Lambda}^2 + A\tilde{\Lambda} = (\tilde{\Lambda}^T)^2G + \tilde{\Lambda}^TA^T. \tag{2.12}$$

It is not difficult to prove that $A_0 = -\tilde{\Lambda}^TG - G\tilde{\Lambda}$ is a particular solution of (2.12). Consider the following homogeneous equation

$$H\tilde{\Lambda} = \tilde{\Lambda}^TH, \quad \text{s.t. } H = H^T. \tag{2.13}$$

When setting

$$\tilde{\Lambda}_1 = \operatorname{diag}\left\{\begin{bmatrix}\alpha_1 & \beta_1\\-\beta_1 & \alpha_1\end{bmatrix}, \ldots, \begin{bmatrix}\alpha_{2l-1} & \beta_{2l-1}\\-\beta_{2l-1} & \alpha_{2l-1}\end{bmatrix}\right\}, \qquad \tilde{\Lambda}_2 = \operatorname{diag}\{\lambda_{2l+1}, \ldots, \lambda_p\}, \tag{2.14}$$

and

$$H = \begin{bmatrix}H_1 & H_2\\H_2^T & H_3\end{bmatrix} \quad \text{with } H_1 = H_1^T \in \mathbf{R}^{2l\times 2l} \text{ and } H_3 = H_3^T \in \mathbf{R}^{(p-2l)\times(p-2l)}, \tag{2.15}$$

we know the equation (2.13) is equivalent to the following three matrix equations:

$$H_3\tilde{\Lambda}_2 = \tilde{\Lambda}_2H_3, \tag{2.16}$$

$$H_2\tilde{\Lambda}_2 = \tilde{\Lambda}_1^TH_2, \tag{2.17}$$

$$H_1\tilde{\Lambda}_1 = \tilde{\Lambda}_1^TH_1. \tag{2.18}$$

Because the matrix H_3 can be commuted with the diagonal matrix $\tilde{\Lambda}_2$ and the diagonal elements of $\tilde{\Lambda}_2$ are distinct, it follows from the equation (2.16) that

$$H_3 = \mathrm{diag}\{\xi_{2l+1}, \ldots, \xi_p\}, \tag{2.19}$$

where the scalars $\xi_j, j = 2l+1, \ldots, p$, are arbitrary real numbers.

Observe that the matrices $\tilde{\Lambda}_2$ and $\tilde{\Lambda}_1^T$ have no eigenvalues in common. Applying an established result [6, Theorem 2, Section 12.3], we conclude that

$$H_2 \equiv 0. \tag{2.20}$$

Now we are ready to deal with the equation (2.18). Let

$$H_1 = [H_{ij}]_{2l\times 2l} \quad \text{with } H_{ij} = \begin{bmatrix} \eta_1^{(ij)} & \eta_2^{(ij)} \\ \eta_3^{(ij)} & \eta_4^{(ij)} \end{bmatrix}_{2\times 2} \quad \text{for } i, j = 1, \ldots, l. \tag{2.21}$$

From the equation (2.18), when $i \neq j, i, j = 1, \ldots, l$, we have

$$\begin{bmatrix} \eta_1^{(ij)} & \eta_2^{(ij)} \\ \eta_3^{(ij)} & \eta_4^{(ij)} \end{bmatrix} \begin{bmatrix} \alpha_{2j-1} & \beta_{2j-1} \\ -\beta_{2j-1} & \alpha_{2j-1} \end{bmatrix} = \begin{bmatrix} \alpha_{2i-1} & -\beta_{2i-1} \\ \beta_{2i-1} & \alpha_{2i-1} \end{bmatrix} \begin{bmatrix} \eta_1^{(ij)} & \eta_2^{(ij)} \\ \eta_3^{(ij)} & \eta_4^{(ij)} \end{bmatrix}.$$

After some manipulations this results in

$$\begin{bmatrix} \zeta & -\beta_{2j-1} & \beta_{2i-1} & 0 \\ \beta_{2j-1} & \zeta & 0 & \beta_{2i-1} \\ -\beta_{2i-1} & 0 & \zeta & -\beta_{2j-1} \\ 0 & -\beta_{2i-1} & \beta_{2j-1} & \zeta \end{bmatrix} \begin{bmatrix} \eta_1^{(ij)} \\ \eta_2^{(ij)} \\ \eta_3^{(ij)} \\ \eta_4^{(ij)} \end{bmatrix} = 0, \tag{2.22}$$

where $\zeta = \alpha_{2j-1} - \alpha_{2i-1}$. It is easily verified that

$$\det \begin{bmatrix} \zeta & -\beta_{2j-1} & \beta_{2i-1} & 0 \\ \beta_{2j-1} & \zeta & 0 & \beta_{2i-1} \\ -\beta_{2i-1} & 0 & \zeta & -\beta_{2j-1} \\ 0 & -\beta_{2i-1} & \beta_{2j-1} & \zeta \end{bmatrix} = (\beta_{2j-1}^2 - \beta_{2i-1}^2)^2 + \zeta^4 + 2\zeta^2(\beta_{2i-1}^2 + \beta_{2j-1}^2) \neq 0.$$

Hence, it follows from the equation (2.22) that

$$\eta_t^{(ij)} = 0 \quad \text{for } t = 1, 2, 3, 4,$$

namely,

$$H_{ij} = 0 \quad \text{for } i \neq j,\ i,\ j = 1, \ldots, l.$$

When $i = j, i = 1, \ldots, l$, from the equation (2.18), it is easy to check that

$$H_{ii} = \begin{bmatrix} \varepsilon_{2i-1} & \delta_{2i-1} \\ \delta_{2i-1} & -\varepsilon_{2i-1} \end{bmatrix}, \quad i = 1, \ldots, l, \tag{2.23}$$

where the scalars $\varepsilon_{2i-1}, \delta_{2i-1},\ i =, 1, \ldots, l$, are arbitrary real numbers.

By now, we know that the general symmetric solution to the equation (2.13) with unknown matrix H can be expressed as

$$H = \mathrm{diag}\left\{ \begin{bmatrix} \varepsilon_1 & \delta_1 \\ \delta_1 & -\varepsilon_1 \end{bmatrix}, \ldots, \begin{bmatrix} \varepsilon_{2l-1} & \delta_{2l-1} \\ \delta_{2l-1} & -\varepsilon_{2l-1} \end{bmatrix}, \xi_{2l+1}, \ldots, \xi_p \right\}, \tag{2.24}$$

then the general symmetric solution to the equation (2.12) with unknown matrix A becomes

$$A = -\tilde{\Lambda}^T G - G\tilde{\Lambda} + H. \tag{2.25}$$

Combining (2.25) with (2.11), we find that

$$B = \tilde{\Lambda}^T G \tilde{\Lambda} - H\tilde{\Lambda}. \tag{2.26}$$

Inserting (2.25) and (2.26) into (2.10) yields

$$D_{11} = -\Sigma^{-1} Q^T (\tilde{\Lambda}^T G + G\tilde{\Lambda}) Q \Sigma^{-1} + \Sigma^{-1} Q^T H Q \Sigma^{-1} \tag{2.27}$$

and

$$K_{11} = \Sigma^{-1} Q^T \tilde{\Lambda}^T G \tilde{\Lambda} Q \Sigma^{-1} - \Sigma^{-1} Q^T H \tilde{\Lambda} Q \Sigma^{-1}. \tag{2.28}$$

From the equation (2.9), it follows that

$$K_{12} = -\Sigma^{-1} Q^T \tilde{\Lambda}^T Q \Sigma D_{12}, \tag{2.29}$$

where D_{12} is an arbitrary matrix.

Summing up above discussion, we can reach our first major result as follows.

Theorem 2.1. *Let $(\Lambda, X) \in \mathbf{C}^{p\times p} \times \mathbf{C}^{n\times p}$ be given as in (1.5) and (1.6). Separate matrices Λ and X into real parts and imaginary parts resulting $\tilde{\Lambda}$ and $\tilde{X}$ expressed as in (2.1) and (2.2). Let the SVD of $\tilde{X}$ be (2.4). Then the solution set of Problem I, denoted by $\mathcal{S}_{DK}$, admits the following representation:*

$$\mathcal{S}_{DK} = \left\{ (D,\ K) \left| \begin{array}{c} D = U\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{bmatrix} U^T, \\ K = U\begin{bmatrix} K_{11} & -\Sigma^{-1}Q^T\tilde{\Lambda}^T Q\Sigma D_{12} \\ -D_{12}^T \Sigma Q^T \tilde{\Lambda} Q \Sigma^{-1} & K_{22} \end{bmatrix} U^T \end{array} \right. \right\}, \tag{2.30}$$

where $D_{12} \in \mathbf{R}^{p\times(n-p)}$ is an arbitrary matrix, $D_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$ and $K_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$ are arbitrary symmetric matrices, and $D_{11},\ K_{11}$ and H are the same as in (2.27), (2.28) and (2.24), respectively.

3. Solving Problem II

In the preceding section, we have shown that the Problem I is always solvable, that is, the solution set $\mathcal{S}_{DK}$ is always nonempty. It is easy to verify that $\mathcal{S}_{DK}$ is a closed convex set. Therefore, for given matrices $\tilde{D}, \tilde{K} \in \mathbf{R}^{n\times n}$, based on the variation principle (see [7]), we know there exists a unique solution $(\hat{D}, \hat{K})$ in $\mathcal{S}_{DK}$ such that the equality (1.8) holds.

Now, we shall focus our attention on seeking the unique solution $(\hat{D}, \hat{K})$ in $\mathcal{S}_{DK}$. For given matrices $\tilde{D},\ \tilde{K} \in \mathbf{R}^{n\times n}$, and any pair of matrices $(D, K) \in \mathcal{S}_{DK}$ given in (2.30), we obtain

$$\begin{aligned} \|D-\tilde{D}\|_W^2 + \|K-\tilde{K}\|_W^2 &= \|WDW - W\tilde{D}W\|^2 + \|WKW - W\tilde{K}W\|^2 \\ &= \left\| \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} U^T D U \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} U^T \tilde{D} U \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \right\|^2 \\ &\quad + \left\| \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} U^T K U \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} U^T \tilde{K} U \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \right\|^2, \end{aligned}$$

where $\|\cdot\|_W$ is defined by (2.5). Let

$$U^T\tilde{D}U = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix}, \qquad U^T\tilde{K}U = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{21} & \tilde{K}_{22} \end{bmatrix} \quad \text{with } \tilde{D}_{11},\ \tilde{K}_{11} \in \mathbf{R}^{p\times p}. \tag{3.1}$$

Hence,

$$\begin{aligned} \|D-\tilde{D}\|_W^2 + \|K-\tilde{K}\|_W^2 &= \|\Sigma D_{11}\Sigma - \Sigma\tilde{D}_{11}\Sigma\|^2 + \|\Sigma D_{12} - \Sigma\tilde{D}_{12}\|^2 + \|D_{12}^T\Sigma - \tilde{D}_{21}\Sigma\|^2 \\ &\quad + \|D_{22} - \tilde{D}_{22}\|^2 + \|\Sigma K_{11}\Sigma - \Sigma\tilde{K}_{11}\Sigma\|^2 + \|Q^T\tilde{\Lambda}^T Q\Sigma D_{12} + \Sigma\tilde{K}_{12}\|^2 \\ &\quad + \|D_{12}^T\Sigma Q^T\tilde{\Lambda}Q + \tilde{K}_{21}\Sigma\|^2 + \|K_{22} - \tilde{K}_{22}\|^2 \\ &= \|H - (Q\Sigma\tilde{D}_{11}\Sigma Q^T + \tilde{\Lambda}^T G + G\tilde{\Lambda})\|^2 + \|H\tilde{\Lambda} + (Q\Sigma\tilde{K}_{11}\Sigma Q^T - \tilde{\Lambda}^T G\tilde{\Lambda})\|^2 \\ &\quad + \|\Sigma D_{12} - \Sigma\tilde{D}_{12}\|^2 + \|\Sigma D_{12} - \Sigma\tilde{D}_{21}^T\|^2 + \|\tilde{\Lambda}^T Q\Sigma D_{12} + Q\Sigma\tilde{K}_{12}\|^2 \\ &\quad + \|\tilde{\Lambda}^T Q\Sigma D_{12} + Q\Sigma\tilde{K}_{21}^T\|^2 + \|D_{22} - \tilde{D}_{22}\|^2 + \|K_{22} - \tilde{K}_{22}\|^2. \end{aligned} \tag{3.2}$$

It is easily seen from the relation (3.2) that $\|D-\tilde{D}\|_W^2 + \|K-\tilde{K}\|_W^2 = \min$ if and only if

$$\|H - (Q\Sigma\tilde{D}_{11}\Sigma Q^T + \tilde{\Lambda}^T G + G\tilde{\Lambda})\|^2 + \|H\tilde{\Lambda} + (Q\Sigma\tilde{K}_{11}\Sigma Q^T - \tilde{\Lambda}^T G\tilde{\Lambda})\|^2 = \min, \tag{3.3}$$

$$\|\Sigma D_{12} - \Sigma\tilde{D}_{12}\|^2 + \|\Sigma D_{12} - \Sigma\tilde{D}_{21}^T\|^2 + \|\tilde{\Lambda}^T Q\Sigma D_{12} + Q\Sigma\tilde{K}_{12}\|^2 + \|\tilde{\Lambda}^T Q\Sigma D_{12} + Q\Sigma\tilde{K}_{21}^T\|^2 = \min, \tag{3.4}$$

$$\|D_{22} - \tilde{D}_{22}\|^2 = \min, \quad \text{s.t. } D_{22} = D_{22}^T, \tag{3.5}$$

$$\|K_{22} - \tilde{K}_{22}\|^2 = \min, \quad \text{s.t. } K_{22} = K_{22}^T. \tag{3.6}$$

Let

$$N = Q\Sigma\tilde{D}_{11}\Sigma Q^T + \tilde{\Lambda}^T G + G\tilde{\Lambda} = [\gamma_{ij}]_{p\times p}, \qquad L = \tilde{\Lambda}^T G\tilde{\Lambda} - Q\Sigma\tilde{K}_{11}\Sigma Q^T = [\mu_{ij}]_{p\times p}. \tag{3.7}$$

Then the minimization problem (3.3) is equivalent to

$$\begin{aligned} &\left\| \begin{bmatrix} \varepsilon_{2i-1} & \delta_{2i-1} \\ \delta_{2i-1} & -\varepsilon_{2i-1} \end{bmatrix} - \begin{bmatrix} \gamma_{2i-1,2i-1} & \gamma_{2i-1,2i} \\ \gamma_{2i,2i-1} & \gamma_{2i,2i} \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} \alpha_{2i-1}\varepsilon_{2i-1} - \beta_{2i-1}\delta_{2i-1} & \beta_{2i-1}\varepsilon_{2i-1} + \alpha_{2i-1}\delta_{2i-1} \\ \alpha_{2i-1}\delta_{2i-1} + \beta_{2i-1}\varepsilon_{2i-1} & \beta_{2i-1}\delta_{2i-1} - \alpha_{2i-1}\varepsilon_{2i-1} \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} \mu_{2i-1,2i-1} & \mu_{2i-1,2i} \\ \mu_{2i,2i-1} & \mu_{2i,2i} \end{bmatrix} \right\|^2 = \min \quad \text{for } i = 1, \ldots, l \end{aligned} \tag{3.8}$$

and

$$\|\xi_j - \gamma_{jj}\|^2 + \|\lambda_j \xi_j - \mu_{jj}\|^2 = \min, \quad \text{for } j = 2l+1, \dots, p. \tag{3.9}$$

After some algebraic manipulations, from (3.8) and (3.9) we conclude that

$$\begin{aligned}\varepsilon_{2i-1} &= \frac{1}{2(1+\alpha_{2i-1}^2+\beta_{2i-1}^2)}(\gamma_{2i-1,2i-1} - \gamma_{2i,2i} + \alpha_{2i-1}\mu_{2i-1,2i-1} \\ &\quad + \beta_{2i-1}\mu_{2i-1,2i} + \beta_{2i-1}\mu_{2i,2i-1} - \alpha_{2i-1}\mu_{2i,2i}), \quad i = 1, \dots, l,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\delta_{2i-1} &= \frac{1}{2(1+\alpha_{2i-1}^2+\beta_{2i-1}^2)}(\gamma_{2i-1,2i} + \gamma_{2i,2i-1} - \beta_{2i-1}\mu_{2i-1,2i-1} \\ &\quad + \alpha_{2i-1}\mu_{2i-1,2i} + \alpha_{2i-1}\mu_{2i,2i-1} + \beta_{2i-1}\mu_{2i,2i}), \quad i = 1, \dots, l,\end{aligned} \tag{3.11}$$

$$\xi_j = \frac{1}{1+\lambda_j^2}(\gamma_{jj} + \lambda_j \mu_{jj}), \quad j = 2l+1, \dots, p. \tag{3.12}$$

Let the SVD of matrix $\tilde{\Lambda}^T Q$ be

$$\tilde{\Lambda}^T Q = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} V^T = P_1 \Omega V_1, \tag{3.13}$$

where $\Omega = \operatorname{diag}\{\theta_1, \theta_2, \dots, \theta_s\} > 0$, $s = \operatorname{rank}(\tilde{\Lambda}^T Q)$, $P = [P_1, P_2] \in \mathbf{OR}^{p\times p}$, $V = [V_1, V_2] \in \mathbf{OR}^{p\times p}$ with $P_1, V_1 \in \mathbf{R}^{p\times s}$. Partition the product $V^T \Sigma D_{12}$ into

$$V^T \Sigma D_{12} = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix}, \quad Z_{11} \in \mathbf{R}^{s\times(n-p)}. \tag{3.14}$$

Clearly, the minimization problem (3.4) is equivalent to

$$\|Z_{11} - V_1^T \Sigma \tilde{D}_{12}\|^2 + \|Z_{11} - V_1^T \Sigma \tilde{D}_{21}^T\|^2 + \|\Omega Z_{11} + P_1^T Q \Sigma \tilde{K}_{12}\|^2 + \|\Omega Z_{11} + P_1^T Q \Sigma \tilde{K}_{21}^T\|^2 = \min,$$
$$\|Z_{21} - V_2^T \Sigma \tilde{D}_{12}\|^2 + \|Z_{21} - V_2^T \Sigma \tilde{D}_{21}^T\|^2 = \min,$$

which can be easily determined as

$$Z_{11} = \Phi * (V_1^T \Sigma \tilde{D}_{12} + V_1^T \Sigma \tilde{D}_{21}^T - \Omega P_1^T Q \Sigma \tilde{K}_{12} - \Omega P_1^T Q \Sigma \tilde{K}_{21}^T), \tag{3.15}$$

where $\Phi = [\varphi_{ij}]_{s\times(n-p)}$ with $\varphi_{ij} = \frac{1}{2(1+\theta_i^2)}$ for $i = 1, \dots, s$, and $j = 1, \dots, (n-p)$, and

$$Z_{21} = \frac{1}{2}(V_2^T \Sigma \tilde{D}_{12} + V_2^T \Sigma \tilde{D}_{21}^T). \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) yields

$$D_{12} = \Sigma^{-1} V_1 Z_{11} + \Sigma^{-1} V_2 Z_{21}. \tag{3.17}$$

Applying the result established by Fan and Hoffman, see [8, Theorem 2], we conclude from the minimization problems (3.5) and (3.6) that

$$D_{22} = \frac{1}{2}(\tilde{D}_{22} + \tilde{D}_{22}^T), \qquad K_{22} = \frac{1}{2}(\tilde{K}_{22} + \tilde{K}_{22}^T).$$

We have now proved the following result.

Theorem 3.1. *Given matrices $\tilde{D}, \tilde{K} \in \mathbf{R}^{n\times n}$. Let $U^T \tilde{D} U = [\tilde{D}_{ij}]_{2\times 2}$, $U^T \tilde{K} U = [\tilde{K}_{ij}]_{2\times 2}$, and $Q \Sigma \tilde{D}_{11} \Sigma Q^T + \tilde{\Lambda}^T G + G\tilde{\Lambda} = [\gamma_{ij}]_{p\times p}$, $\tilde{\Lambda}^T G \tilde{\Lambda} - Q \Sigma \tilde{K}_{11} \Sigma Q^T = [\mu_{ij}]_{p\times p}$ be given as in (3.1) and (3.7). Assume that the SVD of matrix $\tilde{\Lambda}^T Q$ has the form (3.13). Then the unique solution of Problem II can be expressed as*

$$\hat{D} = U \begin{bmatrix} D_{11} & \Sigma^{-1} V_1 Z_{11} + \Sigma^{-1} V_2 Z_{21} \\ (\Sigma^{-1} V_1 Z_{11} + \Sigma^{-1} V_2 Z_{21})^T & \frac{1}{2}(\tilde{D}_{22} + \tilde{D}_{22}^T) \end{bmatrix} U^T, \tag{3.18}$$

$$\hat{K} = U \begin{bmatrix} K_{11} & -\Sigma^{-1} Q^T \tilde{\Lambda}^T Q (V_1 Z_{11} + V_2 Z_{21}) \\ -(\Sigma^{-1} Q^T \tilde{\Lambda}^T Q (V_1 Z_{11} + V_2 Z_{21}))^T & \frac{1}{2}(\tilde{K}_{22} + \tilde{K}_{22}^T) \end{bmatrix} U^T, \tag{3.19}$$

where $Z_{11}, Z_{21}, D_{11}, K_{11}$ and H are the same as in (2.24), (2.27), (2.28), (3.15) and (3.16), respectively, the scalars $\varepsilon_{2i-1}, \delta_{2i-1}, i = 1, \dots, l$, and $\xi_j, j = 2l+1, \dots, p$, which are the elements of matrix H, are given by (3.10)–(3.12), respectively.

4. Numerical algorithm and numerical example

Discussion in the preceding sections offers a constructive way to solve the IMQEP and the approximation problem, which can be formulated as an algorithm in the following steps.

Algorithm 4.1. (1) Input $\Lambda, X, \tilde{D}, \tilde{K}$.
(2) Separate matrices Λ and X into real parts and imaginary parts resulting $\tilde{\Lambda}$ and $\tilde{X}$ given as in (2.1) and (2.2).
(3) Compute the SVD of $\tilde{X}$ according to (2.4).
(4) Form the matrices $\tilde{D}_{ij}, \tilde{K}_{ij}$ for $i, j = 1, 2$ by (3.1).
(5) Compute the matrices N, L by (3.7).
(6) Assemble the matrix H expressed in (2.24) with the scalars in (3.10)–(3.12).
(7) Compute the SVD of $\tilde{\Lambda}^T Q$ according to (3.13).
(8) Calculate the matrices Z_{11} and Z_{21} on the basis of (3.15) and (3.16), and form the matrix D_{12} according to (3.17).
(9) Compute the matrices D_{11} and K_{11} in the light of (2.27) and (2.28).
(10) According to (3.18) and (3.19), calculate the optimal approximation solution $(\hat{D}, \hat{K})$.

Example 4.1. Let $n = 5, p = 3$. Given

$$\Lambda = \text{diag}\{-0.2168 - 4.3159i \quad -0.2168 + 4.3159i \quad -0.3064\},$$

$$X = \begin{bmatrix} -0.4132 + 5.2801i & -0.4132 - 5.2801i & -9.6715 \\ -4.3518 + 3.2758i & -4.3518 - 3.2758i & -9.1357 \\ -0.1336 - 4.0588i & -0.1336 + 4.0588i & -4.4715 \\ -5.1414 + 4.4003i & -5.1414 - 4.4003i & -6.9659 \\ 8.6146 - 4.0112i & 8.6146 + 4.0112i & -4.4708 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 0.9501 & 0.4966 & 0.6111 & 0.44585 & 0.4746 \\ 0.4966 & 0.4565 & 0.4052 & 0.87845 & 0.3988 \\ 0.6111 & 0.4052 & 0.9218 & 0.82755 & 0.49475 \\ 0.44585 & 0.87845 & 0.82755 & 0.4103 & 0.45175 \\ 0.4746 & 0.3988 & 0.49475 & 0.45175 & 0.1389 \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} 0.2028 & 0.107 & 0.5112 & 0.55515 & 0.3508 \\ 0.107 & 0.7468 & 0.64565 & 0.4757 & 0.58775 \\ 0.5112 & 0.64565 & 0.5252 & 0.44195 & 0.5505 \\ 0.55515 & 0.4757 & 0.44195 & 0.3795 & 0.5682 \\ 0.3508 & 0.58775 & 0.5505 & 0.5682 & 0.1897 \end{bmatrix}.$$

Using Algorithm 4.1, we obtain the unique solution $(\hat{D}, \hat{K})$ of Problem II as follows.

$$\hat{D} = \begin{bmatrix} 3.0197 & 1.4619 & -0.66031 & 1.7773 & 0.21842 \\ 1.4619 & -1.0225 & -1.7043 & -0.25371 & 2.0397 \\ -0.66031 & -1.7043 & 1.0113 & -1.6342 & 3.1118 \\ 1.7773 & -0.25371 & -1.6342 & -0.19947 & 1.3446 \\ 0.21842 & 2.0397 & 3.1118 & 1.3446 & 0.7091 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} 21.163 & -8.2495 & -30.471 & -1.9469 & 8.0122 \\ -8.2495 & 4.1853 & 11.534 & 2.0892 & -5.3793 \\ -30.471 & 11.534 & 46.211 & 2.19 & -8.3976 \\ -1.9469 & 2.0892 & 2.19 & 2.1772 & -4.951 \\ 8.0122 & -5.3793 & -8.3976 & -4.951 & 12.913 \end{bmatrix}.$$

We define the residual as

$$\text{res}(\lambda_i, x_i) = \|(\lambda_i^2 I_n + \lambda_i \hat{D} + \hat{K})x_i\|,$$

and the numerical results are shown in the following table.

(λ_i, x_i)	(λ_1, x_1)	(λ_2, x_2)	(λ_3, x_3)
$\text{res}(\lambda_i, x_i)$	2.5487e−013	2.5487e−013	2.5513e−013

Therefore, the given eigenvalues (the diagonal elements of the matrix Λ) and eigenvectors (the column vectors of the matrix X) are embedded in the new system $(\lambda^2 I_n + \lambda \hat{D} + \hat{K})x = 0$.

Example 4.2. Consider a model updating problem. The original model is the statically condensed oil rig model (M_a, D_a, K_a) represented by the triplet BCSSTRUC1 in the Harwell–Boeing collection [9]. In this model, M_a and $K_a \in \mathbf{R}^{66\times 66}$ are symmetric and positive definite, and $D_a = 1.55I_{66}$. There are 132 eigenpairs. Let the Cholesky decomposition of M_a be $M_a = LL^T$, then $\tilde{D} = L^{-1}D_aL^{-T}$, $\tilde{K} = L^{-1}K_aL^{-T}$. Assume that the measured eigenvalues are $\lambda_1 = -34.62+574.48i$, $\lambda_2 = -34.62-574.48i$, $\lambda_3 = -44.503$, $\lambda_4 = -27.554$ and $\lambda_5 = -9.2761$, and the corresponding eigenvectors are the same as those of the original model. By Algorithm 4.1, we can obtain the unique solution $(\hat{D}, \hat{K})$ of Problem II, and it is easily verified that

$$\|X\Lambda^2 + \hat{D}X\Lambda + \hat{K}X\| = 6.1241\text{e} - 009.$$

Observe that the prescribed eigenvalues and eigenvectors have been embedded in the new model.

5. Concluding remarks

Just as the quadratic eigenvalue problems are critical to scores of important applications, the quadratic inverse eigenvalue problems are equally vital in many different fields of disciplines. In a large or complicated system, often it is the case that only partial eigeninformation is available. To understand how a physical system modeled by a quadratic pencil should be modified with only partial eigeninformation in hand, it will be very helpful to first understand how the IMQEP should be solved. The main contribution of this paper is to offer an explicit formula for Problem II. The method is developed using constrained minimization theory. The minimization error function is formulated such that the resulting changes to the given matrices are a minimum under the weighted Frobenius norm sense. The method does not require iteration or eigenanalysis and the resulting system is consistent with the partially prescribed eigenstructure. The approach is demonstrated by two numerical examples and reasonable results are produced. The proposed method seems to have enough generality that, with some suitable modifications, it can be applied to other types of partially described inverse eigenvalue problems as well.

Acknowledgements

The authors are grateful to the referees for very valuable comments and suggestions concerning an earlier version of this paper.

References

[1] F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, SIAM Review 43 (2001) 235–286.
[2] J. Carvalho, B.N. Datta, W.W. Lin, C.S. Wang, Symmetric preserving eigenvalue embedding in finite element model updating of vibrating structures, Journal of Sound and Vibration 290 (2006) 839–864.
[3] W.R. Ferng, W.W. Lin, D. Pierce, C.S. Wang, Nonequivalence transformation of λ-matrix eigenproblems and model embedding approach to model tunning, Numerical Linear Algebra with Applications 8 (2001) 53–70.
[4] D.C. Zimmerman, M. Widengren, Correcting finite element models using a symmetric eigenstructure assignment technique, AIAA Journal 28 (1990) 1670–1676.
[5] L. Starek, D.J. Inman, Symmetric inverse eigenvalue vibration problem and its application, Mechanical Systems and Signal Processing 15 (2001) 11–29.
[6] P. Lancaster, M. Tismenetsky, The Theory of Matrices, 2nd ed., Academic Press, London, 1985.
[7] J.P. Aubin, Applied Functional Analysis, John Wiley, New York, 1979.
[8] Ky Fan, A.J. Hoffman, Some metric inequalities in the space of matrices, Proceedings of the American Mathematical Society 6 (1965) 111–116.
[9] R. Boisvert, R. Pozo, K. Remington, R. Barrett, J.J. Dongarra, Available online at: http://math.nist.gov/MatrixMarket.