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Implicit function theorem over free groups

Olga Kharlampovich [a,*], Alexei Myasnikov [b]

[a] *Department of Mathematics and Statistics, McGill University, Montreal, QC H3A 2K6, Canada*
[b] *Department of Mathematics, City College of New York, New York, NY 10031, USA*

Received 20 October 1999

Available online 9 June 2005

Communicated by Efim Zelmanov

Abstract

We introduce the notion of a regular quadratic equation and a regular NTQ system over a free group. We prove the results that can be described as implicit function theorems for algebraic varieties corresponding to regular quadratic and NTQ systems. We will also show that the implicit function theorem is true only for these varieties. In algebraic geometry such results would be described as lifting solutions of equations into generic points. From the model theoretic view-point we claim the existence of simple Skolem functions for particular ∀∃-formulas over free groups. Proving these theorems we describe in details a new version of the Makanin–Razborov process for solving equations in free groups. We also prove a weak version of the implicit function theorem for NTQ systems which is one of the key results in the solution of the Tarski's problems about the elementary theory of a free group. We call it the parametrization theorem.

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* Corresponding author.
E-mail addresses: olga@math.mcgill.ca (O. Kharlampovich), alexeim@att.net (A. Myasnikov).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.04.001

1. Introduction

In this paper we prove so-called implicit function theorems for regular quadratic and NTQ systems over free groups (Theorems 3, 9, 11). They can be viewed as analogs of the corresponding result from analysis, hence the name. To show this we formulate a very basic version of the implicit function theorem.

Let

$$S(x_1, \dots, x_n, a_1, \dots, a_k) = 1$$

be a "regular" quadratic equation in variables $X = (x_1, \dots, x_n)$ with constants $a_1, \dots, a_k$ in a free group F (roughly speaking "regular" means that the radical of S coincides with the normal closure of S and S is not an equation of one of few very specific types). Suppose now that for each solution of the equation $S(X)=1$ some other equation

$$T(x_1,\dots,x_n,y_1,\dots,y_m,a_1,\dots,a_k)=1$$

has a solution in F, then $T(X,Y)=1$ has a solution $Y=(y_1,\dots,y_m)$ in the coordinate group $G_{R(S)}$ of the equation $S(X)=1$.

This implies, that locally (in terms of Zariski topology), i.e., in the neighborhood defined by the equation $S(X)=1$, the implicit functions $y_1,\dots,y_m$ can be expressed as explicit words in variables $x_1,\dots,x_n$ and constants from F, say $Y=P(X)$. This result allows one to eliminate a quantifier from the following formula

$$\Phi=\forall X\exists Y\bigl(S(X)=1\ \to\ T(X,Y)=1\bigr).$$

Indeed, the sentence Φ is equivalent in F to the following one:

$$\Psi=\forall X\bigl(S(X)=1\ \to\ T\bigl(X,P(X)\bigr)=1\bigr).$$

From model theoretic view-point the theorems claim existence of very simple Skolem functions for particular ∀∃-formulas over free groups. While in algebraic geometry such results would be described as lifting solutions of equations into generic points. We discuss definitions and general properties of liftings in Section 6. We also prove Theorem 12 which is a weak version of the implicit function theorem for NTQ systems. We call it the parametrization theorem. This weak version of the implicit function theorems forms an important part of the solution of Tarski's problems in [16]. All implicit function theorems will be proved in Section 7.

In Sections 4 and 5 we describe a new version of the Makanin–Razborov process for a system of equations with parameters, describe a solution set of such a system (Theorems 5 and 7) and introduce a new type of equations over groups, so-called *cut equations* (see Definition 21 and Theorem 8).

We collect some preliminary results and basic notions of algebraic geometry for free groups in Section 2. In Section 3 we discuss first order formulas over a free group and reduce an arbitrary sentence to a relatively simple form.

This paper is an extended version of the paper [15]; the basic version of the implicit function theorem was announced at the Model Theory Conference at MSRI in 1998 (see [23] and [14]).

2. Preliminaries

2.1. Free monoids and free groups

Let $A=\{a_1,\dots,a_m\}$ be a set. By $F_{\mathrm{mon}}(A)$ we denote the free monoid generated by A which is defined as the set of all words (including the empty word 1) over the alphabet A with concatenation as multiplication. For a word $w=b_1\dots b_n$, where $b_i\in A$, by $|w|$ or $d(w)$ we denote the length n of w.

To each $a \in A$ we associate a symbol a^{-1}. Put $A^{-1} = \{a^{-1} \mid a \in A\}$, and suppose that $A \cap A^{-1} = \emptyset$. We assume that $a^1 = a$, $(a^{-1})^{-1} = a$ and $A^1 = A$. Denote $A^{\pm 1} = A \cup A^{-1}$. If $w = b_1^{\varepsilon_1} \dots b_n^{\varepsilon_n} \in F_{\text{mon}}(A^{\pm 1})$, where $(\varepsilon_i \in \{1, -1\})$, then we put $w^{-1} = b_n^{-\varepsilon_n} \dots b_1^{-\varepsilon_1}$; we see that $w^{-1} \in M(A^{\pm 1})$ and say that w^{-1} is an inverse of w. Furthermore, we put $1^{-1} = 1$.

A word $w \in F_{\text{mon}}(A^{\pm 1})$ is called *reduced* if it does not contain subwords bb^{-1} for $b \in A^{\pm 1}$. If $w = w_1 bb^{-1} w_2$, $w \in F_{\text{mon}}(A^{\pm 1})$ then we say that $w_1 w_2$ is obtained from w by an elementary reduction $bb^{-1} \to 1$. A reduction process for w consists of finitely many reductions which bring w to a reduced word $\bar{w}$. This $\bar{w}$ does not depend on a particular reduction process and is called the *reduced form* of w.

Consider a congruence relation on $F_{\text{mon}}(A^{\pm 1})$, defined the following way: two words are congruent if a reduction process brings them to the same reduced word. The set of congruence classes with respect to this relation forms a free group $F(A)$ with basis A. If not said otherwise, we assume that $F(A)$ is given as the set of all reduced words in $A^{\pm 1}$. Multiplication in $F(A)$ of two words u, w is given by the reduced form of their concatenation, i.e., $u\dot{v} = \bar{uv}$. A word $w \in F_{\text{mon}}(A^{\pm 1})$ determines the element $\bar{w} \in F(A)$, in this event we sometimes say that w is an element of $F(A)$ (even though w may not be reduced).

Words $u, w \in F_{\text{mon}}(A^{\pm 1})$ are *graphically equal* if they are equal in the monoid $F_{\text{mon}}(A^{\pm 1})$ (for example, $a_1 a_2 a_2^{-1}$ is not graphically equal to a_1).

Let $X = \{x_1, \dots, x_n\}$ be a finite set of elements disjoint with A. Let $w(X) = w(x_1, \dots, x_n)$ be a word in the alphabet $(X \cup A)^{\pm 1}$ and $U = (u_1(A), \dots, u_n(A))$ be a tuple of words in the alphabet $A^{\pm 1}$. By $w(U)$ we denote the word which is obtained from w by replacing each x_i by u_i. Similarly, if $W = (w_1(X), \dots, w_m(X))$ is an m-tuple of words in variables X then by $W(U)$ we denote the tuple $(w_1(U), \dots, w_m(U))$. For any set S we denote by S^n the set of all n-tuples of elements from S. Every word $w(X)$ gives rise to a map $p_w : (F_{\text{mon}}(A^{\pm 1}))^n \to F_{\text{mon}}(A^{\pm 1})$ defined by $p_w(U) = w(U)$ for $U \in F_{\text{mon}}(A^{\pm 1})^n$. We call p_w the word map defined by $w(X)$. If $W(X) = (w_1(X), \dots, w_m(X))$ is an m-tuple of words in variables X then we define a word map $P_W : (F_{\text{mon}}(A^{\pm 1}))^n \to F_{\text{mon}}(A^{\pm 1})^m$ by the rule $P_W(U) = W(U)$.

2.2. On G-groups

For the purpose of algebraic geometry over a given fixed group G, one has to consider the category of G-groups, i.e., groups which contain the group G as a distinguished subgroup. If H and K are G-groups then a homomorphism $\phi : H \to K$ is a G-homomorphism if $g^\phi = g$ for every $g \in G$, in this event we write $\phi : H \to_G K$. In this category morphisms are G-homomorphisms; subgroups are G-subgroups, etc. By $\text{Hom}_G(H, K)$ we denote the set of all G-homomorphisms from H into K. It is not hard to see that the free product $G * F(X)$ is a free object in the category of G-groups. This group is called a free G-group with basis X, and we denote it by $G[X]$. A G-group H is termed *finitely generated G-group* if there exists a finite subset $A \subset H$ such that the set $G \cup A$ generates H. We refer to [3] for a general discussion on G-groups.

To deal with cancellation in the group $G[X]$ we need the following notation. Let $u = u_1 \dots u_n \in G[X] = G * F(X)$. We say that u is *reduced* (as written) if $u_i \neq 1$, u_i and u_{i+1} are in different factors of the free product, and if $u_i \in F(X)$ then it is reduced in the free group $F(X)$. By $\text{red}(u)$ we denote the reduced form of u. If $\text{red}(u) = u_1 \dots u_n \in G[X]$, then we define $|u| = n$, so $|u|$ is the syllable length of u in the free product $G[X]$. For reduced $u, v \in G[X]$, we write $u \circ v$ if the product uv is reduced as written. If $u = u_1 \dots u_n$ is reduced and u_1, u_n are in different factors, then we say that u is *cyclically reduced*.

If $u = r \circ s$, $v = s^{-1} \circ t$, and $rt = r \circ t$ then we say that the word s *cancels out in reducing* uv, or, simply, s cancels out in uv. Therefore s corresponds to the *maximal* cancellation in uv.

2.3. Formulas in the language L_A

Let G be a group generated by a set of generators A. The standard first-order language of group theory, which we denote by L, consists of a symbol for multiplication $\cdot$, a symbol for inversion $^{-1}$, and a symbol for the identity 1. To deal with G-groups, we have to enlarge the language L by all non-trivial elements from G as constants. In fact, we do not need to add all the elements of G as constants, it suffices to add only new constants corresponding to the generating set A. By L_A we denote the language L with constants from A.

A group word in variables X and constants A is a word $S(X, A)$ in the alphabet $(X \cup A)^{\pm 1}$. One may consider the word $S(X, A)$ as a term in the language L_A. Observe that every term in the language L_A is equivalent modulo the axioms of group theory to a group word in variables X and constants $A \cup \{1\}$. An *atomic formula* in the language L_A is a formula of the type $S(X, A) = 1$, where $S(X, A)$ is a group word in X and A. With a slight abuse of language we will consider atomic formulas in L_A as equations over G, and vice versa. A *boolean combination* of atomic formulas in the language L_A is a disjunction of conjunctions of atomic formulas or their negations. Thus every boolean combination Φ of atomic formulas in L_A can be written in the form $\Phi = \bigvee_{i=1}^{n} \Psi_i$, where each Ψ_i has one of the following forms:

$$\bigwedge_{j=1}^{n} \big(S_j(X, A) = 1\big), \qquad \bigwedge_{j=1}^{n} \big(T_j(X, A) \neq 1\big),$$

$$\bigwedge_{j=1}^{n} \big(S_j(X, A) = 1\big) \ \& \ \bigwedge_{k=1}^{m} \big(T_k(X, A) \neq 1\big).$$

Observe that if the group G is not trivial, then every formula Ψ, as above, can be written in the from

$$\bigwedge_{j=1}^{n} \big(S_j(X, A) = 1 \ \& \ T_j(X, A) \neq 1\big),$$

where (if necessary) we add into the formula the trivial equality $1 = 1$, or an inequality of the type $a \neq 1$ for a given fixed non-trivial $a \in A$.

It follows from general results on disjunctive normal forms in propositional logic that every quantifier-free formula in the language L_A is logically equivalent (modulo the axioms of group theory) to a boolean combination of atomic ones. Moreover, every formula Φ in L_A with variables $Z\{z_1, \dots, z_k\}$ is logically equivalent to a formula of the type

$$Q_1x_1Q_2x_2\dots Q_nx_n\Psi(X, Z, A),$$

where $Q_i \in \{\forall, \exists\}$, and $\Psi(X, Z, A)$ is a boolean combination of atomic formulas in variables from $X \cup Z$. Using vector notations $QY = Qy_1 \dots Qy_n$ for strings of similar quantifiers we can rewrite such formulas in the form

$$\Phi(Z) = Q_1Z_1\dots Q_kZ_k\Psi(Z_1, \dots, Z_k, X).$$

Introducing fictitious quantifiers, one can always rewrite the formula Φ in the form

$$\Phi(Z) = \forall X_1\exists Y_1\dots\forall X_k\exists Y_k\Psi(X_1, Y_1, \dots, X_k, Y_k, Z).$$

If H is a G-group, then the set $\mathrm{Th}_A(H)$ of all sentences in L_A which are valid in H is called the *elementary theory* of H in the language L_A. Two G-groups H and K are *elementarily equivalent* in the language L_A (or G-elementarily equivalent) if $\mathrm{Th}_A(H) = \mathrm{Th}_A(K)$.

Let T be a set of sentences in the language L_A. For a formula $\Phi(X)$ in the language L_A, we write $T \vdash \Phi$ if Φ is a logical consequence of the theory T. If K is a G-group, then we write $K \models T$ if every sentence from T holds in K (where we interpret constants from A by corresponding elements in the subgroup G of K). Notice, that $\mathrm{Th}_A(H) \vdash \Phi$ holds if and only if $K \models \forall X\Phi(X)$ for every G-group K which is G-elementarily equivalent to H. Two formulas $\Phi(X)$ and $\Psi(X)$ in the language L_A are said to be *equivalent modulo T* (we write $\Phi \sim_T \Psi$) if $T \vdash \forall X(\Phi(X) \leftrightarrow \Psi(X))$. Sometimes, instead of $\Phi \sim_{\mathrm{Th}_A(G)} \Psi$ we write $\Phi \sim_G \Psi$ and say that Φ is equivalent to Ψ over G.

2.4. Elements of algebraic geometry over groups

Here we introduce some basic notions of algebraic geometry over groups. We refer to [3] and [11] for details.

Let G be a group generated by a finite set A, $F(X)$ be a free group with basis $X = \{x_1, x_2, \dots, x_n\}$, $G[X] = G * F(X)$ be a free product of G and $F(X)$. If $S \subset G[X]$ then the expression $S = 1$ is called *a system of equations* over G. As an element of the free product, the left side of every equation in $S = 1$ can be written as a product of some elements from $X \cup X^{-1}$ (which are called *variables*) and some elements from A (*constants*). To emphasize this we sometimes write $S(X, A) = 1$.

A *solution* of the system $S(X) = 1$ over a group G is a tuple of elements $g_1, \dots, g_n \in G$ such that after replacement of each x_i by g_i the left-hand side of every equation in $S = 1$ turns into the trivial element of G. Equivalently, a solution of the system $S = 1$ over G can be described as a G-homomorphism $\phi : G[X] \to G$ such that $\phi(S) = 1$. Denote by $\mathrm{ncl}(S)$ the normal closure of S in $G[X]$, and by G_S the quotient group $G[X]/\mathrm{ncl}(S)$. Then every solution of $S(X) = 1$ in G gives rise to a G-homomorphism $G_S \to G$, and vice versa. By

$V_G(S)$ we denote the set of all solutions in G of the system $S = 1$, it is called the *algebraic set defined by S*. This algebraic set $V_G(S)$ uniquely corresponds to the normal subgroup

$$R(S) = \{T(x) \in G[X] \mid \forall A \in G^n\big(S(A) = 1 \to T(A) = 1\big)\}$$

of the group $G[X]$. Notice that if $V_G(S) = \emptyset$, then $R(S) = G[X]$. The subgroup $R(S)$ contains S, and it is called the *radical of S*. The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the *coordinate group* of the algebraic set $V(S)$. Again, every solution of $S(X) = 1$ in G can be described as a G-homomorphism $G_{R(S)} \to G$.

We recall from [25] that a group G is called a *CSA group* if every maximal abelian subgroup M of G is malnormal, i.e., $M^g \cap M = 1$ for any $g \in G - M$. The class of CSA-groups is quite substantial. It includes all abelian groups, all torsion-free hyperbolic groups [25], all groups acting freely on Λ-trees [2], and many one-relator groups [8].

We define a Zariski topology on G^n by taking algebraic sets in G^n as a sub-basis for the closed sets of this topology. If G is a non-abelian CSA group, in particular, a non-abelian freely discriminated group, then the union of two algebraic sets is again algebraic (see Lemma 4). Therefore the closed sets in the Zariski topology over G are precisely the algebraic sets.

A G-group H is called *equationally Noetherian* if every system $S(X) = 1$ with coefficients from G is equivalent over G to a finite subsystem $S_0 = 1$, where $S_0 \subset S$, i.e., $V_G(S) = V_G(S_0)$. If G is G-equationally Noetherian, then we say that G is equationally Noetherian. It is known that linear groups (in particular, freely discriminated groups) are equationally Noetherian (see [3,5,10]). If G is equationally Noetherian then the Zariski topology over G^n is *Noetherian* for every n, i.e., every proper descending chain of closed sets in G^n is finite. This implies that every algebraic set V in G^n is a finite union of irreducible subsets (called *irreducible components* of V), and such a decomposition of V is unique. Recall that a closed subset V is *irreducible* if it is not a union of two proper closed (in the induced topology) subsets.

Two algebraic sets $V_F(S_1)$ and $V_F(S_2)$ are *rationally equivalent* if there exists an isomorphism between their coordinate groups which is identical on F.

2.5. Discrimination and big powers

Let H and K be G-groups. We say that a family of G-homomorphisms $\mathcal{F} \subset \mathrm{Hom}_G(H, K)$ *separates* (*discriminates*) H into K if for every non-trivial element $h \in H$ (every finite set of non-trivial elements $H_0 \subset H$) there exists $\phi \in \mathcal{F}$ such that $h^\phi \neq 1$ ($h^\phi \neq 1$ for every $h \in H_0$). In this case we say that H is *G-separated* (*G-discriminated*) by K. Sometimes we do not mention G and simply say that H is separated (discriminated) by K. In the event when K is a free group we say that H is *freely separated* (*freely discriminated*).

Below we describe a method of discrimination which is called a *big powers* method. We refer to [25] and [24] for details about BP-groups.

Let G be a group. We say that a tuple $u = (u_1, \ldots, u_k) \in G^k$ has *commutation* if $[u_i, u_{i+1}] = 1$ for some $i = 1, \ldots, k-1$. Otherwise we call u *commutation-free*.

Definition 1. A group G satisfies the *big powers condition* (BP) if for any commutation-free tuple $u = (u_1, \ldots, u_k)$ of elements from G there exists an integer $n(u)$ (called *a boundary of separation* for u) such that

$$u_1^{\alpha_1} \ldots u_k^{\alpha_k} \neq 1$$

for any integers $\alpha_1, \ldots, \alpha_k \geqslant n(u)$. Such groups are called *BP-groups*.

The following provides a host of examples of BP-groups. Obviously, a subgroup of a BP-group is a BP-group; a group discriminated by a BP-group is a BP-group [25]; every torsion-free hyperbolic group is a BP-group [26]. From those facts it follows that every freely discriminated group is a BP-group.

Let G be a non-abelian CSA group and $u \in G$ not be a proper power. The following HNN-extension

$$G(u,t) = \langle G, t \mid g^t = g \ (g \in C_G(u)) \rangle$$

is called a *free extension* of the centralizer $C_G(u)$ by a letter t. It is not hard to see that for any integer k the map $t \to u^k$ can be extended uniquely to a G-homomorphism $\xi_k : G(u,t) \to G$.

The result below is the essence of the big powers method of discrimination.

Theorem [25]. *Let G be a non-abelian CSA BP-group. If $G(u,t)$ is a free extension of the centralizer of the non-proper power u by t, Then the family of G-homomorphisms $\{\xi_k \mid k$ is an integer$\}$ discriminates $G(u,v)$ into G. More precisely, for every $w \in G(u,t)$ there exists an integer N_w such that for every $k \geqslant N_w$, $w^{\xi_k} \neq 1$.*

If G is a non-abelian CSA BP-group and X is a finite set, then the group $G[X]$ is G-embeddable into $G(u,t)$ for any non-proper power $u \in G$. It follows from the theorem above that $G[X]$ is G-discriminated by G.

Unions of chains of extensions of centralizers play an important part in this paper. Let G be a non-abelian CSA BP-group and

$$G = G_0 < G_1 < \cdots < G_n$$

be a chain of extensions of centralizers $G_{i+1} = G_i(u_i, t_i)$. Then every n-tuple of integers $p = (p_1, \ldots, p_n)$ gives rise to a G-homomorphism $\xi_p : G_n \to G$ which is composition of homomorphisms $\xi_{p_i} : G_i \to G_{i-1}$ described above. If a centralizer of u_i is extended several times, we can suppose it is extended on the consecutive steps by letters $t_i, \ldots, t_{i+j}$. Therefore $u_{i+1} = t_i, \ldots, u_{i+j} = t_{i+j-1}$.

A set P of n-tuples of integers is called *unbounded* if for every integer d there exists a tuple $p = (p_1, \ldots, p_n) \in P$ with $p_i \geqslant d$ for each i. The following result is a consequence of the theorem above.

Corollary. *Let G_n be as above. Then for every unbounded set of tuples P the set of G-homomorphisms $\Xi_P = \{\xi_p \mid p \in P\}$ G-discriminates G_n into G.*

Similar results hold for infinite chains of extensions of centralizers (see [25] and [4]). For example, Lyndon's free $Z[x]$-group $F^{Z[x]}$ can be realized as union of a countable chain of extensions of centralizers which starts with the free group F (see [25]), hence there exists a family of F-homomorphisms which discriminates $F^{Z[x]}$ into F.

2.6. Freely discriminated groups

Here we formulate several results on freely discriminated groups which are crucial for our considerations.

It is not hard to see that every freely discriminated group is a torsion-free CSA group [3].

Notice that every CSA group is commutation transitive [25]. A group G is called *commutation transitive* if commutation is transitive on the set of all non-trivial elements of G, i.e., if $a, b, c \in G - \{1\}$ and $[a, b] = 1$, $[b, c] = 1$, then $[a, c] = 1$. Clearly, commutation transitive groups are precisely the groups in which centralizers of non-trivial elements are commutative. It is easy to see that every commutative transitive group G which satisfies the condition $[a, a^b] = 1 \rightarrow [a, b] = 1$ for all $a, b \in G$ is CSA.

Theorem [28]. *Let F be a free non-abelian group. Then a finitely generated F-group G is freely F-discriminated by F if and only if G is F-universally equivalent to F (i.e., G and F satisfy precisely the same universal sentences in the language L_A).*

Theorem [3,11]. *Let F be a free non-abelian group. Then a finitely generated F-group G is the coordinate group of a non-empty irreducible algebraic set over F if and only if G is freely F-discriminated by F.*

Theorem [12]. *Let F be a non-abelian free group. Then a finitely generated F-group is the coordinate group $F_{R(S)}$ of an irreducible non-empty algebraic set $V(S)$ over F if and only if G is F-embeddable into the free Lyndon's $Z[t]$-group $F^{Z[t]}$.*

This theorem implies that finitely generated freely discriminated groups are finitely presented, also it allows one to present such groups as fundamental groups of graphs of groups of a very particular type (see [12] for details).

2.7. Quadratic equations over freely discriminated groups

In this section we collect some known results about quadratic equations over fully residually free groups, which will be in use throughout this paper.

Let $S \subset G[X]$. Denote by $\mathrm{var}(S)$ the set of variables that occur in S.

Definition 2. A set $S \subset G[X]$ is called quadratic if every variable from $\mathrm{var}(S)$ occurs in S not more then twice. The set S is strictly quadratic if every letter from $\mathrm{var}(S)$ occurs in S exactly twice.

A system $S=1$ over G is *quadratic* (*strictly quadratic*) if the corresponding set S is quadratic (strictly quadratic).

Definition 3. A standard quadratic equation over a group G is an equation of the one of the following forms (below d, c_i are non-trivial elements from G):

$$\prod_{i=1}^{n}[x_i, y_i] = 1, \quad n > 0, \tag{1}$$

$$\prod_{i=1}^{n}[x_i, y_i]\prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geqslant 0,\ m+n \geqslant 1, \tag{2}$$

$$\prod_{i=1}^{n} x_i^2 = 1, \quad n > 0, \tag{3}$$

$$\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geqslant 0,\ n+m \geqslant 1. \tag{4}$$

Equations (1), (2) are called *orientable* of genus n, Eqs. (3), (4) are called *non-orientable* of genus n.

Lemma 1. *Let W be a strictly quadratic word over G. Then there is a G-automorphism $f \in \mathrm{Aut}_G(G[X])$ such that W^f is a standard quadratic word over G.*

Proof. See [7]. $\square$

Definition 4. Strictly quadratic words of the type $[x, y]$, x^2, $z^{-1}cz$, where $c \in G$, are called *atomic quadratic words* or simply *atoms*.

By definition a standard quadratic equation $S=1$ over G has the form

$$r_1 r_2 \dots r_k d = 1,$$

where r_i are atoms, $d \in G$. This number k is called the *atomic rank* of this equation, we denote it by $r(S)$. In Section 2.4 we defined the notion of the coordinate group $G_{R(S)}$. Every solution of the system $S=1$ is a homomorphism $\phi : G_{R(S)} \to G$.

Definition 5. Let $S=1$ be a standard quadratic equation written in the atomic form $r_1 r_2 \dots r_k d = 1$ with $k \geqslant 2$. A solution $\phi : G_{R(S)} \to G$ of $S=1$ is called:

(1) degenerate, if $r_i^{\phi} = 1$ for some i, and non-degenerate otherwise;
(2) commutative, if $[r_i^{\phi}, r_{i+1}^{\phi}] = 1$ for all $i = 1, \dots, k-1$, and non-commutative otherwise;
(3) in a general position, if $[r_i^{\phi}, r_{i+1}^{\phi}] \neq 1$ for all $i = 1, \dots, k-1$.

Observe that if a standard quadratic equation $S(X) = 1$ has a degenerate non-commutative solution then it has a non-degenerate non-commutative solution, see [12]).

Theorem 1 [12]. *Let G be a freely discriminated group and $S = 1$ a standard quadratic equation over G which has a solution in G. In the following cases a standard quadratic equation $S = 1$ always has a solution in a general position*:

(1) *$S = 1$ is of the form* (1), *$n > 2$*;
(2) *$S = 1$ is of the form* (2), *$n > 0$, $n + m > 1$*;
(3) *$S = 1$ is of the form* (3), *$n > 3$*;
(4) *$S = 1$ is of the form* (4), *$n > 2$*;
(5) *$r(S) \geqslant 2$ and $S = 1$ has a non-commutative solution.*

The following theorem describes the radical $R(S)$ of a standard quadratic equation $S = 1$ which has at least one solution in a freely discriminated group G.

Theorem 2 [12]. *Let G be a freely discriminated group and let $S = 1$ be a standard quadratic equation over G which has a solution in G. Then*:

(1) *If $S = [x, y]d$ or $S = [x_1, y_1][x_2, y_2]$, then $R(S) = \mathrm{ncl}(S)$*;
(2) *If $S = x^2 d$, then $R(S) = \mathrm{ncl}(xb)$ where $b^2 = d$*;
(3) *If $S = c^z d$, then $R(S) = \mathrm{ncl}([zb^{-1}, c])$ where $d^{-1} = c^b$*;
(4) *If $S = x_1^2 x_2^2$, then $R(S) = \mathrm{ncl}([x_1, x_2])$*;
(5) *If $S = x_1^2 x_2^2 x_3^2$, then $R(S) = \mathrm{ncl}([x_1, x_2], [x_1, x_3], [x_2, x_3])$*;
(6) *If $r(S) \geqslant 2$ and $S = 1$ has a non-commutative solution, then $R(S) = \mathrm{ncl}(S)$*;
(7) *If $S = 1$ is of the type* (4) *and all solutions of $S = 1$ are commutative, then $R(S)$ is the normal closure of the following system*:

$$\bigl\{x_1 \ldots x_n = s_1 \ldots s_n,\ [x_k, x_l] = 1,\ \bigl[a_i^{-1} z_i, x_k\bigr] = 1,\ [x_k, C] = 1,\ \bigl[a_i^{-1} z_i, C\bigr] = 1,$$
$$\bigl[a_i^{-1} z_i, a_j^{-1} z_j\bigr] = 1\ (k, l = 1, \ldots, n,\ i, j = 1, \ldots, m)\bigr\},$$

where $x_k \to s_k$, $z_i \to a_i$ is a solution of $S = 1$ and $C = C_G(c_1^{a_1}, \ldots, c_m^{a_m}, s_1, \ldots, s_n)$ is the corresponding centralizer. The group $G_{R(S)}$ is an extension of the centralizer C.

Definition 6. A standard quadratic equation $S = 1$ over F is called *regular* if either it is an equation of the type $[x, y] = d$ $(d \neq 1)$, or the equation $[x_1, y_1][x_2, y_2] = 1$, or $r(S) \geqslant 2$ and $S(X) = 1$ has a non-commutative solution and it is not an equation of the type $c_1^{z_1} c_2^{z_2} = c_1 c_2$, $x^2 c^z = a^2 c$, $x_1^2 x_2^2 = a_1^2 a_2^2$.

Put

$$\kappa(S) = |X| + \varepsilon(S),$$

where $\varepsilon(S) = 1$ if the coefficient d occurs in S, and $\varepsilon(S) = 0$ otherwise.

Equivalently, a standard quadratic equation $S(X) = 1$ is *regular* if $\kappa(S) \geqslant 4$ and there is a non-commutative solution of $S(X) = 1$ in G, or it is an equation of the type $[x, y]d = 1$.

Notice, that if $S(X) = 1$ has a solution in G, $\kappa(S) \geqslant 4$, and $n > 0$ in the orientable case ($n > 2$ in the non-orientable case), then the equation $S = 1$ has a non-commutative solution, hence regular.

Corollary 1.

(1) *Every consistent orientable quadratic equation $S(X) = 1$ of positive genus is regular, unless it is the equation $[x, y] = 1$;*
(2) *Every consistent non-orientable equation of positive genus is regular, unless it is an equation of the type $x^2c^z = a^2c$, $x_1^2x_2^2 = a_1^2a_2^2$, $x_1^2x_2^2x_3^2 = 1$, or $S(X) = 1$ can be transformed to the form $[\bar{z}_i, \bar{z}_j] = [\bar{z}_i, a] = 1$, $i, j = 1, \ldots, m$, by changing variables.*
(3) *Every standard quadratic equation $S(X) = 1$ of genus 0 is regular unless either it is an equation of the type $c_1^{z_1} = d$, $c_1^{z_1}c_2^{z_2} = c_1c_2$, or $S(X) = 1$ can be transformed to the form $[\bar{z}_i, \bar{z}_j] = [\bar{z}_i, a] = 1$, $i, j = 1, \ldots, m$, by changing variables.*

2.8. Formulation of the basic implicit function theorem

In this section we formulate the implicit function theorem over free groups in its basic simplest form. We refer to Sections 7.2, 7.4 for the proofs and to Section 7.6 for generalizations.

Theorem 3. *Let $S(X) = 1$ be a regular standard quadratic equation over a non-abelian free group F and let $T(X, Y) = 1$ be an equation over F, $|X| = m$, $|Y| = n$. Suppose that for any solution $U \in V_F(S)$ there exists a tuple of elements $W \in F^n$ such that $T(U, W) = 1$. Then there exists a tuple of words $P = (p_1(X), \ldots, p_n(X))$, with constants from F, such that $T(U, P(U)) = 1$ for any $U \in V_F(S)$. Moreover, one can fund a tuple P as above effectively.*

From algebraic geometric view-point the implicit function theorem tells one that (in the notations above) $T(X, Y) = 1$ has a solution at a generic point of the equation $S(X) = 1$.

3. Formulas over freely discriminated groups

In this section we collect some results (old and new) on how to effectively rewrite formulas over a non-abelian freely discriminated group G into more simple or more convenient “normal” forms. Some of these results hold for many other groups beyond the class of freely discriminated ones. We do not present the most general formulations here, instead, we limit our considerations to a class of groups $\mathcal{T}$ which will just suffice for our purposes.

Let us fix a finite set of constants A and the corresponding group theory language L_A, let also a, b be two fixed elements in A.

Definition 7. A group G satisfies Vaught's conjecture if the following universal sentence holds in G

(V) $\forall x \forall y \forall z (x^2 y^2 z^2 = 1 \ \rightarrow \ [x, y] = 1 \ \& \ [x, z] = 1 \ \& \ [y, z] = 1)$.

Lyndon proved that every free group satisfies the condition (V) (see [17]).

Denote by $\mathcal{T}$ the class of all groups G such that:

(1) G is torsion-free;
(2) G satisfies Vaught's conjecture;
(3) G is CSA;
(4) G has two distinguished elements a, b with $[a, b] \neq 1$.

It is easy to write down axioms for the class $\mathcal{T}$ in the language $L_{\{a,b\}}$. Indeed, the following universal sentences describe the conditions (1)–(4) above:

(TF) $x^n = 1 \rightarrow x = 1$ $(n = 2, 3, \ldots)$;
(V) $\forall x \forall y \forall z (x^2 y^2 z^2 = 1 \ \rightarrow \ [x, y] = 1 \ \& \ [x, z] = 1 \ \& \ [y, z] = 1)$;
(CT) $\forall x \forall y \forall z (x \neq 1 \ \& \ y \neq 1 \ \& \ z \neq 1 \ \& \ [x, y] = 1 \ \& \ [x, z] = 1 \ \rightarrow \ [y, z] = 1)$;
(WCSA) $\forall x \forall y ([x, x^y] = 1 \ \rightarrow \ [x, y] = 1)$;
(NA) $[a, b] \neq 1$.

Observe that the condition (WCSA) is a weak form of (CSA) but (WCSA) and (CT) together provide the CSA condition. Let GROUPS be a set of axioms of group theory. Denote by $A_{\mathcal{T}}$ the union of axioms (TF), (V), (CT), (WCSA), (NA) and GROUPS. Notice that the axiom (V) is equivalent modulo GROUPS to the following quasi-identity

$$\forall x \forall y \forall z \big(x^2 y^2 z^2 = 1 \ \rightarrow \ [x, y] = 1\big).$$

It follows that all axioms in $A_{\mathcal{T}}$, with exception of (CT) and (NA), are quasi-identities.

Lemma 2. *The class $\mathcal{T}$ contains all freely discriminated non-abelian groups.*

Proof. We show here that every freely discriminated group G satisfies (V). Similar arguments work for the other conditions. If $u^2 v^2 w^2 = 1$ for some $u, v, w \in G$ and, say, $[u, v] \neq 1$, then there exists a homomorphism $\phi : G \to F$ from G onto a free group F such that $[u^\phi, v^\phi] \neq 1$. This shows that the elements u^ϕ, v^ϕ, w^ϕ in F give a counterexample to Vaught's conjecture. This contradicts to the Lyndon's result. Hence (V) holds in G. This proves the lemma. □

Almost all results in this section state that a formula $\Phi(X)$ in L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula $\Psi(X)$ in L_A. We will use these results in the following particular form. Namely, if G is a group generated by A and H is a G-group from $\mathcal{T}$ then for any tuple of elements $U \in H^n$ (here $n = |X|$) the formula $\Phi(U)$ holds in H if and only if $\Psi(U)$ holds in H.

3.1. Quantifier-free formulas

In this section by letters X, Y, Z we denote finite tuples of variables.

The following result is due to A. Malcev [21]. He proved it for free groups, but his argument is valid in a more general context.

Lemma 3. *Let $G \in \mathcal{T}$. Then the equation*

$$x^2ax^2a^{-1} = \left(ybyb^{-1}\right)^2 \tag{5}$$

has only the trivial solution $x = 1$ and $y = 1$ in G.

Proof. Let G be as above and let x, y be a solution in G of Eq. (5) such that $x \neq 1$. Then

$$\left(x^2a\right)^2a^{-2} = \left((yb)^2b^{-2}\right)^2. \tag{6}$$

In view of the condition (V), we deduce from (6) that $[x^2a, a^{-1}] = 1$, hence $[x^2, a^{-1}] = 1$. By transitivity of commutation $[x, a] = 1$ (here we use inequality $x \neq 1$). Now, we can rewrite (6) in the form

$$x^2x^2 = \left((yb)^2b^{-2}\right)^2,$$

which implies (according to (V)), that $[x^2, (yb)^2b^{-2}] = 1$, and hence (since G is torsion-free)

$$x^2 = (yb)^2b^{-2}. \tag{7}$$

Again, it follows from (V) that $[y, b] = 1$. Henceforth, $x^2 = y^2$ and, by the argument above, $x = y$. We proved that $[x, a] = 1$ and $[x, b] = 1$ therefore, by transitivity of commutation, $[a, b] = 1$, which contradicts to the choice of a, b. This contradiction shows that $x = 1$. In this event, Eq. (6) transforms into

$$\left((yb)^2b^{-2}\right)^2 = 1,$$

which implies $(yb)^2b^{-2} = 1$. Now from (V) we deduce that $[yb, b] = 1$, and hence $[y, b] = 1$. It follows that $y^2 = 1$, so $y = 1$, as desired. $\square$

Corollary 2. *Let $G \in \mathcal{T}$. Then for any finite system of equations $S_1(X) = 1, \ldots, S_k(X) = 1$ over G one can effectively find a single equation $S(X) = 1$ over G such that*

$$V_G(S_1, \ldots, S_n) = V_G(S).$$

Proof. By induction it suffices to prove the result for $k = 2$. In this case, by the lemma above, the following equation (after bringing the right side to the left)

$$S_1(X)^2 a S_1(X)^2 a^{-1} = \left(S_2(X) b S_2(X) b^{-1}\right)^2$$

can be chosen as the equation $S(X) = 1$. □

Corollary 3. *For any finite system of equations*

$$S_1(X) = 1, \ldots, S_k(X) = 1$$

in L_A, one can effectively find a single equation $S(X) = 1$ in L_A such that

$$\left(\bigwedge_{i=1}^{k} S_i(X) = 1\right) \sim_{A_{\mathcal{T}}} S(X) = 1.$$

Remark 1. In the proof of Lemma 3 and Corollaries 2 and 3 we did not use the condition (WCSA) so the results hold for an arbitrary non-abelian torsion-free commutation transitive group satisfying Vought's conjecture.

The next lemma shows how to rewrite finite disjunctions of equations into conjunctions of equations. In the case of free groups this result was known for years (in [20] Makanin attributes this to Y. Gurevich). We give here a different proof.

Lemma 4. *Let G be a CSA group and let a, b be arbitrary non-commuting elements in G. Then for any solution $x, y \in G$ of the system*

$$\left[x, y^a\right] = 1, \qquad \left[x, y^b\right] = 1, \qquad \left[x, y^{ab}\right] = 1, \tag{8}$$

either $x = 1$ or $y = 1$. The converse is also true.

Proof. Suppose x, y are non-trivial elements from G, such that

$$\left[x, y^a\right] = 1, \qquad \left[x, y^b\right] = 1, \qquad \left[x, y^{ab}\right] = 1.$$

Then by the transitivity of commutation $[y^b, y^{ab}] = 1$ and $[y^a, y^b] = 1$. The first relation implies that $[y, y^a] = 1$, and since a maximal abelian subgroup M of G containing y is malnormal in G, we have $[y, a] = 1$. Now from $[y^a, y^b] = 1$ it follows that $[y, y^b] = 1$ and, consequently, $[y, b] = 1$. This implies $[a, b] = 1$, a contradiction, which completes the proof. □

Combining Lemmas 4 and 3 yields an algorithm to encode an arbitrary finite disjunction of equations into a single equation.

Corollary 4. *Let $G \in \mathcal{T}$. Then for any finite set of equations $S_1(X) = 1, \dots, S_k(X) = 1$ over G one can effectively find a single equation $S(X) = 1$ over G such that*

$$V_G(S_1) \cup \cdots \cup V_G(S_k) = V_G(S).$$

Inspection of the proof above shows that the following corollary holds.

Corollary 5. *For any finite set of equations $S_1(X) = 1, \dots, S_k(X) = 1$ in L_A, one can effectively find a single equation $S(X) = 1$ in L_A such that*

$$\left(\bigvee_{i=1}^{k} S_i(X) = 1\right) \sim_{A_{\mathcal{T}}} S(X) = 1.$$

Corollary 6. *Every positive quantifier-free formula $\Phi(X)$ in L_A is equivalent modulo $A_{\mathcal{T}}$ to a single equation $S(X) = 1$.*

The next result shows that one can effectively encode finite conjunctions and finite disjunctions of *inequalities* into a single inequality modulo $A_{\mathcal{T}}$.

Lemma 5. *For any finite set of inequalities*

$$S_1(X) \neq 1, \dots, S_k(X) \neq 1$$

in L_A, one can effectively find an inequality $R(X) \neq 1$ and an inequality $T(X) \neq 1$ in L_A such that

$$\left(\bigwedge_{i=1}^{k} S_i(X) \neq 1\right) \sim_{A_{\mathcal{T}}} R(X) \neq 1$$

and

$$\left(\bigvee_{i=1}^{k} S_i(X) \neq 1\right) \sim_{A_{\mathcal{T}}} T(X) \neq 1.$$

Proof. By Corollary 5 there exists an equation $R(X) = 1$ such that

$$\bigvee_{i=1}^{k} \bigl(S_i(X) = 1\bigr) \sim_{A_{\mathcal{T}}} R(X) = 1.$$

Hence

$$\left(\bigwedge_{i=1}^{k} S_i(X) \neq 1\right) \sim_{A_{\mathcal{T}}} \neg\left(\bigvee_{i=1}^{k} S_i(X) = 1\right) \sim_{A_{\mathcal{T}}} \neg\bigl(R(X) = 1\bigr) \sim_{A_{\mathcal{T}}} R(X) \neq 1.$$

This proves the first part of the result. Similarly, by Corollary 3 there exists an equation $T(X)=1$ such that

$$\left(\bigwedge_{i=1}^{k} S_i(X)=1\right) \sim_{A_{\mathcal{T}}} T(X)=1.$$

Hence

$$\left(\bigvee_{i=1}^{k} S_i(X)\neq 1\right) \sim_{A_{\mathcal{T}}} \neg\left(\bigwedge_{i=1}^{k} S_i(X)=1\right) \sim_{A_{\mathcal{T}}} \neg\big(T(X)=1\big) \sim_{A_{\mathcal{T}}} T(X)\neq 1.$$

This completes the proof. □

Corollary 7. *For every quantifier-free formula $\Phi(X)$ in the language L_A, one can effectively find a formula*

$$\Psi(X)=\bigvee_{i=1}^{n}\big(S_i(X)=1 \;\&\; T_i(X)\neq 1\big)$$

in L_A which is equivalent to $\Phi(X)$ modulo $A_{\mathcal{T}}$. In particular, if $G\in\mathcal{T}$, then every quantifier-free formula $\Phi(X)$ in L_G is equivalent over G to a formula $\Psi(X)$ as above.

3.2. Universal formulas over F

In this section we discuss canonical forms of universal formulas in the language L_A modulo the theory $A_{\mathcal{T}}$ of the class $\mathcal{T}$ of all torsion-free non-abelian CSA groups satisfying Vaught's conjecture. We show that every universal formula in L_A is equivalent modulo $A_{\mathcal{T}}$ to a universal formula in canonical radical form. This implies that if $G\in\mathcal{T}$ is generated by A, then the universal theory of G in the language L_A consists of the axioms describing the diagram of G (multiplication table for G with all the equalities and inequalities between group words in A), the set of axioms $A_{\mathcal{T}}$, and a set of axioms A_R which describes the radicals of finite systems over G.

Also, we describe an effective quantifier elimination for universal positive formulas in L_A modulo $\mathrm{Th}_A(G)$, where $G\in\mathcal{T}$ and G is a BP-group (in particular, a non-abelian freely discriminated group). Notice, that in Section 4.4 in the case when G is a free group, we describe an effective quantifier elimination procedure (due to Merzljakov and Makanin) for arbitrary positive sentences modulo $\mathrm{Th}_A(G)$.

Let $G\in\mathcal{T}$ and A be a generating set for G.

We say that a universal formula in L_A is in *canonical radical form* (is a *radical formula*) if it has the following form

$$\Phi_{S,T}(X)=\forall Y\big(S(X,Y)=1 \;\rightarrow\; T(Y)=1\big) \tag{9}$$

for some $S\in G[X\cup Y]$, $T\in G[Y]$.

For an arbitrary finite system $S(X) = 1$ with coefficients from A denote by $\tilde{S}(X) = 1$ an equation which is equivalent over G to the system $S(X) = 1$ (such $\tilde{S}(X)$ exists by Corollary 3). Then for the radical $R(S)$ of the system $S = 1$ we have

$$R(S) = \{T \in G[X] \mid G \models \Phi_{\tilde{S},T}\}.$$

It follows that the set of radical sentences

$$A_S = \{\Phi_{\tilde{S},T} \mid G \models \Phi_{\tilde{S},T}\}$$

describes precisely the radical $R(S)$ of the system $S = 1$ over G, hence the name.

Lemma 6. *Every universal formula in L_A is equivalent modulo $A_{\mathcal{T}}$ to a radical formula.*

Proof. By Corollary 7 every boolean combination of atomic formulas in the language L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula of the type

$$\bigvee_{i=1}^{n} (S_i = 1 \ \& \ T_i \neq 1).$$

This implies that every existential formula in L_A is equivalent to a formula in the form

$$\exists Y \left(\bigvee_{i=1}^{n} \big(S_i(X,Y) = 1 \ \& \ T_i(X,Y) \neq 1\big) \right).$$

This formula is equivalent modulo $A_{\mathcal{T}}$ to the formula

$$\exists z_1 \ldots \exists z_n \exists Y \left(\left(\bigwedge_{i=1}^{n} z_i \neq 1 \right) \ \& \ \left(\bigvee_{i=1}^{n} \big(S_i(X,Y) = 1 \ \& \ T_i(X,Y) = z_i\big) \right) \right).$$

By Corollaries 5 and 3 one can effectively find $S \in G[X, Y, Z]$ and $T \in G[Z]$ (where $Z = (z_1, \ldots, z_n)$) such that

$$\bigvee_{i=1}^{n} \big(S_i(X,Y) = 1 \ \& \ T_i(X,Y) = z_i\big) \ \sim_{A_{\mathcal{T}}} \ S(X,Y,Z) = 1$$

and

$$\bigwedge_{i=1}^{n} (z_i \neq 1) \ \sim_{A_{\mathcal{T}}} \ T(Z) \neq 1.$$

It follows that every existential formula in L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula of the type

$$\exists Z \exists Y \big(S(X,Y,Z) = 1 \ \& \ T(Z) \neq 1\big).$$

Hence every universal formula in L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula in the form

$$\forall Z \forall Y \big(S(X, Y, Z) \neq 1 \ \vee \ T(Z) = 1\big),$$

which is equivalent to the radical formula

$$\forall Z \forall Y \big(S(X, Y, Z) = 1 \ \rightarrow \ T(Z) = 1\big).$$

This proves the lemma. □

Now we consider universal positive formulas.

Lemma 7. *Let G be a BP-group from $\mathcal{T}$. Then*

$$G \models \forall X \big(U(X) = 1\big) \quad \Leftrightarrow \quad G[X] \models U(X) = 1,$$

i.e., only the trivial equation has the whole set G^n as its solution set.

Proof. The group $G[X]$ is discriminated by G [3]. Therefore, if the word $U(X)$ is a non-trivial element of $G[X]$, then there exists a G-homomorphism $\phi : G[X] \to G$ such that $U^{\phi} \neq 1$. But then $U(X^{\phi}) \neq 1$ in G—contradiction with conditions of the lemma. So $U(X) = 1$ in $G[X]$. □

Remark 2. The proof above holds for every non-abelian group G for which $G[X]$ is discriminated by G.

The next result shows how to eliminate quantifiers from positive universal formulas over non-abelian freely discriminated groups.

Lemma 8. *Let G be a BP-group from $\mathcal{T}$. For a given word $U(X, Y) \in G[X \cup Y]$, one can effectively find a word $W(Y) \in G[Y]$ such that*

$$\forall X \big(U(X, Y) = 1\big) \sim_G \ W(Y) = 1. \tag{10}$$

Proof. By Lemma 7, for any tuple of constants C from G, the following equivalence holds:

$$G \models \forall X \big(U(X, C) = 1\big) \quad \Leftrightarrow \quad G[X] \models U(X, C) = 1.$$

Now it suffices to prove that for a given $U(X, Y) \in G[X \cup Y]$ one can effectively find a word $W(Y) \in G[Y]$ such that for any tuple of constants C over F the following equivalence holds

$$G[X] \models U(X, C) = 1 \quad \Leftrightarrow \quad G \models W(C) = 1.$$

We do this by induction on the syllable length of $U(X, Y)$ which comes from the free product $G[X \cup Y] = G[Y] * F(X)$ (notice that $F(X)$ does not contain constants from G, but

$G[Y]$ does). If $U(X,Y)$ is of the syllable length 1, then either $U(X,Y)=U(X)\in F(X)$ or $U(X,Y)=U(Y)\in G[Y]$. In the first event $F\models U(X)=1$ means exactly that the reduced form of $U(X)$ is trivial, so we can take $W(Y)$ trivial also. In the event $U(X,Y)=U(Y)$ we can take $W(Y)=U(Y)$.

Suppose now that $U(X,Y)\in G[Y]*F(X)$ and it has the following reduced form:

$$U(X,Y)=g_1(Y)v_1(X)g_2(Y)v_2(X)\dots v_m(X)g_{m+1}(Y)$$

where v_i's are reduced non-trivial words in $F(X)$ and $g_i(Y)$'s are reduced words in $G[Y]$ which are all non-trivial except, possibly, $g_1(Y)$ and $g_{m+1}(Y)$.

If for a tuple of constants C over G we have $G[X]\models U(X,C)=1$ then at least one of the elements $g_2(C),\dots,g_m(C)$ must be trivial in G. This observation leads to the following construction. For each $i=2,\dots,m$ delete the subword $g_i(Y)$ from $U(X,Y)$ and reduce the new word to the reduced form in the free product $F(X)*G[Y]$. Denote the resulting word by $U_i(X,Y)$. Notice that the syllable length of $U_i(X,Y)$ is less then the length of $U(X,Y)$. It follows from the argument above that for any tuple of constants C the following equivalence holds:

$$G[X]\models U(X,C)=1 \quad\Leftrightarrow\quad G[X]\models\bigvee_{i=2}^{m}\bigl(g_i(C)=1\ \&\ U_i(X,C)=1\bigr).$$

By induction one can effectively find words $W_2(Y),\dots,W_m(Y)\in G[Y]$ such that for any tuple of constants C we have

$$G[X]\models U_i(X,C)=1 \quad\Leftrightarrow\quad G\models W_i(C)=1,$$

for each $i=2,\dots,m$. Combining the equivalences above we see that

$$G[X]\models U(X,C)=1 \quad\Leftrightarrow\quad G\models\bigvee_{i=2}^{m}\bigl(g_i(C)=1\ \&\ W_i(C)=1\bigr).$$

By Corollaries 3 and 5 from the previous section we can effectively rewrite the disjunction

$$\bigvee_{i=2}^{m}\bigl(g_i(Y)=1\ \&\ W_i(Y)=1\bigr)$$

as a single equation $W(Y)=1$. That finishes the proof. □

3.3. Positive and general formulas

In this section we describe normal forms of general formulas and positive formulas. We show that every positive formula is equivalent modulo $A_{\mathcal{T}}$ to a formula which consists of an equation and a string of quantifiers in front of it; and for an arbitrary formula Φ either Φ or $\neg\Phi$ is equivalent modulo $A_{\mathcal{T}}$ to a formula in a general radical form (it is a radical formula with a string of quantifiers in front of it).

Lemma 9. *Every positive formula $\Phi(X)$ in L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula of the type*

$$Q_1X_1\dots Q_kX_k\big(S(X,X_1,\dots,X_k)=1\big),$$

where $Q_i\in\{\exists,\forall\}$ $(i=1,\dots,k)$.

Proof. The result follows immediately from Corollaries 3 and 5. □

Lemma 10. *Let $\Phi(X)$ be a formula in L_A of the form*

$$\Phi(X)=Q_1X_1\dots Q_kX_k\forall Y\Phi_0(X,X_1,\dots,X_k,Y),$$

where $Q_i\in\{\exists,\forall\}$ and Φ_0 is a quantifier-free formula. Then one can effectively find a formula $\Psi(X)$ of the form

$$\Psi(X)=Q_1X_1\dots Q_kX_k\forall Y\forall Z\big(S(X,X_1,\dots,X_k,Y,Z)=1\ \to\ T(Z)=1\big)$$

such that $\Phi(X)$ is equivalent to $\Psi(X)$ modulo $A_{\mathcal{T}}$.

Proof. Let

$$\Phi(X)=Q_1X_1\dots Q_kX_k\forall Y\Phi_0(X,X_1,\dots,X_k,Y),$$

where $Q_i\in\{\exists,\forall\}$ and Φ_0 is a quantifier-free formula. By Lemma 6 there exists equations $S(X,X_1,\dots,X_k,Y,Z)=1$ and $T(Z)=1$ such that

$$\forall Y\Phi_0(X,X_1,\dots,X,Y)\ \sim_{A_{\mathcal{T}}}\ \forall Y\forall Z\big(S(X_1,\dots X_k,Y,Z)=1\ \to\ T(Z)=1\big).$$

It follows that

$$\begin{aligned}\Phi(X)&=Q_1X_1\dots Q_kX_k\forall Y\Phi_0(X,X_1,\dots,X_k,Y)\\&\sim_{A_{\mathcal{T}}}\ Q_1X_1\dots Q_kX_k\forall Y\forall Z\big(S(X,\dots,X_k,Y,Z)=1\ \to\ T(Z)=1\big),\end{aligned}$$

as desired. □

Lemma 11. *For any formula $\Phi(X)$ in the language L_A, one can effectively find a formula $\Psi(X)$ in the language L_A in the form*

$$\Psi(X)=\exists X_1\forall Y_1\dots\exists X_k\forall Y_k\forall Z\big(S(X,X_1,Y_1,\dots,X_k,Y_k,Z)=1\ \to\ T(Z)=1\big),$$

such that $\Phi(X)$ or its negation $\neg\Phi(X)$ (and we can check effectively which one of them) is equivalent to $\Psi(X)$ modulo $A_{\mathcal{T}}$.

Proof. For any formula $\Phi(X)$ in the language L_A one can effectively find a disjunctive normal form $\Phi_1(X)$ of $\Phi(X)$, as well as a disjunctive normal form Φ_2 of the negation $\neg\Phi(X)$ of $\Phi(X)$ (see, for example, [6]). We can assume that either in $\Phi_1(X)$ or in $\Phi_2(X)$ the quantifier prefix ends with a universal quantifier. Moreover, adding (if necessary) an existential quantifier $\exists v$ in front of the formula (where v does not occur in the formula) we may also assume that the formula begins with an existential quantifier. Now by Lemma 10 one can effectively find a formula Ψ with the required conditions. □

4. Generalized equations and positive theory of free groups

Makanin [19] introduced the concept of a generalized equation constructed for a finite system of equations in a free group $F = F(A)$. Geometrically a generalized equation consists of three kinds of objects: bases, boundaries and items. Roughly it is a long interval with marked division points. The marked division points are the boundaries. Subintervals between division points are items (we assign a variable to each item). Line segments below certain subintervals, beginning at some boundary and ending at some other boundary, are bases. Each base either corresponds to a letter from A or has a double.

This concept becomes crucial to our subsequent work and is difficult to understand. This is one of the main tools used to describe solution sets of systems of equations. In subsequent papers we will use it also to obtain effectively different splittings of groups. Before we give a formal definition we will try to motivate it with a simple example.

Suppose we have the simple equation $xyz = 1$ in a free group. Suppose that we have a solution to this equation denoted by x^ϕ, y^ϕ, z^ϕ where is ϕ is a given homomorphism into a free group $F(A)$. Since x^ϕ, y^ϕ, z^ϕ are reduced words in the generators A there must be complete cancellation. If we take a concatenation of the geodesic subpaths corresponding to x^ϕ, y^ϕ and z^ϕ we obtain a path in the Cayley graph corresponding to this complete cancellation. This is called a cancellation tree (see Fig. 1). In the simplest situation $x = \lambda_1 \circ \lambda_2$, $y = \lambda_2^{-1} \circ \lambda_3$ and $z = \lambda_3^{-1} \circ \lambda_1^{-1}$. The generalized equation would then be the following interval.

The boundaries would be the division points, the bases are the λ's and the items in this simple case are also the λ's. In a more complicated equation where the variables X, Y, Z appear more than one time this basic interval would be extended. Since the solution of any

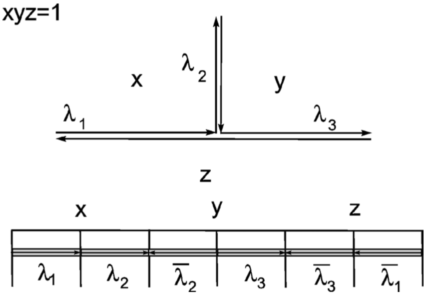


Fig. 1. From the cancellation tree for the equation $xyz = 1$ to the generalized equation ($x = \lambda_1 \circ \lambda_2$, $y = \lambda_2^{-1} \circ \lambda_3$, $z = \lambda_3^{-1} \circ \lambda_1^{-1}$).

equation in a free group must involve complete cancellation this drawing of the interval is essentially the way one would solve such an equation. Our picture depended on one fixed solution ϕ. However for any equation there are only finitely many such cancellation trees and hence only finitely many generalized equations.

4.1. Generalized equations

Let $A = \{a_1, \dots, a_m\}$ be a set of constants and $X = \{x_1, \dots, x_n\}$ be a set of variables. Put $G = F(A)$ and $G[X] = G * F(X)$.

Definition 8. A combinatorial generalized equation Ω (with constants from $A^{\pm 1}$) consists of the following objects.

(1) A finite set of *bases* $BS = BS(\Omega)$. Every base is either a constant base or a variable base. Each constant base is associated with exactly one letter from $A^{\pm 1}$. The set of variable bases $\mathcal{M}$ consists of $2n$ elements $\mathcal{M} = \{\mu_1, \dots, \mu_{2n}\}$. The set $\mathcal{M}$ comes equipped with two functions: a function $\varepsilon : \mathcal{M} \to \{1, -1\}$ and an involution $\Delta : \mathcal{M} \to \mathcal{M}$ (i.e., Δ is a bijection such that Δ^2 is an identity on $\mathcal{M}$). Bases μ and $\Delta(\mu)$ (or $\bar{\mu}$) are called *dual bases*. We denote variable bases by $\mu, \lambda, \dots$.
(2) A set of *boundaries* $BD = BD(\Omega)$. BD is a finite initial segment of the set of positive integers $BD = \{1, 2, \dots, \rho + 1\}$. We use letters $i, j, \dots$ for boundaries.
(3) Two functions $\alpha : BS \to BD$ and $\beta : BS \to BD$. We call $\alpha(\mu)$ and $\beta(\mu)$ the initial and terminal boundaries of the base μ (or endpoints of μ). These functions satisfy the following conditions: $\alpha(b) < \beta(b)$ for every base $b \in BS$; if b is a constant base then $\beta(b) = \alpha(b) + 1$.
(4) A finite set of *boundary connections* $BC = BC(\Omega)$. A boundary connection is a triple (i, μ, j) where $i, j \in BD$, $\mu \in \mathcal{M}$ such that $\alpha(\mu) < i < \beta(\mu)$ and $\alpha(\Delta(\mu)) < j < \beta(\Delta(\mu))$. We will assume for simplicity, that if $(i, \mu, j) \in BC$ then $(j, \Delta(\mu), i) \in BC$. This allows one to identify connections (i, μ, j) and $(j, \Delta(\mu), i)$.

For a combinatorial generalized equation Ω, one can canonically associate a system of equations in *variables* $h_1, \dots, h_\rho$ over $F(A)$ (variables h_i are sometimes called *items*). This system is called a *generalized equation*, and (slightly abusing the language) we denote it by the same symbol Ω. The generalized equation Ω consists of the following three types of equations.

(1) Each pair of dual variable bases $(\lambda, \Delta(\lambda))$ provides an equation

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \dots h_{\beta(\lambda)-1}]^{\varepsilon(\lambda)} = [h_{\alpha(\Delta(\lambda))} h_{\alpha(\Delta(\lambda))+1} \dots h_{\beta(\Delta(\lambda))-1}]^{\varepsilon(\Delta(\lambda))}.$$

These equations are called *basic equations*.
(2) For each constant base b we write down a *coefficient equation*

$$h_{\alpha(b)} = a,$$

where $a \in A^{\pm 1}$ is the constant associated with b.

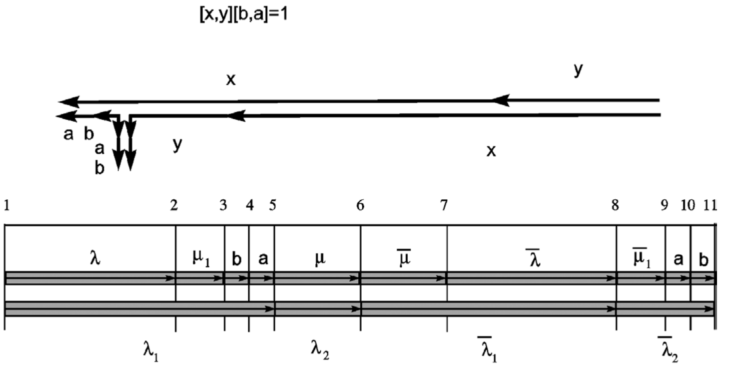


Fig. 2. A cancellation tree and the generalized equation corresponding to this tree for the equation $[x, y][b, a] = 1$.

(3) Every boundary connection (p, λ, q) gives rise to a *boundary equation*

$$[h_{\alpha(\lambda)}h_{\alpha(\lambda)+1}\dots h_{p-1}] = [h_{\alpha(\Delta(\lambda))}h_{\alpha(\Delta(\lambda))+1}\dots h_{q-1}],$$

if $\varepsilon(\lambda) = \varepsilon(\Delta(\lambda))$ and

$$[h_{\alpha(\lambda)}h_{\alpha(\lambda)+1}\dots h_{p-1}] = [h_q h_{q+1}\dots h_{\beta(\Delta(\lambda))-1}]^{-1},$$

if $\varepsilon(\lambda) = -\varepsilon(\Delta(\lambda))$.

Remark 3. We assume that every generalized equation comes associated with a combinatorial one.

Example. Consider as an example the Malcev equation $[x, y][b, a] = 1$, where $a, b \in A$. Consider the following solution of this equation:

$$x^\phi = \left(\left(b^{n_1}a\right)^{n_2}b\right)^{n_3}b^{n_1}a, \qquad y^\phi = \left(b^{n_1}a\right)^{n_2}b.$$

Figure 2 shows the cancellation tree and the generalized equation for this solution. This generalized equation has ten variables $h_1, \dots, h_{10}$ and eleven boundaries. The system of basic equations for this generalized equation is the following

$$h_1 = h_7, \qquad h_2 = h_8, \qquad h_5 = h_6,$$
$$h_1h_2h_3h_4 = h_6h_7, \qquad h_5 = h_8h_9h_{10}.$$

The system of coefficient equations is

$$h_3 = b, \qquad h_4 = a, \qquad h_9 = a, \qquad h_{10} = b.$$

Definition 9. Let $\Omega(h) = \{L_1(h) = R_1(h), \dots, L_s(h) = R_s(h)\}$ be a generalized equation in variables $h = (h_1, \dots, h_\rho)$ with constants from $A^{\pm 1}$. A sequence of reduced non-empty words $U = (U_1(A), \dots, U_\rho(A))$ in the alphabet $A^{\pm 1}$ is a *solution* of Ω if:

(1) all words $L_i(U)$, $R_i(U)$ are reduced as written;
(2) $L_i(U) = R_i(U)$, $i = 1, \ldots, s$.

The notation (Ω, U) means that U is a solution of the generalized equation Ω.

Remark 4. Notice that a solution U of a generalized equation Ω can be viewed as a solution of Ω in the free monoid $F_{\text{mon}}(A^{\pm 1})$ (i.e., the equalities $L_i(U) = R_i(U)$ are graphical) which satisfies an additional condition $U \in F(A) \leqslant F_{\text{mon}}(A^{\pm 1})$.

Obviously, each solution U of Ω gives rise to a solution of Ω in the free group $F(A)$. The converse does not hold in general, i.e., it might happen that U is a solution of Ω in $F(A)$ but not in $F_{\text{mon}}(A^{\pm 1})$, i.e., all equalities $L_i(U) = R_i(U)$ hold only after a free reduction but not graphically. We introduce the following notation which will allow us to distinguish in which structure ($F_{\text{mon}}(A^{\pm 1})$ or $F(A)$) we are looking for solutions for Ω.

If

$$S = \{L_1(h) = R_1(h), \ldots, L_s(h) = R_s(h)\}$$

is an arbitrary system of equations with constants from $A^{\pm 1}$, then by S^* we denote the system of equations

$$S^* = \{L_1(h)R_1(h)^{-1} = 1, \ldots, L_s(h)R_s(h)^{-1} = 1\}$$

over the free group $F(A)$.

Definition 10. A generalized equation Ω is called *formally consistent* if it satisfies the following conditions.

(1) If $\varepsilon(\mu) = -\varepsilon(\Delta(\mu))$, then the bases μ and $\Delta(\mu)$ do not intersect, i.e., none of the items $h_{\alpha(\mu)}, h_{\beta(\mu)-1}$ is contained in $\Delta(\mu)$.
(2) If two boundary equations have respective parameters (p, λ, q) and (p_1, λ, q_1) with $p \leqslant p_1$, then $q \leqslant q_1$ in the case when $\varepsilon(\lambda)\varepsilon(\Delta(\lambda)) = 1$, and $q \geqslant q_1$ in the case $\varepsilon(\lambda)\varepsilon(\Delta(\lambda)) = -1$, in particular, if $p = p_1$ then $q = q_1$.
(3) Let μ be a base such that $\alpha(\mu) = \alpha(\Delta(\mu))$ (in this case we say that bases μ and $\Delta(\mu)$ form a matched pair of dual bases). If (p, μ, q) is a boundary connection related to μ then $p = q$.
(4) A variable cannot occur in two distinct coefficient equations, i.e., any two constant bases with the same left endpoint are labelled by the same letter from $A^{\pm 1}$.
(5) If h_i is a variable from some coefficient equation, and if (i, μ, q_1), $(i + 1, \mu, q_2)$ are boundary connections, then $|q_1 - q_2| = 1$.

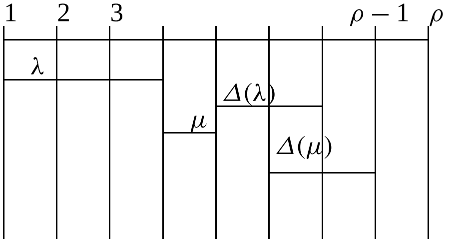
Lemma 12.

(1) *If a generalized equation Ω has a solution then Ω is formally consistent.*
(2) *There is an algorithm which for every generalized equation checks whether it is formally consistent or not.*

The proof is easy and we omit it.

Remark 5. In the sequel we consider only formally consistent generalized equations.

It is convenient to visualize a generalized equation Ω as follows.



4.2. Reduction to generalized equations

In this section, following Makanin [19], we show how for a given finite system of equations $S(X,A)=1$ over a free group $F(A)$ one can canonically associate a finite collection of generalized equations $\mathcal{GE}(S)$ with constants from $A^{\pm 1}$, which to some extent describes all solutions of the system $S(X,A)=1$.

Let $S(X,A)=1$ be a finite system of equations $S_1=1,\ldots,S_m=1$ over a free group $F(A)$. We write $S(X,A)=1$ in the form

$$\begin{aligned} r_{11}r_{12}\ldots r_{1l_1} &= 1,\\ r_{21}r_{22}\ldots r_{2l_2} &= 1,\\ &\ldots\\ r_{m1}r_{m2}\ldots r_{ml_m} &= 1, \end{aligned} \tag{11}$$

where r_{ij} are letters in the alphabet $X^{\pm 1}\cup A^{\pm 1}$.

A *partition table* T of the system above is a set of reduced words

$$T=\{V_{ij}(z_1,\ldots,z_p)\} \quad (1\leqslant i\leqslant m,\ 1\leqslant j\leqslant l_i)$$

from a free group $F[Z]=F(A\cup Z)$, where $Z=\{z_1,\ldots,z_p\}$, which satisfies the following conditions:

(1) the equality $V_{i1}V_{i2}\ldots V_{il_i}=1$, $1\leqslant i\leqslant m$, holds in $F[Z]$;
(2) $|V_{ij}|\leqslant l_i-1$;
(3) if $r_{ij}=a\in A^{\pm 1}$, then $V_{ij}=a$.

Since $|V_{ij}|\leqslant l_i-1$ then at most $|S|=\sum_{i=1}^{m}(l_i-1)l_i$ different letters z_i can occur in a partition table of the equation $S(X,A)=1$. Therefore we will always assume that $p\leqslant |S|$.

Each partition table encodes a particular type of cancellation that happens when one substitutes a particular solution $W(A) \in F(A)$ into $S(X, A) = 1$ and then freely reduces the words in $S(W(A), A)$ into the empty word.

Lemma 13. *Let $S(X, A) = 1$ be a finite system of equations over $F(A)$. Then*:

(1) *the set $PT(S)$ of all partition tables of $S(X, A) = 1$ is finite, and its cardinality is bounded by a number which depends only on $|S(X, A)|$;*
(2) *one can effectively enumerate the set $PT(S)$.*

Proof. Since the words V_{ij} have bounded length, one can effectively enumerate the finite set of all collections of words $\{V_{ij}\}$ in $F[Z]$ which satisfy the conditions (2), (3) above. Now for each such collection $\{V_{ij}\}$, one can effectively check whether the equalities $V_{i1}V_{i2}\dots V_{il_i} = 1$, $1 \leqslant i \leqslant m$, hold in the free group $F[Z]$ or not. This allows one to list effectively all partition tables for $S(X, A) = 1$. □

To each partition table $T = \{V_{ij}\}$ one can assign a generalized equation Ω_T in the following way (below we use the notation $\doteq$ for graphical equality). Consider the following word V in $M(A^{\pm 1} \cup Z^{\pm 1})$:

$$V \doteq V_{11}V_{12}\dots V_{1l_1}\dots V_{m1}V_{m2}\dots V_{ml_m} = y_1\dots y_\rho,$$

where $y_i \in A^{\pm 1} \cup Z^{\pm 1}$ and $\rho = l(V)$ is the length of V. Then the generalized equation $\Omega_T = \Omega_T(h)$ has $\rho + 1$ boundaries and ρ variables $h_1, \dots, h_\rho$ which are denoted by $h = (h_1, \dots, h_\rho)$.

Now we define bases of Ω_T and the functions α, β, ε.

Let $z \in Z$. For any two distinct occurrences of z in V as

$$y_i = z^{\varepsilon_i}, \quad y_j = z^{\varepsilon_j} \quad \bigl(\varepsilon_i, \varepsilon_j \in \{1, -1\}\bigr)$$

we introduce a pair of dual variable bases $\mu_{z,i}, \mu_{z,j}$ such that $\Delta(\mu_{z,i}) = \mu_{z,j}$ (say, if $i < j$). Put

$$\alpha(\mu_{z,i}) = i, \qquad \beta(\mu_{z,i}) = i + 1, \qquad \epsilon(\mu_{z,i}) = \varepsilon_i.$$

The basic equation that corresponds to this pair of dual bases is $h_i^{\varepsilon_i} = h_j^{\varepsilon_j}$.

Let $x \in X$. For any two distinct occurrences of x in $S(X, A) = 1$ as

$$r_{i,j} = x^{\varepsilon_{ij}}, \quad r_{s,t} = x^{\varepsilon_{st}} \quad \bigl(\varepsilon_{ij}, \varepsilon_{st} \in \{1, -1\}\bigr)$$

we introduce a pair of dual bases $\mu_{x,i,j}$ and $\mu_{x,s,t}$ such that $\Delta(\mu_{x,i,j}) = \mu_{x,s,t}$ (say, if $(i, j) < (s, t)$ in the left lexicographic order). Now let V_{ij} occurs in the word V as a subword

$$V_{ij} = y_c \dots y_d.$$

Then we put

$$\alpha(\mu_{x,i,j}) = c, \qquad \beta(\mu_{x,i,j}) = d+1, \qquad \epsilon(\mu_{x,i,j}) = \varepsilon_{ij}.$$

The basic equation which corresponds to these dual bases can be written in the form

$$[h_{\alpha(\mu_{x,i,j})} \dots h_{\beta(\mu_{x,i,j})-1}]^{\varepsilon_{ij}} = [h_{\alpha(\mu_{x,s,t})} \dots h_{\beta(\mu_{x,s,t})-1}]^{\varepsilon_{st}}.$$

Let $r_{ij} = a \in A^{\pm 1}$. In this case we introduce a constant base μ_{ij} with the label a. If V_{ij} occurs in V as $V_{ij} = y_c$, then we put

$$\alpha(\mu_{ij}) = c, \qquad \beta(\mu_{ij}) = c+1.$$

The corresponding coefficient equation is written as $h_c = a$.

The list of boundary connections here (and hence the boundary equations) is empty. This defines the generalized equation Ω_T. Put

$$\mathcal{G}E(S) = \left\{ \Omega_T \;\middle|\; T \text{ is a partition table for } S(X, A) = 1 \right\}.$$

Then $\mathcal{G}E(S)$ is a finite collection of generalized equations which can be effectively constructed for a given $S(X, A) = 1$.

For a generalized equation Ω we can also consider the same system of equations in a free group. We denote this system by Ω^*. By $F_{R(\Omega)}$ we denote the coordinate group of Ω^*. Now we explain relations between the coordinate groups of $S(X, A) = 1$ and Ω_T^*.

For a letter x in X we choose an arbitrary occurrence of x in $S(X, A) = 1$ as

$$r_{ij} = x^{\varepsilon_{ij}}.$$

Let $\mu = \mu_{x,i,j}$ be the base that corresponds to this occurrence of x. Then V_{ij} occurs in V as the subword

$$V_{ij} = y_{\alpha(\mu)} \dots y_{\beta(\mu)-1}.$$

Define a word $P_x(h) \in F[h]$ (where $h = \{h_1, \dots, h_\rho\}$) as

$$P_x(h, A) = h_{\alpha(\mu)} \dots h_{\beta(\mu)-1}^{\varepsilon_{ij}},$$

and put

$$P(h) = (P_{x_1}, \dots, P_{x_n}).$$

The tuple of words $P(h)$ depends on a choice of occurrences of letters from X in V. It follows from the construction above that the map $X \to F[h]$ defined by $x \to P_x(h, A)$ gives rise to an F-homomorphism

$$\pi : F_{R(S)} \to F_{R(\Omega_T)}.$$

Observe that the image $\pi(x)$ in $F_{R(\Omega_T)}$ does not depend on a particular choice of the occurrence of x in $S(X, A)$ (the basic equations of Ω_T make these images equal). Hence π depends only on Ω_T.

Now we relate solutions of $S(X, A) = 1$ with solutions of generalized equations from $\mathcal{GE}(S)$. Let $W(A)$ be a solution of $S(X, A) = 1$ in $F(A)$. If in the system (11) we make the substitution $\sigma : X \to W(A)$, then

$$(r_{i1} r_{i2} \dots r_{il_i})^\sigma = r_{i1}^\sigma r_{i2}^\sigma \dots r_{il_i}^\sigma = 1$$

in $F(A)$ for every $i = 1, \dots, m$. Hence every product $R_i = r_{i1}^\sigma r_{i2}^\sigma \dots r_{il_i}^\sigma$ can be reduced to the empty word by a sequence of free reductions. Let us fix a particular reduction process for each R_i. Denote by $\tilde{z}_1, \dots, \tilde{z}_p$ all the (maximal) non-trivial subwords of r_{ij}^σ that cancel out in some R_i $(i = 1, \dots, m)$ during the chosen reduction process. Since every word r_{ij}^σ in this process cancels out completely, that implies that

$$r_{ij}^\sigma = V_{ij}(\tilde{z}_1, \dots, \tilde{z}_p)$$

for some reduced words $V_{ij}(Z)$ in variables $Z = \{z_1, \dots, z_p\}$. Moreover, the equality above is graphical. Observe also that if $r_{ij} = a \in A^{\pm 1}$ then $r_{ij}^\sigma = a$ and we have $V_{ij} = a$. Since every word r_{ij}^σ in R_i has at most one cancellation with any other word r_{ik}^σ and does not have cancellation with itself, we have $l(V_{ij}) \leqslant l_i - 1$. This shows that the set $T = \{V_{ij}\}$ is a partition table for $S(X, A) = 1$. Obviously,

$$U(A) = (\tilde{z}_1, \dots, \tilde{z}_p)$$

is the solution of the generalized equation Ω_T, which is induced by $W(A)$. From the construction of the map $P(H)$ we deduce that $W(A) = P(U(A))$.

The reverse is also true: if $U(A)$ is an arbitrary solution of the generalized equation Ω_T, then $P(U(A))$ is a solution of $S(X, A) = 1$.

We summarize the discussion above in the following lemma, which is essentially due to Makanin [19].

Lemma 14. *For a given system of equations $S(X, A) = 1$ over a free group $F = F(A)$, one can effectively construct a finite set*

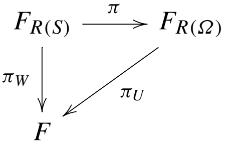
$$\mathcal{GE}(S) = \left\{ \Omega_T \;\middle|\; T \text{ is a partition table for } S(X, A) = 1 \right\}$$

of generalized equations such that:

(1) *if the set $\mathcal{GE}(S)$ is empty, then $S(X, A) = 1$ has no solutions in $F(A)$*;
(2) *for each $\Omega(H) \in \mathcal{GE}(S)$ and for each $x \in X$ one can effectively find a word $P_x(H, A) \in F[H]$ of length at most $|H|$ such that the map $x :\to P_x(H, A)$ $(x \in X)$ gives rise to an F-homomorphism $\pi_\Omega : F_{R(S)} \to F_{R(\Omega)}$*;
(3) *for any solution $W(A) \in F(A)^n$ of the system $S(X, A) = 1$ there exists $\Omega(H) \in \mathcal{GE}(S)$ and a solution $U(A)$ of $\Omega(H)$ such that $W(A) = P(U(A))$, where $P(H) = (P_{x_1}, \dots, P_{x_n})$, and this equality is graphical*;

(4) *for any* F*-group* $\tilde{F}$*, if a generalized equation* $\Omega(H) \in \mathcal{G}E(S)$ *has a solution* $\tilde{U}$ *in* $\tilde{F}$*, then* $P(\tilde{U})$ *is a solution of* $S(X, A) = 1$ *in* $\tilde{F}$*.*

Corollary 8. *In the notations of Lemma* 14 *for any solution* $W(A) \in F(A)^n$ *of the system* $S(X, A) = 1$ *there exists* $\Omega(H) \in \mathcal{G}E(S)$ *and a solution* $U(A)$ *of* $\Omega(H)$ *such that the following diagram commutes.*



4.3. Generalized equations with parameters

In this section, following [27] and [13], we consider generalized equations with *parameters*. This kind of equations appear naturally in Makanin's type rewriting processes and provide a convenient tool to organize induction properly.

Let Ω be a generalized equation. An item h_i *belongs* to a base μ (and, in this event, μ *contains* h_i) if $\alpha(\mu) \leqslant i \leqslant \beta(\mu) - 1$. An item h_i is *constant* if it belongs to a constant base, h_i is *free* if it does not belong to any base. By $\gamma(h_i) = \gamma_i$ we denote the number of bases which contain h_i. We call γ_i the *degree* of h_i.

A boundary i *crosses* (or *intersects*) the base μ if $\alpha(\mu) < i < \beta(\mu)$. A boundary i *touches* the base μ (or i is an endpoint of μ) if $i = \alpha(\mu)$ or $i = \beta(\mu)$. A boundary is said to be *open* if it crosses at least one base, otherwise it is called *closed*. We say that a boundary i is *tied* (or *bound*) by a base μ (or μ*-tied*) if there exists a boundary connection (p, μ, q) such that $i = p$ or $i = q$. A boundary is *free* if it does not touch any base and it is not tied by a boundary connection.

A set of consecutive items $[i, j] = \{h_i, \ldots, h_{i+j-1}\}$ is called a *section*. A section is said to be *closed* if the boundaries i and $i + j$ are closed and all the boundaries between them are open. A base μ is *contained* in a base λ if $\alpha(\lambda) \leqslant \alpha(\mu) < \beta(\mu) \leqslant \beta(\lambda)$. If μ is a base then by $\sigma(\mu)$ we denote the section $[\alpha(\mu), \beta(\mu)]$ and by $h(\mu)$ we denote the product of items $h_{\alpha(\mu)} \ldots h_{\beta(\mu)-1}$. In general for a section $[i, j]$ by $h[i, j]$ we denote the product $h_i \ldots h_{j-1}$.

Definition 11. Let Ω be a generalized equation. If the set $\Sigma = \Sigma_\Omega$ of all closed sections of Ω is partitioned into a disjoint union of subsets

$$\Sigma_\Omega = V\Sigma \cup P\Sigma \cup C\Sigma, \tag{12}$$

then Ω is called a *generalized equation with parameters* or a *parametric* generalized equation. Sections from $V\Sigma$, $P\Sigma$, and $C\Sigma$ are called correspondingly, *variable, parametric,* and *constant* sections. To organize the branching process properly, we usually divide variable sections into two disjoint parts:

$$V\Sigma = A\Sigma \cup NA\Sigma. \tag{13}$$

Sections from $A\Sigma$ are called *active*, and sections from $NA\Sigma$ are *non-active*. In the case when partition (13) is not specified we assume that $A\Sigma = V\Sigma$. Thus, in general, we have a partition

$$\Sigma_\Omega = A\Sigma \cup NA\Sigma \cup P\Sigma \cup C\Sigma. \tag{14}$$

If $\sigma \in \Sigma$, then every base or item from σ is called active, non-active, parametric, or constant, with respect to the type of σ.

We will see later that every parametric generalized equation can be written in a particular *standard* form.

Definition 12. We say that a parametric generalized equation Ω is *in a standard form* if the following conditions hold:

(1) all non-active sections from $NA\Sigma_\Omega$ are located to the right of all active sections from $A\Sigma$, all parametric sections from $P\Sigma_\Omega$ are located to the right of all non-active sections, and all constant sections from $C\Sigma$ are located to the right of all parametric sections; namely, there are numbers $1 \leqslant \rho_A \leqslant \rho_{NA} \leqslant \rho_P \leqslant \rho_C \leqslant \rho = \rho_\Omega$ such that $[1, \rho_A + 1]$, $[\rho_A + 1, \rho_{NA} + 1]$, $[\rho_{NA} + 1, \rho_P + 1]$, and $[\rho_P + 1, \rho_\Omega + 1]$ are, correspondingly, unions of all active, all non-active, all parametric, and all constant sections;
(2) for every letter $a \in A^{\pm 1}$ there is at most one constant base in Ω labelled by a, and all such bases are located in the $C\Sigma$;
(3) every free variable (item) h_i of Ω is located in $C\Sigma$.

Now we describe a typical method for constructing generalized equations with parameters starting with a system of ordinary group equations with constants from A.

Parametric generalized equations corresponding to group equations

Let

$$S(X, Y_1, Y_2, \ldots, Y_k, A) = 1 \tag{15}$$

be a finite system of equations with constants from $A^{\pm 1}$ and with the set of variables partitioned into a disjoint union

$$X \cup Y_1 \cup \cdots \cup Y_k \tag{16}$$

Denote by $\mathcal{GE}(S)$ the set of generalized equations corresponding to $S = 1$ from Lemma 14. Put $Y = Y_1 \cup \cdots \cup Y_k$. Let $\Omega \in \mathcal{GE}(S)$. Recall that every base μ occurs in Ω either related to some occurrence of a variable from $X \cup Y$ in the system $S(X, Y, A) = 1$, or related to an occurrence of a letter $z \in Z$ in the word V (see Lemma 13), or is a constant base. If μ corresponds to a variable $x \in X$ ($y \in Y_i$) then we say that μ is an *X-base* (*Y_i-base*). Sometimes we refer to Y_i-bases as to Y-bases. For a base μ of Ω denote by σ_μ the section $\sigma_\mu = [\alpha(\mu), \beta(\mu)]$. Observe that the section σ_μ is closed in Ω for every X-base, or Y-base.

If μ is an X-base (Y-base or Y_i-base), then the section σ_μ is called an *X-section* (*Y-section* or *Y_i-section*). If μ is a constant base and the section σ_μ is closed then we call σ_μ a *constant* section. Using the derived transformation $D2$ we transport all closed Y_1-sections to the right end of the generalized equations behind all the sections of the equation (in an arbitrary order), then we transport all Y_2-sections an put them behind all Y_1-sections, and so on. Eventually, we transport all Y-sections to the very end of the interval and they appear there with respect to the partition (16). After that we take all the constant sections and put them behind all the parametric sections. Now, let $A\Sigma$ be the set of all X-sections, $NA\Sigma = \emptyset$, $P\Sigma$ be the set of all Y-sections, and $C\Sigma$ be the set of all constant sections. This defines a parametric generalized equation $\Omega = \Omega_Y$ with parameters corresponding to the set of variables Y. If the partition of variables (16) is fixed we will omit Y in the notation above and call Ω the *parameterized* equation obtained from Ω. Denote by

$$\mathcal{G}E_{\text{par}}(\Omega) = \{\Omega_Y \mid \Omega \in \mathcal{G}E(\Omega)\}$$

the set of all parameterized equations of the system (15).

4.4. Positive theory of free groups

In this section we prove first the Merzljakov's result on elimination of quantifiers for positive sentences over free group $F = F(A)$ [22]. This proof is based on the notion of a generalized equation. Combining Merzljakov's theorem with Makanin's result on decidability of equations over free groups we obtain decidability of the positive theory of free groups. This argument is due to Makanin [20].

Recall that every positive formula $\Psi(Z)$ in the language L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula of the type

$$\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \big(S(X, Y, Z, A) = 1\big),$$

where $S(X, Y, Z, A) = 1$ is an equation with constants from $A^{\pm 1}$, $X = (x_1, \ldots, x_k)$, $Y = (y_1, \ldots, y_k)$, $Z = (z_1, \ldots, z_m)$. Indeed, one can insert fictitious quantifiers to ensure the direct alteration of quantifiers in the prefix. In particular, every positive sentence in L_A is equivalent modulo $A_{\mathcal{T}}$ to a formula of the type

$$\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \big(S(X, Y, A) = 1\big).$$

Now we prove the Merzljakov's theorem from [22], though in a slightly different form.

Merzljakov's Theorem. *If*

$$F \models \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \big(S(X, Y, A) = 1\big),$$

then there exist words (*with constants from* F) $q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in F[X]$, *such that*

$$F[X] \models S\big(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k, A)\big) = 1,$$

i.e., the equation

$$S(x_1, y_1, \dots, x_k, y_k, A) = 1$$

(*in variables* Y) *has a solution in the free group* $F[X]$.

Proof. Let $\mathcal{G}E(u) = \{\Omega_1(Z_1), \dots, \Omega_r(Z_r)\}$ be generalized equations associated with equation $S(X, Y, A) = 1$ in Lemma 14. Denote by $\rho_i = |Z_i|$ the number of variables in Ω_i.

Let $a, b \in A$, $[a, b] \neq 1$, and put

$$g_1 = ba^{m_{11}}ba^{m_{12}}b \dots a^{m_{1n_1}}b,$$

where $m_{11} < m_{12} < \dots < m_{1n_1}$ and $\max\{\rho_1, \dots, \rho_r\}|S(X, A)| < n_1$. Then there exists h_1 such that

$$F \models \forall x_2 \exists y_2 \dots \forall x_k \exists y_k \bigl(S(g_1, h_1, x_2, y_2, \dots, x_k, y_k) = 1\bigr).$$

Suppose now that elements $g_1, h_1, \dots, g_{i-1}, h_{i-1} \in F$ are given. We define

$$g_i = ba^{m_{i1}}ba^{m_{i2}}b \dots a^{m_{in_i}}b \tag{17}$$

such that:

(1) $m_{i1} < m_{i2} < \dots < m_{in_i}$;
(2) $\max\{\rho_1, \dots, \rho_r\}|S(X, A)| < n_i$;
(3) no subword of the type $ba^{m_{ij}}b$ occur in any of the words g_l, h_l for $l < i$.

We call words (17) *Merzljakov's words.* Then there exists an element $h_i \in F$ such that

$$F \models \forall x_{i+1} \exists y_{i+1} \dots \forall x_k \exists y_k \bigl(S(g_1, h_1, \dots, g_i, h_i, x_{i+1}, y_{i+1}, \dots, x_k, y_k) = 1\bigr).$$

By induction we have constructed elements $g_1, h_1, \dots, g_k, h_k \in F$ such that

$$S(g_1, h_1, \dots, g_k, h_k) = 1$$

and each g_i has the form (17) and satisfies the conditions (1)–(3).

By Lemma 14 there exists a generalized equation $\Omega(Z) \in \mathcal{G}E(S)$, words $P_i(Z, A)$, $Q_i(Z, A) \in F[Z]$ $(i = 1, \dots, k)$ of length not more then $\rho = |Z|$, and a solution $U = (u_1, \dots, u_\rho)$ of $\Omega(Z)$ in F such that the following words are graphically equal:

$$g_i = P_i(U), \quad h_i = Q_i(U) \quad (i = 1, \dots, k).$$

Since $n_i > \rho|S(X, A)|$ (by condition (2)) and $P_i(U) = y_1 \dots y_q$ with $y_i \in U^{\pm 1}$, $q \leqslant \rho$, the graphical equalities

$$g_i = ba^{m_{i1}}ba^{m_{i2}}b \dots a^{m_{in_i}}b = P_i(U) \quad (i = 1, \dots, k) \tag{18}$$

show that there exists a subword $v_i = ba^{m_{ij}}b$ of g_i such that every occurrence of this subword in (18) is an occurrence inside some $u_j^{\pm 1}$. For each i fix such a subword $v_i = ba^{m_{ij}}b$ in g_i. In view of condition (3) the word v_i does not occur in any of the words g_j $(j \neq i)$, h_s $(s < i)$, moreover, in g_i it occurs precisely once. Denote by $j(i)$ the unique index such that v_i occurs inside $u_{j(i)}^{\pm 1}$ in $P_i(U)$ from (18) (and v_i occurs in it precisely once).

The argument above shows that the variable $z_{j(i)}$ does not occur in words $P_t(Z, A)$ $(t \neq i)$, $Q_s(Z, A)$ $(s < i)$. Moreover, in $P_i(Z)$ it occurs precisely once. It follows that the variable $z_{j(i)}$ in the generalized equation $\Omega(Z)$ does not occur neither in coefficient equations nor in basic equations corresponding to the dual bases related to x_t $(t \neq i)$, y_s $(s < i)$.

We “mark” (or select) the unique occurrence of v_i (as $v_i^{\pm 1}$) in $u_{j(i)}$ $i = 1, \ldots, k$. Now we are going to mark some other occurrences of v_i in words $u_1, \ldots, u_\rho$ as follows. Suppose some u_d has a marked occurrence of some v_i. If Ω contains an equation of the type $z_d^\varepsilon = z_r^\delta$, then $u_d^\varepsilon = u_r^\delta$ graphically. Hence u_r has an occurrence of subword $v_i^{\pm 1}$ which correspond to the marked occurrence of $v_i^{\pm 1}$ in u_d. We mark this occurrence of $v_i^{\pm 1}$ in u_r.

Suppose Ω contains an equation of the type

$$[h_{\alpha_1} \ldots h_{\beta_1 - 1}]^{\varepsilon_1} = [h_{\alpha_2} \ldots h_{\beta_2 - 1}]^{\varepsilon_2}$$

such that z_d occurs in it, say in the left. Then

$$[u_{\alpha_1} \ldots u_{\beta_1 - 1}]^{\varepsilon_1} = [u_{\alpha_2} \ldots u_{\beta_2 - 1}]^{\varepsilon_2}$$

graphically. Since $v_i^{\pm 1}$ is a subword of u_d, it occurs also in the right-hand part of the equality above, say in some u_r. We marked this occurrence of $v_i^{\pm 1}$ in u_r. The marking process will be over in finitely many steps. Observe that one and the same u_r can have several marked occurrences of some $v_i^{\pm 1}$.

Now in all words $u_1, \ldots, u_\rho$ we replace every marked occurrence of $v_i = ba^{m_{ij}}b$ with a new word $ba^{m_{ij}}x_i b$ from the group $F[X]$. Denote the resulting words from $F[X]$ by $\tilde{u}_1, \ldots, \tilde{u}_\rho$. It follows from description of the marking process that the tuple $\tilde{U} = (\tilde{u}_1, \ldots, \tilde{u}_\rho)$ is a solution of the generalized equation Ω in the free group $F[X]$. Indeed, all the equations in Ω are graphically satisfied by the substitution $z_i \to u_i$ hence the substitution $u_i \to \tilde{u}_i$ still makes them graphically equal. Now by Lemma 14, $X = P(\tilde{U})$, $Y = Q(\tilde{U})$ is a solution of the equation $S(X, A) = 1$ over $F[X]$ as desired. □

Corollary 9 [20]. *There is an algorithm which for a given positive sentence*

$$\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \bigl(S(X, Y, A) = 1\bigr)$$

in L_A determines whether or not this formula holds in F, and if it does, the algorithm finds words

$$q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in F[X]$$

such that

$$F[X] \models u\big(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k)\big) = 1.$$

Proof. The proof follows from Proposition 4 and decidability of equations over free groups with constraints $y_i \in F[X_i]$, where $X_i = \{x_1, \ldots, x_i\}$ [19]. □

Definition 13. Let ϕ be a sentence in the language L_A written in the standard form

$$\phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \phi_0(x_1, y_1, \ldots, x_k, y_k),$$

where ϕ_0 is a quantifier-free formula in L_A. We say that G *freely lifts* ϕ if there exist words (with constants from F) $q_1(x_1), \ldots, q_k(x_1, \ldots, x_k) \in F[X]$, such that

$$F[X] \models \phi_0\big(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k, A)\big) = 1.$$

Theorem 4. *F freely lifts every sentence in L_A that is true in F.*

Proof. Suppose a sentence

$$\phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \big(U(x_1, y_1, \ldots, x_k, y_k) = 1 \ \wedge \ V(x_1, y_1, \ldots, x_k, y_k) \neq 1\big), \tag{19}$$

is true in F. We choose $x_1 = g_1,\ y_1 = h_1, \ldots, x_k = g_k,\ y_k = h_k$ precisely like in the Merzljakov's theorem. Then the formula

$$U(g_1, h_1, \ldots, g_k, h_k) = 1 \ \wedge \ V(g_1, h_1, \ldots, g_k, h_k) \neq 1$$

holds in F. In particular, $U(g_1, h_1, \ldots, g_k, h_k) = 1$ in F. It follows from the argument in Theorem 4 that there are words $q_1(x_1) \in F[x_1], \ldots, q_k(x_1, \ldots, x_k) \in F[x_1, \ldots, x_k]$ such that

$$F[X] \models U\big(x_1, q_1(x_1, \ldots, x_k), \ldots, x_k, q_k(x_1, \ldots, x_k)\big) = 1.$$

Moreover, it follows from the construction that $h_1 = q_1(g_1), \ldots, h_k = q_k(g_1, \ldots, g_k)$. We claim that

$$F[X] \models V\big(x_1, q_1(x_1, \ldots, x_k), \ldots, x_k, q_k(x_1, \ldots, x_k)\big) \neq 1.$$

Indeed, if

$$V\big(x_1, q_1(x_1, \ldots, x_k), \ldots, x_k, q_k(x_1, \ldots, x_k)\big) = 1$$

in $F[X]$, then its image in F under any specialization $X \to F$ is also trivial, but this is not the case for specialization $x_1 \to g_1, \ldots, x_k \to g_k$—contradiction. This proves the theorem for sentences ϕ of the form (19). A similar argument works for formulas of the type

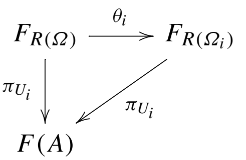
$$\phi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \bigvee_{i=1}^{n} \big(U_i(x_1, y_1, \ldots, x_k, y_k) = 1 \ \wedge \ V_i(x_1, y_1, \ldots, x_k, y_k) \neq 1\big),$$

which is, actually, the general case by Corollary 7. This finishes the proof. □

5. Makanin's process and cut equations

5.1. Elementary transformations

In this section we describe *elementary transformations* of generalized equations which were introduced by Makanin in [19]. Recall that we consider only formally consistent equations. In general, an elementary transformation ET associates to a generalized equation Ω a finite set of generalized equations $ET(\Omega) = \{\Omega_1, \ldots, \Omega_r\}$ and a collection of surjective homomorphisms $\theta_i : G_{R(\Omega)} \to G_{R(\Omega_i)}$ such that for every pair (Ω, U) there exists a unique pair of the type (Ω_i, U_i) for which the following diagram commutes.



Here $\pi_U(X) = U$. Since the pair (Ω_i, U_i) is defined uniquely, we have a well-defined map $ET : (\Omega, U) \to (\Omega_i, U_i)$.

*ET*1 (*Cutting a base*). Suppose Ω contains a boundary connection $\langle p, \lambda, q\rangle$. Then we replace (cut in p) the base λ by two new bases λ_1 and λ_2 and also replace (cut in q) $\Delta(\lambda)$ by two new bases $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$ such that the following conditions hold.

If $\varepsilon(\lambda) = \varepsilon(\Delta(\lambda))$, then

$$\alpha(\lambda_1) = \alpha(\lambda), \quad \beta(\lambda_1) = p, \quad \alpha(\lambda_2) = p, \quad \beta(\lambda_2) = \beta(\lambda),$$
$$\alpha\big(\Delta(\lambda_1)\big) = \alpha\big(\Delta(\lambda)\big), \quad \beta\big(\Delta(\lambda_1)\big) = q, \quad \alpha\big(\Delta(\lambda_2)\big) = q, \quad \beta\big(\Delta(\lambda_2)\big) = \beta\big(\Delta(\lambda)\big).$$

If $\varepsilon(\lambda) = -\varepsilon(\Delta(\lambda))$, then

$$\alpha(\lambda_1) = \alpha(\lambda), \quad \beta(\lambda_1) = p, \quad \alpha(\lambda_2) = p, \quad \beta(\lambda_2) = \beta(\lambda),$$
$$\alpha\big(\Delta(\lambda_1)\big) = q, \quad \beta\big(\Delta(\lambda_1)\big) = \beta\big(\Delta(\lambda)\big), \quad \alpha\big(\Delta(\lambda_2)\big) = \alpha\big(\Delta(\lambda)\big), \quad \beta\big(\Delta(\lambda_2)\big) = q.$$

Put $\varepsilon(\lambda_i) = \varepsilon(\lambda)$, $\varepsilon(\Delta(\lambda_i)) = \varepsilon(\Delta(\lambda))$, $i = 1, 2$.

Let (p', λ, q') be a boundary connection in Ω.

If $p' < p$, then replace (p', λ, q') by (p', λ_1, q').

If $p' > p$, then replace (p', λ, q') by (p', λ_2, q').

Notice, since the equation Ω is formally consistent, then the conditions above define boundary connections in the new generalized equation. The resulting generalized equation Ω' is formally consistent. Put $ET(\Omega) = \{\Omega'\}$. Figure 3(a) explains the name of the transformation $ET1$.

*ET*2 (*Transfer of a base*). Let a base θ of a generalized equation Ω be contained in the base μ, i.e., $\alpha(\mu) \leqslant \alpha(\theta) < \beta(\theta) \leqslant \beta(\mu))$. Suppose that the boundaries $\alpha(\theta)$ and $\beta(\theta))$ are μ-tied, i.e., there are boundary connections of the type $\langle \alpha(\theta), \mu, \gamma_1 \rangle$ and $\langle \beta(\theta), \mu, \gamma_2 \rangle$. Suppose also that every θ-tied boundary is μ-tied. Then we transfer θ from its location on the base μ to the corresponding location on the base $\Delta(\mu)$ and adjust all the basic and boundary equations (see Fig. 3(b)). More formally, we replace θ by a new base θ' such that $\alpha(\theta') = \gamma_1, \beta(\theta') = \gamma_2$ and replace each θ-boundary connection (p, θ, q) with a new one (p', θ', q) where p and p' come from the μ-boundary connection (p, μ, p'). The resulting equation is denoted by $\Omega' = ET2(\Omega)$.

*ET*3 (*Removal of a pair of matched bases* (*see Fig.* 3(c))). Let μ and $\Delta(\mu)$ be a pair of matched bases in Ω. Since Ω is formally consistent one has $\varepsilon(\mu) = \varepsilon(\Delta(\mu))$, $\beta(\mu) = \beta(\Delta(\mu))$ and every μ-boundary connection is of the type (p, μ, p). Remove the pair of bases $\mu, \Delta(\mu)$ with all boundary connections related to μ. Denote the new generalized equation by Ω'.

Remark. Observe, that for $i = 1, 2, 3$, $ETi(\Omega)$ consists of a single equation Ω', such that Ω and Ω' have the same set of variables H, and the identity map $F[H] \to F[H]$ induces an F-isomorphism $F_{R(\Omega)} \to F_{R(\Omega')}$. Moreover, U is a solution of Ω if and only if U is a solution of Ω'.

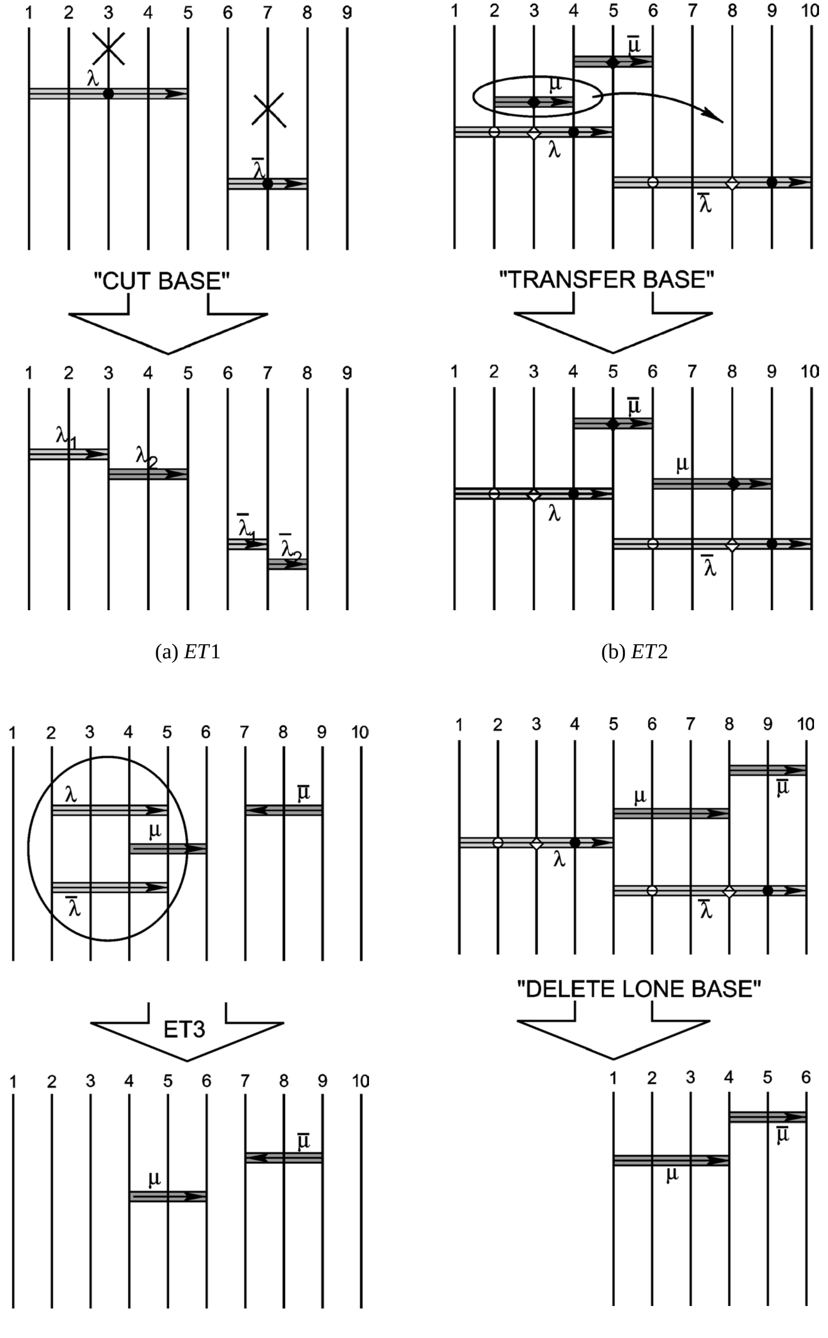
*ET*4 (*Removal of a lonely base* (*see Fig.* 3(d))). Suppose in Ω a variable base μ does not intersect any other variable base, i.e., the items $h_{\alpha(\mu)}, \ldots, h_{\beta(\mu)-1}$ are contained in only one variable base μ. Suppose also that all boundaries in μ are μ-tied, i.e., for every i $(\alpha(\mu) + 1 \leqslant i \leqslant \beta - 1)$ there exists a boundary $b(i)$ such that $(i, \mu, b(i))$ is a boundary connection in Ω. For convenience we define: $b(\alpha(\mu)) = \alpha(\Delta(\mu))$ and $b(\beta(\mu)) = \beta(\Delta(\mu))$ if $\varepsilon(\mu)\varepsilon(\Delta(\mu)) = 1$, and $b(\alpha(\mu)) = \beta(\Delta(\mu))$ and $b(\beta(\mu)) = \alpha(\Delta(\mu))$ if $\varepsilon(\mu)\varepsilon(\Delta(\mu)) = -1$.

The transformation $ET4$ carries Ω into a unique generalized equation Ω_1 which is obtained from Ω by deleting the pair of bases μ and $\Delta(\mu)$; deleting all the boundaries $\alpha(\mu) + 1, \ldots, \beta(\mu) - 1$ (and renaming the rest $\beta(\mu) - \alpha(\mu) - 1$ boundaries) together with all μ-boundary connections; replacing every constant base λ which is contained in μ by a constant base λ' with the same label as λ and such that $\alpha(\lambda') = b(\alpha(\lambda))$, $\beta(\lambda') = b(\beta(\lambda))$.

We define the homomorphism $\pi : F_{R(\Omega)} \to F_{R(\Omega')}$ as follows: $\pi(h_j) = h_j$ if $j < \alpha(\mu)$ or $j \geqslant \beta(\mu)$;

$$\pi(h_i) = \begin{cases} h_{b(i)} \ldots h_{b(i)-1}, & \text{if } \varepsilon(\mu) = \varepsilon(\Delta\mu), \\ h_{b(i)} \ldots h_{b(i-1)-1}, & \text{if } \varepsilon(\mu) = -\varepsilon(\Delta\mu) \end{cases}$$

for $\alpha + 1 \leqslant i \leqslant \beta(\mu) - 1$. It is not hard to see that π is an F-isomorphism.



(a) $ET1$ (b) $ET2$

(c) $ET3$ (d) $ET4$

Fig. 3. Elementary transformations ETi, $i = 1, 2, 3, 4$.

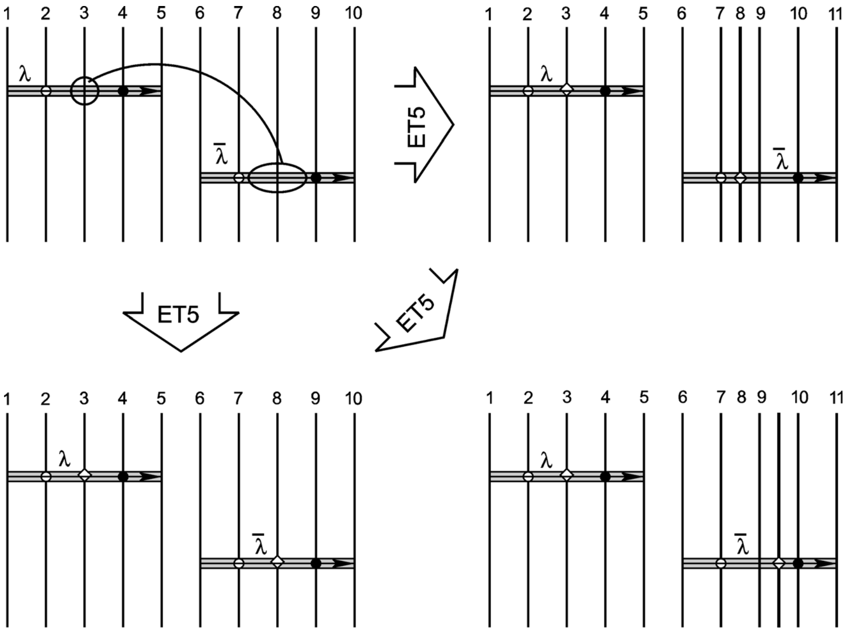


Fig. 4. Elementary transformation *ET*5.

*ET*5 (*Introduction of a boundary* (*see Fig.* 4)). Suppose a point p in a base μ is not μ-tied. The transformation *ET*5 μ-ties it in all possible ways, producing finitely many different generalized equations. To this end, let q be a boundary on $\Delta(\mu)$. Then we perform one of the following two transformations.

(1) Introduce the boundary connection $\langle p, \mu, q\rangle$ if the resulting equation Ω_q is formally consistent. In this case the corresponding F-homomorphism $\pi_q : F_{R(\Omega)} \to F_{R(\Omega_q)}$ is induced by the identity isomorphism on $F[H]$. Observe that π_q is not necessary an isomorphism.
(2) Introduce a new boundary q' between q and $q+1$ (and rename all the boundaries); introduce a new boundary connection (p, μ, q'). Denote the resulting equation by Ω'_q. In this case the corresponding F-homomorphism $\pi_{q'} : F_{R(\Omega)} \to F_{R(\Omega_{q'})}$ is induced by the map $\pi(h) = h$, if $h \neq h_q$, and $\pi(h_q) = h_{q'} h_{q'+1}$. Observe that $\pi_{q'}$ is an F-isomorphism.

Let Ω be a generalized equation and E be an elementary transformation. By $E(\Omega)$ we denote a generalized equation obtained from Ω by elementary transformation E (perhaps several such equations) if E is applicable to Ω, otherwise we put $E(\Omega) = \Omega$. By $\phi_E : F_{R(\Omega)} \to F_{R(E(\Omega))}$ we denote the canonical homomorphism of the coordinate groups (which has been described above in the case $E(\Omega) \neq \Omega$), otherwise, the identical isomorphism.

Lemma 15. *There exists an algorithm which for every generalized equation Ω and every elementary transformation E determines whether the canonical homomorphism $\phi_E : F_{R(\Omega)} \to F_{R(E(\Omega))}$ is an isomorphism or not.*

Proof. The only non-trivial case is when $E = E5$ and no new boundaries were introduced. In this case $E(\Omega)$ is obtained from Ω by adding a new particular equation, say $s = 1$, which is effectively determined by Ω and $E(\Omega)$. In this event, the coordinate group

$$F_{R(E(\Omega))} = F_{R(\Omega\cup\{s\})}$$

is a quotient group of $F_{R(\Omega)}$. Now ϕ_E is an isomorphism if and only if $R(\Omega) = R(\Omega \cup \{s\})$, or, equivalently, $s \in R(\Omega)$. The latter condition holds if and only if s vanishes on all solutions of the system of (group-theoretic) equations $\Omega = 1$ in F, i.e., if the following formula holds in F:

$$\forall x_1 \ldots \forall x_\rho \big(\Omega(x_1, \ldots, x_\rho) = 1 \ \to \ s(x_1, \ldots, x_\rho) = 1\big).$$

This can be checked effectively, since the universal theory of a free group F is decidable [20]. $\square$

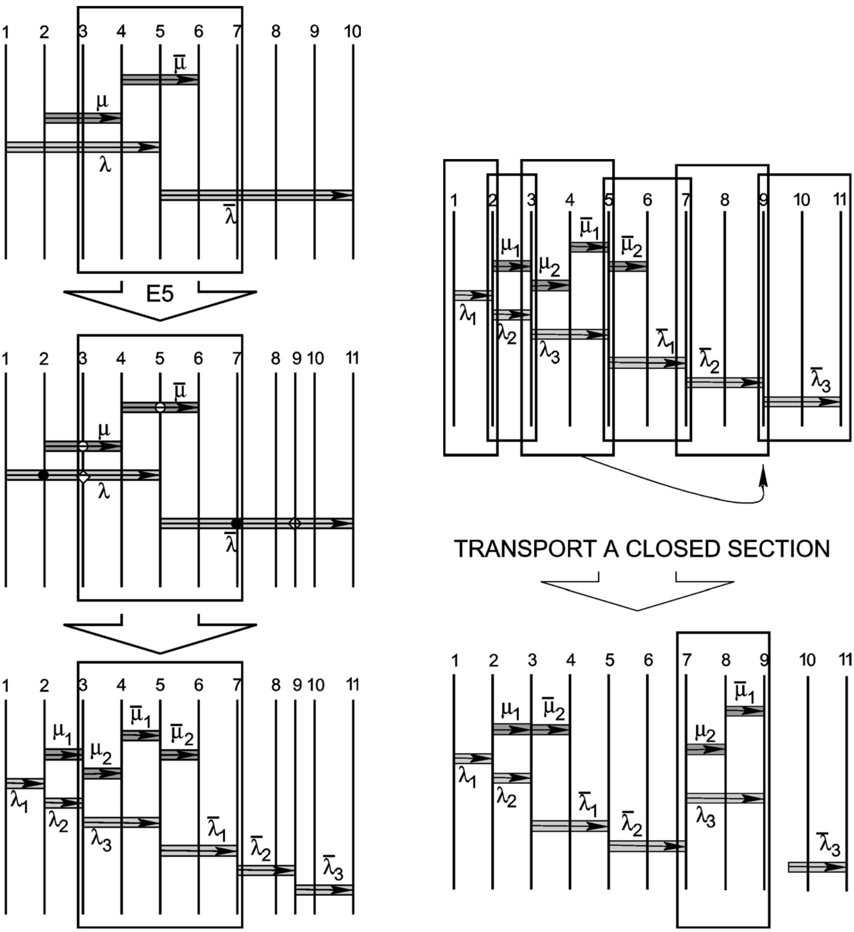
5.2. Derived transformations and auxiliary transformations

In this section we describe several useful transformations of generalized equations. Some of them can be realized as finite sequences of elementary transformations, we call them *derived* transformations. Other transformations result in equivalent generalized equations but cannot be realized by finite sequences of elementary moves.

$D1$ (*Closing a section*). Let σ be a section of Ω. The transformation $D1$ makes the section σ closed. To perform $D1$ we introduce boundary connections (transformations $ET5$) through the endpoints of σ until these endpoints are tied by every base containing them, and then cut through the endpoints all the bases containing them (transformations $ET1$) (see Fig. 5(a)).

$D2$ (*Transporting a closed section*). Let σ be a closed section of a generalized equation Ω. We cut σ out of the interval $[1, \rho_\Omega]$ together with all the bases and boundary connections on σ and put σ at the end of the interval or between any two consecutive closed sections of Ω. After that we correspondingly re-enumerate all the items and boundaries of the latter equation to bring it to the proper form. Clearly, the original equation Ω and the new one Ω' have the same solution sets and their coordinate groups are isomorphic (see Fig. 5(b)).

$D3$ (*Complete cut*). Let Ω be a generalized equation. For every boundary connection (p, μ, q) in Ω we cut the base μ at p applying $ET1$. The resulting generalized equation $\tilde{\Omega}$ is obtained from Ω by a consequent application of all possible $ET1$ transformations. Clearly, $\tilde{\Omega}$ does not depend on a particular choice of the sequence of transformations $ET1$. Since $ET1$ preserves isomorphism between the coordinate groups, equations Ω and $\tilde{\Omega}$



(a) (b)

Fig. 5. Derived transformations *D*1 (a) and *D*2 (b).

have isomorphic coordinate groups, and the isomorphism arises from the identity map $F[H] \to F[H]$.

*D*4 (*Kernel of a generalized equation*). Suppose that a generalized equation Ω does not contain boundary connections. An active base $\mu \in A\Sigma_\Omega$ is called *eliminable* if at least one of the following holds:

(a) μ contains an item h_i with $\gamma(h_i) = 1$;
(b) at least one of the boundaries $\alpha(\mu), \beta(\mu)$ is different from $1, \rho + 1$ and it does not touch any other base (except μ).

An *elimination process* for Ω consists of consequent removals (*eliminations*) of eliminable bases until no eliminable bases left in the equation. The resulting generalized equation is called a *kernel* of Ω and we denote it by $\mathrm{Ker}(\Omega)$. It is easy to see that $\mathrm{Ker}(\Omega)$ does not depend on a particular elimination process. Indeed, if Ω has two different eliminable bases μ_1, μ_2, and deletion of μ_i results in an equation Ω_i then by induction (on the number of eliminations) $\mathrm{Ker}(\Omega_i)$ is uniquely defined for $i = 1, 2$. Obviously, μ_1 is still eliminable in Ω_2, as well as μ_2 is eliminable in Ω_1. Now eliminating μ_1 and μ_2 from Ω_2 and Ω_1 we get one and the same equation Ω_0. By induction $\mathrm{Ker}(\Omega_1) = \mathrm{Ker}(\Omega_0) = \mathrm{Ker}(\Omega_2)$ hence the result. We say that a variable h_i *belongs to the kernel* ($h_i \in \mathrm{Ker}(\Omega)$), if either h_i belongs to at least one base in the kernel, or it is parametric, or it is constant.

Also, for an equation Ω by $\bar{\Omega}$ we denote the equation which is obtained from Ω by deleting all free variables. Obviously,

$$F_{R(\Omega)} = F_{R(\bar{\Omega})} * F(Y)$$

where Y is the set of free variables in Ω.

Let us consider what happens on the group level in the elimination process.

We start with the case when just one base is eliminated. Let μ be an eliminable base in $\Omega = \Omega(h_1, \ldots, h_\rho)$. Denote by Ω_1 the equation resulting from Ω by eliminating μ.

(1) Suppose $h_i \in \mu$ and $\gamma(h_i) = 1$. Then the variable h_i occurs only once in Ω—precisely in the equation $s_\mu = 1$ corresponding to the base μ. Therefore, in the coordinate group $F_{R(\Omega)}$ the relation $s_\mu = 1$ can be written as $h_i = w$, where w does not contain h_i. Using Tietze transformations we can rewrite the presentation of $F_{R(\Omega)}$ as $F_{R(\Omega')}$, where Ω' is obtained from Ω by deleting s_μ and the item h_i. It follows immediately that

$$F_{R(\Omega_1)} \simeq F_{R(\Omega')} * \langle h_i \rangle$$

and

$$F_{R(\Omega)} \simeq F_{R(\Omega')} \simeq F_{R(\bar{\Omega}_1)} * F(Z) \tag{20}$$

for some free group $F(Z)$. Notice that all the groups and equations which occur above can be found effectively.

(2) Suppose now that μ satisfies case (b) above with respect to a boundary i. Then in the equation $s_\mu = 1$ the variable h_{i-1} either occurs only once or it occurs precisely twice and in this event the second occurrence of h_{i-1} (in $\Delta(\mu)$) is a part of the subword $(h_{i-1}h_i)^{\pm 1}$. In both cases it is easy to see that the tuple

$$(h_1, \ldots, h_{i-2}, s_\mu, h_{i-1}h_i, h_{i+1}, \ldots, h_\rho)$$

forms a basis of the ambient free group generated by $(h_1, \ldots, h_\rho)$ and constants from A. Therefore, eliminating the relation $s_\mu = 1$, we can rewrite the presentation of $F_{R(\Omega)}$ in generators $Y = (h_1, \ldots, h_{i-2}, h_{i-1}h_i, h_{i+1}, \ldots, h_\rho)$. Observe also that any other equation $s_\lambda = 1$ ($\lambda \neq \mu$) of Ω either does not contain variables h_{i-1}, h_i or it contains them as parts of the subword $(h_{i-1}h_i)^{\pm 1}$, i.e., any such a word s_λ can be expressed as a word $w_\lambda(Y)$ in terms of generators Y and constants from A. This shows that

$$F_{R(\Omega)} \simeq F(Y \cup A)_{R(w_\lambda(Y) \mid \lambda \neq \mu)} \simeq F_{R(\Omega')},$$

where Ω' is a generalized equation obtained from Ω_1 by deleting the boundary i. Denote by Ω'' an equation obtained from Ω' by adding a free variable z to the right end of Ω'. It follows now that

$$F_{R(\Omega_1)} \simeq F_{R(\Omega'')} \simeq F_{R(\Omega)} * \langle z \rangle$$

and

$$F_{R(\Omega)} \simeq F_{R(\bar{\Omega}')} * F(Z) \tag{21}$$

for some free group $F(Z)$. Notice that all the groups and equations which occur above can be found effectively.

By induction on the number of steps in elimination process we obtain the following lemma.

Lemma 16.

$$F_{R(\Omega)} \simeq F_{R(\overline{\operatorname{Ker}\Omega})} * F(Z)$$

where $F(Z)$ is a free group on Z. Moreover, all the groups and equations which occur above can be found effectively.

Proof. Let

$$\Omega = \Omega_0 \to \Omega_1 \to \cdots \to \Omega_l = \operatorname{Ker}\Omega$$

be an elimination process for Ω. It is easy to see (by induction on l) that for every $j = 0, \dots, l-1$

$$\overline{\operatorname{Ker}\Omega_j} = \overline{\operatorname{Ker}\bar{\Omega}_j}.$$

Moreover, if Ω_{j+1} is obtained from Ω_j as in the case (2) above, then (in the notations above)

$$\overline{\operatorname{Ker}(\Omega_j)_1} = \overline{\operatorname{Ker}\Omega'_j}.$$

Now the statement of the lemma follows from the remarks above and equalities (20) and (21). □

*D*5 (*Entire transformation*). We need a few further definitions. A base μ of the equation Ω is called a *leading* base if $\alpha(\mu) = 1$. A leading base is said to be *maximal* (or a *carrier*) if $\beta(\lambda) \leqslant \beta(\mu)$, for any other leading base λ. Let μ be a carrier base of Ω. Any active base $\lambda \neq \mu$ with $\beta(\lambda) \leqslant \beta(\mu)$ is called a *transfer* base (with respect to μ).

Suppose now that Ω is a generalized equation with $\gamma(h_i) \geqslant 2$ for each h_i in the active part of Ω. An *entire transformation* is a sequence of elementary transformations which are performed as follows. We fix a carrier base μ of Ω. For any transfer base λ we μ-tie (applying $ET5$) all boundaries in λ. Using $ET2$ we transfer all transfer bases from μ onto $\Delta(\mu)$. Now, there exists some $i < \beta(\mu)$ such that $h_1, \ldots, h_i$ belong to only one base μ, while h_{i+1} belongs to at least two bases. Applying $ET1$ we cut μ along the boundary $i+1$. Finally, applying $ET4$ we delete the section $[1, i+1]$.

*D*6 (*Identifying closed constant sections*). Let λ and μ be two constant bases in Ω with labels a^{ε_λ} and a^{ε_μ}, where $a \in A$ and $\varepsilon_\lambda, \varepsilon_\mu \in \{1, -1\}$. Suppose that the sections $\sigma(\lambda) = [i, i+1]$ and $\sigma(\mu) = [j, j+1]$ are closed. Then we introduce a new variable base δ with its dual $\Delta(\delta)$ such that $\sigma(\delta) = [i, i+1]$, $\sigma(\Delta(\delta)) = [j, j+1]$, $\varepsilon(\delta) = \varepsilon_\lambda$, $\varepsilon(\Delta(\delta)) = \varepsilon_\mu$. After that we transfer all bases from δ onto $\Delta(\delta)$ using $ET2$, remove the bases δ and $\Delta(\delta)$, remove the item h_i, and enumerate the items in a proper order. Obviously, the coordinate group of the resulting equation is isomorphic to the coordinate group of the original equation.

5.3. *Construction of the tree $T(\Omega)$*

In this section we describe a branching rewrite process for a generalized equation Ω. This process results in an (infinite) tree $T(\Omega)$. At the end of the section we describe infinite paths in $T(\Omega)$.

Complexity of a parametric generalized equation

Denote by ρ_A the number of variables h_i in all active sections of Ω, by $n_A = n_A(\Omega)$ the number of bases in active sections of Ω, by ν'—the number of open boundaries in the active sections, by σ'—the number of closed boundaries in the active sections.

The number of closed active sections containing no bases, precisely one base, or more than one base are denoted by t_{A0}, t_{A1}, t_{A2} respectively. For a closed section $\sigma \in \Sigma_\Omega$ denote by $n(\sigma)$, $\rho(\sigma)$ the number of bases and, respectively, variables in σ.

$$\rho_A = \rho_A(\Omega) = \sum_{\sigma \in A\Sigma_\Omega} \rho(\sigma),$$

$$n_A = n_A(\Omega) = \sum_{\sigma \in A\Sigma_\Omega} n(\sigma).$$

The *complexity* of a parametric equation Ω is the number

$$\tau = \tau(\Omega) = \sum_{\sigma \in A\Sigma_\Omega} \max\bigl\{0, n(\sigma) - 2\bigr\}.$$

Notice that the entire transformation ($D5$) as well as the cleaning process ($D4$) do not increase complexity of equations.

Let Ω be a parametric generalized equation. We construct a tree $T(\Omega)$ (with associated structures), as a directed tree oriented from a root v_0, starting at v_0 and proceeding by induction from vertices at distance n from the root to vertices at distance $n+1$ from the root.

We start with a general description of the tree $T(\Omega)$. For each vertex v in $T(\Omega)$ there exists a unique generalized equation Ω_v associated with v. The initial equation Ω is associated with the root v_0, $\Omega_{v_0} = \Omega$. For each edge $v \to v'$ (here v and v' are the origin and the terminus of the edge) there exists a unique surjective homomorphism $\pi(v, v') : F_{R(\Omega_v)} \to F_{R(\Omega'_v)}$ associated with $v \to v'$.

If

$$v \to v_1 \to \cdots \to v_s \to u$$

is a path in $T(\Omega)$, then by $\pi(v, u)$ we denote composition of corresponding homomorphisms

$$\pi(v, u) = \pi(v, v_1) \dots \pi(v_s, u).$$

The set of edges of $T(\Omega)$ is subdivided into two classes: *principal* and *auxiliary*. Every newly constructed edge is principle, if not said otherwise. If $v \to v'$ is a principle edge then there exists a finite sequence of elementary or derived transformations from Ω_v to $\Omega_{v'}$ and the homomorphism $\pi(v, v')$ is composition of the homomorphisms corresponding to these transformations. We also assume that active (non-active) sections in $\Omega_{v'}$ are naturally inherited from Ω_v, if not said otherwise.

Suppose the tree $T(\Omega)$ is constructed by induction up to a level n, and suppose v is a vertex at distance n from the root v_0. We describe now how to extend the tree from v. The construction of the outgoing edges at v depends on which case described below takes place at the vertex v. We always assume that if we have Case i, then all Cases j, with $j \leqslant i - 1$, do not take place at v. We will see from the description below that there is an effective procedure to check whether or not a given case takes place at a given vertex. It will be obvious for all cases, except Case 1. We treat this case below.

Preprocessing

Case 0. In Ω_v we transport closed sections using $D2$ in such a way that all active sections are at the left end of the interval (the active part of the equation), then come all non-active sections (the non-active part of the equation), then come parametric sections (the parametric part of the equation), and behind them all constant sections are located (the constant part of the equation).

Termination conditions

Case 1. The homomorphism $\pi(v_0, v)$ is not an isomorphism (or equivalently, the homomorphism $\pi(v_1, v)$, where v_1 is the parent of v, is not an isomorphism). The vertex v is called a *leaf* or an *end* vertex. There are no outgoing edges from v.

Lemma 17. *There is an algorithm to verify whether the homomorphism $\pi(v, u)$, associated with an edge $v \to u$ in $T(\Omega)$ is an isomorphism or not.*

Proof. We will see below (by a straightforward inspection of Cases 1–15 below) that every homomorphism of the type $\pi(v, u)$ is a composition of the canonical homomorphisms

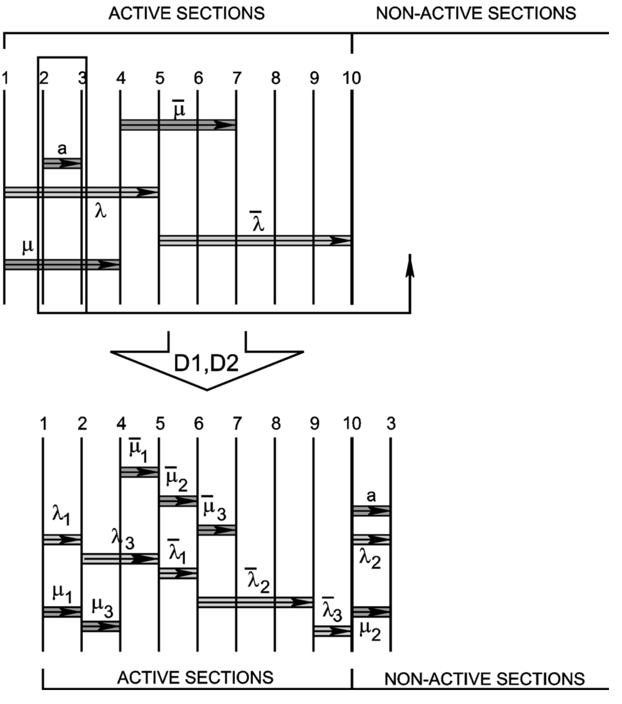


Fig. 6. Cases 3, 4: Moving constant bases.

corresponding to the elementary (derived) transformations. Moreover, this composition is effectively given. Now the result follows from Lemma 15. □

Case 2. Ω_v does not contain active sections. The vertex v is called a *leaf* or an *end* vertex. There are no outgoing edges from v.

Moving constants to the right

Case 3. Ω_v contains a constant base λ in an active section such that the section $\sigma(\lambda)$ is not closed.

Here we close the section $\sigma(\lambda)$ using the derived transformation $D1$.

Case 4. Ω_v contains a constant base λ with a label $a \in A^{\pm 1}$ such that the section $\sigma(\lambda)$ is closed.

Here we transport the section $\sigma(\lambda)$ to the location right after all variable and parametric sections in Ω_v using the derived transformation $D2$. Then we identify all closed sections of the type $[i, i+1]$, which contain a constant base with the label $a^{\pm 1}$, with the transported section $\sigma(\lambda)$, using the derived transformation $D6$. In the resulting generalized equation $\Omega_{v'}$ the section $\sigma(\lambda)$ becomes a constant section, and the corresponding edge (v, v') is auxiliary. See Fig. 6.

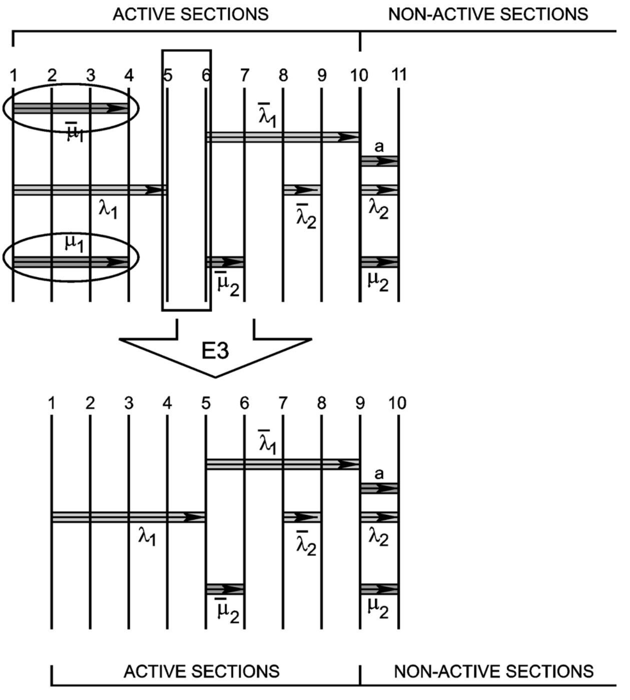


Fig. 7. Cases 5, 6: Trivial equations and useless variables.

Moving free variables to the right

Case 5. Ω_v contains a free variable h_q in an active section.

Here we close the section $[q, q+1]$ using $D1$, transport it to the very end of the interval behind all items in Ω_v using $D2$. In the resulting generalized equation $\Omega_{v'}$ the transported section becomes a constant section, and the corresponding edge (v, v') is auxiliary.

Remark 6. If Cases 0–5 are not possible at v then the parametric generalized equation Ω_v is in standard form.

Case 6. Ω_v contains a pair of matched bases in an active section.

Here we perform $ET3$ and delete it. See Fig. 7.

Eliminating linear variables

Case 7. In Ω_v there is h_i in an active section with $\gamma_i = 1$ and such that both boundaries i and $i+1$ are closed.

Here we remove the closed section $[i, i+1]$ together with the lone base using $ET4$.

Case 8. In Ω_v there is h_i in an active section with $\gamma_i = 1$ and such that one of the boundaries $i, i+1$ is open, say $i+1$, and the other is closed.

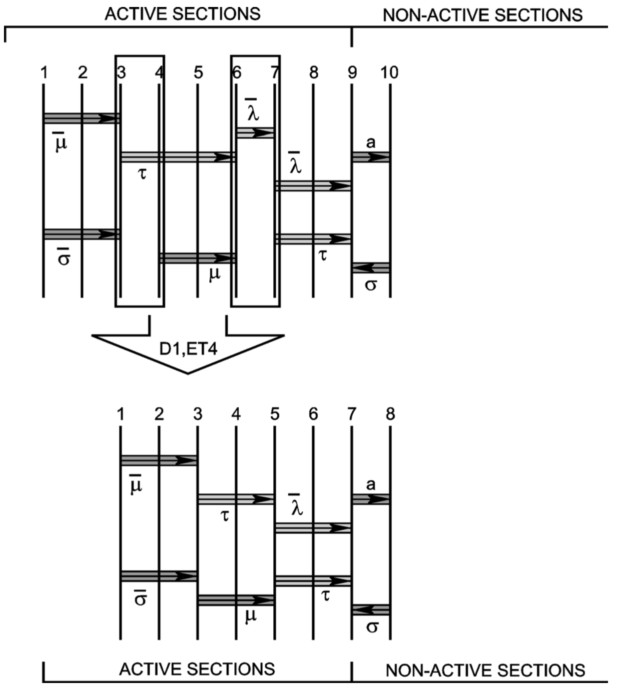


Fig. 8. Cases 7–10: Linear variables.

Here we perform $ET5$ and μ-tie $i+1$ through the only base μ it intersects; using $ET1$ we cut μ in $i+1$; and then we delete the closed section $[i, i+1]$ by $ET4$. See Fig. 8.

Case 9. In Ω_v there is h_i in an active section with $\gamma_i = 1$ and such that both boundaries i and $i+1$ are open. In addition, assume that there is a closed section σ containing exactly two (not matched) bases μ_1 and μ_2, such that $\sigma = \sigma(\mu_1) = \sigma(\mu_2)$ and in the generalized equation $\tilde{\Omega}_v$ (see the derived transformation $D3$) all the bases obtained from μ_1, μ_2 by $ET1$ in constructing $\tilde{\Omega}_v$ from Ω_v, do not belong to the kernel of $\tilde{\Omega}_v$.

Here, using $ET5$, we μ_1-tie all the boundaries inside μ_1; using $ET2$, we transfer μ_2 onto $\Delta(\mu_1)$; and remove μ_1 together with the closed section σ using $ET4$.

Case 10. Ω_v satisfies the first assumption of Case 9 and does not satisfy the second one.

In this event we close the section $[i, i+1]$ using $D1$ and remove it using $ET4$.

Tying a free boundary

Case 11. Some boundary i in the active part of Ω_v is free. Since we do not have Case 5 the boundary i intersects at least one base, say, μ.

Here we μ-tie i using $ET5$.

Quadratic case

Case 12. Ω_v satisfies the condition $\gamma_i = 2$ for each h_i in the active part.
We apply the entire transformation *D*5.

Case 13. Ω_v satisfies the condition $\gamma_i \geqslant 2$ for each h_i in the active part, and $\gamma_i > 2$ or at least one such h_i. In addition, for some active base μ section $\sigma(\mu) = [\alpha(\mu), \beta(\mu)]$ is closed.

In this case using *ET*5, we μ-tie every boundary inside μ; using *ET*2, we transfer all bases from μ to $\Delta(\mu)$; using *ET*4, we remove the lone base μ together with the section $\sigma(\mu)$.

Case 14. Ω_v satisfies the condition $\gamma_i \geqslant 2$ for each h_i in the active part, and $\gamma_i > 2$ for at least one such h_i. In addition, some boundary j in the active part touches some base λ, intersects some base μ, and j is not μ-tied.
Here we μ-tie j.

General JSJ-case

Case 15. Ω_v satisfies the condition $\gamma_i \geqslant 2$ for each h_i in the active part, and $\gamma_i > 2$ for at least one such h_i. We apply, first, the entire transformation *D*5.

Here for every boundary j in the active part that touches at least one base, we μ-tie j by every base μ containing j. This results in finitely many new vertices $\Omega_{v'}$ with principle edges (v, v').

If, in addition, Ω_v satisfies the following condition (we called it Case 15.1 in [13]) then we construct the principle edges as was described above, and also construct a few more auxiliary edges outgoing from the vertex v:

Case 15.1. The carrier base μ of the equation Ω_v intersects with its dual $\Delta(\mu)$.

Here we construct an auxiliary equation $\hat{\Omega}_v$ (which does not occur in $T(\Omega)$) as follows. Firstly, we add a new constant section $[\rho_v + 1, \rho_v + 2]$ to the right of all sections in Ω_v (in particular, h_{ρ_v+1} is a new free variable). Secondly, we introduce a new pair of bases $(\lambda, \Delta(\lambda))$ such that

$$\alpha(\lambda) = 1, \qquad \beta(\lambda) = \beta\big(\Delta(\mu)\big), \qquad \alpha\big(\Delta(\lambda)\big) = \rho_v + 1, \qquad \beta\big(\Delta(\lambda)\big) = \rho_v + 2.$$

Notice that Ω_v can be obtained from $\hat{\Omega}_v$ by *ET*4: deleting $\delta(\lambda)$ together with the closed section $[\rho_v + 1, \rho_v + 2]$.

Let

$$\hat{\pi}_v : F_{R(\Omega_v)} \to F_{R(\hat{\Omega}_v)}$$

be the isomorphism induced by *ET*4. Case 15 still holds for $\hat{\Omega}_v$, but now λ is the carrier base. Applying to $\hat{\Omega}_v$ transformations described in Case 15, we obtain a list of new vertices

$\Omega_{v'}$ together with isomorphisms

$$\eta_{v'}: F_{R(\hat{\Omega}_v)} \to F_{R(\Omega_{v'})}.$$

Now for each such v' we add to $T(\Omega)$ an auxiliary edge (v, v') equipped with composition of homomorphisms $\pi(v, v') = \eta_{v'} \circ \hat{\pi}_v$ and assign $\Omega_{v'}$ to the vertex v'.

If none of Cases 0–15 is possible, then we stop, and the tree $T(\Omega)$ is constructed. In any case, the tree $T(\Omega)$ is constructed by induction. Observe that, in general, $T(\Omega)$ is an infinite locally finite tree.

If Case i $(0 \leqslant i \leqslant 15)$ takes place at a vertex v then we say that v has type i and write $\mathrm{tp}(v) = i$.

Lemma 18 [27, Lemma 3.1]. *If $u \to v$ is a principal edge of the tree $T(\Omega)$, then*:

(1) $n_A(\Omega_v) \leqslant n_A(\Omega_u)$, *if* $\mathrm{tp}(v_1) \neq 3, 10$, *this inequality is proper if* $\mathrm{tp}(v_1) = 6, 7, 9, 13$;
(2) *if* $\mathrm{tp}(v_1) = 10$, *then* $n_A(\Omega_v) \leqslant n_A(\Omega_u) + 2$;
(3) $\nu'(\Omega_v) \leqslant \nu'(\Omega_u)$ *if* $\mathrm{tp}(v_1) \leqslant 13$ *and* $\mathrm{tp}(v_1) \neq 3, 11$;
(4) $\tau(\Omega_v) \leqslant \tau(\Omega_u)$, *if* $\mathrm{tp}(v_1) \neq 3$.

Proof. Straightforward verification. □

Lemma 19. *Let*

$$v_1 \to v_2 \to \cdots \to v_r \to \cdots$$

be an infinite path in the tree $T(\Omega)$. Then there exists a natural number N such that all the edges $v_n \to v_{n+1}$ of this path with $n \geqslant N$ are principal edges, and one of the following situations holds:

(1) (*linear case*) $7 \leqslant \mathrm{tp}(v_n) \leqslant 10$ *for all* $n \geqslant N$;
(2) (*quadratic case*) $\mathrm{tp}(v_n) = 12$ *for all* $n \geqslant N$;
(3) (*general JSJ-case*) $\mathrm{tp}(v_n) = 15$ *for all* $n \geqslant N$.

Proof. Observe that starting with a generalized equation Ω we can have Case 0 only once, afterward in all other equations the active part is at the left, then comes the non-active part, then—the parametric part, and at the end—the constant part. Obviously, Cases 1 and 2 do not occur on an infinite path. Notice also that Cases 3 and 4 can only occur finitely many times, namely, not more then $2t$ times where t is the number of constant bases in the original equation Ω. Therefore, there exists a natural number N_1 such that $\mathrm{tp}(v_i) \geqslant 5$ for all $i \geqslant N_1$.

Now we show that the number of vertices v_i $(i \geqslant N)$ for which $\mathrm{tp}(v_i) = 5$ is not more than the minimal number of generators of the group $F_{R(\Omega)}$, in particular, it cannot be greater than $\rho + 1 + |A|$, where $\rho = \rho(\Omega)$. Indeed, if a path from the root v_0 to a vertex v contains k vertices of type 5, then Ω_v has at least k free variables in the constant part. This implies that the coordinate group $F_{R(\Omega_v)}$ has a free group of rank k as a free factor, hence it cannot be generated by less than k elements. Since $\pi(v_0, v): F_{R(\Omega)} \to F_{R(\Omega_v)}$ is a surjective homomorphism, the group $F_{R(\Omega)}$ cannot be generated by less then k elements. This shows that $k \leqslant \rho + 1 + |A|$. It follows that there exists a number $N_2 \geqslant N_1$ such that $\mathrm{tp}(v_i) > 5$ for every $i \geqslant N_2$.

Suppose $i > N_2$. If $\mathrm{tp}(v_i) = 12$, then it is easy to see that $\mathrm{tp}(v_{i+1}) = 6$ or $\mathrm{tp}(v_{i+1}) = 12$. But if $\mathrm{tp}(v_{i+1}) = 6$, then $\mathrm{tp}(v_{i+2}) = 5$—contradiction with $i > N_2$. Therefore, $\mathrm{tp}(v_{i+1}) = \mathrm{tp}(v_{i+2}) = \cdots = \mathrm{tp}(v_{i+j}) = 12$ for every $j > 0$ and we have situation (2) of the lemma.

Suppose now $\mathrm{tp}(v_i) \neq 12$ for all $i \geqslant N_2$. By Lemma 18, $\tau(\Omega_{v_{j+1}}) \leqslant \tau(\Omega_{v_j})$ for every principle edge $v_j \to v_{j+1}$ where $j \geqslant N_2$. If $v_j \to v_{j+1}$, where $j \geqslant N_2$, is an auxiliary edge then $\mathrm{tp}(v_j) = 15$ and, in fact, Case 15.1 takes place at v_j. In the notation of Case 15.1, $\Omega_{v_{j+1}}$ is obtained from $\hat{\Omega}_{v_j}$ by transformations from Case 15. In this event, both bases μ and $\Delta(\mu)$ will be transferred from the new carrier base λ to the constant part, so the complexity will be decreased at least by two: $\tau(\Omega_{v_{j+1}}) \leqslant \tau(\hat{\Omega}_{v_j}) - 2$. Observe also that $\tau(\hat{\Omega}_{v_j}) = \tau(\Omega_{v_j}) + 1$. Hence $\tau(\Omega_{v_{j+1}}) < \tau(\Omega_{v_j})$.

It follows that there exists a number $N_3 \geqslant N_2$ such that $\tau(\Omega_{v_j}) = \tau(\Omega_{v_{N_3}})$ for every $j \geqslant N_3$, i.e., complexity stabilizes. Since every auxiliary edge gives a decrease of complexity, this implies that for every $j \geqslant N_3$ the edge $v_j \to v_{j+1}$ is principle.

Suppose now that $i \geqslant N_3$. We claim that $\mathrm{tp}(v_i) \neq 6$. Indeed, if $\mathrm{tp}(v_i) = 6$, then the closed section, containing the matched bases μ, $\Delta(\mu)$, does not contain any other bases (otherwise the complexity of $\Omega_{v_{i+1}}$ would decrease). But in this event $\mathrm{tp}(v_{i+1}) = 5$ which is impossible.

So $\mathrm{tp}(v_i) \geqslant 7$ for every $i \geqslant N_3$. Observe that $ET3$ (deleting match bases) is the only elementary transformation that can produce new free boundaries. Observe also that $ET3$ can be applied only in Case 6. Since Case 6 does not occur anymore along the path for $i \geqslant N_3$, one can see that no new free boundaries occur in equations Ω_{v_j} for $j \geqslant N_3$. It follows that there exists a number $N_4 \geqslant N_3$ such that $\mathrm{tp}(v_i) \neq 11$ for every $j \geqslant N_4$.

Suppose now that for some $i \geqslant N_4$, $13 \leqslant \mathrm{tp}(v_i) \leqslant 15$. It is easy to see from the description of these cases that in this event $\mathrm{tp}(v_{i+1}) \in \{6, 13, 14, 15\}$. Since $\mathrm{tp}(v_{i+1}) \neq 6$, this implies that $13 \leqslant \mathrm{tp}(v_j) \leqslant 15$ for every $j \geqslant i$. In this case the sequence $n_A(\Omega_{v_j})$ stabilizes by Lemma 18. In addition, if $\mathrm{tp}(v_j) = 13$, then $n_A(\Omega_{v_{j+1}}) < n_A(\Omega_{v_j})$. Hence there exists a number $N_5 \geqslant N_4$ such that $\mathrm{tp}(v_j) \neq 13$ for all $j \geqslant N_5$.

Suppose $i \geqslant N_5$. There cannot be more than $8(n_A(\Omega_{v_i}))^2$ vertices of type 14 in a row starting at a vertex v_i; hence there exists $j \geqslant i$ such that $\mathrm{tp}(v_j) = 15$. The series of transformations $ET5$ in Case 15 guarantees the inequality $\mathrm{tp}(v_{j+1}) \neq 14$; hence $\mathrm{tp}(v_{j+1}) = 15$, and we have situation (3) of the lemma.

So we can suppose $\mathrm{tp}(v_i) \leqslant 10$ for all the vertices of our path. Then we have situation (1) of the lemma. □

5.4. Periodized equations

In this section we introduce a notion of a periodic structure which allows one to describe periodic solutions of generalized equations. Recall that a reduced word P in a free group F is called a *period* if it is cyclically reduced and not a proper power. A word $w \in F$ is called *P-periodic* if $|w| \geqslant |P|$ and it is a subword of P^n for some n. Every P-periodic word w can be presented in the form

$$w = A^r A_1 \tag{22}$$

where A is a cyclic permutation of $P^{\pm 1}$, $r \geqslant 1$, $A = A_1 \circ A_2$, and $A_2 \neq 1$. This representation is unique if $r \geqslant 2$. The number r is called the *exponent* of w. A maximal exponent of P-periodic subword in a word u is called the *exponent of P-periodicity in u*. We denote it $e_P(u)$.

Definition 14. Let Ω be a standard generalized equation. A solution $H: h_i \to H_i$ of Ω is called *periodic with respect to a period P*, if for every variable section σ of Ω one of the following conditions hold:

(1) $H(\sigma)$ is P-periodic with exponent $r \geqslant 2$;
(2) $|H(\sigma)| \leqslant |P|$;
(3) $H(\sigma)$ is A-periodic and $|A| \leqslant |P|$.

Moreover, condition (1) holds at least for one such σ.

Let H be a P-periodic solution of Ω. Then a section σ satisfying (1) is called *P-periodic* (with respect to H).

5.4.1. Periodic structure

Let Ω be a parametrized generalized equation. It turns out that every periodic solution of Ω is a composition of a canonical automorphism of the coordinate group $F_{R(\Omega)}$ with either a solution with bounded exponent of periodicity (modulo parameters) or a solution of a "proper" equation. These canonical automorphisms correspond to Dehn twists of $F_{R(\Omega)}$ which are related to the splitting of this group (which comes from the periodic structure) over an abelian edge group.

We fix till the end of the section a generalized equation Ω in standard form. Recall that in Ω all closed sections σ, bases μ, and variables h_i belong to either the variable part $V\Sigma$, or the parametric part $P\Sigma$, or the constant part $C\Sigma$ of Ω.

Definition 15. Let Ω be a generalized equation in standard form with no boundary connections. A *periodic structure* on Ω is a pair $\langle \mathcal{P}, R \rangle$, where:

(1) $\mathcal{P}$ is a set consisting of some variables h_i, some bases μ, and some closed sections σ from $V\Sigma$ and such that the following conditions are satisfied:
   (a) if $h_i \in \mathcal{P}$ and $h_i \in \mu$, and $\Delta(\mu) \in V\Sigma$, then $\mu \in \mathcal{P}$;
   (b) if $\mu \in \mathcal{P}$, then $\Delta(\mu) \in \mathcal{P}$;
   (c) if $\mu \in \mathcal{P}$ and $\mu \in \sigma$, then $\sigma \in \mathcal{P}$;
   (d) there exists a function $\mathcal{X}$ mapping the set of closed sections from $\mathcal{P}$ into $\{-1, +1\}$ such that for every $\mu, \sigma_1, \sigma_2 \in \mathcal{P}$, the condition that $\mu \in \sigma_1$ and $\Delta(\mu) \in \sigma_2$ implies $\varepsilon(\mu) \cdot \varepsilon(\Delta(\mu)) = \mathcal{X}(\sigma_1) \cdot \mathcal{X}(\sigma_2)$;

(2) R is an equivalence relation on a certain set $\mathcal{B}$ (defined below) such that the following conditions are satisfied.
   (e) Notice, that for every boundary l belonging to a closed section in $\mathcal{P}$ either there exists a unique closed section $\sigma(l)$ in $\mathcal{P}$ containing l, or there exist precisely two closed section $\sigma_{\text{left}}(l) = [i, l]$, $\sigma_{\text{right}} = [l, j]$ in $\mathcal{P}$ containing l. The set of boundaries of the first type we denote by $\mathcal{B}_1$, and of the second type—by $\mathcal{B}_2$. Put

$$\mathcal{B} = \mathcal{B}_1 \cup \{l_{\text{left}}, l_{\text{right}} \mid l \in \mathcal{B}_2\}$$

   here $l_{\text{left}}, l_{\text{right}}$ are two "formal copies" of l. We will use the following agreement: for any base μ if $\alpha(\mu) \in \mathcal{B}_2$ then by $\alpha(\mu)$ we mean $\alpha(\mu)_{\text{right}}$ and, similarly, if $\beta(\mu) \in \mathcal{B}_2$ then by $\beta(\mu)$ we mean $\beta(\mu)_{\text{left}}$.
   (f) Now, we define R as follows. If $\mu \in \mathcal{P}$ then

$$\alpha(\mu) \sim_R \alpha\big(\Delta(\mu)\big), \quad \beta(\mu) \sim_R \beta\big(\Delta(\mu)\big) \quad \text{if } \varepsilon(\mu) = \varepsilon\big(\Delta(\mu)\big),$$
$$\alpha(\mu) \sim_R \beta\big(\Delta(\mu)\big), \quad \beta(\mu) \sim_R \alpha\big(\Delta(\mu)\big) \quad \text{if } \varepsilon(\mu) = -\varepsilon\big(\Delta(\mu)\big).$$

Remark 7. This definition coincides with the definition of a periodic structure given in [13] in the case of empty set of parameters $P\Sigma$. For a given Ω one can effectively find all periodic structures on Ω.

Let $\langle \mathcal{P}, R\rangle$ be a periodic structure of Ω. Put

$$N\mathcal{P} = \big\{\mu \in B\Omega \;\big|\; \exists h_i \in \mathcal{P} \text{ such that } h_i \in \mu \text{ and } \Delta(\mu) \text{ is parametric or constant}\big\}.$$

Now we will show how one can associate with a P-periodic solution H of Ω a periodic structure $\mathcal{P}(H, P) = \langle \mathcal{P}, R\rangle$. We define $\mathcal{P}$ as follows. A closed section σ is in $\mathcal{P}$ if and only if σ is P-periodic. A variable h_i is in $\mathcal{P}$ if and only if $h_i \in \sigma$ for some $\sigma \in \mathcal{P}$ and $|H_i| \geqslant 2|P|$. A base μ is in $\mathcal{P}$ if and only if both μ and $\Delta(\mu)$ are in $V\Sigma$ and one of them contains h_i from $\mathcal{P}$.

Put $\mathcal{X}([i, j]) = \pm 1$ depending on whether in (22) the word A is conjugate to P or to P^{-1}.

Now let $[i, j] \in \mathcal{P}$ and $i \leqslant l \leqslant j$. Then there exists a subdivision $P = P_1P_2$ such that if $\mathcal{X}([i, j]) = 1$, then the word $H[i, l]$ is the end of the word $(P^\infty)P_1$, where P^∞ is the infinite word obtained by concatenations of powers of P, and $H[l, j]$ is the beginning of the word $P_2(P^\infty)$, and if $\mathcal{X}([i, j]) = -1$, then the word $H[i, l]$ is the end of the word $(P^{-1})^\infty P_2^{-1}$ and $H[l, j]$ is the beginning of $P_1^{-1}(P^{-1})^\infty$. Lemma 1.2.9 of [1] implies that the subdivision $P = P_1P_2$ with the indicated properties is unique; denote it by $\delta(l)$. Let us define a relation R in the following way: $R(l_1, l_2) \rightleftharpoons \delta(l_1) = \delta(l_2)$.

Lemma 20. *Let H be a periodic solution of Ω. Then $\mathcal{P}(H, P)$ is a periodic structure on Ω.*

Proof. Let $\mathcal{P}(H, P) = \langle \mathcal{P}, R\rangle$. Obviously, $\mathcal{P}$ satisfies (a) and (b) from Definition 15.

Let $\mu \in \mathcal{P}$ and $\mu \in [i, j]$. There exists an unknown $h_k \in \mathcal{P}$ such that $h_k \in \mu$ or $h_k \in \Delta(\mu)$. If $h_k \in \mu$, then, obviously, $[i, j] \in \mathcal{P}$. If $h_k \in \Delta(\mu)$ and $\Delta(\mu) \in [i', j']$, then $[i', j'] \in \mathcal{P}$, and hence, the word $H[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]$ can be written in the form $Q^{r'}Q_1$, where $Q = Q_1Q_2$; Q is a cyclic shift of the word $P^{\pm 1}$ and $r' \geqslant 2$. Now let (22) be a presentation for the section $[i, j]$. Then $H[\alpha(\mu), \beta(\mu)] = B^s B_1$, where B is a cyclic shift of the word $A^{\pm 1}$, $|B| \leqslant |P|$, $B = B_1B_2$, and $s \geqslant 0$. From the equality $H[\alpha(\mu), \beta(\mu)]^{\varepsilon(\mu)} = H[\alpha(\Delta(\mu)), \beta(\Delta(\mu)))]^{\varepsilon(\Delta(\mu))}$ and [1, Lemma 1.2.9] it follows that B is a cyclic shift of the word $Q^{\pm 1}$. Consequently, A is a cyclic shift of the word $P^{\pm 1}$, and $r \geqslant 2$ in (22), since $|H[i, j]| \geqslant |H[\alpha(\mu), \beta(\mu)]| \geqslant 2|P|$. Therefore, $[i, j] \in \mathcal{P}$; i.e, part (c) of Definition 15 holds.

If $\mu \in [i_1, j_1]$, $\Delta(\mu) \in [i_2, j_2]$, and $\mu \in \mathcal{P}$, then the equality $\varepsilon(\mu) \cdot \varepsilon(\Delta(\mu)) = \mathcal{X}([i_1, j_1]) \cdot \mathcal{X}([i_2, j_2])$ follows from the fact that given $A^r A_1 = B^s B_1$ and $r, s \geqslant 2$, the word A cannot be a cyclic shift of the word B^{-1}. Hence part (d) also holds.

Condition (e) of the definition of a periodic structure obviously holds.

Condition (f) follows from the graphic equality $H[\alpha(\mu), \beta(\mu)]^{\varepsilon(\mu)} = H[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]^{\varepsilon(\Delta(\mu))}$ and [1, Lemma 1.2.9].

This proves the lemma. □

Now let us fix a non-empty periodic structure $\langle \mathcal{P}, R\rangle$. Item (d) allows us to assume (after replacing the variables $h_i, \ldots, h_{j-1}$ by $h_{j-1}^{-1}, \ldots, h_i^{-1}$ on those sections $[i, j] \in \mathcal{P}$ for which $\mathcal{X}([i, j]) = -1$) that $\varepsilon(\mu) = 1$ for all $\mu \in \mathcal{P}$. For a boundary k, we will denote by (k) the equivalence class of the relation R to which it belongs.

Let us construct an oriented graph Γ whose set of vertices is the set of R-equivalence classes. For each unknown h_k lying on a certain closed section from $\mathcal{P}$, we introduce an oriented edge e leading from (k) to $(k + 1)$ and an inverse edge e^{-1} leading from $(k + 1)$ to (k). This edge e is assigned the label $h(e) \rightleftharpoons h_k$ (respectively, $h(e^{-1}) \rightleftharpoons h_k^{-1}$). For every path $r = e_1^{\pm 1} \ldots e_s^{\pm 1}$ in the graph Γ, we denote by $h(r)$ its label $h(e_1^{\pm 1}) \ldots h(e_j^{\pm 1})$. The periodic structure $\langle \mathcal{P}, R\rangle$ is called *connected*, if the graph Γ is connected. Suppose first that $\langle \mathcal{P}, R\rangle$ is connected. Suppose that some boundary k (between h_{k-1} and h_k) in the variable part of Ω is not a boundary between two bases. Since h_{k-1} and h_k appear in all the basic equations together, and there is no boundary equations, one can consider a generalized equation Ω_1 obtained from Ω by replacing the product $h_{k-1}h_k$ in all basic equations by one variable h_k'. The group $F_{R(\Omega)}$ splits as a free product of the cyclic group generated by h_{k-1} and $F_{R(\Omega_1)}$. In this case we can consider Ω_1 instead of Ω. Therefore we suppose now that each boundary of Ω is a boundary between two bases.

Lemma 21. *Let H be a P-periodic solution of a generalized equation Ω, $\langle \mathcal{P}, R\rangle = \mathcal{P}(H, P)$; c be a cycle in the graph Γ at the vertex (l); $\delta(l) = P_1P_2$. Then there exists $n \in \mathbf{Z}$ such that $H(c) = (P_2P_1)^n$.*

Proof. If e is an edge in the graph Γ with initial vertex V' and terminal vertex V'', and $P = P_1'P_2'$, $P = P_1''P_2''$ are two subdivisions corresponding to the boundaries from V', V'' respectively, then, obviously, $H(e) = P_2'P^{n_k}P_1''$ $(n_k \in \mathbf{Z})$. The claim is easily proven by multiplying together the values $H(E)$ for all the edges e taking part in the cycle c. □

Definition 16. A generalized equation Ω is called *periodized* with respect to a given periodic structure $\langle \mathcal{P}, R\rangle$ of Ω, if for every two cycles c_1 and c_2 with the same initial vertex in the graph Γ, there is a relation $[h(c_1), h(c_2)] = 1$ in $F_{R(\Omega)}$.

5.4.2. Case 1. Set $N\mathcal{P}$ is empty

Let Γ_0 be the subgraph of the graph Γ having the same set of vertices and consisting of the edges e whose labels do not belong to $\mathcal{P}$. Choose a maximal subforest T_0 in the graph Γ_0 and extend it to a maximal subforest T of the graph Γ. Since $\langle \mathcal{P}, R\rangle$ is connected by assumption, it follows that T is a tree. Let v_0 be an arbitrary vertex of the graph Γ and $r(v_0, v)$ the (unique) path from v_0 to v all of whose vertices belong to T. For every edge $e : v \to v'$ not lying in T, we introduce a cycle $c_e = r(v_0, v)e(r(v_0, v'))^{-1}$. Then the fundamental group $\pi_1(\Gamma, v_0)$ is generated by the cycles c_e (see, for example, the proof of Proposition 3.2.1 in [18]). This and decidability of the universal theory of a free group imply that the property of a generalized equation “to be periodized with respect to a given periodic structure” is algorithmically decidable.

Furthermore, the set of elements

$$\bigl\{h(e) \bigm| e \in T\bigr\} \cup \bigl\{h(c_e) \bigm| e \notin T\bigr\} \tag{23}$$

forms a basis of the free group with the set of generators $\{h_k \mid h_k$ is an unknown lying on a closed section from $\mathcal{P}\}$. If $\mu \in \mathcal{P}$, then $(\beta(\mu)) = (\beta(\Delta(\mu)))$, $(\alpha(\mu)) = (\alpha(\Delta(\mu)))$ by part (f) from Definition 15 and, consequently, the word $h[\alpha(\mu), \beta(\mu)]h[\alpha(\Delta(\mu)), \beta(\Delta(\mu))]^{-1}$ is the label of a cycle $c'(\mu)$ from $\pi_1(\Gamma, (\alpha(\mu)))$. Let

$$c(\mu) = r\bigl(v_0, \bigl(\alpha(\mu)\bigr)\bigr)c'(\mu)r\bigl(v_0, \bigl(\alpha(\mu)\bigr)\bigr)^{-1}.$$

Then

$$h\bigl(c(\mu)\bigr) = uh\bigl[\alpha(\mu), \beta(\mu)\bigr]h\bigl[\alpha\bigl(\Delta(\mu)\bigr), \beta\bigl(\Delta(\mu)\bigr)\bigr]^{-1}u^{-1}, \tag{24}$$

where u is a certain word. Since $c(\mu) \in \pi_1(\Gamma, v_0)$, it follows that $c(\mu) = b_\mu(\{c_e \mid e \notin T\})$, where b_μ is a certain word in the indicated generators which can be effectively constructed (see [18, Proposition 3.2.1]).

Let $\tilde{b}_\mu$ denote the image of the word b_μ in the abelianization of $\pi(\Gamma, v_0)$. Denote by $\tilde{Z}$ the free abelian group consisting of formal linear combinations $\sum_{e\notin T} n_e\tilde{c}_e$ $(n_e \in \mathbf{Z})$, and by $\tilde{B}$ its subgroup generated by the elements $\tilde{b}_\mu$ $(\mu \in \mathcal{P})$ and the elements $\tilde{c}_e$ $(e \notin T$, $h(e) \notin \mathcal{P})$. Let $\tilde{A} = \tilde{Z}/\tilde{B}$, $T(\tilde{A})$ the torsion subgroups of the group $\tilde{A}$, and $\tilde{Z}_1$ the preimage of $T(\tilde{A})$ in $\tilde{Z}$. The group $\tilde{Z}/\tilde{Z}_1$ is free; therefore, there exists a decomposition of the form

$$\tilde{Z} = \tilde{Z}_1 \oplus \tilde{Z}_2, \qquad \tilde{B} \subseteq \tilde{Z}_1, \qquad \bigl(\tilde{Z}_1 : \tilde{B}\bigr) < \infty. \tag{25}$$

Note that it is possible to express effectively a certain basis $\tilde{\tilde{c}}^{(1)}$, $\tilde{\tilde{c}}^{(2)}$ of the group $\tilde{Z}$ in terms of the generators $\tilde{c}_e$ so that for the subgroups $\tilde{Z}_1$, $\tilde{Z}_2$ generated by the sets $\tilde{\tilde{c}}^{(1)}$, $\tilde{\tilde{c}}^{(2)}$ respectively, relation (25) holds. For this it suffices, for instance, to look through the bases one by one, using the fact that under the condition $\tilde{Z} = \tilde{Z}_1 \oplus \tilde{Z}_2$ the relations $\tilde{B} \subseteq \tilde{Z}_1$,

$(\tilde{Z}_1 : \tilde{B}) < \infty$ hold if and only if the generators of the groups $\tilde{B}$ and $\tilde{Z}_1$ generate the same linear subspace over $\mathbf{Q}$, and the latter is easily verified algorithmically. Notice, that a more economical algorithm can be constructed by analyzing the proof of the classification theorem for finitely generated abelian groups. By [18, Proposition 1.4.4], one can effectively construct a basis $\bar{c}^{(1)}$, $\bar{c}^{(2)}$ of the free (non-abelian) group $\pi_1(\Gamma, v_0)$ so that $\tilde{\bar{c}}^{(1)}$, $\tilde{\bar{c}}^{(2)}$ are the natural images of the elements $\bar{c}^{(1)}$, $\bar{c}^{(2)}$ in $\tilde{Z}$.

Now assume that $\langle \mathcal{P}, R\rangle$ is an arbitrary periodic structure of a periodized generalized equation Ω, not necessarily connected. Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of the graph Γ. The labels of edges of the component Γ_i form in the equation Ω a union of closed sections from $\mathcal{P}$; moreover, if a base $\mu \in \mathcal{P}$ belongs to such a section, then its dual $\Delta(\mu)$, by condition (f) of Definition 15, also possesses this property. Therefore, by taking for $\mathcal{P}_i$ the set of labels of edges from Γ_i belonging to $\mathcal{P}$, sections to which these labels belong, and bases $\mu \in \mathcal{P}$ belonging to these sections, and restricting in the corresponding way the relation R, we obtain a periodic connected structure $\langle \mathcal{P}_i, R_i\rangle$ with the graph Γ_i.

The notation $\langle \mathcal{P}', R'\rangle \subseteq \langle \mathcal{P}, R\rangle$ means that $\mathcal{P}' \subseteq \mathcal{P}$, and the relation R' is a restriction of the relation R. In particular, $\langle \mathcal{P}_i, R_i\rangle \subseteq \langle \mathcal{P}, R\rangle$ in the situation described in the previous paragraph. Since Ω is periodized, the periodic structure must be connected.

Let $e_1, \ldots, e_m$ be all the edges of the graph Γ from $T \setminus T_0$. Since T_0 is the spanning forest of the graph Γ_0, it follows that $h(e_1), \ldots, h(e_m) \in \mathcal{P}$. Let $F(\Omega)$ be a free group generated by the variables of Ω. Consider in the group $F(\Omega)$ a new basis $A \cup \bar{x}$ consisting of A, variables not belonging to the closed sections from $\mathcal{P}$ (we denote by $\bar{t}$ the family of these variables), variables $\{h(e) \mid e \in T\}$ and words $h(\bar{c}^{(1)}), h(\bar{c}^{(2)})$. Let v_i be the initial vertex of the edge e_i. We introduce new variables $\bar{u}^{(i)} = \{u_{ie} \mid e \notin T, e \notin \mathcal{P}\}$, $\bar{z}^{(i)} = \{z_{ie} \mid e \notin T, e \notin \mathcal{P}\}$ for $1 \leqslant i \leqslant m$, as follows

$$u_{ie} = h(r(v_0, v_i))^{-1} h(c_e) h\big(r(v_0, v_i)\big), \tag{26}$$

$$h(e_i)^{-1} u_{ie} h(e_i) = z_{ie}. \tag{27}$$

Notice, that without loss of generality we can assume that v_0 corresponds to the beginning of the period P.

Lemma 22. *Let Ω be a consistent generalized equation periodized with respect to a periodic structure $\langle \mathcal{P}, R\rangle$ with empty set $N\mathcal{P}$. Then the following is true.*

(1) *One can choose the basis $\bar{c}^{(1)}$ so that for any solution H of Ω periodic with respect to a period P and $\mathcal{P}(H, P) = \langle \mathcal{P}, R\rangle$ and any $c \in \bar{c}^{(1)}$, $H(c) = P^n$, where $|n| < 2\rho$.*
(2) *In a fully residually free quotient of $F_{R(\Omega)}$ discriminated by solutions from* (1) *the image of $\langle h(\bar{c}^{(1)})\rangle$ is either trivial or a cyclic subgroup.*
(3) *Let K be the subgroup of $F_{R(\Omega)}$ generated by $\bar{t}$, $h(e)$, $e \in T_0$, $h(\bar{c}^{(1)})$, $\bar{u}^{(i)}$ and $\bar{z}^{(i)}$, $i = 1, \ldots, m$. If $|\bar{c}^{(2)}| = s \geqslant 1$, then the group $F_{R(\Omega)}$ splits as a fundamental group of a graph of groups with two vertices, where one vertex group is K and the other is a free abelian group generated by $h(\bar{c}^{(2)})$ and $h(\bar{c}^{(1)})$. The corresponding edge group is generated by $h(\bar{c}^{(1)})$. The other edges are loops at the vertex with vertex group K, have*

stable letters $h(e_i)$, $i = 1, \ldots, m$, and associated subgroups $\langle \bar{u}^i \rangle$, $\langle \bar{z}^i \rangle$. If $\bar{c}^{(2)} = \emptyset$, then there is no vertex with abelian vertex group.

(4) *Let $A \cup \bar{x}$ be the generators of the group $F_{R(\Omega)}$ constructed above. If $e_i \in \mathcal{P} \cap T$, then the mapping defined as $h(e_i) \to u_{ie}^k h(e_i)$ (k is any integer) on the generator $h(e_i)$ and fixing all the other generators can be extended to an automorphism of $F_{R(\Omega)}$.*

(5) *If $c \in \bar{c}^{(2)}$ and c' is a cycle with initial vertex v_0, then the mapping defined by $h(c) \to h(c')^k h(c)$ and fixing all the other generators can be extended to an automorphism of $F_{R(\Omega)}$.*

Proof. To prove assertion (1) we have to show that each simple cycle in the graph Γ_0 has length less than 2ρ. This is obvious, because the total number of edges in Γ_0 is not more than ρ and corresponding variables do not belong to $\mathcal{P}$.

(2) The image of the group $\langle h(\bar{c}^{(1)}) \rangle$ in F is cyclic, therefore one of the finite number of equalities $h(c_1)^n = h(c_2)^m$, where $c_1, c_2 \in c^{(1)}$, $n, m < 2\rho$, must hold for any solution. Therefore in a fully residually free quotient the group generated by the image of $\langle h(\bar{c}^{(1)}) \rangle$ is a cyclic subgroup.

To prove (3) we are to study in more detail how the unknowns $h(e_i)$ ($1 \leqslant i \leqslant m$) can participate in the equations from Ω^* rewritten in the set of variables $\bar{x} \cup A$.

If h_k does not lie on a closed section from $\mathcal{P}$, or $h_k \notin \mathcal{P}$, but $e \in T$ (where $h(e) = h_k$), then h_k belongs to the basis $\bar{x} \cup A$ and is distinct from each of $h(e_1), \ldots, h(e_m)$. Now let $h(e) = h_k$, $h_k \notin \mathcal{P}$ and $e \notin T$. Then $e = r_1 c_e r_2$, where r_1, r_2 are paths in T. Since $e \in \Gamma_0$, $h(c_e)$ belongs to $\langle c^{(1)} \rangle$ modulo commutation of cycles. The vertices (k) and $(k+1)$ lie in the same connected component of the graph Γ_0, and hence they are connected by a path s in the forest T_0. Furthermore, r_1 and sr_2^{-1} are paths in the tree T connecting the vertices (k) and v_0; consequently, $r_1 = sr_2^{-1}$. Thus, $e = sr_2^{-1} c_e r_2$ and $h_k = h(s)h(r_2)^{-1}h(c_e)h(r_2)$. The unknown $h(e_i)$ ($1 \leqslant i \leqslant m$) can occur in the right-hand side of the expression obtained (written in the basis $\bar{x} \cup A$) only in $h(r_2)$ and at most once. Moreover, the sign of this occurrence (if it exists) depends only on the orientation of the edge e_i with respect to the root v_0 of the tree T. If $r_2 = r_2' e_i^{\pm 1} r_2''$, then all the occurrences of the unknown $h(e_i)$ in the words h_k written in the basis $\bar{x} \cup A$, with $h_k \notin \mathcal{P}$, are contained in the occurrences of words of the form $h(e_i)^{\mp 1} h((r_2')^{-1} c_e r_2') h(e_i)^{\pm 1}$, i.e., in occurrences of the form $h(e_i)^{\mp 1} h(c) h(e_i)^{\pm 1}$, where c is a certain cycle of the graph Γ starting at the initial vertex of the edge $e_i^{\pm 1}$.

Therefore all the occurrences of $h(e_i)$, $i = 1, \ldots, m$, in the equations corresponding to $\mu \notin \mathcal{P}$ are of the form $h(e_i^{-1})h(c)h(e_i)$. Also, $h(e_i)$ does not occur in the equations corresponding to $\mu \in \mathcal{P}$ in the basis $A \cup \bar{x}$. The system Ω^* is equivalent to the following system in the variables $\bar{x}$, $\bar{z}^{(i)}$, $\bar{u}^{(i)}$, A, $i = 1, \ldots, m$: Eqs. (26), (27),

$$[u_{ie_1}, u_{ie_2}] = 1, \tag{28}$$

$$\bigl[h(c_1), h(c_2)\bigr] = 1, \quad c_1, c_2 \in c^{(1)}, c^{(2)}, \tag{29}$$

and a system $\bar{\psi}(h(e), e \in T \setminus \mathcal{P}, h(\bar{c}^{(1)}), \bar{t}, \bar{z}^{(i)}, \bar{u}^{(i)}, A) = 1$, such that either $h(e_i)$ or $\bar{c}^{(2)}$ do not occur in $\bar{\psi}$. Let $K = F_{R(\bar{\psi})}$. Then to obtain $F_{R(\Omega)}$ we fist take an HNN extension of the group K with abelian associated subgroups generated by $\bar{u}^{(i)}$ and $\bar{z}^{(i)}$ and stable letters

$h(e_i)$, and then extend the centralizer of the image of $\langle \bar{c}^{(1)} \rangle$ by the free abelian subgroup generated by the images of $\bar{c}^{(2)}$.

Statements (4) and (5) follow from (3). □

We now introduce the notion of a *canonical group of automorphisms corresponding to a connected periodic structure*.

Definition 17. In the case when the family of bases $N\mathcal{P}$ is empty automorphisms described in Lemma 22 for $e_1, \ldots, e_m \in T \setminus T_0$ and all c_e for $e \in \mathcal{P} \setminus T$ generate the *canonical group of automorphisms* P_0 corresponding to a connected periodic structure.

Lemma 23. *Let Ω be a non-degenerate generalized equation with no boundary connections, periodized with respect to the periodic structure $\langle \mathcal{P}, R \rangle$. Suppose that the set $N\mathcal{P}$ is empty. Let H be a solution of Ω periodic with respect to a period P and $\mathcal{P}(H, P) = \langle \mathcal{P}, R \rangle$. Combining canonical automorphisms of $F_{R(\Omega)}$ one can get a solution H^+ of Ω with the property that for any $h_k \in \mathcal{P}$ such that $H_k = P_2 P^{n_k} P_1$ (P_2 and P_1 are an end and a beginning of P), $H_k^+ = P_2 P^{n_k^+} P_1$, where $n_k, n_k^+ > 0$ and the numbers n_k^+'s are bounded by a certain computable function $f_2(\Omega, \mathcal{P}, R)$. For all $h_k \notin \mathcal{P}$, $H_k = H_k^+$.*

Proof. Let $\delta((k)) = P_1^{(k)} P_2^{(k)}$. Denote by $t(\mu, h_k)$ the number of occurrences of the edge with label h_k in the cycle c_μ, calculated taking into account the orientation. Let

$$H_k = P_2^{(k)} P^{n_k} P_1^{(k+1)} \tag{30}$$

(h_k lies on a closed section from $\mathcal{P}$), where the equality in (30) is graphic whenever $h_k \in \mathcal{P}$. Direct calculations show that

$$H(b_\mu) = P^{\sum_k t(\mu, h_k)(n_k+1)}. \tag{31}$$

This equation implies that the vector $\{n_k\}$ is a solution to the following system of Diophantine equations in variables $\{z_k \mid h_k \in \mathcal{P}\}$:

$$\sum_{h_k \in \mathcal{P}} t(\mu, h_k) z_k + \sum_{h_k \notin \mathcal{P}} t(\mu, h_k) n_k = 0, \tag{32}$$

$\mu \in \mathcal{P}$. Note that the number of unknowns is bounded, and coefficients of this system are bounded from above ($|n_k| \leqslant 2$ for $h_k \notin \mathcal{P}$) by a certain computable function of $\Omega, \mathcal{P}$, and R. Obviously, $(P_2^{(k)})^{-1} H_k^+ H_k^{-1} P_2^{(k)} = P^{n_k^+ - n_k}$ commutes with $H(c)$, where c is a cycle such that $H(c) = P^{n_0}$, $n_0 < 2\rho$.

If system (32) has only one solution, then it is bounded. Suppose it has infinitely many solutions. Then $(z_1, \ldots, z_k, \ldots)$ is a composition of a bounded solution of (32) and a linear combination of independent solutions of the corresponding homogeneous system. Applying canonical automorphisms from Lemma 22 we can decrease the coefficients of this linear combination to obtain a bounded solution H^+. Hence for $h_k = h(e_i)$, $e_i \in \mathcal{P}$, the value H_k can be obtained by a composition of a canonical automorphism (Lemma 22) and a suitable bounded solution H^+ of Ω. □

5.4.3. Case 2. Set $N\mathcal{P}$ is non-empty

We construct an oriented graph $B\Gamma$ with the same set of vertices as Γ. For each item $h_k \notin \mathcal{P}$ such that h_k lie on a certain closed section from $\mathcal{P}$ introduce an edge e leading from (k) to $(k+1)$ and e^{-1} leading from $(k+1)$ to (k). For each pair of bases μ, $\Delta(\mu) \in \mathcal{P}$ introduce an edge e leading from $(\alpha(\mu)) = (\alpha(\Delta(\mu)))$ to $(\beta(\mu)) = (\beta(\delta(\mu)))$ and e^{-1} leading from $(\beta(\mu))$ to $(\alpha(\mu))$. For each base $\mu \in N\mathcal{P}$ introduce an edge e leading from $(\alpha(\mu)$ to $(\beta(\mu))$ and e^{-1} leading from $(\beta(\mu))$ to $(\alpha(\mu))$. Denote by $B\Gamma_0$ the subgraph with the same set of vertices and edges corresponding to items not from $\mathcal{P}$ and bases from $\mu \in N\mathcal{P}$. Choose a maximal subforest BT_0 in the graph $B\Gamma_0$ and extend it to a maximal subforest BT of the graph $B\Gamma$. Since $\mathcal{P}$ is connected, BT is a tree. The proof of the following lemma is similar to the proof of Lemma 21.

Lemma 24. *Let H be a solution of a generalized equation Ω periodic with respect to a period P, $\langle \mathcal{P}, R\rangle = \mathcal{P}(H, P)$; c be a cycle in the graph $B\Gamma$ at the vertex (l); $\delta(l) = P_1P_2$. Then there exists $n \in \mathbf{Z}$ such that $H(c) = (P_2P_1)^n$.*

As we did in the graph Γ, we choose a vertex v_0. Let $r(v_0, v)$ be the unique path in BT from v_0 to v. For every edge $e = e(\mu): v \to v'$ not lying in BT, introduce a cycle $c_\mu = r(v_0, v)e(\mu)r(v_0, v')^{-1}$. For every edge $e = e(h_k): V \to V'$ not lying in BT, introduce a cycle $c_{h_k} = r(v_0, v)e(h_k)r(v_0, v')^{-1}$.

It suffices to restrict ourselves to the case of a connected periodic structure. If $e = e(h_k)$, we denote $h(e) = h_k$; if $e = e(\mu)$, then $h(e) = \mu$. Let $e_1, \dots, e_m$ be all the edges of the graph $B\Gamma$ from $BT \setminus BT_0$. Since BT_0 is the spanning forest of the graph $B\Gamma_0$, it follows that $h(e_1), \dots, h(e_m) \in \mathcal{P}$. Consider in the free group $F(\Omega)$ a new basis $A \cup \bar{x}$ consisting of A, items h_k such that h_k does not belong to closed sections from $\mathcal{P}$ (denote this set by $\bar{t}$), variables $\{h(e) \mid e \in T\}$ and words from $h(C^{(1)})$, $h(C^{(2)})$, where the set $C^{(1)}$, $C^{(2)}$ form a basis of the free group $\pi(B\Gamma, v_0)$, $C^{(1)}$ correspond to the cycles that represent the identity in $F_{R(\Omega)}$ (if v and v' are initial and terminal vertices of some closed section in $\mathcal{P}$ and r and r_1 are different paths from v to v', then $r(v_0, v)rr_1^{-1}r(v_0, v)^{-1}$ represents the identity), cycles $c_\mu, \mu \in N\mathcal{P}$ and c_{h_k}, $h_k \notin \mathcal{P}$; and $C^{(2)}$ contains the rest of the basis of $\pi(B\Gamma, v_0)$.

We study in more detail how the unknowns $h(e_i)$ $(1 \leqslant i \leqslant m)$ can participate in the equations from Ω^* rewritten in this basis.

If h_k does not lie on a closed section from $\mathcal{P}$, or $h_k = h(e)$, $h(\mu) = h(e) \notin \mathcal{P}$, but $e \in T$, then $h(\mu)$ or h_k belongs to the basis $\bar{x} \cup A$ and is distinct from each of $h(e_1), \dots, h(e_m)$. Now let $h(e) = h(\mu)$, $h(\mu) \notin \mathcal{P}$ and $e \notin T$. Then $e = r_1c_er_2$, where r_1, r_2 are path in BT from $(\alpha(\mu))$ to v_0 and from $(\beta(\mu))$ to v_0. Since $e \in B\Gamma_0$, the vertices $(\alpha(\mu))$ and $(\beta(\mu))$ lie in the same connected component of the graph $B\Gamma_0$, and hence are connected by a path s in the forest BT_0. Furthermore, r_1 and sr_2^{-1} are paths in the tree BT connecting the vertices $(\alpha(\mu))$ and v_0; consequently, $r_1 = sr_2^{-1}$. Thus, $e = sr_2^{-1}c_er_2$ and $h(\mu) = h(s)h(r_2)^{-1}h(c_e)h(r_2)$. The unknown $h(e_i)$ $(1 \leqslant i \leqslant m)$ can occur in the right-hand side of the expression obtained (written in the basis $\bar{x} \cup A$) only in $h(r_2)$ and at most once. Moreover, the sign of this occurrence (if it exists) depends only on the orientation of the edge e_i with respect to the root v_0 of the tree T. If $r_2 = r_2' e_i^{\pm 1} r_2''$, then all the occurrences of the unknown $h(e_i)$ in the words $h(\mu)$ written in the basis $\bar{x} \cup A$, with $h(\mu) \notin \mathcal{P}$, are contained in the occurrences of words of the form $h(e_i)^{\mp 1} h((r_2')^{-1} c_e r_2') h(e_i)^{\pm 1}$, i.e., in occurrences of the form $h(e_i)^{\mp 1} h(c) h(e_i)^{\pm 1}$, where c is a certain cycle of the graph $B\Gamma$ starting at the initial vertex of the edge $e_i^{\pm 1}$. Similarly, all the occurrences of the unknown $h(e_i)$ in the words h_k written in the basis $\bar{x}$, A, with $h_k \notin \mathcal{P}$, are contained in occurrences of words of the form $h(e_i)^{\mp 1} h(c) h(e_i)^{\pm 1}$.

Therefore all the occurrences of $h(e_i)$, $i = 1, \dots, m$, in the equations corresponding to $\mu \notin \mathcal{P}$ are of the form $h(e_i^{-1}) h(c) h(e_i)$. Also, cycles from $C^{(1)}$ that represent the identity and not in $B\Gamma_0$ are basis elements themselves. This implies

Lemma 25.

(1) *Let K be the subgroup of $F_{R(\Omega)}$ generated by $\bar{t}$, $h(e)$, $e \in BT_0$, $h(C^{(1)})$ and $\bar{u}^{(i)}$, $\bar{z}^{(i)}$, $i = 1, \dots, m$, where elements $\bar{z}^{(i)}$ are defined similarly to the case of empty $N\mathcal{P}$.*
If $|C^{(2)}| = s \geqslant 1$, then the group $F_{R(\Omega)}$ splits as a fundamental group of a graph of groups with two vertices, where one vertex group is K and the other is a free abelian group generated by $h(C^{(2)})$ and $h(C^{(1)})$. The edge group is generated by $h(C^{(1)})$. The other edges are loops at the vertex with vertex group K and have stable letters $h(e)$, $e \in BT \setminus BT_0$. If $C^{(2)} = \emptyset$, then there is no vertex with abelian vertex group.

(2) *Let H be a solution of Ω periodic with respect to a period P and $\langle \mathcal{P}, R \rangle = \mathcal{P}(H, P)$. Let $P_1 P_2$ be a partition of P corresponding to the initial vertex of e_i. A transformation $H(e_i) \to P_2 P_1 H(e_i)$, $i \in \{1, \dots, m\}$, which is identical on all the other elements from A, $H(\bar{x})$, can be extended to another solution of Ω^*. If c is a cycle beginning at the initial vertex of e_i, then the transformation $h(e_i) \to h(c) h(e_i)$ which is identical on all other elements from $A \cup \bar{x}$, is an automorphism of $F_{R(\Omega)}$.*

(3) *If $c(e) \in C^{(2)}$, then the transformation $H(c(e)) \to P H(c(e))$ which is identical on all other elements from A, $H(\bar{x})$, can be extended to another solution of Ω^*. A transformation $h(c(e)) \to h(c) h(c(e))$ which is identical on all other elements from $A \cup \bar{x}$, is an automorphism of $F_{R(\Omega)}$.*

Definition 18. If Ω is a non-degenerate generalized equation periodic with respect to a connected periodic structure $\langle \mathcal{P}, R \rangle$ and the set $N\mathcal{P}$ is non-empty, we consider the group $\bar{A}(\Omega)$ of transformations of solutions of Ω^*, where $\bar{A}(\Omega)$ is generated by the transformations defined in Lemma 25. If these transformations are automorphisms, the group will be denoted $A(\Omega)$.

Definition 19. In the case when for a connected periodic structure $\langle \mathcal{P}, R \rangle$, the set $C^{(2)}$ has more than one element or $C^{(2)}$ has one element, and $C^{(1)}$ contains a cycle formed by edges e such that variables $h_k = h(e)$ are not from $\mathcal{P}$, the periodic structure will be called *singular*.

This definition coincides with the definition of singular periodic structure given in [13]) in the case of empty set Λ.

Lemma 25 implies the following

Lemma 26. *Let Ω be a non-degenerate generalized equation with no boundary connections, periodized with respect to a singular periodic structure $\langle \mathcal{P}, R\rangle$. Let H be a solution of Ω periodic with respect to a period P and $\langle \mathcal{P}, R\rangle = \mathcal{P}(H, P)$. Combining canonical automorphisms from $A(\Omega)$ one can get a solution H^+ of Ω^* with the following properties*:

(1) *for any $h_k \in \mathcal{P}$ such that $H_k = P_2 P^{n_k} P_1$ (P_2 and P_1 are an end and a beginning of P) $H_k^+ = P_2 P^{n_k^+} P_1$, where $n_k, n_k^+ \in \mathbb{Z}$;*
(2) *for any $h_k \notin \mathcal{P}$, $H_k = H_k^+$;*
(3) *for any base $\mu \notin \mathcal{P}$, $H(\mu) = H^+(\mu)$;*
(4) *there exists a cycle c such that $h(c) \neq 1$ in $F_{R(\Omega)}$ but $H^+(c) = 1$.*

Notice, that in the case described in the lemma, solution H^+ satisfies a proper equation. Solution H^+ is not necessarily a solution of the generalized equation Ω, but we will modify Ω into a generalized equation $\Omega(\mathcal{P}, BT)$. This modification will be called the *first minimal replacement*. Equation $\Omega(\mathcal{P}, BT)$ will have the following properties:

(1) $\Omega(\mathcal{P}, BT)$ contains all the same parameter sections and closed sections which are not in $\mathcal{P}$, as Ω;
(2) H^+ is a solution of $\Omega(\mathcal{P}, BT)$;
(3) group $F_{R(\Omega(\mathcal{P},BT))}$ is generated by the same set of variables $h_1, \ldots, h_\delta$;
(4) $\Omega(\mathcal{P}, BT)$ has the same set of bases as Ω and possibly some new bases, but each new base is a product of bases from Ω;
(5) the mapping $h_i \to h_i$ is a proper homomorphism from $F_{R(\Omega)}$ onto $F_{R(\Omega(\mathcal{P},BT))}$.

To obtain $\Omega(\mathcal{P}, BT)$ we have to modify the closed sections from $\mathcal{P}$.

The label of each cycle in $B\Gamma$ is a product of some bases $\mu_1 \ldots \mu_k$. Write a generalized equation $\tilde{\Omega}$ for the equations that say that $\mu_1 \ldots \mu_k = 1$ for each cycle from $C^{(1)}$ representing the trivial element and for each cycle from $C^{(2)}$. Each μ_i is a product $\mu_i = h_{i1} \ldots h_{it}$. Due to the first statement of Lemma 26, in each product $H_{ij}^+ H_{i,j+1}^+$ either there is no cancellations between H_{ij}^+ and $H_{i,j+1}^+$, or one of them is completely cancelled in the other. Therefore the same can be said about each pair $H^+(\mu_i)H^+(\mu_{i+1})$, and we can make a cancellation table without cutting items or bases of Ω.

Let $\hat{\Omega}$ be a generalized equation obtained from Ω by deleting bases from $\mathcal{P} \cup N\mathcal{P}$ and items from $\mathcal{P}$ from the closed sections from $\mathcal{P}$. Take a union of $\tilde{\Omega}$ and $\hat{\Omega}$ on the disjoint set of variables, and add basic equations identifying in $\hat{\Omega}$ and $\tilde{\Omega}$ the same bases that don't belong to $\mathcal{P}$. This gives us $\Omega(\mathcal{P}, BT)$.

Suppose that $C^{(2)}$ for the equation Ω is either empty or contains one cycle. Suppose also that for each closed section from $\mathcal{P}$ in Ω there exists a base μ such that the initial boundary of this section is $\alpha(\mu)$ and the terminal boundary is $\beta(\Delta(\mu))$.

Lemma 27. *Suppose that the generalized equation Ω is periodized with respect to a non-singular periodic structure $\mathcal{P}$. Then for any periodic solution H of Ω we can choose a tree BT, some set of variables $S = \{h_{j_1}, \ldots, h_{j_s}\}$ and a solution H^+ of Ω equivalent to H with respect to the group of canonical transformations $\bar{A}(\Omega)$ in such a way that each of the bases $\lambda_i \in BT \setminus BT_0$ can be represented as $\lambda_i = \lambda_{i1} h_{k_i} \lambda_{i2}$, where $h_{k_i} \in S$ and for* *any $h_j \in S$, $|H_j^+| < f_3|P|$, where f_3 is some constructible function depending on Ω. This representation gives a new generalized equation Ω' periodic with respect to a periodic structure $\mathcal{P}'$ with the same period P and all $h_j \in S$ considered as variables not from $\mathcal{P}'$. The graph $B\Gamma'$ for the periodic structure $\mathcal{P}'$ has the same set of vertices as $B\Gamma$, has empty set $C^{(2)}$ and $BT' = BT'_0$.*

Let c be a cycle from $C^{(1)}$ of minimal length, then $H(c) = P^{n_c}$, where $|n_c| \leqslant 2\rho$. Using canonical automorphisms from $A(\Omega)$ one can transform any solution H of Ω into a solution H^+ such that for any $h_j \in S$, $|H_j^+| \leqslant f_3|c|$. Let $\mathcal{P}'$ be a periodic structure, in which all $h_i \in S$ are considered as variables not from $\mathcal{P}'$, then $B\Gamma'$ has empty set $C^{(2)}$ and $BT' = BT'_0$.

Proof. Suppose first that $C^{(2)}$ is empty. We prove the statement of the lemma by induction on the number of edges in $BT \setminus BT_0$. It is true, when this set is empty. Consider temporarily all the edges in $BT \setminus BT_0$ except one edge $e(\lambda)$ as edges corresponding to bases from $N\mathcal{P}$. Then the difference between BT_0 and BT is one edge.

Changing $H(e(\lambda))$ by a transformation from $\bar{A}(\Omega)$ we can change only $H(e')$ for $e' \in B\Gamma$ that could be included into $BT \setminus BT_0$ instead of e. For each base $\mu \in N\mathcal{P}$, $H(\mu) = P_2(\mu)P^{n(\mu)}P_1(\mu)$, for each base $\mu \in \mathcal{P}$, $H(\mu) = P_2(\mu)P^{x(\mu)}P_1(\mu)$. For each cycle c in $C^{(1)}$ such that $h(c)$ represents the identity element we have a linear equation in variables $x(\mu)$ with coefficients depending on $n(\mu)$. We also know that this system has a solution for arbitrary $x(\lambda)$ (where $\lambda \in BT \setminus BT_0$) and the other $x(\nu)$ are uniquely determined by the value of $x(\lambda)$.

If we write for each variable $h_k \in \mathcal{P}$, $H_k = P_{2k}P^{y_k}P_{1k}$, then the positive unknowns y_k's satisfy the system of equations saying that $H(\mu) = H(\Delta(\mu))$ for bases $\mu \in \mathcal{P}$ and equations saying that μ is a constant for bases $\mu \in N\mathcal{P}$. Fixing $x(\lambda)$ we automatically fix all the y_k's. Therefore at least one of the y_k belonging to λ can be taken arbitrary. So there exist some elements y_k which can be taken as free variables for the second system of linear equations. Using elementary transformations over $\mathbb{Z}$ we can write the system of equations for y_k's in the form:

$$\begin{array}{cccc} n_1y_1 & 0 & \cdots & = m_1y_k + C_1, \\ & n_2y_2 & \cdots & = m_2y_k + C_2, \\ & & \ddots & \\ \vdots & \vdots & & \ddots \\ & \cdots & n_{k-1}y_{k-1} & = m_{k-1}y_k + C_{k-1}, \end{array} \tag{33}$$

where $C_1, \ldots, C_k$ are constants depending on parameters, we can suppose that they are sufficiently large positive or negative (small constants we can treat as constants not depending on parameters). Notice that integers $n_1, m_1, \ldots, n_{k-1}, m_{k-1}$ in this system do not depend on parameters. We can always suppose that all $n_1, \ldots, n_{k-1}$ are positive. Notice that m_i and C_i cannot be simultaneously negative, because in this case it would not be a positive solution of the system. Changing the order of the equations we can write first all equation with m_i, C_i positive, then equations with negative m_i and positive C_i and, finally, equations with negative C_i and positive m_i. The system will have the form:

$$\begin{array}{llll} n_1y_1 & 0 & \cdots & = |m_1|y_k + |C_1|, \\ & & \ddots & \\ & n_ty_t & \cdots & = -|m_t|y_k + |C_t|, \\ & & \ddots & \\ & & & \ddots \\ & & n_sy_s & = |m_s|y_k - |C_s|. \end{array} \tag{34}$$

If the last block (with negative C_s) is non-empty, we can take a minimal y_s of bounded value. Indeed, instead of y_s we can always take a remainder of the division of y_s by the product $n_1 \dots n_{k-1}|m_1 \dots m_{k-1}|$, which is less than this product (or by the product $n_1 \dots n_{k-1}|m_1 \dots m_{k-1}|n_c$ if we wish to decrease y_s by a multiple of n_c). We respectively decrease y_k and adjust y_i's in the blocks with positive C_i's. If the third block is not present, we decrease y_k taking a remainder of the division of y_k by $n_1 \dots n_{k-1}$ (or by $n_1 \dots n_{k-1}n_c$) and adjust y_i's. Therefore for some h_i belonging to a base which can be included into $BT \setminus BT_0$, $|H^+(h_i)| < f_3|P|$. Suppose this base is λ, represent $\lambda = \lambda_1 h_i \lambda_2$. Suppose $e(\lambda): v \to v_1$ in $B\Gamma$. Let v_2, v_3 be the vertices in $B\Gamma$ corresponding to the initial and terminal boundary of h_k. They would be the vertices in Γ, and Γ and $B\Gamma$ have the same set of vertices. To obtain the graph $B\Gamma'$ from $B\Gamma$ we have to replace $e(\lambda)$ by three edges $e(\lambda_1): v \to v_2$, $e(h_k): v_2 \to v_3$ and $e(\lambda_2): v_3 \to v_1$. There is no path in BT_0 from v_2 to v_3, because if there were such a path p, then we would have the equality $h_k = h(c_1)h(p)h(c_2)$, in $F_{R(\Omega)}$, where c_1 and c_2 are cycles in $B\Gamma$ beginning in vertices v_2 and v_3 respectively. Changing H_k we do not change $H(c_1)$, $H(c_2)$ and $H(p)$, because all the cycles are generated by cycles in $C^{(1)}$. Therefore there are paths $r: v \to v_2$ and $r_1: v_3 \to v_1$ in BT_0, and edges $e(\lambda_1), e(\lambda_2)$ cannot be included in $BT' \setminus BT'_0$ in $B\Gamma'$. Therefore $BT' = BT'_0$. Now we can recall that all the edges except one in $BT \setminus BT_0$ were temporarily considered as edges in $N\mathcal{P}$. We managed to decrease the number of such edges by one. Induction finishes the proof.

If the set $C^{(2)}$ contains one cycle, we can temporarily consider all the bases from BT as parameters, and consider the same system of linear equations for y_i's. Similarly, as above, at least one y_t can be bounded. We will bound as many y_i's as we can. For the new periodic structure either BT contains less elements or the set $C^{(2)}$ is empty.

The second part of the lemma follows from the remark that for $\mu \in T$ left multiplication of $h(\mu)$ by $h(rcr^{-1})$, where r is the path in T from v_0 to the initial vertex of μ, is an automorphism from $A(\Omega)$. □

We call a solution H^+ constructed in Lemma 27 a *solution equivalent to H with maximal number of short variables*.

Consider now variables from S as variables not from $\mathcal{P}'$, so that for the equation Ω the sets $C^{(2)}$ and $BT' \setminus BT'_0$ are both empty. In this case we make the *second minimal replacement*, which we will describe in the lemma below.

Definition 20. A pair of bases $\mu, \Delta(\mu)$ is called an overlapping pair if $\epsilon(\mu) = 1$ and $\beta(\mu) > \alpha(\Delta(\mu)) > \alpha(\mu)$ or $\epsilon(\mu) = -1$ and $\beta(\mu) < \beta(\Delta(\mu)) < \alpha(\mu)$. If a closed section begins with $\alpha(\mu)$ and ends with $\beta(\Delta(\mu))$ for an overlapping pair of bases we call such a pair of bases a *principal overlapping pair* and say that a section is in *overlapping form*.

Notice, that if $\lambda \in N\mathcal{P}$, then $H(\lambda)$ is the same for any solution H, and we just write λ instead of $H(\lambda)$.

Lemma 28. *Suppose that for the generalized equation Ω' obtained in Lemma* 27 *the sets $C^{(2)}$ and $BT' \setminus BT'_0$ are empty, $\mathcal{P}'$ is a non-empty periodic structure, and each closed section from $\mathcal{P}'$ has a principal overlapping pair. Then for each base $\mu \in \mathcal{P}'$ there is a fixed presentation for $h(\mu) = \prod(parameters)$ as a product of elements $h(\lambda)$, $\lambda \in N\mathcal{P}$, $h_k \notin \mathcal{P}'$ corresponding to a path in $B\Gamma'_0$. The maximal number of terms in this presentation is bounded by a computable function of Ω.*

Proof. Let e be the edge in the graph $B\Gamma'$ corresponding to a base μ and suppose $e : v \to v'$. There is a path s in BT' joining v and v', and a cycle $\bar{c}$ which is a product of cycles from $C^{(1)}$ such that $h(\mu) = h(\bar{c})h(s)$. For each cycle c from $C^{(1)}$ either $h(c) = 1$ or c can be written using only edges with labels not from $\mathcal{P}'$; therefore, $\bar{c}$ contains only edges with labels not from $\mathcal{P}'$. Therefore

$$h(\mu) = \prod(\text{parameters}) = h(\lambda_{i_1})\Pi_1 \ldots h(\lambda_{s_i})\Pi_s, \tag{35}$$

where the doubles of all λ_i are parameters, and $\Pi_1, \ldots, \Pi_s$ are products of variables $h_{k_i} \notin \mathcal{P}'$. □

In the equality

$$H(\mu) = H(\lambda_{i_1})\bar{\Pi}_1 \ldots H(\lambda_{s_i})\bar{\Pi}_s, \tag{36}$$

where $\bar{\Pi}_1, \ldots, \bar{\Pi}_s$ are products of H_{k_i} for variables $h_{k_i} \notin \mathcal{P}'$, the cancellations between two terms in the right side are complete because the equality corresponds to a path in $B\Gamma'_0$. Therefore the cancellation tree for the equality (36) can be situated on a horizontal axis with intervals corresponding to λ_i's directed either to the right or to the left. This tree can be drawn on a P-scaled axis. We call this one-dimensional tree a μ*-tree*. Denote by $I(\lambda)$ the interval corresponding to λ in the μ-tree. If $I(\mu) \subseteq \bigcup_{\lambda_i \in N\mathcal{P}} I(\lambda_i)$, then we say that μ is covered by parameters. In this case a generalized equation corresponding to (36) can be situated on the intervals corresponding to bases from $N\mathcal{P}$.

We can shift the whole μ-tree to the left or to the right so that in the new situation the uncovered part becomes covered by the bases from $N\mathcal{P}$. Certainly, we have to make sure that the shift is through the interval corresponding to a cycle in $C^{(1)}$. Equivalently, we can shift any base belonging to the μ-tree through such an interval.

If c is a cycle from $C^{(1)}$ with shortest $H(c)$, then there is a corresponding c-tree. Shifting this c-tree to the right or to the left through the intervals corresponding to

$H(c)$ bounded number of times we can cover every H_i, where $h_i \in S$ by a product $H(\lambda_{j_1})\bar{\Pi}_1 \ldots H(\lambda_{j_t})\bar{\Pi}_t$, where $\bar{\Pi}_1, \ldots, \bar{\Pi}_t$ are products of values of variables not from $\mathcal{P}$ and $\lambda_{j_1}, \ldots \lambda_{j_t}$ are bases from $N\mathcal{P}$. Combining this covering together with the covering of $H(\mu)$ by the product (36), we obtain that $H([\alpha(\mu), \beta(\Delta(\mu))])$ is almost covered by parameters, except for the short products $\bar{\Pi}$. Let $h(\mu)$ be covered by

$$h(\Lambda_1)\Pi_1, \ldots, h(\Lambda_s)\Pi_s, \tag{37}$$

where $h(\Lambda_1), \ldots, h(\Lambda_s)$ are parts completely covered by parameters, and $\Pi_1, \ldots, \Pi_s$ are products of variables not in $\mathcal{P}$. We also remove those bases from $N\mathcal{P}$ from each Λ_i which do not overlap with $h(\mu)$. Denote by f_4 the maximal number of bases in $N\mathcal{P}$ and $h_i \notin \mathcal{P}$ in the covering (37).

If $\lambda_{i_1}, \ldots, \lambda_{i_s}$ are parametric bases, then for any solution H and any pair $\lambda_i, \lambda_j \in \{\lambda_{i_1}, \ldots, \lambda_{i_s}\}$ we have either $|H(\lambda_i)| < |H(\lambda_j)|$ or $|H(\lambda_i)| = |H(\lambda_j)|$ or $|H(\lambda_i)| > |H(\lambda_j)|$. We call a *relationship between lengths of parametric bases* a collection that consists of one such inequality or equality for each pair of bases. There is only a finite number of possible relationships between lengths of parametric bases. Therefore we can talk about a parametric base λ of maximal length meaning that we consider the family of solutions for which $H(\lambda)$ has maximal length.

Lemma 29. *Let $\lambda_\mu \in N\mathcal{P}$ be a base of max length in the covering* (37) *for $\mu \in \mathcal{P}$. If for a solution H of Ω, and for each closed section $[\alpha(\mu), \beta(\Delta(\mu)]$ in $\mathcal{P}$, $\min|H[\alpha(\nu), \alpha(\Delta(\nu))]| \leqslant |H(\lambda_\mu)|$, where the minimum is taken for all pairs of overlapping bases for this section, then one can transform Ω into one of the finite number* (*depending on Ω*) *of generalized equations $\Omega(\mathcal{P})$ which do not contain closed sections from $\mathcal{P}$ but contain the same other closed sections except for parametric sections. The content of closed sections from $\mathcal{P}$ is transferred using bases from $N\mathcal{P}$ to the parametric part. This transformation is called the* second minimal replacement.

Proof. Suppose for a closed section $[\alpha(\mu), \beta(\Delta(\mu))]$ that there exists a base λ in (37) such that $|H(\lambda)| \geqslant \min(H(\alpha(\nu), \alpha(\Delta(\nu)))$, where the minimum is taken for all pairs of overlapping bases for this section. We can shift the cover $H(\Lambda_1)\bar{\Pi}_1, \ldots, H(\Lambda_s)\bar{\Pi}_s$ through the distance $d_1 = |H[\alpha(\mu), \alpha(\Delta(\mu))]|$. Consider first the case when $d_1 \leqslant |H(\lambda)|$ for the largest base in (37). Suppose the part of $H(\mu)$ corresponding to $\bar{\Pi}_i$ is not covered by parameters. Take the first base λ_j in (37) to the right or to the left of $\bar{\Pi}_i$ such that $|H(\lambda_j)| \geqslant d_1$. Suppose λ_j is situated to the left from $\bar{\Pi}_i$. Shifting λ_j to the right through a bounded by f_4 multiple of d_1 we will cover $\bar{\Pi}_i$.

Consider now the case when $d_1 > |H(\lambda)|$, but there exists an overlapping pair $\nu, \Delta(\nu)$ such that

$$d_2 = \left|H\left[\alpha(\nu), \alpha\left(\Delta(\nu)\right)\right]\right| \leqslant \left|H(\lambda)\right|.$$

If the part of $H(\mu)$ corresponding to $\bar{\Pi}_i$ is not covered by parameters, we take the first base λ_j in (37) to the right or to the left of $\bar{\Pi}_i$ such that $|H(\lambda_j)| \geqslant d_2$. Without loss of generality we can suppose that λ_j is situated to the left of $\bar{\Pi}_i$. Shifting λ_j to the right through a bounded by f_4 multiple of d_2 we will cover $\bar{\Pi}_i$.

Therefore, if the first alternative in the lemma does not take place, we can cover the whole section $[\alpha(\mu), \beta(\Delta(\mu))]$ by the bases from $N\mathcal{P}$, and transform Ω into one of the finite number of generalized equations which do not contain the closed section $[\alpha(\mu), \beta(\Delta(\mu))]$ and have all the other non-parametric sections the same. All the cancellations between two neighboring terms of any equality that we have gotten are complete, therefore the coordinate groups of new equations are quotients of $F_{R(\Omega)}$. □

5.5. *Minimal solutions and tree $T_0(\Omega)$*

5.5.1. *Minimal solutions*

Let $F = F(A \cup B)$ be a free group with basis $A \cup B$, Ω be a generalized equation with constants from $(A \cup B)^{\pm 1}$, and parameters Λ. Let $A(\Omega)$ be an arbitrary group of $(A \cup \Lambda)$-automorphisms of $F_{R(\Omega)}$. For solutions $H^{(1)}$ and $H^{(2)}$ of the equation Ω in the group F we write $H^{(1)} <_{A(\Omega)} H^{(2)}$ if there exists an endomorphism π of the group F which is an (A, Λ)-homomorphism, and an automorphism $\sigma \in A(\Omega)$ such that the following conditions hold:

(1) $\pi_{H^{(2)}} = \sigma \pi_{H^{(1)}} \pi$,
(2) for all active variables $d(H_k^{(1)}) \leqslant d(H_k^{(2)})$ for all $1 \leqslant k \leqslant \rho$ and $d(H_k^{(1)}) < d(H_k^{(2)})$ at least for one such k (here $d(H)$ is an alternative notation for the length $|H|$).

We also define a relation $<_{cA(\Omega)}$ by the same way as $<_{A(\Omega)}$ but with extra property:

(3) for any k, j, if $(H_k^{(2)})^{\epsilon}(H_j^{(2)})^{\delta}$ in non-cancellable, then $(H_k^{(1)})^{\epsilon}(H_j^{(1)})^{\delta}$ in non-cancellable ($\epsilon, \delta = \pm 1$).

Obviously, both relations are transitive.

A solution $\bar{H}$ of Ω is called $A(\Omega)$-*minimal* if there is no any solution $\bar{H}'$ of the equation Ω such that $\bar{H}' <_{A(\Omega)} \bar{H}$. Since the total length $\sum_{i=1}^{\rho} l(H_i)$ of a solution $\bar{H}$ is a non-negative integer, every strictly decreasing chain of solutions $\bar{H} > \bar{H}^1 > \cdots > \bar{H}^k >_{A(\Omega)} \cdots$ is finite. It follows that for every solution $\bar{H}$ of Ω there exists a minimal solution $\bar{H}^0$ such that $\bar{H}^0 <_{A(\Omega)} \bar{H}$.

5.5.2. *Automorphisms*

Assign to some vertices v of the tree $T(\Omega)$ the groups of automorphisms of groups $F_{R(\Omega_v)}$. For each vertex v such that $\mathrm{tp}(v) = 12$ the canonical group of automorphisms $A(\Omega_v)$ assigned to it is the group of automorphisms of $F_{R(\Omega_v)}$ identical on Λ. For each vertex v such that $7 \leqslant \mathrm{tp}(v) \leqslant 10$ we assign the group of automorphisms invariant with respect to the kernel.

For each vertex v such that $\mathrm{tp}(v) = 2$, assign the group $\bar{A}_v$ generated by the groups of automorphisms constructed in Lemma 25 that applied to Ω_v and all possible non-singular periodic structures of this equation.

Let $\mathrm{tp}(v) = 15$. Apply transformation D_3 and consider $\Omega = \tilde{\Omega}_v$. Notice that the function γ_i is constant when h_i belongs to some closed section of $\tilde{\Omega}_v$. Applying D_2, we can suppose that the section $[1, j+1]$ is covered exactly twice. We say now that this is a quadratic section. Assign to the vertex v the group of automorphisms of $F_{R(\Omega)}$ acting identically on the non-quadratic part.

5.5.3. The finite subtree $T_0(\Omega)$: cutting off long branches

For a generalized equation Ω with parameters we construct a finite tree $T_0(\Omega)$. Then we show that the subtree of $T(\Omega)$ obtained by tracing those path in $T(\Omega)$ which actually can happen for "short" solutions is a subtree of $T_0(\Omega)$.

According to Lemma 19, along an infinite path in $T(\Omega)$ one can either have $7 \leqslant \mathrm{tp}(v_k) \leqslant 10$ for all k or $\mathrm{tp}(v_k) = 12$ for all k, or $\mathrm{tp}(v_k) = 15$ for all k.

Lemma 30 (Lemma 15 from [13]). *Let $v_1 \to v_2 \to \cdots \to v_k \to \cdots$ be an infinite path in the tree $T(\Omega)$, and $7 \leqslant \mathrm{tp}(v_k) \leqslant 10$ for all k. Then among $\{\Omega_k\}$ some generalized equation occurs infinitely many times. If $\Omega_{v_k} = \Omega_{v_l}$, then $\pi(v_k, v_l)$ is an isomorphism invariant with respect to the kernel.*

Lemma 31. *Let $\mathrm{tp}(v) = 12$. If a solution $\bar{H}$ of Ω_v is minimal with respect to the canonical group of automorphisms, then there is a recursive function f_0 such that in the sequence*

$$(\Omega_v, \bar{H}) \to (\Omega_{v_1}, \bar{H}^1) \to \cdots \to (\Omega_{v_i}, \bar{H}^i) \to \cdots \tag{38}$$

corresponding to the path in $T(\Omega_v)$ and for the solution $\bar{H}$, Case 12 cannot be repeated more than f_0 times.

Proof. If μ and $\Delta\mu$ both belong to the quadratic section, then μ is called a *quadratic base*. Consider the following set of generators for $F_{R(\Omega_v)}$: variables from Λ and quadratic bases from the active part. Relations in this set of generators consist of the following three families.

(1) Relations between variables in Λ.
(2) If μ is an active base and $\Delta(\mu)$ is a parametric base, and $\Delta(\mu) = h_i \dots h_{i+t}$, then there is a relation $\mu = h_i \dots h_{i+t}$.
(3) Since $\gamma_i = 2$ for each h_i in the active part the product of $h_i \dots h_j$, where $[i, j+1]$ is a closed active section, can be written in two different ways w_1 and w_2 as a product of active bases. We write the relations $w_1 w_2^{-1} = 1$. These relations give a quadratic system of equations with coefficients in the subgroup generated by Λ.

When we apply the entire transformation in Case 12, the number of variables is not increasing and the complexity of the generalized equation is not increasing. Suppose the same generalized equation is repeated twice in the sequence (38), for example, $\Omega_j = \Omega_{j+k}$. Then $\pi(v_j, v_{j+k})$ is an automorphism of $F_{R(\Omega_j)}$ induced by the automorphism of the free product $\langle\Lambda\rangle * B$, where B is a free group generated by quadratic bases, identical on $\langle\Lambda\rangle$ and fixing all words $w_1 w_2^{-1}$. Therefore, $\bar{H}^j > \bar{H}^{j+k}$, which contradicts to the minimality of $\bar{H}$. Therefore there is only a finite number (bounded by f_0) of possible generalized equations that can appear in the sequence (38). $\square$

Let $\bar{H}$ be a solution of the equation Ω with quadratic part $[1, j+1]$. If μ belongs and $\Delta\mu$ does not belong to the quadratic section, then μ is called a *quadratic-coefficient base*. Define the following numbers:

$$d_1(\bar{H}) = \sum_{i=1}^{j} d(H_i), \tag{39}$$

$$d_2(\bar{H}) = \sum_{\mu} d\big(H\big[\alpha(\mu), \beta(\mu)\big]\big), \tag{40}$$

where μ is a quadratic-coefficient base.

Lemma 32. *Let* $\mathrm{tp}(v) = 15$. *For any solution* $\bar{H}$ *of* Ω_v *there is a minimal solution* $\bar{H}^+$, *which is an automorphic image of* $\bar{H}$ *with respect to the group of automorphisms defined in the beginning of this section, such that*

$$d_1(\bar{H}^+) \leqslant f_1(\Omega_v) \max\big\{d_2(\bar{H}^+), 1\big\},$$

where $f_1(\Omega)$ *is some recursive function.*

Proof. Consider instead of Ω_v equation $\Omega = (\tilde{\Omega}_v)$ which does not have any boundary connections, $F_{R(\Omega_v)}$ is isomorphic to $F_{R(\Omega)}$. Consider a presentation of $F_{R(\Omega_v)}$ in the set of generators consisting of variables in the non-quadratic part and active bases. Relations in this generating set consist of the following three families.

(1) Relations between variables in the non-quadratic part.
(2) If μ is a quadratic-coefficient base and $\Delta(\mu) = h_i \dots h_{i+t}$ in the non-quadratic part, then there is a relation $\mu = h_i \dots h_{i+t}$.
(3) Since $\gamma_i = 2$ for each h_i in the active part the product $h_i \dots h_j$, where $[i, j+1]$ is a closed active section, can be written in two different ways w_1 and w_2 as a product of quadratic and quadratic-coefficient bases. We write the relations $w_1 w_2^{-1} = 1$.

Let $\bar{H}$ be a solution of Ω_v minimal with respect to the canonical group of automorphisms of the free product $B_1 * B$, where B is a free group generated by quadratic bases, and B_1 is a subgroup of $F_{R(\Omega_v)}$ generated by variables in the non-quadratic part, identical on $\langle \Lambda \rangle$ and fixing all words $w_1 w_2^{-1}$.

Consider the sequence

$$(\Omega, \bar{H}) \to (\Omega_{v_1}, \bar{H}^1) \to \dots \to (\Omega_{v_i}, \bar{H}^i) \to \cdots. \tag{41}$$

Apply now the entire transformations to the quadratic section of Ω. As in the proof of the previous lemma, each time we apply the entire transformation, we do not increase complexity and do not increase the total number of items in the whole interval. Since $\bar{H}$ is a solution of Ω_v, if the same generalized equation appear in this sequence $2^{4^{j^2}}+1$ times then for some $j, j+k$ we have $\bar{H}^j >_c \bar{H}^{j+k}$, therefore the same equation can only appear a bounded number of times. Every quadratic base (except those that become matching bases of length 1) in the quadratic part can be transferred to the non-quadratic part with the use of some quadratic-coefficient base as a carrier base. This means that the length of the transferred base is equal to the length of the part of the quadratic-coefficient carrier base, which will then be deleted. The double of the transferred base becomes a quadratic-coefficient base. Because there are not more than n_A bases in the active part, this would give

$$d_1(\bar{H}') \leqslant n_A d_2(\bar{H}'),$$

for some solution $\bar{H}^+$ of the equation $\tilde{\Omega}_v$. But $\bar{H}^+$ is obtained from the minimal solution $\bar{H}$ in a bounded number of steps. □

We call a path $v_1 \to v_2 \to \cdots \to v_k \to \cdots$ in $T(\Omega)$ for which $7 \leqslant \mathrm{tp}(v_k) \leqslant 10$ for all k or type 12 *prohibited* if some generalized equation with ρ variables occurs among $\{\Omega_{v_i} \mid 1 \leqslant i \leqslant \ell\}$ at least $2^{4\rho^2}+1$ times. We will define below also prohibited paths in $T(\Omega)$, for which $\mathrm{tp}(v_k) = 15$ for all k. We will need some auxiliary definitions.

Introduce a new parameter

$$\tau_v' = \tau_v + \rho - \rho_v',$$

where ρ is the number of variables of the initial equation Ω and ρ_v' the number of free variables belonging to the non-active sections of the equation Ω_v. We have $\rho_v' \leqslant \rho$ (see the proof of Lemma 19), hence $\tau_v' \geqslant 0$. In addition if $v_1 \to v_2$ is an auxiliary edge, then $\tau_2' < \tau_1'$.

Define by the joint induction on τ_v' a finite subtree $T_0(\Omega_v)$ and a natural number $s(\Omega_v)$. The tree $T_0(\Omega_v)$ will have v as a root and consist of some vertices and edges of $T(\Omega)$ that lie higher than v. Let $\tau_v' = 0$; then in $T(\Omega)$ there can not be auxiliary edges and vertices of type 15 higher than v. Hence a subtree $T_0(\Omega_v)$ consisting of vertices v_1 of $T(\Omega)$ that are higher than v, and for which the path from v to v_1 does not contain prohibited subpaths, is finite.

Let now

$$s(\Omega_v) = \max_w \max_{\langle \mathcal{P}, R\rangle} \bigl\{\rho_w f_2(\Omega_w, \mathcal{P}, R),\ f_4\bigl(\Omega_w', \mathcal{P}, R\bigr)\bigr\}, \tag{42}$$

where w runs through all the vertices of $T_0(v)$ for which $\mathrm{tp}(w) = 2$, Ω_w contains non-trivial non-parametric sections, $\langle \mathcal{P}, R\rangle$ is the set of non-singular periodic structures of the equation $\tilde{\Omega}_w$, f_2 is a function appearing in Lemma 23 (f_2 is present only if a periodic structure has empty set $N\mathcal{P}$) and Ω_w' is constructed as in Lemma 27, where f_4 is a function appearing in covering (37).

Suppose now that $\tau'_v > 0$ and that for all v_1 with $\tau'_{v_1} < \tau'_v$ the tree $T_0(\Omega_{v_1})$ and the number $s(\Omega_{v_1})$ are already defined. We begin with the consideration of the paths

$$r = v_1 \to v_2 \to \cdots \to v_m, \tag{43}$$

where $\operatorname{tp}(v_i) = 15$ $(1 \leqslant i \leqslant m)$. We have

$$\tau'_{v_i} = \tau'_v.$$

Denote by μ_i the carrier base of the equation Ω_{v_i}. The path (43) will be called μ-reducing if $\mu_1 = \mu$ and either there are no auxiliary edges from the vertex v_2 and μ occurs in the sequence $\mu_1, \ldots, \mu_{m-1}$ at least twice, or there are auxiliary edges $v_2 \to w_1$, $v_2 \to w_2, \ldots, v_2 \to w_k$ from v_2 and μ occurs in the sequence $\mu_1, \ldots, \mu_{m-1}$ at least $\max_{1\leqslant i\leqslant k} s(\Omega_{w_i})$ times.

The path (43) will be called *prohibited*, if it can be represented in the form

$$r = r_1 s_1 \ldots r_l s_l r', \tag{44}$$

such that for some sequence of bases $\eta_1, \ldots, \eta_l$ the following three properties hold:

(1) every base occurring at least once in the sequence $\mu_1, \ldots, \mu_{m-1}$ occurs at least $40n^2 f_1(\Omega_{v_2}) + 20n + 1$ times in the sequence $\eta_1, \ldots, \eta_l$, where n is the number of pairs of bases in equations Ω_{v_i};
(2) the path r_i is η_i-reducing;
(3) every transfer base of some equation of path r is a transfer base of some equation of path r'.

The property of path (43) of being prohibited is algorithmically decidable. Every infinite path (43) contains a prohibited subpath. Indeed, let ω be the set of all bases occurring in the sequence $\mu_1, \ldots, \mu_m, \ldots$ infinitely many times, and $\tilde{\omega}$ the set of all bases, that are transfer bases of infinitely many equations Ω_{v_i}. If one cuts out some finite part in the beginning of this infinite path, one can suppose that all the bases in the sequence $\mu_1, \ldots, \mu_m, \ldots$ belong to ω and each base that is a transfer base of at least one equation, belongs to $\tilde{\omega}$. Such an infinite path for any $\mu \in \omega$ contains infinitely many non-intersecting μ-reducing finite subpaths. Hence it is possible to construct a subpath (44) of this path satisfying the first two conditions in the definition of a prohibited subpath. Making r' longer, one obtains a prohibited subpath.

Let $T'(\Omega_v)$ be a subtree of $T(\Omega_v)$ consisting of the vertices v_1 for which the path from v to v_1 in $T(\Omega)$ contains neither prohibited subpaths nor vertices v_2 with $\tau'_{v_2} < \tau'_v$, except perhaps v_1. So the terminal vertices of $T'(\Omega_v)$ are either vertices v_1 such that $\tau'_{v_1} < \tau'_v$, or terminal vertices of $T(\Omega_v)$. A subtree $T'(\Omega_v)$ can be effectively constructed. $T_0(\Omega_v)$ is obtained by attaching of $T_0(\Omega_{v_1})$ (already constructed by the induction hypothesis) to those terminal vertices v_1 of $T'(\Omega_v)$ for which $\tau'_{v_1} < \tau'_v$. The function $s(\Omega_v)$ is defined by (42). Let now $T_0(\Omega) = T_0(\Omega_{v_0})$. This tree is finite by construction.

5.5.4. Paths corresponding to minimal solutions of Ω are in $T_0(\Omega)$

Notice, that if $\mathrm{tp}(v) \geqslant 6$ and $v \to w_1, \dots, v \to w_m$ is the list of principal outgoing edges from v, then the generalized equations $\Omega_{w_1}, \dots, \Omega_{w_m}$ are obtained from Ω_v by the application of several elementary transformations. Denote by e a function that assigns a pair $(\Omega_{w_i}, \bar{H}^{(i)})$ to the pair $(\Omega_v, \bar{H})$. For $\mathrm{tp}(v) = 4, 5$ this function is identical.

If $\mathrm{tp}(v) = 15$ and there are auxiliary edges from the vertex v, then the carrier base μ of the equation Ω_v intersects $\Delta(\mu)$. For any solution $\bar{H}$ of the equation Ω_v one can construct a solution $\bar{H}'$ of the equation $\Omega_{v'}$ by $H'_{\rho_v+1} = H[1, \beta(\Delta(\mu))]$. Let $e'(\Omega_v, \bar{H}) = e(\Omega_{v'}, \bar{H}')$.

In the beginning of this section we assigned to vertices v of types 12, 15, 2 and such that $7 \leqslant \mathrm{tp}(v) \leqslant 10$ of $T(\Omega)$ the groups of automorphisms $A(\Omega_v)$. Denote by $\mathrm{Aut}(\Omega)$ the group of automorphisms of $F_{R(\Omega)}$, generated by all groups $\pi(v_0, v)A(\Omega_v)\pi(v_0, v)^{-1}$, $v \in T_0(\Omega)$. (Here $\pi(v_0, v)$ is an isomorphism, because $\mathrm{tp}(v) \neq 1$.) We are to formulate the main technical result of this section. The following proposition states that every minimal solution of a generalized equation Ω with respect to the group $A(\Omega)$ either factors through one of the finite family of proper quotients of the group $F_{R(\Omega)}$ or (in the case of a non-empty parametric part) can be transferred to the parametric part.

Proposition 1. *For any solution $\bar{H}$ of a generalized equation Ω there exists a terminal vertex w of the tree $T_0(\Omega)$ having type* 1 *or* 2, *and a solution $\bar{H}^{(w)}$ of a generalized equation Ω_w such that*:

(1) $\pi_{\bar{H}} = \sigma\pi(v_0, w)\pi_{\bar{H}^{(w)}}\pi$ *where π is an endomorphism of a free group F, $\sigma \in \mathrm{Aut}(\Omega)$*;
(2) *if* $\mathrm{tp}(w) = 2$ *and the equation Ω_w contains non-trivial non-parametric sections, then there exists a primitive cyclically reduced word P such that $\bar{H}^{(w)}$ is periodic with respect to $\mathcal{P}$ and one of the following conditions holds*:
   (a) *the equation Ω_w is singular with respect to a periodic structure $\mathcal{P}(\bar{H}^{(w)}, P)$ and the first minimal replacement can be applied,*
   (b) *it is possible to apply the second minimal replacement and make the family of closed sections in $\mathcal{P}$ empty.*

Construct a directed tree with paths from the initial vertex

$$\big(\Omega, \bar{H}\big) = \big(\Omega_{v_0}, \bar{H}^{(0)}\big) \to \big(\Omega_{v_1}, \bar{H}^{(1)}\big) \to \cdots \to \big(\Omega_{v_u}, \bar{H}^{(u)}\big) \to \cdots \tag{45}$$

in which the v_i are the vertices of the tree $T(\Omega)$ in the following way. Let $v_1 = v_0$ and let $\bar{H}^{(1)}$ be some solution of the equation Ω, minimal with respect to the group of automorphisms $A(\Omega v_0)$ with the property $\bar{H} \geqslant \bar{H}^{(1)}$.

Let $i \geqslant 1$ and suppose the term $(\Omega_{v_i}, \bar{H}^{(i)})$ of the sequence (45) has been already constructed. If $7 \leqslant \mathrm{tp}(v_i) \leqslant 10$ or $\mathrm{tp}(v_i) = 12$ and there exists a minimal solution $\bar{H}^+$ of Ω_{v_i} such that $\bar{H}^+ < \bar{H}^{(i)}$, then we set $v_{i+1} = v_i$, $\bar{H}^{(i+1)} = \bar{H}^+$.

If $\mathrm{tp}(v_i) = 15$, $v_i \neq v_{i-1}$ and there are auxiliary edges from vertex $v_i : v_i \to w_1, \dots,$ $v_i \to w_k$ (the carrier base μ intersects with its double $\Delta(\mu)$), then there exists a primitive word P such that

$$H^{(i)}\big[1, \beta\big(\Delta(\mu)\big)\big] \equiv P^r P_1, \quad r \geqslant 2, \qquad P \equiv P_1 P_2, \tag{46}$$

where $\equiv$ denotes a graphical equality. In this case the path (45) can be continued along several possible edges of $T(\Omega)$.

For each group of automorphisms assigned to vertices of type 2 in the trees $T_0(\Omega_{w_i})$, $i = 1, \ldots, k$, and non-singular periodic structure including the closed section $[1, \beta(\Delta(\mu)]$ of the equation Ω_{v_i} and corresponding to solution $\bar{H}^{(i)}$ we replace $\bar{H}^{(i)}$ by a solution $\bar{H}^{(i)+}$ with maximal number of short variables (see the definition after Lemma 27). This collection of short variables can be different for different periodic structures. Either all the variables in $\bar{H}^{(i)+}$ are short or there exists a parametric base $\lambda_{\max}$ of maximal length in the covering (37). Suppose there is a μ-reducing path (43) beginning at v_i and corresponding to $\bar{H}^{(i)+}$. Let $\mu_1, \ldots, \mu_m$ be the leading bases of this path. Let $\tilde{H}^1 = H^{(i)+}, \ldots, \tilde{H}^j$ be solutions of the generalized equations corresponding to the vertices of this path. If for some μ_i there is an inequality $d(\tilde{H}^j[\alpha(\mu_i), \alpha(\Delta(\mu_i))]) \leqslant d(\lambda_{\max})$, we set $(\Omega_{v_{i+1}}, \bar{H}^{(i+1)}) = e'(\Omega_{v_i}, \bar{H}^{(i)})$ and call the section $[1, \beta(\Delta(\mu))]$ which becomes non-active, *potentially transferable*.

If there is a singular periodic structure in a vertex of type 2 of some tree $T_0(\Omega_{w_i}), i \in \{1, \ldots, k\}$, including the closed section $[1, \beta(\Delta(\mu)]$ of the equation Ω_{v_i} and corresponding to the solution $\bar{H}^{(i)}$, we also include the possibility $(\Omega_{v_{i+1}}, \bar{H}^{(i+1)}) = e'(\Omega_{v_i}, \bar{H}^{(i)})$.

In all of the other cases we set $(\Omega_{v_{i+1}}, \bar{H}^{(i+1)}) = e(\Omega_{v_i}, \bar{H}^{(i)+})$, where $\bar{H}^{(i)+}$ is a solution with maximal number of short variables and minimal solution of Ω_{v_i} with respect to the canonical group of automorphisms P_{v_i} (if it exists). The path (45) ends if $\mathrm{tp}(v_i) \leqslant 2$.

We will show that in the path (45) $v_i \in T_0(\Omega)$. We use induction on τ'. Suppose $v_i \notin T_0(\Omega)$, and let i_0 be the first of such numbers. It follows from the construction of $T_0(\Omega)$ that there exists $i_1 < i_0$ such that the path from v_{i_1} into v_{i_0} contains a subpath prohibited in the construction of $T_2(\Omega_{v_{i_1}})$. From the minimality of i_0 it follows that this subpath goes from v_{i_2} ($i_1 \leqslant i_2 < i_0$) to v_{i_0}. It cannot be that $7 \leqslant \mathrm{tp}(v_i) \leqslant 10$ or $\mathrm{tp}(v_i) = 12$ for all $i_2 \leqslant i \leqslant i_1$, because there will be two indices $p < q$ between i_2 and i_0 such that $\bar{H}^{(p)} = \bar{H}^{(q)}$, and this gives a contradiction, because in this case it must be by construction $v_{p+1} = v_p$. So $\mathrm{tp}(v_i) = 15$ ($i_2 \leqslant i \leqslant i_0$).

Suppose we have a subpath (43) corresponding to the fragment

$$\left(\Omega_{v_1}, \bar{H}^{(1)}\right) \to \left(\Omega_{v_2}, \bar{H}^{(2)}\right) \to \cdots \to \left(\Omega_{v_m}, \bar{H}^{(m)}\right) \to \cdots \tag{47}$$

of the sequence (45). Here $v_1, v_2, \ldots, v_{m-1}$ are vertices of the tree $T_0(\Omega)$, and for all vertices v_i the edge $v_i \to v_{i+1}$ is principal.

As before, let μ_i denote the carrier base of Ω_{v_i}, and $\omega = \{\mu_1, \ldots, \mu_{m-1}\}$, and $\tilde{\omega}$ denote the set of such bases which are transfer bases for at least one equation in (47). By ω_1 denote the set of such bases μ for which either μ or $\Delta(\mu)$ belongs to $\omega \cup \tilde{\omega}$; by ω_2 denote the set of all the other bases. Let

$$\alpha(\omega) = \min\left(\min_{\mu \in \omega_2} \alpha(\mu), j\right),$$

where j is the boundary between active and non-active sections. Let $X_\mu \doteq H[\alpha(\mu), \beta(\mu)]$. If $(\Omega, \bar{H})$ is a member of sequence (47), then denote

$$d_\omega(\bar{H}) = \sum_{i=1}^{\alpha(\omega)-1} d(H_i), \tag{48}$$

$$\psi_\omega(\bar{H}) = \sum_{\mu\in\omega_1} d(X_\mu) - 2d_\omega(\bar{H}). \tag{49}$$

Every item h_i of the section $[1, \alpha(\omega)]$ belongs to at least two bases, and both bases are in ω_1, hence $\psi_\omega(\bar{H}) \geqslant 0$.

Consider the quadratic part of $\tilde{\Omega}_{v_1}$ which is situated to the left of $\alpha(\omega)$. The solution $\bar{H}^{(1)}$ is minimal with respect to the canonical group of automorphisms corresponding to this vertex. By Lemma 32 we have

$$d_1(\bar{H}^{(1)}) \leqslant f_1(\Omega_{v_1}) d_2(\bar{H}^{(1)}). \tag{50}$$

Using this inequality we estimate the length of the interval participating in the process $d_\omega(\bar{H}^{(1)})$ from above by a product of ψ_ω and some function depending on f_1. This will be inequality (55). Then we will show that for a prohibited subpath the length of the participating interval must be reduced by more than this figure (equalities (65), (66)). This will imply that there is no prohibited subpath in the path (47).

Denote by $\gamma_i(\omega)$ the number of bases $\mu \in \omega_1$ containing h_i. Then

$$\sum_{\mu\in\omega_1} d(X_\mu^{(1)}) = \sum_{i=1}^{\rho} d(H_i^{(1)}) \gamma_i(\omega), \tag{51}$$

where $\rho = \rho(\Omega_{v_1})$. Let $I = \{i \mid 1 \leqslant i \leqslant \alpha(\omega) - 1 \ \&\ \gamma_i = 2\}$ and $J = \{i \mid 1 \leqslant i \leqslant \alpha(\omega) - 1 \ \&\ \gamma_i > 2\}$. By (48)

$$d_\omega(\bar{H}^{(1)}) = \sum_{i\in I} d(H_i^{(1)}) + \sum_{i\in J} d(H_i^{(1)}) = d_1(\bar{H}^{(1)}) + \sum_{i\in J} d(H_i^{(1)}). \tag{52}$$

Let $(\lambda, \Delta(\lambda))$ be a pair of quadratic-coefficient bases of the equation $\tilde{\Omega}_{v_1}$, where λ belongs to the non-quadratic part. This pair can appear only from the bases $\mu \in \omega_1$. There are two types of quadratic-coefficient bases.

Type 1. λ is situated to the left of the boundary $\alpha(\omega)$. Then λ is formed by items $\{h_i \mid i \in J\}$ and hence

$$d(X_\lambda) \leqslant \sum_{i\in J} d(H_i^{(1)}).$$

Thus the sum of the lengths $d(X_\lambda) + d(X_{\Delta(\lambda)})$ for quadratic-coefficient bases of this type is not more than $2n \sum_{i\in J} d(H_i^{(1)})$.

Type 2. λ is situated to the right of the boundary $\alpha(\omega)$. The sum of length of the quadratic-coefficient bases of the second type is not more than $2 \sum_{i=\alpha(\omega)}^{\rho} d(H_i^{(1)}) \gamma_i(\omega)$.

We have

$$d_2\big(\bar{H}^{(1)}\big) \leqslant 2n \sum_{i\in J} d\big(H_i^{(1)}\big) + 2 \sum_{i=\alpha(\omega)}^{\rho} d\big(H_i^{(1)}\big)\gamma_i(\omega). \tag{53}$$

Now (49) and (51) imply

$$\psi_\omega\big(\bar{H}_i^{(1)}\big) \geqslant \sum_{i\in J} d\big(H_i^{(1)}\big) + \sum_{i=\alpha(\omega)}^{\rho} d\big(H_i^{(1)}\big)\gamma_i(\omega). \tag{54}$$

Inequalities (50), (52), (53), (54) imply

$$d_\omega\big(\bar{H}^{(1)}\big) \leqslant \max\big\{\psi_\omega\big(\bar{H}^{(1)}\big)\big(2nf_1(\Omega_{v_1}) + 1\big),\ f_1(\Omega_{v_1})\big\}. \tag{55}$$

From the definition of Case 15 it follows that all the words $H^{(i)}[1, \rho_i + 1]$ are the ends of the word $H^{(1)}[1, \rho_1 + 1]$, that is

$$H^{(1)}[1, \rho_1 + 1] \doteq U_i H^{(i)}[1, \rho_i + 1]. \tag{56}$$

On the other hand bases $\mu \in \omega_2$ participate in these transformations neither as carrier bases nor as transfer bases; hence $H^{(1)}[\alpha(\omega), \rho_1 + 1]$ is the end of the word $H^{(i)}[1, \rho_i + 1]$, that is

$$H^{(i)}[1, \rho_i + 1] \doteq V_i H^{(1)}\big[\alpha(\omega), \rho_1 + 1\big]. \tag{57}$$

So we have

$$\begin{aligned} d_\omega\big(\bar{H}^{(i)}\big) - d_\omega\big(\bar{H}^{(i+1)}\big) &= d(V_i) - d(V_{i+1}) = d(U_{i+1}) - d(U_i) \\ &= d\big(X_{\mu_i}^{(i)}\big) - d\big(X_{\mu_i}^{(i+1)}\big). \end{aligned} \tag{58}$$

In particular (49), (58) imply that $\psi_\omega(\bar{H}^{(1)}) = \psi_\omega(\bar{H}^{(2)}) = \cdots = \psi_\omega(\bar{H}^{(m)}) = \psi_\omega$. Denote the number (58) by δ_i.

Let the path (43) be μ-reducing, that is either $\mu_1 = \mu$ and v_2 does not have auxiliary edges and μ occurs in the sequence $\mu_1, \ldots, \mu_{m-1}$ at least twice, or v_2 does have auxiliary edges $v_2 \to w_1, \ldots, v_2 \to w_k$ and the base μ occurs in the sequence $\mu_1, \ldots, \mu_{m-1}$ at least $\max_{1\leqslant i\leqslant k} s(\Omega_{w_i})$ times. Estimate $d(U_m) = \sum_{i=1}^{m-1} \delta_i$ from below. First notice that if $\mu_{i_1} = \mu_{i_2} = \mu (i_1 < i_2)$ and $\mu_i \neq \mu$ for $i_1 < i < i_2$, then

$$\sum_{i=i_1}^{i_2-1} \delta_i \geqslant d\big(H^{i_1+1}\big[1, \alpha\big(\Delta(\mu_{i_1+1})\big)\big]\big). \tag{59}$$

Indeed, if $i_2 = i_1 + 1$, then $\delta_{i_1} = d(H^{(i_1)}[1, \alpha(\Delta(\mu))] = d(H^{(i_1+1)}[1, \alpha(\Delta(\mu))]$. If $i_2 > i_1 + 1$, then $\mu_{i_1+1} \neq \mu$ and μ is a transfer base in the equation $\Omega_{v_{i_1+1}}$. Hence

$$\delta_{i_1+1} + d\big(H^{(i_1+2)}\big[1, \alpha(\mu)\big]\big) = d\big(H^{(i_1+1)}\big[1, \alpha(\mu_{i_1+1})\big]\big).$$

Now (59) follows from

$$\sum_{i=i_1+2}^{i_2-1} \delta_i \geqslant d\big(H^{(i_1+2)}\big[1, \alpha(\mu)\big]\big).$$

So if v_2 does not have outgoing auxiliary edges, that is the bases μ_2 and $\Delta(\mu_2)$ do not intersect in the equation Ω_{v_2}; then (59) implies that

$$\sum_{i=1}^{m-1} \delta_i \geqslant d\big(H^{(2)}\big[1, \alpha(\Delta\mu_2)\big]\big) \geqslant d\big(X_{\mu_2}^{(2)}\big) \geqslant d\big(X_{\mu}^{(2)}\big) = d\big(X_{\mu}^{(1)}\big) - \delta_1,$$

which implies that

$$\sum_{i=1}^{m-1} \delta_i \geqslant \frac{1}{2} d\big(X_{\mu}^{(1)}\big). \tag{60}$$

Suppose now there are outgoing auxiliary edges from the vertex v_2: $v_2 \to w_1, \ldots, v_2 \to w_k$. The equation Ω_{v_1} has some solution. Let

$$H^{(2)}\big[1, \alpha\big(\Delta(\mu_2)\big)\big] \doteq Q,$$

and P a word (in the final h's) such that $Q \doteq P^d$, then $X_{\mu_2}^{(2)}$ and $X_{\mu}^{(2)}$ are beginnings of the word $H^{(2)}[1, \beta(\Delta(\mu_2))]$, which is a beginning of P^{∞}. Denote $M = \max_{1 \leqslant j \leqslant k} s(\Omega_{w_j})$.

By the construction of (45)we either have

$$X_{\mu}^{(2)} \doteq P^r P_1, \quad P \doteq P_1 P_2, \quad r < M. \tag{61}$$

or for each base μ_i, $i \geqslant 2$, there is an inequality $d(H^{(i)}(\alpha(\mu_i), \alpha(\Delta(\mu_i)))) \geqslant d(\lambda)$ and therefore

$$d\big(X_{\mu}^{(2)}\big) < M d\big(H^{(i)}\big[\alpha(\mu_i), \alpha\big(\Delta(\mu_i)\big)\big]\big). \tag{62}$$

Let $\mu_{i_1} = \mu_{i_2} = \mu$, $i_1 < i_2$, $\mu_i \neq \mu$ for $i_1 < i < i_2$. If

$$d\big(X_{\mu_{i_1+1}}^{(i_1+1)}\big) \geqslant 2d(P) \tag{63}$$

and $H^{(i_1+1)}[1, \rho_{i_1+1} + 1]$ begins with a cyclic permutation of P^3, then

$$d\big(H^{(i_1+1)}\big[1, \alpha\big(\Delta(\mu_{i_1+1})\big)\big]\big) > d\big(X_{\mu}^{(2)}\big)/M.$$

Together with (59) this gives

$$\sum_{i=i_1}^{i_2-1} \delta_i > d\bigl(X_\mu^{(2)}\bigr)/M.$$

The base μ occurs in the sequence $\mu_1, \dots, \mu_{m-1}$ at least M times, so either (63) fails for some $i_1 \leqslant m-1$ or $\sum_{i=1}^{m-1} \delta_i (M-3) d(X_\mu^{(2)})/M$.

If (63) fails, then the inequality $d(X_{\mu_i}^{(i+1)}) \leqslant d(X_{\mu_{i+1}}^{(i+1)})$, and the definition (58) imply that

$$\sum_{i=1}^{i_1} \delta_i \geqslant d\bigl(X_\mu^{(1)}\bigr) - d\bigl(X_{\mu_{i_1+1}}^{(i_1+1)}\bigr) \geqslant (M-2) d\bigl(X_\mu^{(2)}\bigr)/M;$$

so everything is reduced to the second case.

Let

$$\sum_{i=1}^{m-1} \delta_i \geqslant (M-3) d\bigl(X_\mu^{(1)}\bigr)/M.$$

Notice that (59) implies for $i_1 = 1$, $\sum_{i=1}^{m-1} \delta_i \geqslant d(Q) \geqslant d(P)$; so

$$\sum_{i=1}^{m-1} \delta_i \geqslant \max\{1, M-3\} d\bigl(X_\mu^{(2)}\bigr)/M.$$

Together with (61) this implies

$$\sum_{i=1}^{m-1} \delta_i \geqslant \frac{1}{5} d\bigl(X_\mu^{(2)}\bigr) = \frac{1}{5}\bigl(d\bigl(X_\mu^{(1)}\bigr) - \delta_1\bigr).$$

Finally,

$$\sum_{i=1}^{m-1} \delta_i \geqslant \frac{1}{10} d\bigl(X_\mu^{(1)}\bigr). \tag{64}$$

Comparing (60) and (64) we can see that for the μ-reducing path (43) inequality (64) always holds.

Suppose now that the path (43) is prohibited; hence it can be represented in the form (44). From definition (49) we have

$$\sum_{\mu \in \omega_1} d\bigl(X_\mu^{(m)}\bigr) \geqslant \psi_\omega;$$

so at least for one base $\mu \in \omega_1$ the inequality $d(X_\mu^{(m)}) \geqslant \frac{1}{2n}\psi_\omega$ holds. Because $X_\mu^{(m)} \doteq (X_{\Delta(\mu)}^{(m)})^{\pm 1}$, we can suppose that $\mu \in \omega \cup \tilde{\omega}$. Let m_1 be the length of the path $r_1 s_1 \ldots r_l s_l$ in (44). If $\mu \in \tilde{\omega}$ then by the third part of the definition of a prohibited path there exists $m_1 \leqslant i \leqslant m$ such that μ is a transfer base of Ω_{v_i}. Hence,

$$d\big(X_{\mu_i}^{(m_1)}\big) \geqslant d\big(X_{\mu_i}^{(i)}\big) \geqslant d\big(X_{\mu}^{(i)}\big) \geqslant d\big(X_{\mu}^{(m)}\big) \geqslant \frac{1}{2n}\psi_\omega.$$

If $\mu \in \omega$, then take μ instead of μ_i. We proved the existence of a base $\mu \in \omega$ such that

$$d\big(X_{\mu}^{(m_1)}\big) \geqslant \frac{1}{2n}\psi_\omega. \tag{65}$$

By the definition of a prohibited path, the inequality $d(X_\mu^{(i)}) \geqslant d(X_\mu^{(m_1)})$ $(1 \leqslant i \leqslant m_1)$, (64), and (65) we obtain

$$\sum_{i=1}^{m_1-1} \delta_i \geqslant \max\left\{\frac{1}{20n}\psi_\omega, 1\right\}\big(40n^2 f_1 + 20n + 1\big). \tag{66}$$

By (58) the sum in the left part of the inequality (66) equals $d_\omega(\bar{H}^{(1)}) - d_\omega(\bar{H}^{(m_1)})$; hence

$$d_\omega\big(\bar{H}^{(1)}\big) \geqslant \max\left\{\frac{1}{20n}\psi_\omega, 1\right\}\big(40n^2 f_1 + 20n + 1\big),$$

which contradicts (55).

This contradiction was obtained from the supposition that there are prohibited paths (47) in the path (45). Hence (45) does not contain prohibited paths. This implies that $v_i \in T_0(\Omega)$ for all v_i in (45). For all i, $v_i \to v_{i+1}$ is an edge of a finite tree. Hence the path (45) is finite. Let $(\Omega_w, \bar{H}^w)$ be the final term of this sequence. We show that $(\Omega_w, \bar{H}^w)$ satisfies all the properties formulated in the lemma.

The first property is obvious.

Let $\mathrm{tp}(w) = 2$ and let Ω_w have non-trivial non-parametric part. It follows from the construction of (45) that if $[j,k]$ is a non-active section for Ω_{v_i} then $H^{(i)}[j,k] \doteq H^{(i+1)}[j,k] \doteq \cdots \doteq H^{(w)}[j,k]$. Hence (46) and the definition of $s(\Omega_v)$ imply that the word $h_1 \ldots h_{\rho_w}$ can be subdivided into subwords $h[i_1, i_2], \ldots, h[i_{k-1}, i_k]$, such that for any a either $H^{(w)}$ has length 1, or $h[i_a, i_{a+1}]$ does not participate in basic and coefficient equations, or $H^{(w)}[i_a, i_{a+1}]$ can be written as

$$H^{(w)}[i_a, i_{a+1}] \doteq P_a^r P_a', \quad P_a \doteq P_a' P_a'', \quad r \geqslant \max_{\langle \mathcal{P}, R\rangle} \max\big\{\rho_w f_2(\Omega_w, P, R), f_4\big(\Omega_w'\big)\big\}, \tag{67}$$

where P_a is a primitive word, and $\langle \mathcal{P}, R\rangle$ runs through all the periodic structures of $\tilde{\Omega}_w$ such that either one of them is singular or for a solution with maximal number of short variables with respect to the group of extended automorphisms all the closed sections are potentially transferable. The proof of Proposition 1 will be completed after we prove the following statement.

Lemma 33. *If* $\mathrm{tp}(w) = 2$ *and every closed section belonging to a periodic structure* $\mathcal{P}$ *is potentially transferable* (*the definition is given in the construction of* T_0 *in Case* 15), *one can apply the second minimal replacement and get a finite number* (*depending on periodic structures containing this section in the vertices of type* 2 *in the trees* $T_0(w_i)$, $i = 1, \dots, m$) *of possible generalized equations containing the same closed sections not from* $\mathcal{P}$ *and not containing closed sections from* $\mathcal{P}$.

Proof. From the definition of potentially transferable section it follows that after finite number of transformations depending on $f_4(\Omega'_u, \mathcal{P})$, where u runs through the vertices of type 2 in the trees $T_0(w_i)$, $i = 1, \dots, m$, we obtain a cycle that is shorter than or equal to $d(\lambda_{\max})$. This cycle is exactly $h[\alpha(\mu_i), \alpha(\Delta(\mu_i)]$ for the base μ_i in the μ-reducing subpath. The rest of the proof of Lemma 33 is a repetition of the proof of Lemma 29. □

5.5.5. The decomposition tree $T_{\mathrm{dec}}(\Omega)$

We can define now a decomposition tree $T_{\mathrm{dec}}(\Omega)$. To obtain $T_{\mathrm{dec}}(\Omega)$ we add some edges to the terminal vertices of type 2 of $T_0(\Omega)$. Let v be a vertex of type 2 in $T_0(\Omega)$. If there is no periodic structures in Ω_v then this is a terminal vertex of $T_{\mathrm{dec}}(\Omega)$. Suppose there exists a finite number of combinations of different periodic structures $\mathcal{P}_1, \dots, \mathcal{P}_s$ in Ω_v. If some $\mathcal{P}_i$ is singular, we consider a generalized equation $\Omega_{u(\mathcal{P}_1, \dots, \mathcal{P}_s)}$ obtained from $\Omega_v(\mathcal{P}_1, \dots, \mathcal{P}_s)$ by the first minimal replacement corresponding to $\mathcal{P}_i$. We also draw the edge $v \to u = u(\mathcal{P}_1, \dots, \mathcal{P}_s)$. This vertex u is a terminal vertex of $T_{\mathrm{dec}}(\Omega)$. If all $\mathcal{P}_1, \dots, \mathcal{P}_s$ in Ω_v are not singular, we can suppose that for each periodic structure $\mathcal{P}_i$ with period P_i some values of variables in $\mathcal{P}_i$ are shorter than $2|P_i|$ and values of some other variables are shorter than $f_3(\Omega_v)|P_i|$, where f_3 is the function from Lemma 27. Then we apply the second minimal replacement. The resulting generalized equations $\Omega_{u_1}, \dots, \Omega_{u_t}$ will have empty non-parametric part. We draw the edges $v \to u_1, \dots, v \to u_t$ in $T_{\mathrm{dec}}(\Omega)$. Vertices $u_1, \dots, u_t$ are terminal vertices of $T_{\mathrm{dec}}(\Omega)$.

5.6. The solution tree $T_{\mathrm{sol}}(\Omega, \Lambda)$

Let $\Omega = \Omega(H)$ be a generalized equation in variables H with the set of bases $B_\Omega = B \cup \Lambda$. Let $T_{\mathrm{dec}}(\Omega)$ be the tree constructed in Section 5.5.5 for a generalized equation Ω with parameters Λ.

Recall that in a leaf-vertex v of $T_{\mathrm{dec}}(\Omega)$ we have the coordinate group $F_{R(\Omega_v)}$ which is a proper homomorphic image of $F_{R(\Omega)}$. We define a new transformation R_v (we call it *leaf-extension*) of the tree $T_{\mathrm{dec}}(\Omega)$ at the leaf vertex v. We take the union of two trees $T_{\mathrm{dec}}(\Omega)$ and $T_{\mathrm{dec}}(\Omega_v)$ and identify the vertices v in both trees (i.e., we extend the tree $T_{\mathrm{dec}}(\Omega)$ by gluing the tree $T_{\mathrm{dec}}(\Omega_v)$ to the vertex v). Observe that if the equation Ω_v has non-parametric non-constant sections (in this event we call v *a terminal vertex*), then $T_{\mathrm{dec}}(\Omega_v)$ consists of a single vertex, namely v.

Now we construct a solution tree $T_{\mathrm{sol}}(\Omega)$ by induction starting at $T_{\mathrm{dec}}(\Omega)$. Let v be a leaf non-terminal vertex of $T^{(0)} = T_{\mathrm{dec}}(\Omega)$. Then we apply the transformation R_v and obtain a new tree $T^{(1)} = R_v(T_{\text{dec}}(\Omega))$. If there exists a leaf non-terminal vertex v_1 of $T^{(1)}$, then we apply the transformation R_{v_1}, and so on. By induction we construct a strictly increasing sequence of trees

$$T^{(0)} \subset T^{(1)} \subset \cdots \subset T^{(i)} \subset \cdots. \tag{68}$$

This sequence is finite. Indeed, suppose to the contrary that the sequence is infinite and hence the union $T^{(\infty)}$ of this sequence is an infinite tree in which every vertex has a finite degree. By Konig's lemma there is an infinite branch B in $T^{(\infty)}$. Observe that along any infinite branch in $T^{(\infty)}$ one has to encounter infinitely many proper epimorphisms. This contradicts the fact that F is equationally Noetherian.

Denote the union of the sequence of the trees (68) by $T_{\text{sol}}(\Omega, \Lambda)$. We call $T_{\text{sol}}(\Omega, \Lambda)$ the *solution tree of Ω with parameters Λ*. Recall that with every edge e in $T_{\text{dec}}(\Omega)$ (as well as in $T_{\text{sol}}(\Omega, \Lambda)$) with the initial vertex v and the terminal vertex w we associate an epimorphism

$$\pi_e : F_{R(\Omega_v)} \to F_{R(\Omega_v)}.$$

It follows that every connected (directed) path p in the graph gives rise to a composition of homomorphisms which we denote by π_p. Since $T_{\text{sol}}(\Omega, \Lambda)$ is a tree the path p is completely defined by its initial and terminal vertices u, v; in this case we sometimes write $\pi_{u,v}$ instead of π_p. Let π_v be the homomorphism corresponding to the path from the initial vertex v_0 to a given vertex v, we call it the *canonical epimorphism* from $F_{R(\Omega)}$ onto $F_{R(\Omega_v)}$.

Also, with some vertices v in the tree $T_{\text{dec}}(\Omega)$, as well as in the tree $T_{\text{sol}}(\Omega, \Lambda)$, we associate groups of canonical automorphisms $A(\Omega_v)$ or extended automorphisms $\bar{A}(\Omega_v)$ of the coordinate group $F_{R(\Omega_v)}$ which, in particular, fix all variables in the non-active part of Ω_v. We can suppose that the group $\bar{A}(\Omega_v)$ is associated to every vertex, but for some vertices it is trivial. Observe also, that canonical epimorphisms map parametric parts into parametric parts (i.e., subgroups generated by variables in parametric parts).

Recall that writing (Ω, U) means that U is a solution of Ω. If (Ω, U) and $\mu \in B_\Omega$, then by μ_U we denote the element

$$\mu_U = [u_{\alpha(\mu)} \ldots u_{\beta(\mu)-1}]^{\varepsilon(\mu)}. \tag{69}$$

Let $B_U = \{\mu_U \mid \mu \in B\}$ and $\Lambda_U = \{\mu_U \mid \mu \in \Lambda\}$. We refer to these sets as the set of values of bases from B and the set of values of parameters from Λ with respect to the solution U. Notice, that the value μ_U is given in (69) as a value of one fixed word mapping

$$P_\mu(H) = [h_{\alpha(\mu)} \ldots h_{\beta(\mu)-1}]^{\varepsilon(\mu)}.$$

In vector notation we can write that

$$B_U = P_B(U), \qquad \Lambda_U = P_\Lambda(U),$$

where $P_B(H)$ and $P_\Lambda(H)$ are corresponding word mappings.

The following result explains the name of the tree $T_{\text{sol}}(\Omega, \Lambda)$.

Theorem 5. *Let $\Omega = \Omega(H, \Lambda)$ be a generalized equation in variables H with parameters Λ. Let $T_{\text{sol}}(\Omega, \Lambda)$ be the solution tree for Ω with parameters. Then the following conditions hold.*

(1) *For any solution U of the generalized equation Ω there exists a path $v_0, v_1, \dots, v_n = v$ in $T_{\text{sol}}(\Omega, \Lambda)$ from the root vertex v_0 to a terminal vertex v, a sequence of canonical automorphisms $\sigma = (\sigma_0, \dots, \sigma_n)$, $\sigma_i \in A(\Omega_{v_i})$, and a solution U_v of the generalized equation Ω_v such that the solution U (viewed as a homomorphism $F_{R(\Omega)} \to F$) is equal to the following composition of homomorphisms*

$$U = \Phi_{\sigma, U_v} = \sigma_0 \pi_{v_0, v_1} \sigma_1 \dots \pi_{v_{n-1}, v_n} \sigma_n U_v. \tag{70}$$

(2) *For any path $v_0, v_1, \dots, v_n = v$ in $T_{\text{sol}}(\Omega, \Lambda)$ from the root vertex v_0 to a terminal vertex v, a sequence of canonical automorphisms $\sigma = (\sigma_0, \dots, \sigma_n)$, $\sigma_i \in A(\Omega_{v_i})$, and a solution U_v of the generalized equation Ω_v, Φ_{σ, U_v} gives a solution of the group equation $\Omega^* = 1$; moreover, every solution of $\Omega^* = 1$ can be obtained this way.*
(3) *For each terminal vertex v in $T_{\text{sol}}(\Omega, \Lambda)$ there exists a word mapping $Q_v(H_v)$ such that for any solution U_v of Ω_v and any solution $U = \Phi_{\sigma, U_v}$ from (70) the values of the parameters Λ with respect to U can be written as $\Lambda_U = Q_v(U_v)$ (i.e., these values do not depend on σ) and the word $Q_v(U_v)$ is reduced as written.*

Proof. Statements (1) and (2) follow from the construction of the tree $T_{\text{sol}}(\Omega, \Lambda)$. To verify (3) we need to invoke the argument above this theorem which claims that the canonical automorphisms associated with generalized equations in $T_{\text{sol}}(\Omega, \Lambda)$ fix all variables in the parametric part and, also, that the canonical epimorphisms map variables from the parametric part into themselves. $\square$

The set of homomorphisms having form (70) is called a *fundamental sequence*.

Theorem 6. *For any finite system $S(X) = 1$ over a free group F, one can find effectively a finite family of non-degenerate triangular quasi-quadratic systems $U_1, \dots, U_k$ and word mappings $p_i : V_F(U_i) \to V_F(S)$ $(i = 1, \dots, k)$ such that for every $b \in V_F(S)$ there exists i and $c \in V_F(U_i)$ for which $b = p_i(c)$, i.e.,*

$$V_F(S) = p_1\big(V_F(U_1)\big) \cup \dots \cup p_k\big(V_F(U_k)\big)$$

and all sets $p_i(V_F(U_i))$ are irreducible; moreover, every irreducible component of $V_F(S)$ can be obtained as a closure of some $p_i(V_F(U_i))$ in the Zariski topology.

Proof. Each solution of the system $S(X) = 1$ can be obtained as $X = p_i(Y_i)$, where Y_i are variables of $\Omega = \Omega_i$ for a finite number of generalized equations. We have to show that all solutions of Ω^* are solutions of some NTQ system. We can use Theorem 5 without parameters. In this case Ω_v is an empty equation with non-empty set of variables. In other words $F_{R(\Omega_v)} = F * F(h_1, \dots, h_\rho)$. To each of the branches of T_{sol} we assign an NTQ system from the formulation of the theorem. Let Ω_w be a leaf vertex in T_{dec}. Then $F_{R(\Omega_w)}$ is a proper quotient of $F_{R(\Omega)}$. Consider the path $v_0, v_1, \dots, v_n = w$ in $T_{\text{dec}}(\Omega)$ from the root vertex v_0 to a terminal vertex w. All the groups $F_{R(\Omega_{v_i})}$ are isomorphic. There are the following four possibilities.

(1) $\text{tp}(v_{n-1}) = 2$. In this case there is a singular periodic structure on $\Omega_{v_{n-1}}$. By Lemma 22, $F_{R(\Omega_{v_{n-1}})}$ is a fundamental group of a graph of groups with one vertex group K, some free abelian vertex groups, and some edges defining HNN extensions of K. Recall that making the first minimal replacement we first replaced $F_{R(\Omega_{v_{n-1}})}$ by a finite number of proper quotients in which the edge groups corresponding to abelian vertex groups and HNN extensions are maximal cyclic in K. Extend the centralizers of the edge groups of $\Omega_{v_{n-1}}$ corresponding to HNN extensions by stable letters $t_1, \dots, t_k$. This new group that we denote by N is the coordinate group of a quadratic equation over $F_{R(\Omega_w)}$ which has a solution in $F_{R(\Omega_w)}$.

In all the other cases $\text{tp}(v_{n-1}) \neq 2$.

(2) There were no auxiliary edges from vertices $v_0, v_1, \dots, v_n = w$, and if one of Cases 7–10 appeared at one of these vertices, then it only appeared a bounded (the boundary depends on Ω_{v_0}) number of times in the sequence. In this case we replace $F_{R(\Omega)}$ by $F_{R(\Omega_w)}$.

(3) $F_{R(\Omega_w)}$ is a term in a free decomposition of $F_{R(\Omega_{v_{n-1}})}$ (Ω_w is a kernel of a generalized equation $\Omega_{v_{n-1}}$). In this case we also consider $F_{R(\Omega_w)}$ instead of $F_{R(\Omega)}$.

(4) For some i, $\text{tp}(v_i) = 12$ and the path $v_i, \dots, v_n = w$ does not contain vertices of types 7–10, 12 or 15. In this case $F_{R(\Omega)}$ is the coordinate group of a quadratic equation.

(5) The path $v_0, v_1, \dots, v_n = w$ contains vertices of type 15. Suppose $v_{i_j}, \dots, v_{i_j+k_j}$, $j = 1, \dots, l$, are all blocks of consecutive vertices of type 15. This means that $\text{tp}(v_{i_j+k_j+1}) \neq 15$ and $i_j + k_j + 1 < i_{j+1}$. Suppose also that none of the previous cases takes place. To each v_{i_j} we assigned a quadratic equation and a group of canonical automorphisms corresponding to this equation. Going along the path $v_{i_j}, \dots, v_{i_j+k_j}$, we take minimal solutions corresponding to some non-singular periodic structures. Each such structure corresponds to a representation of

$$F_{R(\Omega_{v_{i_j}})}$$

as an HNN extension. As in the case of a singular periodic structure, we can suppose that the edge groups corresponding to HNN extensions are maximal cyclic and not conjugated in K. Extend the centralizers of the edge groups corresponding to HNN extensions by stable letters $t_1, \dots, t_k$. Let N be the new group. Then N is the coordinate group of a quadratic system of equations over

$$F_{R(\Omega_{v_{i_j+k_j+1}})}.$$

Repeating this construction for each $j = 1, \dots, l$, we construct NTQ system over $F_{R(\Omega_w)}$.

Since $F_{R(\Omega_w)}$ is a proper quotient of $F_{R(\Omega)}$, the theorem can now be proved by induction. □

Theorem 7. *For any finitely generated group G and a free group F the set* $\mathrm{Hom}(G, F)$ $[\mathrm{Hom}_F(G, F)]$ *can be effectively described by a finite rooted tree oriented from the root, all vertices except for the root vertex are labelled by coordinate groups of generalized equations. Edges from the root vertex correspond to a finite number of homomorphisms from G into coordinate groups of generalized equations. Leaf vertices are labelled by free groups. To each vertex group we assign the group of canonical automorphisms. Each edge* (*except for the edges from the root*) *in this tree is labelled by a quotient map, and all quotients are proper. Every homomorphism from G to F can be written as a composition of the homomorphisms corresponding to edges, canonical automorphisms of the groups assigned to vertices, and some homomorphism* (*retract*) *from a free group in a leaf vertex into F.*

5.7. Cut equations

In the proof of the implicit function theorems it will be convenient to use a modification of the notion of a generalized equation. The following definition provides a framework for such a modification.

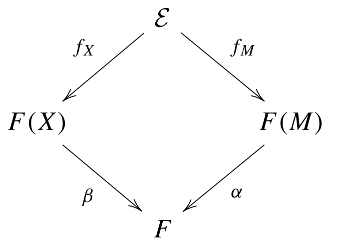
Definition 21. A cut equation $\Pi = (\mathcal{E}, M, X, f_M, f_X)$ consists of a set of intervals $\mathcal{E}$, a set of variables M, a set of parameters X, and two labeling functions

$$f_X : \mathcal{E} \to F[X], \qquad f_M : \mathcal{E} \to F[M].$$

For an interval $\sigma \in \mathcal{E}$ the image $f_M(\sigma) = f_M(\sigma)(M)$ is a reduced word in variables $M^{\pm 1}$ and constants from F, we call it a *partition* of $f_X(\sigma)$.

Sometimes we write $\Pi = (\mathcal{E}, f_M, f_X)$ omitting M and X.

Definition 22. A solution of a cut equation $\Pi = (\mathcal{E}, f_M, f_X)$ with respect to an F-homomorphism $\beta : F[X] \to F$ is an F-homomorphism $\alpha : F[M] \to F$ such that: (1) for every $\mu \in M$ $\alpha(\mu)$ is a reduced non-empty word; (2) for every reduced word $f_M(\sigma)(M)$ $(\sigma \in \mathcal{E})$ the replacement $m \to \alpha(m)$ $(m \in M)$ results in a word $f_M(\sigma)(\alpha(M))$ which is a reduced word as written and such that $f_M(\sigma)(\alpha(M))$ is graphically equal to the reduced form of $\beta(f_X(\sigma))$; in particular, the following diagram is commutative.



If $\alpha : F[M] \to F$ is a solution of a cut equation $\Pi = (\mathcal{E}, f_M, f_X)$ with respect to an F-homomorphism $\beta : F[X] \to F$, then we write (Π, β, α) and refer to α as a *solution of*

Π *modulo* β. In this event, for a given $\sigma \in \mathcal{E}$ we say that $f_M(\sigma)(\alpha(M))$ is a *partition* of $\beta(f_X(\sigma))$. Sometimes we also consider homomorphisms $\alpha : F[M] \to F$, for which the diagram above is still commutative, but cancellation may occur in the words $f_M(\sigma)(\alpha(M))$. In this event we refer to α as a *group* solution of Π with respect to β.

Lemma 34. *For a generalized equation $\Omega(H)$ one can effectively construct a cut equation $\Pi_\Omega = (\mathcal{E}, f_X, f_M)$ such that the following conditions hold*:

(1) *X is a partition of the whole interval $[1, \rho_\Omega]$ into disjoint closed subintervals*;
(2) *M contains the set of variables H*;
(3) *for any solution $U = (u_1, \ldots, u_\rho)$ of Ω the cut equation Π_Ω has a solution α modulo the canonical homomorphism $\beta_U : F(X) \to F$ ($\beta_U(x) = u_i u_{i+1} \ldots u_j$ where i, j are, correspondingly, the left and the right endpoints of the interval x)*;
(4) *for any solution (β, α) of the cut equation Π_Ω the restriction of α on H gives a solution of the generalized equation Ω.*

Proof. We begin with defining the sets X and M. Recall that a closed interval of Ω is a union of closed sections of Ω. Let X be an arbitrary partition of the whole interval $[1, \rho_\Omega]$ into closed subintervals (i.e., any two intervals in X are disjoint and the union of X is the whole interval $[1, \rho_\Omega]$).

Let B be a set of representatives of dual bases of Ω, i.e., for every base μ of Ω either μ or $\Delta(\mu)$ belongs to B, but not both. Put $M = H \cup B$.

Now let $\sigma \in X$. We denote by B_σ the set of all bases over σ and by H_σ the set of all items in σ. Put $S_\sigma = B_\sigma \cup H_\sigma$. For $e \in S_\sigma$ let $s(e)$ be the interval $[i, j]$, where $i < j$ are the endpoints of e. A sequence $P = (e_1, \ldots, e_k)$ of elements from S_σ is called a *partition* of σ if $s(e_1) \cup \cdots \cup s(e_k) = \sigma$ and $s(e_i) \cap s(e_j) = \emptyset$ for $i \neq j$. Let Part_σ be the set of all partitions of σ. Now put

$$\mathcal{E} = \{P \mid P \in \mathrm{Part}_\sigma,\ \sigma \in X\}.$$

Then for every $P \in \mathcal{E}$ there exists one and only one $\sigma \in X$ such that $P \in \mathrm{Part}_\sigma$. Denote this σ by $f_X(P)$. The map $f_X : P \to f_X(P)$ is a well-defined function from $\mathcal{E}$ into $F(X)$.

Each partition $P = (e_1, \ldots, e_k) \in \mathrm{Part}_\sigma$ gives rise to a word $w_P(M) = w_1 \ldots w_k$ as follows. If $e_i \in H_\sigma$ then $w_i = e_i$. If $e_i = \mu \in B_\sigma$ then $w_i = \mu^{\varepsilon(\mu)}$. If $e_i = \mu$ and $\Delta(\mu) \in B_\sigma$ then $w_i = \Delta(\mu)^{\varepsilon(\mu)}$. The map $f_M(P) = w_P(M)$ is a well-defined function from $\mathcal{E}$ into $F(M)$.

Now set $\Pi_\Omega = (\mathcal{E}, f_X, f_M)$. It is not hard to see from the construction that the cut equation Π_Ω satisfies all the required properties. Indeed, (1) and (2) follow directly from the construction.

To verify (3), let us consider a solution $U = (u_1, \ldots, u_{\rho_\Omega})$ of Ω. To define corresponding functions β_U and α, observe that the function $s(e)$ (see above) is defined for every $e \in X \cup M$. Now for $\sigma \in X$ put $\beta_U(\sigma) = u_i \ldots u_j$, where $s(\sigma) = [i, j]$, and for $m \in M$ put $\alpha(m) = u_i \ldots u_j$, where $s(m) = [i, j]$. Clearly, α is a solution of Π_Ω modulo β.

To verify (4) observe that if α is a solution of Π_Ω modulo β, then the restriction of α onto the subset $H \subset M$ gives a solution of the generalized equation Ω. This follows from the construction of the words w_p and the fact that the words $w_p(\alpha(M))$ are reduced as written (see definition of a solution of a cut equation). Indeed, if a base μ occurs in a partition $P \in \mathcal{E}$, then there is a partition $P' \in \mathcal{E}$ which is obtained from P by replacing μ by the sequence $h_i \dots h_j$. Since there is no cancellation in words $w_P(\alpha(M))$ and $w_{P'}(\alpha(M))$, this implies that $\alpha(\mu)^{\varepsilon(\mu)} = \alpha(h_i \dots h_j)$. This shows that α_H is a solution of Ω. □

Theorem 8. *Let $S(X, Y, A)) = 1$ be a system of equations over $F = F(A)$. Then one can effectively construct a finite set of cut equations*

$$\mathcal{C}E(S) = \{\Pi_i \mid \Pi_i = (\mathcal{E}_i, f_{X_i}, f_{M_i}),\ i = 1, \dots, k\}$$

and a finite set of tuples of words $\{Q_i(M_i) \mid i = 1, \dots, k\}$ such that:

(1) *for every equation $\Pi_i = (\mathcal{E}_i, f_{X_i}, f_{M_i}) \in \mathcal{C}E(S)$, one has $X_i = X$ and $f_{X_i}(\mathcal{E}_i) \subset X^{\pm 1}$;*
(2) *for any solution (U, V) of $S(X, Y, A) = 1$ in $F(A)$, there exists a number i and a tuple of words $P_{i,V}$ such that the cut equation $\Pi_i \in \mathcal{C}E(S)$ has a solution $\alpha : M_i \to F$ with respect to the F-homomorphism $\beta_U : F[X] \to F$ which is induced by the map $X \to U$. Moreover, $U = Q_i(\alpha(M_i))$, the word $Q_i(\alpha(M_i))$ is reduced as written, and $V = P_{i,V}(\alpha(M_i))$;*
(3) *for any $\Pi_i \in \mathcal{C}E(S)$ there exists a tuple of words $P_{i,V}$ such that for any solution (group solution) (β, α) of Π_i the pair (U, V), where $U = Q_i(\alpha(M_i))$ and $V = P_{i,V}(\alpha(M_i))$, is a solution of $S(X, Y) = 1$ in F.*

Proof. Let $S(X, Y) = 1$ be a system of equations over a free group F. In Section 4.3 we have constructed a set of initial parameterized generalized equations $\mathcal{G}E_{\mathrm{par}}(S) = \{\Omega_1, \dots, \Omega_r\}$ for $S(X, Y) = 1$ with respect to the set of parameters X. For each $\Omega \in \mathcal{G}E_{\mathrm{par}}(S)$ in Section 5.6 we constructed the finite tree $T_{\mathrm{sol}}(\Omega)$ with respect to parameters X. Observe that parametric part $[j_{v_0}, \rho_{v_0}]$ in the root equation $\Omega = \Omega_{v_0}$ of the tree $T_{\mathrm{sol}}(\Omega)$ is partitioned into a disjoint union of closed sections corresponding to X-bases and constant bases (this follows from the construction of the initial equations in the set $\mathcal{G}E_{\mathrm{par}}(S)$). We label every closed section σ corresponding to a variable $x \in X^{\pm 1}$ by x, and every constant section corresponding to a constant a by a. Due to our construction of the tree $T_{\mathrm{sol}}(\Omega)$ moving along a branch B from the initial vertex v_0 to a terminal vertex v, we transfer all the bases from the active and non-active parts into parametric parts until, eventually, in Ω_v the whole interval consists of the parametric part. Observe also that, moving along B in the parametric part, we neither introduce new closed sections nor delete any. All we do is we split (sometimes) an item in a closed parametric section into two new ones. In any event we keep the same label of the section.

Now for a terminal vertex v in $T_{\mathrm{sol}}(\Omega)$ we construct a cut equation $\Pi'_v = (\mathcal{E}_v, f_{X_v}, f_{M_v})$ as in Lemma 34 taking the set of all closed sections of Ω_v as the partition X_v. The set of cut equations

$$\mathcal{C}E'(S) = \{\Pi'_v \mid \Omega \in \mathcal{G}E_{\mathrm{par}}(S),\ v \in \mathrm{VTerm}(T_{\mathrm{sol}}(\Omega))\}$$

satisfies all the requirements of the theorem except X_v might not be equal to X. To satisfy this condition we adjust slightly the equations Π'_v.

To do this, we denote by $l : X_v \to X^{\pm 1} \cup A^{\pm 1}$ the labelling function on the set of closed sections of Ω_v. Put $\Pi_v = (\mathcal{E}_v, f_X, f_{M_v})$ where f_X is the composition of f_{X_v} and l. The set of cut equations

$$\mathcal{C}E(S) = \{\Pi_v \mid \Omega \in \mathcal{G}E_{\text{par}}(S),\ v \in \text{VTerm}(T_{\text{sol}}(\Omega))\}$$

satisfies all the conditions of the theorem. This follows from Theorem 5 and from Lemma 34. Indeed, to satisfy (3) one can take the words $P_{i,V}$ that correspond to a minimal solution of Π_i, i.e., the words $P_{i,V}$ can be obtained from a given particular way to transfer all bases from Y-part onto X-part. □

The next result shows that for every cut equation Π one can effectively and canonically associate a generalized equation Ω_Π.

For every cut equation $\Pi = (\mathcal{E}, X, M, f_X, f_M)$ one can canonically associate a generalized equation $\Omega_\Pi(M, X)$ as follows. Consider the following word

$$V = f_X(\sigma_1) f_M(\sigma_1) \dots f_X(\sigma_k) f_M(\sigma_k).$$

Now we are going to mimic the construction of the generalized equation in Lemma 13. The set of boundaries *BD* of Ω_Π consists of positive integers $1, \dots, |V| + 1$. The set of bases *BS* is union of the following sets.

(a) Every letter μ in the word V. Letters $X^{\pm 1} \cup M^{\pm 1}$ are variable bases, for every two different occurrences $\mu^{\varepsilon_1}, \mu^{\varepsilon_2}$ of a letter $\mu \in X^{\pm 1} \cup M^{\pm 1}$ in V we say that these bases are dual and they have the same orientation if $\varepsilon_1 \varepsilon_2 = 1$, and different orientation otherwise. Each occurrence of a letter $a \in A^{\pm 1}$ provides a constant base with the label a. Endpoints of these bases correspond to their positions in the word V (see Lemma 14).
(b) Every pair of subwords $f_X(\sigma_i), f_M(\sigma_i)$ provides a pair of dual bases λ_i, $\Delta(\lambda_i)$, the base λ_i is located above the subword $f_X(\sigma_i)$, and $\Delta(\lambda_i)$ is located above $f_M(\sigma_i)$ (this defines the endpoints of the bases).

Informally, one can visualize the generalized equation Ω_Π as follows. Let $\mathcal{E} = \{\sigma_1, \dots, \sigma_k\}$ and let $\mathcal{E}' = \{\sigma' \mid \sigma \in \mathcal{E}\}$ be another disjoint copy of the set $\mathcal{E}$. Locate intervals from $\mathcal{E} \cup \mathcal{E}'$ on a segment I of a straight line from left to the right in the following order $\sigma_1, \sigma_1', \dots, \sigma_k, \sigma_k'$; then put bases over I according to the word V. The next result summarizes the discussion above.

Lemma 35. *For every cut equation $\Pi = (\mathcal{E}, X, M, f_x, f_M)$, one can canonically associate a generalized equation $\Omega_\Pi(M, X)$ such that if $\alpha_\beta : F[M] \to F$ is a solution of the cut equation Π, then the maps $\alpha : F[M] \to F$ and $\beta : F[X] \to F$ give rise to a solution of the group equation (not generalized!) $\Omega^*_\Pi = 1$ in such a way that for every $\sigma \in \mathcal{E}$ $f_M(\sigma)(\alpha(M))$ is a reduced word which is graphically equal to $\beta(f_X(\sigma)(X))$, and vice versa.*

6. Definitions and elementary properties of liftings

In this section we give necessary definitions for the further discussion of liftings of equations and inequalities into coordinate groups.

Let G be a group and let $S(X) = 1$ be a system of equations over G. Recall that by G_S we denote the quotient group $G[X]/\operatorname{ncl}(S)$, where $\operatorname{ncl}(S)$ is the normal closure of S in $G[X]$. In particular, $G_{R(S)} = G[X]/R(S)$ is the coordinate group defined by $S(X) = 1$. The radical $R(S)$ can be described as follows. Consider a set of G-homomorphisms

$$\Phi_{G,S} = \left\{\phi \in \operatorname{Hom}_G\left(G[S], G\right) \,\middle|\, \phi(S) = 1\right\}.$$

Then

$$R(S) = \begin{cases} \bigcap_{\phi \in \Phi_{G,S}} \ker\phi, & \text{if } \Phi_{G,S} \neq \emptyset, \\ G[X], & \text{otherwise.} \end{cases}$$

Now we put these definitions in a more general framework. Let H and K be G-groups and $M \subset H$. Put

$$\Phi_{K,M} = \left\{\phi \in \operatorname{Hom}_G(H, K) \,\middle|\, \phi(M) = 1\right\}.$$

Then the following subgroup is termed the *G-radical of M with respect to K*:

$$\operatorname{Rad}_K(M) = \begin{cases} \bigcap_{\phi \in \Phi_{K,M}} \ker\phi, & \text{if } \Phi_{K,M} \neq \emptyset, \\ G[X], & \text{otherwise.} \end{cases}$$

Sometimes, to emphasize that M is a subset of H, we write $\operatorname{Rad}_K(M, H)$. Clearly, if $K = G$, then $R(S) = \operatorname{Rad}_G(S, G[X])$.

Let

$$H_K^* = H/\operatorname{Rad}_K(1).$$

Then H_K^* is either a G-group or trivial. If $H_K^* \neq 1$, then it is G-separated by K. In the case $K = G$ we omit K in the notation above and simply write H^*. Notice that

$$\left(H/\operatorname{ncl}(M)\right)_K^* \simeq H/\operatorname{Rad}_K(M),$$

in particular, $(G_S)^* = G_{R(S)}$.

Lemma 36. *Let $\alpha : H_1 \to H_2$ be a G-homomorphism and suppose $\Phi = \{\phi : H_2 \to K\}$ be a separating family of G-homomorphisms. Then*

$$\ker\alpha = \bigcap\left\{\ker(\alpha \circ \phi) \,\middle|\, \phi \in \Phi\right\}.$$

Proof. Suppose $h \in H_1$ and $h \notin \ker(\alpha)$. Then $\alpha(h) \neq 1$ in H_2. Hence there exists $\phi \in \Phi$ such that $\phi(\alpha(h)) \neq 1$. This shows that $\ker\alpha \supset \bigcap\{\ker(\alpha \circ \phi) \mid \phi \in \Phi\}$. The other inclusion is obvious. $\square$

Lemma 37. *Let H_1, H_2, and K be G-groups.*

(1) *Let $\alpha : H_1 \to H_2$ be a G-homomorphism and let H_2 be G-separated by K. If $M \subset \ker\alpha$, then $\mathrm{Rad}_K(M) \subseteq \ker\alpha$.*
(2) *Every G-homomorphism $\phi : H_1 \to H_2$ gives rise to a unique homomorphism*

$$\phi^* : (H_1)^*_K \to (H_2)^*_K$$

such that $\eta_2 \circ \phi = \phi^* \circ \eta_1$, where $\eta_i : H_i \to H_i^*$ is the canonical epimorphism.

Proof. (1) We have

$$\begin{aligned}\mathrm{Rad}_K(M, H_1) &= \bigcap \bigl\{\ker\phi \bigm| \phi : H_1 \to_G K \ \wedge\ \phi(M) = 1\bigr\}\\ &\subseteq \bigcap \bigl\{\ker(\alpha \circ \beta) \bigm| \beta : H_2 \to_G K\bigr\} = \ker\alpha.\end{aligned}$$

(2) Let $\alpha : H_1 \to (H_2)^*_K$ be the composition of the following homomorphisms

$$H_1 \xrightarrow{\phi} H_2 \xrightarrow{\eta_2} (H_2)^*_K.$$

Then by assertion (1), $\mathrm{Rad}_K(1, H_1) \subseteq \ker\alpha$, therefore α induces the canonical G-homomorphism $\phi^* : (H_1)^*_K \to (H_2)^*_K$. □

Lemma 38.

(1) *The canonical map $\lambda : G \to G_S$ is an embedding $\Leftrightarrow$ $S(X) = 1$ has a solution in some G-group H.*
(2) *The canonical map $\mu : G \to G_{R(S)}$ is an embedding $\Leftrightarrow$ $S(X) = 1$ has a solution in some G-group H which is G-separated by G.*

Proof. (1) If $S(x_1, \ldots, x_m) = 1$ has a solution $(h_1, \ldots, h_m)$ in some G-group H, then the G-homomorphism $x_i \to h_i$ $(i = 1, \ldots, m)$ from $G[x_1, \ldots, x_m]$ into H induces a homomorphism $\phi : G_S \to H$. Since H is a G-group all non-trivial elements from G are also non-trivial in the factor-group G_S, therefore $\lambda : G \to G_S$ is an embedding. The converse is obvious.

(2) Let $S(x_1, \ldots, x_m) = 1$ have a solution $(h_1, \ldots, h_m)$ in some G-group H which is G-separated by G. Then there exists the canonical G-homomorphism $\alpha : G_S \to H$ defined as in the proof of the first assertion. Hence $R(S) \subseteq \ker\alpha$ by Lemma 37, and α induces a homomorphism from $G_{R(S)}$ into H, which is monic on G. Therefore G embeds into $G_{R(S)}$. The converse is obvious. □

Now we apply Lemma 37 to coordinate groups of non-empty algebraic sets.

Lemma 39. *Let subsets S and T from $G[X]$ define non-empty algebraic sets in a group G. Then every G-homomorphism $\phi : G_S \to G_T$ gives rise to a G-homomorphism $\phi^* : G_{R(S)} \to G_{R(T)}$.*

Proof. The result follows from Lemmas 37 and 38. □

Now we are in a position to give the following

Definition 23. Let $S(X) = 1$ be a system of equations over a group G which has a solution in G. We say that a system of equations $T(X, Y) = 1$ is compatible with $S(X) = 1$ over G if for every solution U of $S(X) = 1$ in G the equation $T(U, Y) = 1$ also has a solution in G, i.e., the algebraic set $V_G(S)$ is a projection of the algebraic set $V_G(S \cup T)$.

The next proposition describes compatibility of two equations in terms of their coordinate groups.

Proposition 2. *Let $S(X) = 1$ be a system of equations over a group G which has a solution in G. Then $T(X, Y) = 1$ is compatible with $S(X) = 1$ over G if and only if $G_{R(S)}$ is canonically embedded into $G_{R(S\cup T)}$, and every G-homomorphism $\alpha : G_{R(S)} \to G$ extends to a G-homomorphisms $\alpha' : G_{R(S\cup T)} \to G$.*

Proof. Suppose first that $T(X, Y) = 1$ is compatible with $S(X) = 1$ over G and suppose that $V_G(S) \neq \emptyset$. The identity map $X \to X$ gives rise to a G-homomorphism

$$\lambda : G_S \to G_{S\cup T}$$

(notice that both G_S and $G_{S\cup T}$ are G-groups by Lemma 38), which by Lemma 39 induces a G-homomorphism

$$\lambda^* : G_{R(S)} \to G_{R(S\cup T)}.$$

We claim that λ^* is an embedding. To show this we need to prove first the statement about the extensions of homomorphisms. Let $\alpha : G_{R(S)} \to G$ be an arbitrary G-homomorphism. It follows that $\alpha(X)$ is a solution of $S(X) = 1$ in G. Since $T(X, Y) = 1$ is compatible with $S(X) = 1$ over G, there exists a solution, say $\beta(Y)$, of $T(\alpha(X), Y) = 1$ in G. The map

$$X \to \alpha(X), \qquad Y \to \beta(Y)$$

gives rise to a G-homomorphism $G[X, Y] \to G$, which induces a G-homomorphism $\phi : G_{S\cup T} \to G$. By Lemma 39, ϕ induces a G-homomorphism

$$\phi^* : G_{R(S\cup T)} \to G.$$

Clearly, ϕ^* makes the following diagram to commute.

$$\begin{array}{ccc} G_{R(S)} & \xrightarrow{\lambda^*} & G_{R(S\cup T)} \\ {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\phi^*} & \\ G & & \end{array}$$

Now to prove that λ^* is an embedding, observe that $G_{R(S)}$ is G-separated by G. Therefore for every non-trivial $h \in G_{R(S)}$ there exists a G-homomorphism $\alpha : G_{R(S)} \to G$ such that $\alpha(h) \neq 1$. But then $\phi^*(\lambda^*(h)) \neq 1$ and consequently $h \notin \ker \lambda^*$. The converse statement is obvious. □

Let $S(X) = 1$ be a system of equations over G and suppose $V_G(S) \neq \emptyset$. The canonical embedding $X \to G[X]$ induces the canonical map

$$\mu : X \to G_{R(S)}.$$

We are ready to formulate the main definition.

Definition 24. Let $S(X) = 1$ be a system of equations over G with $V_G(S) \neq \emptyset$ and let $\mu : X \to G_{R(S)}$ be the canonical map. Let a system $T(X, Y) = 1$ be compatible with $S(X) = 1$ over G. We say that $T(X, Y) = 1$ admits a lift to a generic point of $S = 1$ over G (or, shortly, S-lift over G) if $T(X^\mu, Y) = 1$ has a solution in $G_{R(S)}$ (here Y are variables and X^μ are constants from $G_{R(S)}$).

Lemma 40. *Let $T(X, Y) = 1$ be compatible with $S(X) = 1$ over G. If $T(X, Y) = 1$ admits an S-lift, then the identity map $Y \to Y$ gives rise to a canonical $G_{R(S)}$-epimorphism from $G_{R(S\cup T)}$ onto the coordinate group of $T(X^\mu, Y) = 1$ over $G_{R(S)}$:*

$$\psi^* : G_{R(S\cup T)} \to G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\big(T\big(X^\mu, Y\big)\big).$$

Moreover, every solution U of $T(X^\mu, Y) = 1$ in $G_{R(S)}$ gives rise to a $G_{R(S)}$-homomorphism $\phi_U : G_{R(S\cup T)} \to G_{R(S)}$, where $\phi_U(Y) = U$.

Proof. Observe that the following chain of isomorphisms hold:

$$\begin{aligned} G_{R(S\cup T)} &\simeq_G G[X][Y]/\operatorname{Rad}_G(S \cup T) \simeq_G G[X][Y]/\operatorname{Rad}_G\big(\operatorname{Rad}_G\big(S, G[X]\big) \cup T\big) \\ &\simeq_G \big(G[X][Y]/\operatorname{ncl}\big(\operatorname{Rad}_G\big(S, G[X]\big) \cup T\big)\big)^* \simeq_G \big(G_{R(S)}[Y]/\operatorname{ncl}\big(T\big(X^\mu, Y\big)\big)\big)^*. \end{aligned}$$

Denote by $\overline{G_{R(S)}}$ the canonical image of $G_{R(S)}$ in $(G_{R(S)}[Y]/\operatorname{ncl}(T(X^\mu, Y)))^*$.

Since $\operatorname{Rad}_{G_{R(S)}}(T(X^\mu, Y))$ is a normal subgroup in $G_{R(S)}[Y]$ containing $T(X^\mu, Y)$ there exists a canonical G-epimorphism

$$\psi : G_{R(S)}[Y]/\operatorname{ncl}\big(T\big(X^\mu, Y\big)\big) \to G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\big(T\big(X^\mu, Y\big)\big).$$

By Lemma 37 the homomorphism ψ gives rise to a canonical G-homomorphism

$$\psi^*: \left(G_{R(S)}[Y]/\operatorname{ncl}\left(T\left(X^\mu, Y\right)\right)\right)^* \to \left(G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\left(T\left(X^\mu, Y\right)\right)\right)^*.$$

Notice that the group $G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}(T(X^\mu, Y))$ is the coordinate group of the system $T(X^\mu, Y) = 1$ over $G_{R(S)}$ and this system has a solution in $G_{R(S)}$. Therefore this group is a $G_{R(S)}$-group and it is $G_{R(S)}$-separated by $G_{R(S)}$. Now since $G_{R(S)}$ is the coordinate group of $S(X) = 1$ over G and this system has a solution in G, we see that $G_{R(S)}$ is G-separated by G. It follows that the group $G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}(T(X^\mu, Y))$ is G-separated by G. Therefore

$$G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\left(T\left(X^\mu, Y\right)\right) = \left(G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\left(T\left(X^\mu, Y\right)\right)\right)^*.$$

Now we can see that

$$\psi^*: G_{R(S\cup T)} \to G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\left(T\left(X^\mu, Y\right)\right)$$

is a G-homomorphism which maps the subgroup $\overline{G_{R(S)}}$ from $G_{R(S\cup T)}$ onto the subgroup $G_{R(S)}$ in $G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}(T(X^\mu, Y))$.

This shows that $\overline{G_{R(S)}} \simeq_G G_{R(S)}$ and ψ^* is a $G_{R(S)}$-homomorphism. If U is a solution of $T(X^\mu, Y) = 1$ in $G_{R(S)}$, then there exists a $G_{R(S)}$-homomorphism

$$\phi_U : G_{R(S)}[Y]/\operatorname{Rad}_{G_{R(S)}}\left(T\left(X^\mu, Y\right)\right) \to G_{R(S)}.$$

such that $\phi_U(Y) = U$. Obviously, composition of ϕ_U and ψ^* gives a $G_{R(S)}$-homomorphism from $G_{R(S\cup T)}$ into $G_{R(S)}$, as desired. □

The next result characterizes lifts in terms of the coordinate groups of the corresponding equations.

Proposition 3. *Let $S(X) = 1$ be an equation over G which has a solution in G. Then for an arbitrary equation $T(X, Y) = 1$ over G the following conditions are equivalent*:

(1) $T(X, Y) = 1$ *is compatible with* $S(X) = 1$ *and* $T(X, Y) = 1$ *admits S-lift over G*;
(2) $G_{R(S)}$ *is a retract of* $G_{R(S,T)}$, *i.e.,* $G_{R(S)}$ *is a subgroup of* $G_{R(S,T)}$ *and there exists a* $G_{R(S)}$*-homomorphism* $G_{R(S,T)} \to G_{R(S)}$.

Proof. (1) ⇒ (2). By Proposition 2, $G_{R(S)}$ is a subgroup of $G_{R(S,T)}$. Moreover, $T(X^\mu, Y) = 1$ has a solution in $G_{R(S)}$, so by Lemma 40 there exists a $G_{R(S)}$-homomorphism $G_{R(S,T)} \to G_{R(S)}$, i.e., $G_{R(S)}$ is a retract of $G_{R(S,T)}$.

(2) ⇒ (1). If $\phi: G_{R(S,T)} \to G_{R(S)}$ is a retract then every G-homomorphism $\alpha: G_{R(S)} \to G$ extends to a G-homomorphism $\alpha \circ \phi : G_{R(S,T)} \to G$. It follows from Proposition 2 that $T(X, Y) = 1$ is compatible with $S(X) = 1$ and ϕ gives a solution of $T(X^\mu, Y) = 1$ in $G_{R(S)}$, as desired. □

One can ask whether it is possible to lift a system of equations and inequalities into a generic point of some equation $S = 1$? This is the question that we are going to address below. We start with very general definitions.

Definition 25. Let $S(X) = 1$ be an equation over a group G which has a solution in G. We say that a formula $\Phi(X, Y)$ in the language L_A is compatible with $S(X) = 1$ over G, if for every solution $\bar{a}$ of $S(X) = 1$ in G there exists a tuple $\bar{b}$ over G such that the formula $\Phi(\bar{a}, \bar{b})$ is true in G, i.e., the algebraic set $V_G(S)$ is a projection of the truth set of the formula $\Phi(X, Y) \wedge (S(X) = 1)$.

Definition 26. Let a formula $\Phi(X, Y)$ be compatible with $S(X) = 1$ over G. We say that $\Phi(X, Y)$ admits a lift to a generic point of $S = 1$ over G (or shortly S-lift over G), if $\exists Y \Phi(X^\mu, Y)$ is true in $G_{R(S)}$ (here Y are variables and X^μ are constants from $G_{R(S)}$).

Definition 27. Let $S(X) = 1$ be an equation over G which has a solution in G, and let $T(X, Y) = 1$ be compatible with $S(X) = 1$. We say that an equation $T(X, Y) = 1$ admits a complete S-lift if every formula $T(X, Y) = 1$ & $W(X, Y) \neq 1$, which is compatible with $S(X) = 1$ over G, admits an S-lift.

7. Implicit function theorem: lifting solutions into generic points

Now we are ready to formulate and prove the main results of this paper, Theorems 9, 11, and 12. Let $F(A)$ be a free non-abelian group.

Theorem 9. *Let $S(X, A) = 1$ be a regular standard quadratic equation over $F(A)$. Every equation $T(X, Y, A) = 1$ compatible with $S(X, A) = 1$ admits a complete S-lift.*

We divide the proof of this theorem into two parts: for orientable $S(X, A) = 1$, and for a non-orientable one.

7.1. Basic automorphisms of orientable quadratic equations

In this section, for a finitely generated fully residually free group G we introduce some particular G-automorphisms of a free G-group $G[X]$ which fix a given standard orientable quadratic word with coefficients in G. Then we describe some cancellation properties of these automorphisms.

Let G be a group and let $S(X) = 1$ be a regular standard orientable quadratic equation over G:

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] d^{-1} = 1, \tag{71}$$

where c_i, d are non-trivial constants from G, and

$$X = \{x_i, y_i, z_j \mid i = 1, \ldots, n,\ j = 1, \ldots, m\}$$

is the set of variables. Observe that if $n=0$, then $m \geqslant 3$ by definition of a regular quadratic equation (Definition 6). Sometimes we omit X and write simply $S=1$. Denote by

$$C_S = \{c_1, \dots, c_m, d\}$$

the set of constants which occur in the equation $S=1$.

Below we define a *basic sequence*

$$\Gamma = (\gamma_1, \gamma_2, \dots, \gamma_{K(m,n)})$$

of G-automorphisms of the free G-group $G[X]$, each of which fixes the element

$$S_0 = \prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] \in G[X].$$

We assume that each $\gamma \in \Gamma$ acts identically on all the generators from X that are not mentioned in the description of γ.

Let $m \geqslant 3$, $n=0$. In this case $K(m,0) = m-1$. Put

$$\gamma_i : z_i \to z_i \left(c_i^{z_i} c_{i+1}^{z_{i+1}}\right), \ z_{i+1} \to z_{i+1} \left(c_i^{z_i} c_{i+1}^{z_{i+1}}\right), \quad \text{for } i = 1, \dots, m-1.$$

Let $m=0$, $n \geqslant 1$. In this case $K(0,n) = 4n-1$. Put

$$\begin{aligned}
\gamma_{4i-3} &: y_i \to x_i y_i, \quad \text{for } i = 1, \dots, n,\\
\gamma_{4i-2} &: x_i \to y_i x_i, \quad \text{for } i = 1, \dots, n,\\
\gamma_{4i-1} &: y_i \to x_i y_i, \quad \text{for } i = 1, \dots, n,\\
\gamma_{4i} &: x_i \to \left(y_i x_{i+1}^{-1}\right)^{-1} x_i, \ y_i \to y_i^{y_i x_{i+1}^{-1}}, \ x_{i+1} \to x_{i+1}^{y_i x_{i+1}^{-1}},\\
&\quad y_{i+1} \to \left(y_i x_{i+1}^{-1}\right)^{-1} y_{i+1}, \quad \text{for } i = 1, \dots, n-1.
\end{aligned}$$

Let $m \geqslant 1$, $n \geqslant 1$. In this case $K(m,n) = m + 4n - 1$. Put

$$\begin{aligned}
\gamma_i &: z_i \to z_i \left(c_i^{z_i} c_{i+1}^{z_{i+1}}\right), \ z_{i+1} \to z_{i+1} \left(c_i^{z_i} c_{i+1}^{z_{i+1}}\right), \quad \text{for } i = 1, \dots, m-1,\\
\gamma_m &: z_m \to z_m \left(c_m^{z_m} x_1^{-1}\right), \ x_1 \to x_1^{c_m^{z_m} x_1^{-1}}, \ y_1 \to \left(c_m^{z_m} x_1^{-1}\right)^{-1} y_1,\\
\gamma_{m+4i-3} &: y_i \to x_i y_i, \quad \text{for } i = 1, \dots, n,\\
\gamma_{m+4i-2} &: x_i \to y_i x_i, \quad \text{for } i = 1, \dots, n,\\
\gamma_{m+4i-1} &: y_i \to x_i y_i, \quad \text{for } i = 1, \dots, n,\\
\gamma_{m+4i} &: x_i \to \left(y_i x_{i+1}^{-1}\right)^{-1} x_i, \ y_i \to y_i^{y_i x_{i+1}^{-1}}, \ x_{i+1} \to x_{i+1}^{y_i x_{i+1}^{-1}},\\
&\quad y_{i+1} \to \left(y_i x_{i+1}^{-1}\right)^{-1} y_{i+1}, \quad \text{for } i = 1, \dots, n-1.
\end{aligned}$$

It is easy to check that each $\gamma \in \Gamma$ fixes the word S_0 as well as the word S. This shows that γ induces a G-automorphism on the group $G_S = G[X]/\mathrm{ncl}(S)$. We denote the induced automorphism again by γ, so $\Gamma \subset \mathrm{Aut}_G(G_S)$. Since $S = 1$ is regular, $G_S = G_{R(S)}$. It follows that composition of any product of automorphisms from Γ and a particular solution β of $S = 1$ is again a solution of $S = 1$.

Observe, that in the case $m \neq 0, n \neq 0$ the basic sequence of automorphisms Γ contains the basic automorphisms from the other two cases. This allows us, without loss of generality, to formulate some of the results below only for the case $K(m,n) = m + 4n - 1$. Obvious adjustments provide the proper argument in the other cases. From now on we order elements of the set X in the following way

$$z_1 < \cdots < z_m < x_1 < y_1 < \cdots < x_n < y_n.$$

For a word $w \in F(X)$ we denote by $v(w)$ the *leading* variable (the highest variable with respect to the order introduced above) that occurs in w. For $v = v(w)$ denote by $j(v)$ the following number

$$j(v) = \begin{cases} m + 4i, & \text{if } v = x_i \text{ or } v = y_i \text{ and } i < n, \\ m + 4i - 1, & \text{if } v = x_i \text{ or } v = y_i \text{ and } i = n, \\ i, & \text{if } v = z_i \text{ and } n \neq 0, \\ m - 1, & \text{if } v = z_m,\ n = 0. \end{cases}$$

The following lemma describes the action of powers of basic automorphisms from Γ on X. The proof is obvious, and we omit it.

Lemma 41. *Let $\Gamma = (\gamma_1, \ldots, \gamma_{m+4n-1})$ be the basic sequence of automorphisms and p be a positive integer. Then the following holds*:

$$\gamma_i^p : z_i \to z_i\big(c_i^{z_i} c_{i+1}^{z_{i+1}}\big)^p,\ z_{i+1} \to z_{i+1}\big(c_i^{z_i} c_{i+1}^{z_{i+1}}\big)^p, \quad \textit{for } i = 1, \ldots, m-1,$$

$$\gamma_m^p : z_m \to z_m\big(c_m^{z_m} x_1^{-1}\big)^p,\ x_1 \to x_1^{(c_m^{z_m} x_1^{-1})^p},\ y_1 \to \big(c_m^{z_m} x_1^{-1}\big)^{-p} y_1,$$

$$\gamma_{m+4i-3}^p : y_i \to x_i^p y_i, \quad \textit{for } i = 1, \ldots, n,$$

$$\gamma_{m+4i-2}^p : x_i \to y_i^p x_i, \quad \textit{for } i = 1, \ldots, n,$$

$$\gamma_{m+4i-1}^p : y_i \to x_i^p y_i, \quad \textit{for } i = 1, \ldots, n,$$

$$\gamma_{m+4i}^p : x_i \to \big(y_i x_{i+1}^{-1}\big)^{-p} x_i,\ y_i \to y_i^{(y_i x_{i+1}^{-1})^p},\ x_{i+1} \to x_{i+1}^{(y_i x_{i+1}^{-1})^p},$$
$$y_{i+1} \to \big(y_i x_{i+1}^{-1}\big)^{-p} y_{i+1}, \quad \textit{for } i = 1, \ldots, n-1.$$

The p-powers of elements that occur in Lemma 41 play an important part in what follows, so we describe them in a separate definition.

Definition 28. Let $\Gamma = (\gamma_1, \ldots, \gamma_{m+4n-1})$ be the basic sequence of automorphism for $S = 1$. For every $\gamma \in \Gamma$ we define the leading term $A(\gamma)$ as follows:

$$A(\gamma_i) = c_i^{z_i} c_{i+1}^{z_{i+1}}, \quad \text{for } i = 1, \dots, m-1,$$
$$A(\gamma_m) = c_m^{z_m} x_1^{-1},$$
$$A(\gamma_{m+4i-3}) = x_i, \quad \text{for } i = 1, \dots, n,$$
$$A(\gamma_{m+4i-2}) = y_i, \quad \text{for } i = 1, \dots, n,$$
$$A(\gamma_{m+4i-1}) = x_i, \quad \text{for } i = 1, \dots, n,$$
$$A(\gamma_{m+4i}) = y_i x_{i+1}^{-1}, \quad \text{for } i = 1, \dots, n-1.$$

Now we introduce vector notations for automorphisms of particular type.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}^k$ the set of all k-tuples of elements from $\mathbb{N}$. For $s \in \mathbb{N}$ and $p \in \mathbb{N}^k$ we say that the tuple p is s-*large* if every coordinate of p is greater then s. Similarly, a subset $P \subset \mathbb{N}^k$ is s-*large* if every tuple in P is s-large. We say that the set P is *unbounded* if for any $s \in \mathbb{N}$ there exists an s-large tuple in P.

Let $\delta = (\delta_1, \dots, \delta_k)$ be a sequence of G-automorphisms of the group $G[X]$, and $p = (p_1, \dots, p_k) \in \mathbb{N}^k$. Then by δ^p we denote the following automorphism of $G[X]$:

$$\delta^p = \delta_1^{p_1} \dots \delta_k^{p_k}.$$

Notation 42. Let $\Gamma = (\gamma_1, \dots, \gamma_K)$ be the basic sequence of automorphisms for $S = 1$. Denote by Γ_∞ the infinite periodic sequence with period Γ, i.e., $\Gamma_\infty = \{\gamma_i\}_{i \geqslant 1}$ with $\gamma_{i+K} = \gamma_i$. For $j \in \mathbb{N}$ denote by Γ_j the initial segment of Γ_∞ of length j. Then for a given j and $p \in \mathbb{N}^j$ put

$$\phi_{j,p} = \overleftarrow{\Gamma}_j^{\overleftarrow{p}} = \gamma_j^{p_j} \gamma_{j-1}^{p_{j-1}} \dots \gamma_1^{p_1}.$$

Sometimes we omit p from $\phi_{j,p}$ and write simply ϕ_j.

Agreement. From now on we fix an arbitrary positive multiple L of the number $K = K(m,n)$, a 2-large tuple $p \in \mathbb{N}^L$, and the automorphism $\phi = \phi_{L,p}$ (as well as all the automorphism ϕ_j, $j \leqslant L$).

Definition 29. The leading term $A_j = A(\phi_j)$ of the automorphism ϕ_j is defined to be the cyclically reduced form of the word

$$\begin{cases} A(\gamma_j)^{\phi_{j-1}}, & \text{if } j \leqslant K,\ j \neq m+4i-1 \text{ for any } i = 1, \dots, n, \\ y_i^{-\phi_{j-2}} A(\gamma_j)^{\phi_{j-1}} y_i^{\phi_{j-2}}, & \text{if } j = m+4i-1 \text{ for some } i = 1, \dots, n, \\ A_r^{\phi_{sK}}, & \text{if } j = r + sK,\ r \leqslant K,\ s \in \mathbb{N}. \end{cases}$$

Lemma 43. *For every $j \leqslant L$ the element $A(\phi_j)$ is not a proper power in $G[X]$.*

Proof. It is easy to check that $A(\gamma_s)$ from Definition 28 is not a proper power for $s = 1, \dots, K$. Since $A(\phi_j)$ is the image of some $A(\gamma_s)$ under an automorphism of $G[X]$ it is not a proper power in $G[X]$. $\square$

For words $w, u, v \in G[X]$, the notation

$$\begin{array}{|cc|} \multicolumn{2}{|c|}{w} \\ \hline u & v \end{array}$$

means that $w = u \circ w' \circ v$ for some $w' \in G[X]$, where the length of elements and reduced form defined as in the free product $G * \langle X \rangle$. Similarly, notations

$$\begin{array}{|l} w \\ \hline u \end{array} \qquad \text{and} \qquad \begin{array}{r|} w \\ \hline v \end{array}$$

mean that $w = u \circ w'$ and $w = w' \circ v$. Sometimes we write

$$\begin{array}{|cc|} \multicolumn{2}{|c|}{w} \\ \hline u & * \end{array} \qquad \text{or} \qquad \begin{array}{|cc|} \multicolumn{2}{|c|}{w} \\ \hline * & v \end{array}$$

when the corresponding words are irrelevant.

If n is a positive integer and $w \in G[X]$, then by $\mathrm{Sub}_n(w)$ we denote the set of all n-subwords of w, i.e.,

$$\mathrm{Sub}_n(w) = \bigl\{u \bigm| |u| = n \text{ and } w = w_1 \circ u \circ w_2 \text{ for some } w_1, w_2 \in G[X]\bigr\}.$$

Similarly, by $\mathrm{SubC}_n(w)$ we denote all n-subwords of the *cyclic* word w. More generally, if $W \subseteq G[X]$, then

$$\mathrm{Sub}_n(W) = \bigcup_{w \in W} \mathrm{Sub}_n(w), \qquad \mathrm{SubC}_n(W) = \bigcup_{w \in W} \mathrm{SubC}_n(w).$$

Obviously, the set $\mathrm{Sub}_i(w)$ ($\mathrm{SubC}_i(w)$) can be effectively reconstructed from $\mathrm{Sub}_n(w)$ ($\mathrm{SubC}_n(w)$) for $i \leqslant n$.

In the following series of lemmas we write down explicit expressions for images of elements of X under the automorphism

$$\phi_K = \gamma_K^{p_K} \dots \gamma_1^{p_1}, \qquad K = K(m, n).$$

These lemmas are very easy and straightforward, though tiresome in terms of notations. They provide basic data needed to prove the implicit function theorem. All elements that occur in the lemmas below can be viewed as elements (words) from the free group $F(X \cup C_S)$. In particular, the notations

$$\circ, \qquad \begin{array}{|cc|} \multicolumn{2}{|c|}{w} \\ \hline u & v \end{array} \qquad \text{and} \qquad \mathrm{Sub}_n(W)$$

correspond to the standard length function on $F(X \cup C_S)$. Furthermore, until the end of this section we assume that the elements $c_1, \dots, c_m$ are *pairwise different*.

Lemma 44. *Let $m \neq 0$, $K = K(m,n)$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \ldots \gamma_1^{p_1}.$$

The following statements hold.

(1) *All automorphisms from Γ, except for γ_{i-1}, γ_i (if defined), fix z_i, $i = 1, \ldots, m$. It follows that*

$$z_i^{\phi_K} = \cdots = z_i^{\phi_i} \quad (i = 1, \ldots, m-1).$$

(2) *Let $\tilde{z}_i = z_i^{\phi_{i-1}}$ $(i = 2, \ldots, m)$, $\tilde{z}_1 = z_1$. Then*

$$\tilde{z}_i = \left|\begin{array}{ll} z_i \circ (c_{i-1}^{\tilde{z}_{i-1}} \circ c_i^{z_i})^{p_{i-1}} & \\ \hline {\scriptstyle z_i z_{i-1}^{-1}} & {\scriptstyle c_i z_i} \end{array}\right| \quad (i = 2, \ldots, m).$$

(3) *The reduced forms of the leading terms of the corresponding automorphisms are listed below*:

$$A_1 = \left|\begin{array}{ll} c_1^{z_1} \circ c_2^{z_2} & \\ \hline {\scriptstyle z_1^{-1} c_1} & {\scriptstyle c_2 z_2} \end{array}\right| \quad (m \geqslant 2),$$

$$A_2 = \left(c_2^{z_2} x_1^{-1}\right)^{\phi_1} = A_1^{-p_1} c_2^{z_2} A_1^{p_1} x_1^{-1} \quad (n \neq 0,\ m = 2),$$

$$A_2 = A_1^{-p_1} c_2^{z_2} A_1^{p_1} c_3^{z_3} \quad (n \neq 0,\ m > 2),$$

$$\mathrm{SubC}_3(A_1) = \left\{z_1^{-1} c_1 z_1, c_1 z_1 z_2^{-1}, z_1 z_2^{-1} c_2, z_2^{-1} c_2 z_2, c_2 z_2 z_1^{-1}, z_2 z_1^{-1} c_1\right\},$$

$$A_i = \left|\begin{array}{ll} A_{i-1}^{-p_{i-1}} & \\ \hline {\scriptstyle z_i^{-1} c_i^{-1}} & {\scriptstyle c_{i-1} z_{i-1}} \end{array}\right| c_i^{z_i} \left|\begin{array}{ll} A_{i-1}^{p_{i-1}} & \\ \hline {\scriptstyle z_{i-1}^{-1} c_{i-1}^{-1}} & {\scriptstyle c_i z_i} \end{array}\right| \left.\begin{array}{ll} c_{i+1}^{z_{i+1}} & \\ \hline {\scriptstyle z_{i+1}^{-1}} & {\scriptstyle c_{i+1} z_{i+1}} \end{array}\right| \quad (i = 3, \ldots, m-1),$$

$$\begin{aligned} \mathrm{SubC}_3(A_i) = {} & \mathrm{SubC}_3(A_{i-1})^{\pm 1} \\ & \cup \left\{c_{i-1} z_{i-1} z_i^{-1}, z_{i-1} z_i^{-1} c_i, z_i^{-1} c_i z_i, c_i z_i z_{i-1}^{-1}, z_i z_{i-1}^{-1} c_{i-1}^{-1}, c_i z_i z_{i+1}^{-1},\right. \\ & \quad \left. z_i z_{i+1}^{-1} c_{i+1}, z_{i+1}^{-1} c_{i+1} z_{i+1}, c_{i+1} z_{i+1} z_i^{-1}, z_{i+1} z_i^{-1} c_i^{-1}\right\}, \end{aligned}$$

$$A_m = \left(c_m^{z_m} x_1^{-1}\right)^{\phi_{m-1}} = \left|\begin{array}{ll} A_{m-1}^{-p_{m-1}} & \\ \hline {\scriptstyle z_m^{-1} c_m^{-1}} & {\scriptstyle c_{m-1} z_{m-1}} \end{array}\right| c_m^{z_m} \left|\begin{array}{ll} A_{m-1}^{p_{m-1}} & \\ \hline {\scriptstyle z_{m-1}^{-1} c_{m-1}^{-1}} & {\scriptstyle c_m z_m} \end{array}\right| x_1^{-1} \quad (n \neq 0,\ m \geqslant 3),$$

$$\begin{aligned} \mathrm{SubC}_3(A_m) = {} & \mathrm{SubC}_3(A_{m-1})^{\pm 1} \\ & \cup \left\{c_{m-1} z_{m-1} z_m^{-1}, z_{m-1} z_m^{-1} c_m, z_{m-1}^{-1} c_m z_m, c_m z_m z_{m-1}^{-1}, c_m z_m x_1^{-1},\right. \\ & \quad \left. z_m x_1^{-1} z_m^{-1}, x_1^{-1} z_m^{-1} c_m^{-1}\right\}. \end{aligned}$$

(4) *The reduced forms of $z_i^{\phi_{i-1}}$, $z_i^{\phi_i}$ are listed below*:

$$z_1^{\phi_K} = z_1^{\phi_1} = c_1 \left|\begin{array}{c|c} z_1 c_2^{z_2} & A_1^{p_1-1} \\ \hline \scriptstyle z_1z_2^{-1} \quad c_2z_2 & \scriptstyle z_1^{-1}c_1 \quad c_2z_2 \end{array}\right| \quad (m \geqslant 2),$$

$$\mathrm{SubC}_3\bigl(z_1^{\phi_K}\bigr) = \bigl\{c_1z_1z_2^{-1}, z_1z_2^{-1}c_2, z_2^{-1}c_2z_2, c_2z_2z_1^{-1}, z_2z_1^{-1}c_1, z_1^{-1}c_1z_1\bigr\},$$

$$z_i^{\phi_{i-1}} = z_i \left|\begin{array}{c} A_{i-1}^{p_{i-1}} \\ \hline \scriptstyle z_{i-1}^{-1}c_{i-1}^{-1} \quad c_iz_i \end{array}\right| = \tilde{z}_i,$$

$$z_i^{\phi_K} = z_i^{\phi_i} = c_iz_i \left|\begin{array}{c} A_{i-1}^{p_{i-1}} \\ \hline \scriptstyle z_{i-1}^{-1}c_{i-1}^{-1} \quad c_iz_i \end{array}\right| c_{i+1}^{z_{i+1}} \left|\begin{array}{c} A_i^{p_i-1} \\ \hline \scriptstyle z_i^{-1}c_i^{-1} \quad c_{i+1}z_{i+1} \end{array}\right| \quad (i = 2, \ldots, m-1),$$

$$\begin{aligned} \mathrm{Sub}_3\bigl(z_i^{\phi_K}\bigr) = {} & \mathrm{SubC}_3(A_{i-1}) \cup \mathrm{SubC}_3(A_i) \\ & \cup \bigl\{c_iz_iz_{i-1}^{-1}, z_iz_{i-1}^{-1}c_{i-1}^{-1}, c_iz_iz_{i+1}^{-1}, z_iz_{i+1}^{-1}c_{i+1}, z_{i+1}^{-1}c_{i+1}z_{i+1}, \\ & \quad c_{i+1}z_{i+1}z_i^{-1}, z_{i+1}z_i^{-1}c_i^{-1}\bigr\}, \end{aligned}$$

$$z_m^{\phi_K} = z_m^{\phi_{m-1}} = z_m \left|\begin{array}{c} A_{m-1}^{p_{m-1}} \\ \hline \scriptstyle z_{m-1}^{-1}c_{m-1}^{-1} \quad c_mz_m \end{array}\right| \quad (n = 0),$$

$$\mathrm{Sub}_3\bigl(z_m^{\phi_K}\bigr)_{(when\ n=0)} = \mathrm{SubC}_3(A_{m-1}) \cup \bigl\{z_mz_{m-1}^{-1}c_{m-1}^{-1}\bigr\},$$

$$z_m^{\phi_K} = z_m^{\phi_m} = c_mz_m \left|\begin{array}{c} A_{m-1}^{p_{m-1}} \\ \hline \scriptstyle z_{m-1}^{-1}c_{m-1}^{-1} \quad c_mz_m \end{array}\right| x_1^{-1} \left|\begin{array}{c} A_m^{p_m-1} \\ \hline \scriptstyle z_m^{-1}c_m^{-1} \quad z_mx_1^{-1} \end{array}\right| \quad (n \neq 0,\ m \geqslant 2),$$

$$\begin{aligned} \mathrm{Sub}_3\bigl(z_m^{\phi_K}\bigr) = {} & \mathrm{Sub}_3\bigl(z_m^{\phi_K}\bigr)_{(when\ n=0)} \cup \mathrm{SubC}_3(A_m) \\ & \cup \bigl\{c_mz_mx_1^{-1}, z_mx_1^{-1}z_m^{-1}, x_1^{-1}z_m^{-1}c_m^{-1}\bigr\}. \end{aligned}$$

(5) *The elements* $z_i^{\phi_K}$ *have the following properties*:

$$z_i^{\phi_K} = c_iz_i\hat{z}_i \quad (i = 1, \ldots, m-1),$$

where $\hat{z}_i$ *is a word in the alphabet* $\{c_1^{z_1}, \ldots, c_{i+1}^{z_{i+1}}\}$ *which begins with* $c_{i-1}^{-z_{i-1}}$, *if* $i \neq 1$, *and with* $c_2^{z_2}$, *if* $i = 1$;

$$z_m^{\phi_K} = z_m\hat{z}_m \quad (n = 0),$$

where $\hat{z}_m$ *is a word in the alphabet* $\{c_1^{z_1}, \ldots, c_m^{z_m}\}$;

$$z_m^{\phi_K} = c_mz_m\hat{z}_m \quad (n \neq 0),$$

where $\hat{z}_m$ *is a word in the alphabet* $\{c_1^{z_1}, \ldots, c_m^{z_m}, x_1\}$.
Moreover, if $m \geqslant 3$ *the word* $(c_m^{z_m})^{\pm 1}$ *occurs in* $z_i^{\phi_K}$ $(i = m-1, m)$ *only as a part of the subword* $(\prod_{i=1}^m c_i^{z_i})^{\pm 1}$.

Proof. (1) is obvious. We prove (2) by induction. For $i \geqslant 2$

$$\tilde{z}_i = z_i^{\phi_{i-1}} = z_i^{\gamma_{i-1}^{p_{i-1}}\phi_{i-2}}.$$

Therefore, by induction,

$$\tilde{z}_i = z_i \left(c_{i-1}^{\tilde{z}_{i-1}} c_i^{z_i}\right)^{p_{i-1}} = z_i \circ \left(c_{i-1}^{\tilde{z}_{i-1}} \circ c_i^{z_i}\right)^{p_{i-1}}.$$

Now we prove (3) and (4) simultaneously. Let $m \geqslant 2$. By the straightforward verification one has:

$$A_1 = \left|\frac{c_1^{z_1} \circ c_2^{z_2}}{z_1^{-1} \qquad z_2}\right|,$$

$$z_1^{\phi_1} = z_1^{\gamma_1^{p_1}} = z_1\left(c_1^{z_1} c_2^{z_2}\right)^{p_1} = \left|\frac{c_1 \circ z_1 \circ c_2^{z_2} \circ A_1^{p_1-1}}{c_1 \qquad\qquad\qquad z_2}\right|.$$

Denote by $\operatorname{cycred}(w)$ the cyclically reduced form of w.

$$A_i = \operatorname{cycred}\left(\left(c_i^{z_i} c_{i+1}^{z_{i+1}}\right)^{\phi_{i-1}}\right) = \left|\frac{c_i^{\tilde{z}_i} \circ c_{i+1}^{z_{i+1}}}{z_i^{-1} \qquad z_{i+1}}\right| \quad (i \leqslant m-1).$$

Observe that in the notation above

$$\tilde{z}_i = z_i A_{i-1}^{p_{i-1}} \quad (i \geqslant 2).$$

This shows that we can rewrite $A(\phi_i)$ as follows:

$$A_i = A_{i-1}^{-p_{i-1}} \circ c_i^{z_i} \circ A_{i-1}^{p_{i-1}} \circ c_{i+1}^{z_{i+1}},$$

beginning with z_i^{-1} and ending with z_{i+1} ($i = 2, \ldots, m-1$);

$$A_m = \operatorname{cycred}\left(c_m^{\tilde{z}_m} x_1^{-1}\right) = c_m^{\tilde{z}_m} x_1^{-1} = A_{m-1}^{-p_{m-1}} \circ c_m^{z_m} \circ A_{m-1}^{p_{m-1}} \circ x_1^{-1} \quad (m \geqslant 2),$$

beginning with z_m^{-1} and ending with x_1^{-1} ($n \neq 0$);

$$z_i^{\phi_{i-1}} = \left(z_i \left(c_{i-1}^{z_{i-1}} c_i^{z_i}\right)^{p_{i-1}}\right)^{\phi_{i-2}} = z_i \left(c_{i-1}^{\tilde{z}_{i-1}} c_i^{z_i}\right)^{p_{i-1}} = z_i \circ A_{i-1}^{p_{i-1}},$$

beginning with z_i and ending with z_i;

$$\begin{aligned} z_i^{\phi_i} &= \left(z_i \left(c_i^{z_i} c_{i+1}^{z_{i+1}}\right)^{p_i}\right)^{\phi_{i-1}} = \tilde{z}_i \left(c_i^{\tilde{z}_i} c_{i+1}^{z_{i+1}}\right)^{p_i} = c_i \circ \tilde{z}_i \circ c_{i+1}^{z_{i+1}} \circ \left(c_i^{\tilde{z}_i} c_{i+1}^{z_{i+1}}\right)^{p_i - 1} \\ &= c_i \circ z_i \circ A_{i-1}^{p_{i-1}} \circ c_{i+1}^{z_{i+1}} \circ A_i^{p_i - 1}, \end{aligned}$$

beginning with c_i and ending with z_{i+1} $(i = 2, \ldots, m-1)$;

$$\begin{aligned} z_m^{\phi_m} &= \bigl(z_m (c_m^{z_m} x_1^{-1})^{p_m}\bigr)^{\phi_{m-1}} = \tilde{z}_m \bigl(c_m^{\tilde{z}_m} x_1^{-1}\bigr)^{p_m} = c_m \tilde{z}_m x_1^{-1} \bigl(c_m^{\tilde{z}_m} x_1^{-1}\bigr)^{p_m - 1} \\ &= c_m \circ z_m \circ A_{m-1}^{p_{m-1}} \circ x_1^{-1} \circ A_m^{p_m - 1} \quad (n \neq 0), \end{aligned}$$

beginning with c_m and ending with x_1^{-1}. This proves the lemma.

(5) Direct verification using formulas in (3) and (4). □

In the following two lemmas we describe the reduced expressions of the elements $x_1^{\phi_K}$ and $y_1^{\phi_K}$.

Lemma 45. *Let $m = 0$, $K = 4n - 1$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \ldots \gamma_1^{p_1}.$$

(1) *All automorphisms from Γ, except for γ_2, γ_4, fix x_1, and all automorphisms from Γ, except for $\gamma_1, \gamma_3, \gamma_4$, fix y_1. It follows that*

$$x_1^{\phi_K} = x_1^{\phi_4}, \quad y_1^{\phi_K} = y_1^{\phi_4} \quad (n \geqslant 2).$$

(2) *Below we list the reduced forms of the leading terms of the corresponding automorphisms* (*the words on the right are reduced as written*):

$$\begin{aligned} &A_1 = x_1, \qquad A_2 = x_1^{p_1} y_1 = A_1^{p_1} \circ y_1, \\ &A_3 = \underbrace{A_2^{p_2-1}}_{x_1^2 \ \cdots\ x_1 y_1} x_1^{p_1+1} y_1, \qquad \mathrm{SubC}_3(A_3) = \mathrm{SubC}_3(A_2) = \bigl\{x_1^3, x_1^2 y_1, x_1 y_1 x_1, y_1 x_1^2\bigr\}, \\ &A_4 = \underbrace{\Bigl(\underbrace{A_2^{p_2}}_{x_1^2 \ \cdots\ x_1 y_1} x_1\Bigr)^{p_3}}_{x_1^2 \ \cdots\ y_1 x_1} \underbrace{A_2}_{x_1^2 \ \cdots\ x_1 y_1} x_2^{-1} \quad (n \geqslant 2), \\ &\mathrm{SubC}_3(A_4) = \mathrm{SubC}_3(A_2) \cup \bigl\{x_1 y_1 x_2^{-1}, y_1 x_2^{-1} x_1, x_2^{-1} x_1^2\bigr\} \quad (n \geqslant 2). \end{aligned}$$

(3) *Below we list reduced forms of $x_1^{\phi_j}$, $y_1^{\phi_j}$ for $j = 1, \ldots, 4$:*

$$\begin{aligned} &x_1^{\phi_1} = x_1, \qquad y_1^{\phi_1} = x_1^{p_1} y_1, \\ &x_1^{\phi_2} = \underbrace{A_2^{p_2}}_{x_1^2 \ \cdots\ x_1 y_1} x_1, \qquad y_1^{\phi_2} = x_1^{p_1} y_1, \\ &x_1^{\phi_3} = x_1^{\phi_2} = \underbrace{A_2^{p_2}}_{x_1^2 \ \cdots\ x_1 y_1} x_1 =_{(when\ n=1)} x_1^{\phi_K}, \qquad \mathrm{Sub}_3\bigl(x_1^{\phi_K}\bigr)_{(when\ n=1)} = \mathrm{SubC}_3(A_2), \end{aligned}$$

$$y_1^{\phi_3} = \left|\begin{array}{c} \left(\left|\begin{array}{c} A_2^{p_2} \\ \hline {\scriptstyle x_1^2 \quad x_1y_1} \end{array}\right| x_1\right)^{p_3} \\ \hline {\scriptstyle x_1^2 \qquad\qquad x_1y_1} \end{array}\right| x_1^{p_1} y_1 =_{(when\ n=1)} y_1^{\phi_K},$$

$$\mathrm{Sub}_3\big(y_1^{\phi_K}\big)_{(when\ n=1)} = \mathrm{SubC}_3(A_2),$$

$$x_1^{\phi_4} = x_1^{\phi_K} = \left|\begin{array}{c} A_4^{-(p_4-1)} \\ \hline {\scriptstyle x_2y_1^{-1} \quad x_1^{-2}} \end{array}\right| x_2 \left|\begin{array}{c} A_2^{-1} \\ \hline {\scriptstyle y_1^{-1}x_1^{-1} \quad x_1^{-2}} \end{array}\right| \left|\begin{array}{c} \left(x_1^{-1}\left|\begin{array}{c} A_2^{-p_2} \\ \hline {\scriptstyle y_1^{-1}x_1^{-1} \quad x_1^{-2}} \end{array}\right|\right)^{p_3-1} \\ \hline {\scriptstyle x_1^{-1}y_1^{-1}x_1^{-1} \qquad\qquad x_1^{-2}} \end{array}\right| \quad (n \geqslant 2),$$

$$\mathrm{Sub}_3\big(x_1^{\phi_K}\big) = \mathrm{SubC}_3(A_4)^{-1} \cup \mathrm{SubC}_3(A_2)^{-1}$$
$$\cup \big\{x_1^{-2}x_2, x_1^{-1}x_2y_1^{-1}, x_2y_1^{-1}x_1^{-1}, x_1^{-3}, x_1^{-2}y_1^{-1}, x_1^{-1}y_1^{-1}x_1^{-1}\big\} \quad (n \geqslant 2),$$

$$y_1^{\phi_4} = \left|\begin{array}{c} A_4^{-(p_4-1)} \\ \hline {\scriptstyle x_2y_1^{-1} \quad x_1^{-2}} \end{array}\right| x_2 \left|\begin{array}{c} A_4^{p_4} \\ \hline {\scriptstyle x_1^{2} \quad y_1x_2^{-1}} \end{array}\right| \quad (n \geqslant 2),$$

$$\mathrm{Sub}_3\big(y_1^{\phi_K}\big) = \mathrm{SubC}_3(A_4)^{\pm 1} \cup \big\{x_1^{-2}x_2, x_1^{-1}x_2x_1, x_2x_1^2\big\} \quad (n \geqslant 2).$$

Proof. (1) follows directly from definitions.

To show (2) observe that

$$A_1 = A(\gamma_1) = x_1,$$
$$x_1^{\phi_1} = x_1, \qquad y_1^{\phi_1} = x_1^{p_1}y_1 = A_1^{p_1} \circ y_1.$$

Then

$$A_2 = \mathrm{cycred}\big(A(\gamma_2)^{\phi_1}\big) = \mathrm{cycred}\big(y_1^{\phi_1}\big) = x_1^{p_1} \circ y_1 = A_1^{p_1} \circ y_1,$$
$$x_1^{\phi_2} = \Big(x_1^{\gamma_2^{p_2}}\Big)^{\gamma_1^{p_1}} = \big(y_1^{p_2}x_1\big)^{\gamma_1^{p_1}} = \big(x_1^{p_1}y_1\big)^{p_2}x_1 = A_2^{p_2} \circ x_1,$$
$$y_1^{\phi_2} = \Big(y_1^{\gamma_2^{p_2}}\Big)^{\gamma_1^{p_1}} = y_1^{\gamma_1^{p_1}} = x_1^{p_1}y_1 = A_2.$$

Now

$$A_3 = \mathrm{cycred}\big(y_1^{-\phi_1}A(\gamma_3)^{\phi_2}y_1^{\phi_1}\big) = \mathrm{cycred}\big(\big(x_1^{p_1}y_1\big)^{-1}x_1^{\phi_2}\big(x_1^{p_1}y_1\big)\big)$$
$$= \mathrm{cycred}\big(\big(x_1^{p_1}y_1\big)^{-1}\big(x_1^{p_1}y_1\big)^{p_2}x_1\big(x_1^{p_1}y_1\big)\big)$$
$$= \big(x_1^{p_1}y_1\big)^{p_2-1}x_1^{p_1+1}y_1 = A_2^{p_2-1} \circ A_1^{p_1+1} \circ y_1.$$

It follows that

$$x_1^{\phi_3} = \Big(x_1^{\gamma_3^{p_3}}\Big)^{\phi_2} = x_1^{\phi_2},$$
$$y_1^{\phi_3} = \Big(y_1^{\gamma_3^{p_3}}\Big)^{\phi_2}\big(x_1^{p_3}y_1\big)^{\phi_2} = \big(x_1^{\phi_2}\big)^{p_3}y_1^{\phi_2}\big(A_2^{p_2} \circ x_1\big)^{p_3} \circ A_2.$$

Hence

$$\begin{aligned} A_4 &= \mathrm{cycred}\big(A(\gamma_4)^{\phi_3}\big) = \mathrm{cycred}\big(\big(y_1x_2^{-1}\big)^{\phi_3}\big) = \mathrm{cycred}\big(y_1^{\phi_3}x_2^{-\phi_3}\big) \\ &= \big(A_2^{p_2} \circ x_1\big)^{p_3} \circ A_2 \circ x_2^{-1}. \end{aligned}$$

Finally:

$$\begin{aligned} x_1^{\phi_4} &= \Big(x_1^{\gamma_4^{p_4}}\Big)^{\phi_3} = \big(\big(y_1x_2^{-1}\big)^{-p_4}x_1\big)^{\phi_3} = \big(\big(y_1x_2^{-1}\big)^{\phi_3}\big)^{-p_4}x_1^{\phi_3} = A_4^{-p_4}A_2^{p_2}x_1 \\ &= A_4^{-(p_4-1)} \circ x_2 \circ A_2^{-1} \circ \big(x_1^{-1} \circ A_2^{-p_2}\big)^{p_3-1}, \end{aligned}$$

$$\begin{aligned} y_1^{\phi_4} &= \Big(y_1^{\gamma_4^{p_4}}\Big)^{\phi_3} = \Big(y_1^{(y_1x_2^{-1})^{p_4}}\Big)^{\phi_3} = \big(\big(y_1x_2^{-1}\big)^{\phi_3}\big)^{-p_4}y_1^{\phi_3}\big(\big(y_1x_2^{-1}\big)^{\phi_3}\big)^{p_4} \\ &= A_4^{-p_4}y_1^{\phi_3}A_4^{p_4} = A_4^{-(p_4-1)}A_4^{-1}y_1^{\phi_3}A_4^{p_4} = A_4^{-(p_4-1)} \circ x_2 \circ A_4^{p_4}. \end{aligned}$$

This proves the lemma. □

Lemma 46. *Let $m \neq 0$, $n \neq 0$, $K = m + 4n - 1$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \ldots \gamma_1^{p_1}.$$

(1) *All automorphisms from Γ except for γ_m, γ_{m+2}, γ_{m+4}, fix x_1; and all automorphisms from Γ except for γ_m, γ_{m+1}, γ_{m+3}, γ_{m+4}, fix y_1. It follows that*

$$x_1^{\phi_K} = x_1^{\phi_{m+4}}, \quad y_1^{\phi_K} = y_1^{\phi_{m+4}} \quad (n \geqslant 2).$$

(2) *Below we list the reduced forms of the leading terms of the corresponding automorphisms (the words on the right are reduced as written):*

$$A_{m+1} = x_1, \qquad A_{m+2} = y_1^{\phi_{m+1}} = \left|\begin{array}{c} A_m^{-p_m} \\ \hline x_1z_m^{-1} \quad c_mz_m \end{array}\right| x_1^{p_{m+1}}y_1,$$

$$\mathrm{SubC}_3(A_{m+2}) = \mathrm{SubC}_3(A_m)^{-1} \cup \big\{c_mz_mx_1, z_mx_1^2, x_1^3, x_1^2y_1, x_1y_1x_1, y_1x_1z_m^{-1}\big\},$$

$$A_{m+3} = \left|\begin{array}{c|c} A_{m+2}^{p_{m+2}-1} & A_m^{-p_m} \\ \hline x_1z_m^{-1} \quad x_1y_1 & x_1z_m^{-1} \quad c_mz_m \end{array}\right| x_1^{p_{m+1}+1}y_1, \qquad \mathrm{SubC}_3(A_{m+3}) = \mathrm{SubC}_3(A_{m+2}),$$

$$A_{m+4} = \left|\begin{array}{c} A_m^{-p_m} \\ \hline x_1z_m^{-1} \quad c_mz_m \end{array}\right| \circ \left(x_1^{p_{m+1}}y_1\left|\begin{array}{c|c} A_{m+2}^{p_{m+2}-1} & A_m^{-p_m} \\ \hline x_1z_m^{-1} \quad x_1y_1 & x_1z_m^{-1} \quad c_mz_m \end{array}\right| x_1\right)^{p_{m+3}} x_1^{p_{m+1}}y_1x_2^{-1} \quad (n \geqslant 2),$$

$$\mathrm{SubC}_3(A_{m+4}) = \mathrm{SubC}_3(A_{m+2}) \cup \big\{x_1y_1x_2^{-1}, y_1x_2^{-1}x_1, x_2^{-1}x_1z_m^{-1}\big\} \quad (n \geqslant 2).$$

(3) *Below we list reduced forms of* $x_1^{\phi_j}$, $y_1^{\phi_j}$ *for* $j = m, \ldots, m+4$ *and their expressions via the leading terms*:

$$
\begin{aligned}
&x_1^{\phi_m} = A_m^{-p_m} \circ x_1 \circ A_m^{p_m}, \qquad y_1^{\phi_m} = A_m^{-p_m} \circ y_1,\\
&x_1^{\phi_{m+1}} = x_1^{\phi_m}, \qquad y_1^{\phi_{m+1}} = A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1,\\
&x_1^{\phi_{m+2}} =_{(\textit{when } n=1)} x_1^{\phi_K} = \left|\begin{array}{c} A_{m+2}^{p_{m+2}} \\ \hline x_1 z_m^{-1} \quad x_1 y_1 \end{array}\right| \begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\left| x_1 \right|\begin{array}{c} A_m^{p_m} \\ \hline z_m^{-1} c_m^{-1} \quad z_m x_1^{-1} \end{array}\Bigg|,\\
&\mathrm{Sub}_3\bigl(x_1^{\phi_K}\bigr)_{(\textit{when } n=1)} = \mathrm{SubC}_3(A_{m+2}) \cup \mathrm{SubC}_3(A_m) \cup \bigl\{z_m x_1 z_m^{-1}, x_1 z_m^{-1} c_m^{-1}\bigr\},\\
&y_1^{\phi_{m+2}} = y_1^{\phi_{m+1}},\\
&x_1^{\phi_{m+3}} = x_1^{\phi_{m+2}},\\
&y_1^{\phi_{m+3}} =_{(\textit{when } n=1)} y_1^{\phi_K} = \left|\begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\right| \left(x_1^{p_{m+1}} y_1 \left|\begin{array}{c} A_{m+2}^{p_{m+2}-1} \\ \hline x_1 z_m^{-1} \quad x_1 y_1 \end{array}\right|\begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\Bigg| x_1 \right)^{p_{m+3}} x_1^{p_{m+1}} y_1,\\
&\mathrm{Sub}_3\bigl(y_1^{\phi_K}\bigr) =_{(\textit{when } n=1)} \mathrm{Sub}_3\bigl(y_1^{\phi_{m+3}}\bigr) = \mathrm{SubC}_3(A_{m+2}),\\
&x_1^{\phi_{m+4}} = x_1^{\phi_K}{}_{(\textit{when } n\geqslant 2)}\\
&\quad = \left|\begin{array}{c} A_{m+4}^{-p_{m+4}+1} \\ \hline x_2 y_1^{-1} \quad z_m x_1^{-1} \end{array}\right| x_2 y_1^{-1} x_1^{-p_{m+1}}\\
&\qquad \circ \left(x_1^{-1} \left|\begin{array}{c} A_m^{p_m} \\ \hline z_m^{-1} c_m^{-1} \quad z_m x_1^{-1} \end{array}\right|\begin{array}{c} A_{m+2}^{-p_{m+2}} \\ \hline y_1^{-1} x_1^{-1} \quad z_m x_1^{-1} \end{array}\Bigg| y_1^{-1} x_1^{-p_{m+1}} \right)^{p_{m+3}-1} \left|\begin{array}{c} A_m^{p_m} \\ \hline z_m^{-1} c_m^{-1} \quad z_m x_1^{-1} \end{array}\right| \quad (n \geqslant 2),\\
&\mathrm{Sub}_3\bigl(x_1^{\phi_K}\bigr) = \mathrm{SubC}_3(A_{m+2})^{-1} \cup \bigl\{z_m x_1^{-1} x_2, x_1^{-1} x_2 y_1^{-1}, x_2 y_1^{-1} x_1^{-1}\bigr\} \quad (n \geqslant 2),\\
&y_1^{\phi_{m+4}} = y_1^{\phi_K}{}_{(\textit{when } n\geqslant 2)} = \left|\begin{array}{c} A_{m+4}^{-(p_{m+4}-1)} \\ \hline x_2 y_1^{-1} \quad z_m x_1^{-1} \end{array}\right| x_2 \left|\begin{array}{c} A_{m+4}^{p_{m+4}} \\ \hline x_1 z_m^{-1} \quad y_1 x_2^{-1} \end{array}\right| \quad (n \geqslant 2),\\
&\mathrm{Sub}_3\bigl(y_1^{\phi_K}\bigr) = \mathrm{SubC}_3(A_{m+4})^{\pm 1} \cup \bigl\{z_m x_1^{-1} x_2, x_1^{-1} x_2 x_1, x_2 x_1 z_m^{-1}\bigr\} \quad (n \geqslant 2).
\end{aligned}
$$

Proof. Statement (1) follows immediately from definitions of automorphisms of Γ.

We prove formulas in the second and third statements simultaneously using Lemma 44:

$$
x_1^{\phi_m} = \Bigl(x_1^{(c_m^{z_m} x_1^{-1})^{p_m}}\Bigr)^{\phi_{m-1}} = x_1^{A(\phi_m)^{p_m}} = A_m^{-p_m} \circ x_1 \circ A_m^{p_m},
$$

beginning with x_1 and ending with x_1^{-1};

$$
y_1^{\phi_m} = \bigl(\bigl(c_m^{z_m} x_1^{-1}\bigr)^{-p_m} y_1\bigr)^{\phi_{m-1}} = A(\phi_m)^{-p_m} \circ y_1,
$$

beginning with x_1 and ending with y_1. Now

$$A_{m+1} = \mathrm{cycred}\big(A(\gamma_{m+1})^{\phi_m}\big) = x_1^{\phi_m} = A_m^{-p_m} \circ x_1 \circ A_m^{p_m}, \qquad A_{m+1} = x_1,$$
$$x_1^{\phi_{m+1}} = x_1^{\phi_m},$$
$$y_1^{\phi_{m+1}} = \left(y_1^{\gamma_{m+1}^{p_{m+1}}}\right)^{\phi_m} = \left(x_1^{p_{m+1}} y_1\right)^{\phi_m} = \left(x_1^{\phi_m}\right)^{p_{m+1}} y_1^{\phi_m} = A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1,$$

beginning with x_1 and ending with y_1; moreover, the element that cancels in reducing $A_{m+1}^{p_{m+1}} A_m^{-p_m} y_1$ is equal to $A_m^{p_m}$;

$$A_{m+2} = \mathrm{cycred}\big(A(\gamma_{m+2})^{\phi_{m+1}}\big) = \mathrm{cycred}\big(y_1^{\phi_{m+1}}\big) = A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1,$$

beginning with x_1 and ending with y_1;

$$\begin{aligned} x_1^{\phi_{m+2}} &= \left(x_1^{\gamma_{m+2}^{p_{m+2}}}\right)^{\phi_{m+1}} = \left(y_1^{\phi_{m+1}}\right)^{p_{m+2}} x_1^{\phi_{m+1}} \\ &= A_{m+2}^{p_{m+2}} \circ A_m^{-p_m} \circ x_1 \circ A_m^{p_m} \\ &= A_m^{-p_m} \circ \left(x_1^{p_{m+1}} \circ y_1 \circ A_{m+2}^{p_{m+2}-1} \circ A_m^{-p_m} \circ x_1\right) \circ A_m^{p_m}, \end{aligned}$$

beginning with x_1 and ending with x_1^{-1};

$$y_1^{\phi_{m+2}} = y_1^{\phi_{m+1}},$$
$$A_{m+3} = \mathrm{cycred}\big(y_1^{-\phi_{m+1}} x_1^{\phi_{m+2}} y_1^{\phi_{m+1}}\big) = A_{m+2}^{p_{m+2}-1} \circ A_m^{-p_m} \circ x_1^{p_{m+1}+1} \circ y_1,$$

beginning with x_1 and ending with y_1;

$$x_1^{\phi_{m+3}} = x_1^{\phi_{m+2}},$$
$$\begin{aligned} y_1^{\phi_{m+3}} &= \left(x_1^{\phi_{m+2}}\right)^{p_{m+3}} y_1^{\phi_{m+1}} \\ &= A_m^{-p_m} \circ \left(x_1^{p_{m+1}} \circ y_1 \circ A_{m+2}^{p_{m+2}-1} \circ A_m^{-p_m} \circ x_1\right)^{p_{m+3}} \circ x_1^{p_{m+1}} \circ y_1, \end{aligned}$$

beginning with x_1 and ending with y_1. Finally, for $n \geqslant 2$,

$$A_{m+4} = \mathrm{cycred}\big(A(\gamma_{m+4})^{\phi_{m+3}}\big) = \mathrm{cycred}\big(\big(y_1 x_2^{-1}\big)^{\phi_{m+3}}\big) = y_1^{\phi_{m+3}} x_2^{-1} = y_1^{\phi_{m+3}} \circ x_2^{-1},$$

beginning with x_1 and ending with x_2^{-1};

$$\begin{aligned} x_1^{\phi_{m+4}} &= \left(\left(y_1 x_2^{-1}\right)^{-p_{m+4}} x_1\right)^{\phi_{m+3}} = \left(x_2 y_1^{-\phi_{m+3}}\right)^{p_{m+4}} x_1^{\phi_{m+3}} \\ &= \left(x_2 y_1^{-\phi_{m+1}} \left(x_1^{\phi_{m+2}}\right)^{-p_{m+3}}\right)^{p_{m+4}} x_1^{\phi_{m+2}} \\ &= \left(x_2 y_1^{-\phi_{m+3}}\right)^{p_{m+4}-1} \circ x_2 \circ y_1^{-1} \circ x_1^{-p_{m+1}} \\ &\quad \circ \left(x_1^{-1} \circ A_m^{p_m} \circ A_{m+2}^{-p_{m+2}} \circ y_1^{-1} \circ x_1^{-p_{m+1}}\right)^{p_{m+3}-1} \circ A_m^{p_m}, \end{aligned}$$

beginning with x_2 and ending with x_1^{-1}; moreover, the element that is cancelled out is $x_1^{\phi_{m+2}}$. Similarly,

$$\begin{aligned} y_1^{\phi_{m+4}} &= \left(x_2 y_1^{-\phi_{m+3}}\right)^{p_{m+4}} y_1^{\phi_{m+3}} \left(y_1^{\phi_{m+3}} x_2^{-1}\right)^{p_{m+4}} \\ &= \left(x_2 y_1^{-\phi_{m+3}}\right)^{p_{m+4}-1} \circ x_2 \circ \left(y_1^{\phi_{m+3}} x_2^{-1}\right)^{p_{m+4}} = A_{m+4}^{-(p_{m+4}-1)} \circ x_2 \circ A_{m+4}^{p_{m+4}}, \end{aligned}$$

beginning with x_2 and ending with x_2^{-1}, moreover, the element that is cancelled out is $y_1^{\phi_{m+3}}$.

This proves the lemma. □

In the following lemma we describe the reduced expressions of the elements $x_i^{\phi_j}$ and $y_i^{\phi_j}$ for $i \geqslant 2$.

Lemma 47. *Let $n \geqslant 2$, $K = K(m,n)$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \cdots \gamma_1^{p_1}.$$

Then for any i, $n \geqslant i \geqslant 2$, the following holds:

(1) *All automorphisms from Γ, except for $\gamma_{m+4(i-1)}, \gamma_{m+4i-2}, \gamma_{m+4i}$, fix x_i, and all automorphisms from Γ, except for $\gamma_{m+4(i-1)}, \gamma_{m+4i-3}, \gamma_{m+4i-1}, \gamma_{m+4i}$, fix y_i. It follows that*

$$\begin{aligned} x_i^{\phi_K} &= x_i^{\phi_{K-1}} = \cdots = x_i^{\phi_{m+4i}}, \\ y_i^{\phi_K} &= y_i^{\phi_{K-1}} = \cdots = y_i^{\phi_{m+4i}}. \end{aligned}$$

(2) *Let $\tilde{y}_i = y_i^{\phi_{m+4i-1}}$. Then*

$$\tilde{y}_i = \left| \frac{\tilde{y}_i}{x_i y_{i-1}^{-1} \qquad x_i y_i} \right|$$

where (for $i = 1$) we assume that $y_0 = x_1^{-1}$ for $m = 0$, and $y_0 = z_m$ for $m \neq 0$.

(3) *Below we list the reduced forms of the leading terms of the corresponding automorphisms. Put $q_j = p_{m+4(i-1)+j}$ for $j = 0, \ldots, 4$. In the formulas below we assume that $y_0 = x_1^{-1}$ for $m = 0$, and $y_0 = z_m$ for $m \neq 0$.*

$$A_{m+4i-4} = \left| \frac{\tilde{y}_{i-1}}{x_{i-1} y_{i-2}^{-1} \qquad x_{i-1} y_{i-1}} \right| \circ x_i^{-1},$$

$$\mathrm{SubC}_3(A_{m+4i-4}) = \mathrm{Sub}_3\left(\tilde{y}_{i-1}\right) \cup \left\{x_{i-1} y_{i-1} x_i^{-1}, y_{i-1} x_i^{-1} x_{i-1}, x_i^{-1} x_{i-1} y_{i-2}^{-1}\right\},$$

$$A_{m+4i-3} = x_i, \qquad A_{m+4i-2} = \left| \frac{A_{m+4i-4}^{-q_0}}{x_i y_{i-1}^{-1} \qquad y_{i-2} x_{i-1}^{-1}} \right| x_i^{q_1} y_i,$$

$$\mathrm{SubC}_3(A_{m+4i-2}) = \mathrm{SubC}_3(A_{m+4i-4}) \cup \left\{y_{i-2}x_{i-1}^{-1}x_i,\, x_{i-1}^{-1}x_i^2,\, x_i^2 y_i,\, x_i y_i x_i,\, y_i x_i y_{i-1}^{-1},\, x_i^3\right\},$$

$$A_{m+4i-1} = \left|\frac{A_{m+4i-2}^{q_2-1}}{x_i y_{i-1}^{-1} \quad x_i y_i}\right|\frac{A_{m+4i-4}^{-q_0}}{x_i y_{i-1}^{-1} \quad y_{i-2}x_{i-1}^{-1}}\right| x_i^{q_1+1} y_i,$$

$$\mathrm{SubC}_3(A_{m+4i-1}) = \mathrm{SubC}_3(A_{m+4i-2}).$$

(4) *Below we list the reduced forms of elements* $x_i^{\phi_{m+4(i-1)+j}}$, $y_i^{\phi_{m+4(i-1)+j}}$ *for* $j = 0, \ldots, 4$. *Again, in the formulas below we assume that* $y_0 = x_1^{-1}$ *for* $m = 0$, *and* $y_0 = z_m$ *for* $m \neq 0$.

$$x_i^{\phi_{m+4i-4}} = A_{m+4i-4}^{-q_0} \circ x_i \circ A_{m+4i-4}^{q_0}, \qquad y_i^{\phi_{m+4i-4}} = A_{m+4i-4}^{-q_0} \circ y_i,$$

$$x_i^{\phi_{m+4i-3}} = x_i^{\phi_{m+4i-4}}, \qquad y_i^{\phi_{m+4i-3}} = A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ y_i,$$

$$x_i^{\phi_{m+4i-2}} = \left|\frac{A_{m+4i-2}^{q_2}}{x_i y_{i-1}^{-1} \quad x_i y_i}\right|\frac{A_{m+4i-4}^{-q_0}}{x_i y_{i-1}^{-1} \quad y_{i-2}x_{i-1}^{-1}}\right| x_i \left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1} \quad y_{i-1}x_i^{-1}}\right|,$$

$$y_i^{\phi_{m+4i-2}} = y_i^{\phi_{m+4i-3}},$$

$$x_i^{\phi_{m+4i-1}} = x_i^{\phi_{m+4i-2}} =_{(when\ i=n)} x_i^{\phi_K},$$

$$\mathrm{Sub}_3\left(x_i^{\phi_K}\right) =_{(when\ i=n)} \mathrm{SubC}_3(A_{m+4i-2}) \cup \mathrm{SubC}_3(A_{m+4i-4})^{\pm 1} \cup \left\{y_{i-2}x_{i-1}^{-1}x_i,\, x_{i-1}^{-1}x_i x_{i-1},\, x_i x_{i-1} y_{i-2}^{-1}\right\},$$

$$y_i^{\phi_{m+4i-1}} = \tilde{y}_i =_{(when\ i=n)} y_i^{\phi_K}$$
$$= \left|\frac{A_{m+4i-4}^{-q_0}}{x_i y_{i-1}^{-1} \quad y_{i-2}x_{i-1}^{-1}}\right|\left(x_i^{q_1} y_i \left|\frac{A_{m+4i-2}^{q_2-1}}{x_i y_{i-1}^{-1} \quad x_i y_i}\right|\frac{A_{m+4i-4}^{-q_0}}{x_i y_{i-1}^{-1} \quad y_{i-2}x_{i-1}^{-1}}\right| x_i\right)^{q_3} x_i^{q_1} y_i,$$

$$\mathrm{Sub}_3\left(\tilde{y}_i\right) = \mathrm{SubC}_3(A_{m+4i-2}) \cup \mathrm{SubC}_3(A_{m+4i-4})^{-1} \cup \left\{y_{i-2}x_{i-1}^{-1}x_i,\, x_{i-1}^{-1}x_i^2,\, x_i^3,\, x_i y_i x_i,\, y_i x_i y_{i-1}^{-1},\, x_i^2 y_i\right\}$$

$$x_i^{\phi_{m+4i}} =_{(when\ i\neq n)} x_i^{\phi_K}$$
$$= \left|\frac{A_{m+4i}^{-q_4+1}}{x_{i+1}y_i^{-1} \quad y_{i-1}x_i^{-1}}\right| x_{i+1} \circ y_i^{-1} x_i^{-q_1}$$
$$\circ \left(x_i^{-1}\left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1} \quad y_{i-1}x_i^{-1}}\right|\frac{A_{m+4i-2}^{-q_2+1}}{y_i^{-1}x_i^{-1} \quad y_{i-1}x_i^{-1}}\right| y_i^{-1} x_i^{-q_1}\right)^{q_3-1}\left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1} \quad y_{i-1}x_i^{-1}}\right|,$$

$$\mathrm{Sub}_3\left(x_i^{\phi_K}\right) = \mathrm{SubC}_3(A_{m+4i})^{-1} \cup \mathrm{SubC}_3(A_{m+4i-2})^{-1} \cup \mathrm{SubC}_3(A_{m+4i-4}) \cup \left\{y_{i-1}x_i^{-1}x_{i+1},\, x_i^{-1}x_{i+1}y_i^{-1},\, x_{i+1}y_i^{-1}x_i^{-1},\, y_i^{-1}x_i^{-2},\, x_i^{-3},\, x_i^{-2}x_{i-1},\, x_i^{-1}x_{i-1}y_{i-2}^{-1},\, y_{i-1}x_i^{-1}y_i^{-1},\, x_i^{-1}y_i^{-1}x_i^{-1}\right\},$$

$$y_i^{\phi_{m+4i}} =_{(\text{when } i\neq n)} y_i^{\phi_K} = \left|\underset{x_{i+1}y_i^{-1} \quad y_{i-1}x_i^{-1}}{A_{m+4i}^{-q_4+1}}\right| x_{i+1} \left|\underset{x_i y_{i-1}^{-1} \quad x_i y_i}{\tilde{y}_i}\right| x_{i+1}^{-1} \left|\underset{x_i y_{i-1}^{-1} \quad y_i x_{i+1}^{-1}}{A_{m+4i}^{q_4-1}}\right|,$$

$$\begin{aligned}&\mathrm{Sub}_3\big(y_i^{\phi_K}\big)\\ &\quad = \mathrm{SubC}_3(A_{m+4i})^{\pm 1} \cup \mathrm{Sub}_3\big(\tilde{y}_i\big)\\ &\qquad \cup \big\{y_{i-1}x_i^{-1}x_{i+1}, x_i^{-1}x_{i+1}x_i, x_{i+1}x_i y_{i-1}^{-1}, x_i y_i x_{i+1}^{-1}, y_i x_{i+1}^{-1}x_i, x_{i+1}^{-1}x_i y_{i-1}^{-1}\big\}.\end{aligned}$$

$$(5)\quad A_j = \begin{cases} A(\gamma_j)^{\phi_{j-1}}, & \text{if } j \neq m+4i-1, m+4i-3 \\ & \text{for any } i = 1, \dots, n, \\ A_{m+4i-4}^{p_{m+4i-4}} A(\gamma_j)^{\phi_{j-1}} A_{m+4i-4}^{-p_{m+4i-4}}, & \text{if } j = m+4i-3 \text{ for some } i = 1, \dots, n \\ & (m+4i-4 \neq 0), \\ y_i^{-\phi_{j-1}} A(\gamma_j)^{\phi_{j-1}} y_i^{\phi_{j-1}}, & \text{if } j = m+4i-1 \text{ for some } i = 1, \dots, n. \end{cases}$$

Proof. Statement (1) is obvious. We prove statement (2) by induction on $i \geqslant 2$. Notice that by Lemmas 45 and 46, $\tilde{y}_1 = y_1^{\phi_{m+3}}$ begins with $x_1 z_m^{-1}$ and ends with $x_1 y_1$. Now let $i \geqslant 2$. Then denoting exponents by q_i as in (3), we have

$$\tilde{y}_i = y_i^{\phi_{m+4i-1}} = \big(x_i^{q_3} y_i\big)^{\phi_{m+4i-2}} = \big(\big(y_i^{q_2} x_i\big)^{q_3} y_i\big)^{\phi_{m+4i-3}} = \big(\big(\big(x_i^{q_1} y_i\big)^{q_2} x_i\big)^{q_3} x_i^{q_1} y_i\big)^{\phi_{m+4i-4}}.$$

Before we continue, and to avoid huge formulas, we compute separately $x_i^{\phi_{m+4i-4}}$ and $y_i^{\phi_{m+4i-4}}$:

$$x_i^{\phi_{m+4i-4}} = \Big(x_i^{(y_{i-1}x_i^{-1})^{q_0}}\Big)^{\phi_{m+4(i-1)-1}} = x_i^{(\tilde{y}_{i-1}x_i^{-1})^{q_0}} = \left|\underset{x_i y_{i-1}^{-1} \qquad y_{i-1}x_i^{-1}}{(x_i \tilde{y}_{i-1}^{-1})^{q_0} \circ x_i \circ (\tilde{y}_{i-1}x_i^{-1})^{q_0}}\right|,$$

by induction (by Lemmas 45 and 46 in the case $i = 2$);

$$y_i^{\phi_{m+4i-4}} = \big(\big(y_{i-1}x_i^{-1}\big)^{-q_0} y_i\big)^{\phi_{m+4(i-1)-1}} = \big(\tilde{y}_{i-1}x_i^{-1}\big)^{-q_0} y_i = \big(x_i \circ \tilde{y}_{i-1}^{-1}\big)^{q_0} \circ y_i,$$

beginning with $x_i y_{i-1}^{-1}$ and ending with $x_{i-1}^{-1} y_i$. It follows that

$$\big(x_i^{q_1} y_i\big)^{\phi_{m+4i-4}} = \big(x_i \tilde{y}_{i-1}^{-1}\big)^{q_0} x_i^{q_1} \big(\tilde{y}_{i-1}x_i^{-1}\big)^{q_0} \big(x_i \tilde{y}_{i-1}^{-1}\big)^{q_0} y_i = \big(x_i \tilde{y}_{i-1}^{-1}\big)^{q_0} \circ x_i^{q_1} \circ y_i,$$

beginning with $x_i y_{i-1}^{-1}$ and ending with $x_i y_i$. Now looking at the formula

$$\tilde{y}_i = \big(\big(\big(x_i^{q_1} y_i\big)^{q_2} x_i\big)^{q_3} x_i^{q_1} y_i\big)^{\phi_{m+4i-4}}$$

it is obvious that $\tilde{y}_i$ begins with $x_i y_{i-1}^{-1}$ and ends with $x_i y_i$, as required.

Now we prove statements (3) and (4) simultaneously.

$$A_{m+4i-4} = \mathrm{cycred}\big(\big(y_{i-1}x_i^{-1}\big)^{\phi_{m+4(i-1)-1}}\big) = \tilde{y}_{i-1} \circ x_i^{-1},$$

beginning with x_{i-1} and ending with x_i^{-1}. As we have observed in proving (2)

$$x_i^{\phi_{m+4i-4}} = \big(x_i \tilde{y}_{i-1}^{-1}\big)^{q_0} \circ x_i \circ \big(\tilde{y}_{i-1}x_i^{-1}\big)^{q_0} = A_{m+4i-4}^{-q_0} \circ x_i \circ A_{m+4i-4}^{q_0},$$

beginning with x_i and ending with x_i^{-1};

$$y_i^{\phi_{m+4i-4}} = \big(x_i \circ \tilde{y}_{i-1}^{-1}\big)^{q_0} \circ y_i = A_{m+4i-4}^{-q_0} \circ y_i,$$

beginning with x_i and ending with y_i. Now

$$A_{m+4i-3} = \mathrm{cycred}\big(x_i^{\phi_{m+4i-4}}\big) = x_i,$$

beginning with x_i and ending with x_i;

$$x_i^{\phi_{m+4i-3}} = x_i^{\phi_{m+4i-4}},$$

$$\begin{aligned} y_i^{\phi_{m+4i-3}} &= \big(x_i^{q_1} y_i\big)^{\phi_{m+4i-4}} = A_{m+4i-4}^{-q_0} x_i^{q_1} A_{m+4i-4}^{q_0} A(\phi_{m+4i-4})^{-q_0} y_i \\ &= A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ y_i, \end{aligned}$$

beginning with x_i and ending with y_i.

Now

$$A_{m+4i-2} = y_i^{\phi_{m+4i-3}},$$
$$x_i^{\phi_{m+4i-2}} = \big(y_i^{q_2} x_i\big)^{\phi_{m+4i-3}} = A_{m+4i-2}^{q_2} \circ A_{m+4i-4}^{-q_0} \circ x_i \circ A_{m+4i-4}^{q_0},$$

beginning with x_i and ending with x_i^{-1}. It is also convenient to rewrite $x_i^{\phi_{m+4i-2}}$ (by rewriting the subword A_{m+4i-2}) to show its cyclically reduced form:

$$x_i^{\phi_{m+4i-2}} = A_{m+4i-4}^{-q_0} \circ \big(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i\big) \circ A_{m+4i-4}^{q_0},$$
$$y_i^{\phi_{m+4i-2}} = y_i^{\phi_{m+4i-3}}.$$

Now we can write down the next set of formulas:

$$\begin{aligned} A_{m+4i-1} &= \mathrm{cycred}\big(y_i^{-\phi_{m+4i-3}} x_i^{\phi_{m+4i-2}} y_i^{\phi_{m+4i-3}}\big) \\ &= \mathrm{cycred}\big(A_{m+4i-2}^{-1} A_{m+4i-2}^{q_2} A_{m+4i-4}^{-q_0} x_i A_{m+4i-4}^{q_0} A_{m+4i-2}\big) \\ &= A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i^{q_1+1} \circ y_i, \end{aligned}$$

beginning with x_i and ending with y_i;

$$x_i^{\phi_{m+4i-1}} = x_i^{\phi_{m+4i-2}},$$
$$y_i^{\phi_{m+4i-1}} = \tilde{y}_i = \left(x_i^{q_3} y_i\right)^{\phi_{m+4i-2}} = \left(x_i^{\phi_{m+4i-2}}\right)^{q_3} y_i^{\phi_{m+4i-2}}$$

substituting the cyclic decomposition of $x_i^{\phi_{m+4i-2}}$ from above one has

$$= A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i\right)^{q_3} \circ x_i^{q_1} \circ y_i,$$

beginning with x_i and ending with y_i.

Finally

$$A_{m+4i} = \operatorname{cycred}\left(\left(y_i x_{i+1}^{-1}\right)^{\phi_{m+4i-1}}\right) = \tilde{y}_i \circ x_{i+1}^{-1},$$

beginning with x_i and ending with x_{i+1}^{-1};

$$x_i^{\phi_{m+4i}} = \left(\left(y_i x_{i+1}^{-1}\right)^{-q_4} x_i\right)^{\phi_{m+4i-1}} = \left(\tilde{y}_i x_{i+1}^{-1}\right)^{-q_4} x_i^{\phi_{m+4i-1}} = A_{m+4i}^{-q_4+1} x_{i+1} \tilde{y}_i^{-1} x_i^{\phi_{m+4i-1}}$$
$$= A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ \left(\left(x_i^{\phi_{m+4i-2}}\right)^{q_3-1} y_i^{\phi_{m+4i-2}}\right)^{-1}.$$

Observe that computations similar to that for $y_i^{\phi_{m+4i-1}}$ show that

$$\left(\left(x_i^{\phi_{m+4i-2}}\right)^{q_3-1} y_i^{\phi_{m+4i-2}}\right)^{-1}$$
$$= \left(A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i\right)^{q_3-1} \circ x_i^{q_1} \circ y_i\right)^{-1}.$$

Therefore

$$x_i^{\phi_{m+4i}} = A_{m+4i}^{-q_4+1} \circ x_{i+1}$$
$$\circ \left(A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i\right)^{q_3-1} \circ x_i^{q_1} \circ y_i\right)^{-1},$$

beginning with x_{i+1} and ending with x_i^{-1};

$$y_i^{\phi_{m+4i}} = \left(y_i^{(y_i x_{i+1}^{-1})^{q_4}}\right)^{\phi_{m+4i-1}} = \left(x_{i+1} \tilde{y}_i^{-1}\right)^{q_4} \tilde{y}_i \left(\tilde{y}_i x_{i+1}^{-1}\right)^{q_4}$$
$$= A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ \tilde{y}_i \circ x_{i+1}^{-1} \circ A_{m+4i}^{q_4-1},$$

beginning with x_{i+1} and ending with x_{i+1}^{-1}.

(5) If $j \neq m+4i-1, m+4i-3$, A_j is either $(c_j^{z_j^{\phi_j}} c_{j+1}^{z_{j+1}^{\phi_j}})$ or $y_i^{\phi_j}$ $(j = m+4i-3)$ or $(y_i x_{i+1}^{-1})^{\phi_j}$ $(j = m+4i-1)$. In all these cases

$$A_j = A(\gamma_j)^{\phi_{j-1}}.$$

The formulas for the other two cases can be found in the proof of statement (2). This finishes the proof of the lemma. □

Lemma 48. *Let* $m \geqslant 1$, $K = K(m,n)$, $p = (p_1, \dots, p_K)$ *be a* 3*-large tuple*, $\phi_K = \gamma_K^{p_K} \dots \gamma_1^{p_1}$, *and* $X^{\pm\phi_K} = \{x^{\phi_K} \mid x \in X^{\pm 1}\}$. *Then the following holds*:

$$(1)\ \mathrm{Sub}_2\big(X^{\pm\phi_K}\big) = \left\{\begin{array}{ll} c_j z_j, z_j^{-1} c_j, & 1 \leqslant j \leqslant m, \\ z_j z_{j+1}^{-1}, & 1 \leqslant j \leqslant m-1, \\ z_m x_1^{-1}, z_m x_1, & \text{if } m \neq 0,\ n \neq 0, \\ x_i^2, x_i y_i, y_i x_i, & 1 \leqslant i \leqslant n, \\ x_{i+1} y_i^{-1}, x_i^{-1} x_{i+1}, x_{i+1} x_i & 1 \leqslant i \leqslant n-1 \end{array}\right\}^{\pm 1};$$

moreover, the word $z_j^{-1} c_j$, *as well as* $c_j z_j$, *occurs only as a part of the subword* $(z_j^{-1} c_j z_j)^{\pm 1}$ *in* x^{ϕ_K} $(x \in X^{\pm 1})$;

$$(2)\ \mathrm{Sub}_3\big(X^{\pm\phi_K}\big) = \left\{\begin{array}{ll} z_j^{-1} c_j z_j, & 1 \leqslant j \leqslant m, \\ c_j z_j z_{j+1}^{-1}, z_j z_{j+1}^{-1} c_{j+1}, & 1 \leqslant j \leqslant m-1, \\ z_j z_{j+1}^{-1} c_{j+1}^{-1}, & 2 \leqslant j \leqslant m-1, \\ y_1 x_1^2, & m = 0,\ n = 1, \\ x_2^{-1} x_1^2, x_2 x_1^2, & m = 0,\ n \geqslant 2, \\ c_m^{-1} z_m x_1, & m = 1,\ n \neq 0, \\ c_m z_m x_1^{-1}, z_m x_1^{-1} z_m^{-1}, z_m x_1^2, z_m x_1^{-1} y_1^{-1}, & m \neq 0,\ n \neq 0, \\ c_m z_m x_1, & m \geqslant 2,\ n \neq 0, \\ z_m x_1^{-1} x_2, z_m x_1^{-1} x_2^{-1}, & m \neq 0,\ n \geqslant 2, \\ c_1^{-1} z_1 z_2^{-1}, & m \geqslant 2, \\ x_i^3, x_i^2 y_i, x_i y_i x_i, & 1 \leqslant i \leqslant n, \\ x_i^{-1} x_{i+1} x_i, y_i x_{i+1}^{-1} x_i, x_i y_i x_{i+1}^{-1}, & 1 \leqslant i \leqslant n-1, \\ x_{i-1}^{-1} x_i^2, y_i x_i y_{i-1}^{-1}, & 2 \leqslant i \leqslant n, \\ y_{i-2} x_{i-1}^{-1} x_i^{-1}, y_{i-2} x_{i-1}^{-1} x_i, & 3 \leqslant i \leqslant n \end{array}\right\}^{\pm 1};$$

(3) *for any* 2*-letter word* $uv \in \mathrm{Sub}_2(X^{\pm\phi_K})$ *one has*

$$\mathrm{Sub}_2\big(u^{\phi_K} v^{\phi_K}\big) \subseteq \mathrm{Sub}_2\big(X^{\pm\phi_K}\big) \cup \big\{c_i^2\big\}, \qquad \mathrm{Sub}_3\big(u^{\phi_K} v^{\phi_K}\big) \subseteq \mathrm{Sub}_3\big(X^{\pm\phi_K}\big) \cup \big\{c_i^2 z_i\big\}.$$

Proof. (1) and (2) follow by straightforward inspection of the reduced forms of elements x^{ϕ_K} in Lemmas 44–47.

To prove (3) it suffices for every word $uv \in \mathrm{Sub}_2(X^{\pm\phi_K})$ to write down the product $u^{\phi_K} v^{\phi_K}$ (using formulas from the lemmas mentioned above), then make all possible cancellations and check whether 3-subwords of the resulting word all lie in $\mathrm{Sub}_3(X^{\pm\phi_K})$. Now we do the checking one by one for all possible words from $\mathrm{Sub}_2(X^{\pm\phi_K})$.

(1) For $uv \in \{c_j z_j, z_j^{-1} c_j\}$ the checking is obvious and we omit it.

(2) Let $uv = z_j z_{j+1}^{-1}$. Then there are three cases to consider.

(a) Let $j \leqslant m-2$, then

$$\left(z_j z_{j+1}^{-1}\right)^{\phi_K} = \begin{array}{|c|c|} z_j^{\phi_K} & z_{j+1}^{-\phi_K} \\ \hline * \quad c_{j+1} z_{j+1} & z_{j+2}^{-1} c_{j+2}^{-1} \quad * \end{array};$$

in this case there is no cancellation in $u^{\phi_K} v^{\phi_K}$. All 3-subwords of u^{ϕ_K} and v^{ϕ_K} are obviously in $\mathrm{Sub}_3(X^{\pm\phi_K})$. So one needs only to check the new 3-subwords which arise "in between" u^{ϕ_K} and v^{ϕ_K} (below we will check only subwords of this type). These subwords are $c_{j+1} z_{j+1} z_{j+2}^{-1}$ and $z_{j+1} z_{j+2}^{-1} c_{j+2}^{-1}$ which both lie in $\mathrm{Sub}_3(X^{\pm\phi_K})$.

(b) Let $j = m-1$ and $n \neq 0$. Then

$$\left(z_{m-1} z_m^{-1}\right)^{\phi_K} = \begin{array}{|c|c|} z_{m-1}^{\phi_K} & z_m^{-\phi_K} \\ \hline * \quad c_m z_m & x_1 z_m^{-1} \quad * \end{array};$$

again, there is no cancellation in this case and the words "in between" are $c_m z_m x_1$ and $z_m x_1 z_m^{-1}$, which are in $\mathrm{Sub}_3(X^{\pm\phi_K})$.

(c) Let $j = m-1$ and $n = 0$. Then (below we put $\cdot$ at the place where the corresponding initial segment of u^{ϕ_K} and the corresponding terminal segment of v^{ϕ_K} meet)

$$\left(z_{m-1} z_m^{-1}\right)^{\phi_K} = z_{m-1}^{\phi_K} \cdot z_m^{-\phi_K} = c_{m-1} z_{m-1} A_{m-2}^{p_{m-2}} c_m^{z_m} A_{m-1}^{p_{m-1}-1} \cdot A_{m-1}^{-p_{m-1}} z_m^{-1}$$

(cancelling $A_{m-1}^{p_{m-1}-1}$ and substituting for A_{m-1}^{-1} its expression via the leading terms)

$$\begin{aligned} &= c_{m-1} z_{m-1} A_{m-2}^{p_{m-2}} c_m^{z_m} \cdot \left(c_m^{-z_m} A_{m-2}^{-p_{m-2}} c_{m-1}^{-z_{m-1}} A_{m-2}^{p_{m-2}}\right) z_m^{-1} \\ &= z_{m-1} \begin{array}{|c|} A_{m-2}^{p_{m-2}} \\ \hline z_{m-2}^{-1} \quad * \end{array} z_m^{-1}. \end{aligned}$$

Here $z_{m-1}^{\phi_K}$ is completely cancelled.

(3)(a) Let $n = 1$. Then

$$\begin{aligned} \left(z_m x_1^{-1}\right)^{\phi_K} &= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m-1} \cdot A_m^{-p_m} x_1^{-1} A_m^{p_m} A_{m+2}^{p_{m+2}} \\ &= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} \cdot x_1 A_{m-1}^{-p_{m-1}} c_m^{-z_m} A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m} A_{m+2}^{p_{m+2}} \\ &= \begin{array}{|l|} z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m} A_{m+2}^{p_{m+2}} \\ \hline z_m z_{m-1}^{-1} \end{array}, \end{aligned}$$

and $z_m^{\phi_K}$ is completely cancelled.

(b) Let $n > 1$. Then

$$
\begin{aligned}
&\left(z_m x_1^{-1}\right)^{\phi_K} \\
&\quad = c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m - 1} \\
&\qquad \times A_m^{-p_m} \left(x_1^{-1} A_m^{p_m} A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}}\right)^{-p_{m+3}+1} x_1^{p_{m+1}} y_1 x_2^{-1} A_{m+4}^{p_{m+4}-1} \\
&\quad = c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{-1} \left(x_1^{-1} A_m^{p_m} A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}}\right)^{-p_{m+3}+1} x_1^{p_{m+1}} y_1 x_2^{-1} A_{m+4}^{p_{m+4}-1} \\
&\quad = c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} \cdot x_1 A_{m-1}^{-p_{m-1}} c_m^{-z_m} \\
&\qquad \times A_{m-1}^{p_{m-1}} \left(x_1^{-1} A_m^{p_m} A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}}\right)^{-p_{m+3}+1} x_1^{p_{m+1}} y_1 x_2^{-1} A_{m+4}^{p_{m+4}-1} \\
&\quad = \left|\begin{array}{c} z_m A_{m-1}^{p_{m-1}} \\ \hline z_m z_{m-1}^{-1} c_{m-1}^{-1} \end{array}\right|,
\end{aligned}
$$

and $z_m^{\phi_K}$ is completely cancelled.

(4)(a) Let $n = 1$. Then

$$
(z_m x_1)^{\phi_K} = c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m - 1} \cdot A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} = c_m \left|\begin{array}{c} z_m A_{m-1}^{p_{m-1}} ** \\ \hline z_m z_{m-1}^{-1} c_{m-1}^{-1} \quad * \end{array}\right|,
$$

and $z_m^{\phi_K}$ is completely cancelled.

(b) Let $n > 1$. Then

$$
(z_m x_1)^{\phi_K} = \left|\begin{array}{c|c} z_m^{\phi_K} & x_1^{\phi_K} \\ \hline * \quad z_m x_1^{-1} & x_2 y_1^{-1} \quad * \end{array}\right|.
$$

(5)(a) Let $n = 1$. Then

$$
\begin{aligned}
x_1^{2\phi_K} &= A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \cdot A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \\
&= A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \cdot \left(A_m^{-p_m} x_1^{p_{m+1}} y_1\right) A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1 A_m^{p_m} \\
&= A_{m+2}^{p_{m+2}} \left|\begin{array}{c} A_m^{-p_m} x_1 \\ \hline * \quad z_m x_1 \end{array}\right| \cdot x_1^{p_{m+1}} y_1 **.
\end{aligned}
$$

(b) Let $n > 1$. Then

$$
x_1^{2\phi_K} = \left|\begin{array}{c|c} x_1^{\phi_K} & x_1^{\phi_K} \\ \hline z_m x_1^{-1} & x_2 y_1^{-1} \end{array}\right|.
$$

(6)(a) Let $1 < i < n$. Then

$$\begin{aligned}
x_i^{2\phi_K} &= A_{m+4i}^{-q_4+1} x_{i+1} y_i^{-1} x_i^{-q_1} \left(x^{-1} A_{m+4i-4}^{q_0} A_{m+4i-2}^{-q_2+1} y_i^{-1} x_i^{-q_1}\right)^{q_3-1} \\
&\quad \times \left|\begin{array}{c} A_{m+4i-4}^{q_0} \\ \hline {\scriptstyle y_{i-1} x_i^{-1}} \end{array}\right| \cdot \left|\begin{array}{c} A_{m+4i}^{-q_4+1} \\ \hline {\scriptstyle x_{i+1} y_i^{-1}} \end{array}\right| **.
\end{aligned}$$

$$\begin{aligned}
\text{(b) } x_n^{2\phi_K} &= A_{m+4n-2}^{q_2} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \cdot A_{m+4n-2}^{q_2} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&= A_{m+4n-2}^{q_2} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&\quad \cdot A_{m+4n-4}^{-q_0} x_n^{q_1} y_n A_{m+4n-2}^{q_2-1} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&= A_{m+4n-2}^{q_2} \left|\begin{array}{c} A_{m+4n-4}^{-q_0} x_n \\ \hline {\scriptstyle x_{n-1}^{-1} x_n} \end{array}\right| \cdot x_n^{q_1} **.
\end{aligned}$$

(7)(a) Let $n = 1$. Then

$$(x_1 y_1)^{\phi_K} = A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 \cdot x_1^{p_{m+1}} **.$$

(b) Let $n > 1$. Then

$$(x_1 y_1)^{\phi_K} = \left|\begin{array}{c|c} x_1^{\phi_K} & y_1^{\phi_K} \\ \hline {\scriptstyle z_m x_1^{-1}} & {\scriptstyle x_2 y_1^{-1}} \end{array}\right|.$$

(c) Let $1 < i < n$. Then

$$(x_i y_i)^{\phi_K} = \left|\begin{array}{c|c} x_i^{\phi_K} & y_i^{\phi_K} \\ \hline {\scriptstyle y_{i-1} x_i^{-1}} & {\scriptstyle x_{i+1} y_i^{-1}} \end{array}\right|.$$

(d) Let $n > 1$. Then

$$(x_n y_n)^{\phi_K} = \left|\begin{array}{c|c} x_n^{\phi_K} & y_n^{\phi_K} \\ \hline {\scriptstyle x_{n-1}^{-1} x_n} & {\scriptstyle x_n^2} \end{array}\right|.$$

(8)(a) Let $n = 1$. Then

$$(y_1 x_1)^{\phi_K} = \left|\begin{array}{c|c} y_1^{\phi_K} & x_1^{\phi_K} \\ \hline {\scriptstyle x_1 y_1} & {\scriptstyle x_1 z_m^{-1}} \end{array}\right|.$$

(b) Let $n > 1$. Then

$$\begin{aligned}
(y_1 x_1)^{\phi_K} &= A_{m+4}^{-p_{m+4}+1} x_2 A_{m+4}^{p_{m+4}} \cdot A_{m+4}^{-p_{m+4}+1} x_2 y_1^{-1} x_1^{-p_{m+1}} \circ ** \\
&= A_{m+4}^{-p_{m+4}+1} x_2 A_{m+4} \cdot x_2 y_1^{-1} x_1^{-p_{m+1}} \circ ** \\
&= A_{m+4}^{-p_{m+4}+1} x_2 A_m^{-p_m} \left(x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1\right)^{p_{m+3}} x_1^{p_{m+1}} y_1 x_2^{-1} \\
&\quad \cdot x_2 y_1^{-1} x_1^{-p_{m+1}} (\,)^{p_{m+3}-1} A_m^{p_m} \\
&= A_{m+4}^{-p_{m+4}+1} x_2 A_m^{-p_m} \left(x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1}\right) \left|\begin{array}{c|c} A_m^{-p_m} x_1 & A_m^{p_m} \\ \hline {\scriptstyle z_m x_1} & {\scriptstyle z_m^{-1} c_m^{-1}} \end{array}\right|.
\end{aligned}$$

(c) $(y_n x_n)^{\phi_K} = \left|\begin{array}{c|c} y_n^{\phi_K} & x_n^{\phi_K} \\ \hline {\scriptstyle x_n y_n} & {\scriptstyle x_n y_{n-1}^{-1}} \end{array}\right|.$

(9)(a) If $n = 2$, then $(x_2 y_1^{-1})^{\phi_K} = A_{m+6}^{q_2} A_{m+4}^{-1}$.
(b) If $n > 2$, $1 < i < n$. Then

$$
\begin{aligned}
\left(x_i y_{i-1}^{-1}\right)^{\phi_K} = {} & \left|\begin{array}{c} A_{m+4i}^{-q_4+1} \\ \hline {\scriptstyle x_{i+1} y_i^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \\
& \circ \left(x_i^{-1} \left|\begin{array}{c|c} A_{m+4i-4}^{q_0} & A_{m+4i-2}^{-q_2+1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} & {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \\
& \circ \left|\begin{array}{c} A_{m+4i-4}^{q_0} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| \cdot A_{m+4i-4}^{-q_0+1} \circ x_i \circ \tilde{y}_{i-1} \circ x_i^{-1} \left|\begin{array}{c} A_{m+4i-4}^{q_0-1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| \\
= {} & \left|\begin{array}{c} A_{m+4i}^{-q_4+1} \\ \hline {\scriptstyle x_{i+1} y_i^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \\
& \circ \left(x_i^{-1} \left|\begin{array}{c|c} A_{m+4i-4}^{q_0} & A_{m+4i-2}^{-q_2+1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} & {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \\
& \circ x_i^{-1} \left|\begin{array}{c} A_{m+4i-4}^{q_0-1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right|.
\end{aligned}
$$

(c) $\left(x_n y_{n-1}^{-1}\right)^{\phi_K} = \left|\begin{array}{c|c} A_{m+4n-2}^{q_2} & A_{m+4n-4}^{-1} \\ \hline {\scriptstyle x_n y_n} & {\scriptstyle x_n y_{n-1}^{-1}} \end{array}\right|.$

(10)(a) Let $n = 2$, then

$$
\begin{aligned}
\left(x_1^{-1} x_2\right)^{\phi_K} = {} & A_m^{-p_m} \left(x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1\right)^{p_{m+3}-1} x_1^{p_{m+1}} y_1 x_2^{-1} \\
& \times A_{m+4}^{p_{m+4}-1} A_{m+6}^{p_{m+6}} A_{m+4}^{-p_{m+4}} x_2 A_{m+4}^{p_{m+4}} \\
= {} & A_m^{-p_m} \left(x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1\right)^{p_{m+3}-1} x_1^{p_{m+1}} y_1 x_2^{-1} \\
& \times A_{m+4}^{p_{m+4}-1} \left(A_{m+4}^{-p_{m+4}} x_2^{p_{m+5}} y_2\right)^{p_{m+6}} A_{m+4}^{-p_{m+4}} x_2 A_{m+4}^{p_{m+4}} \\
= {} & A_m^{-p_m} \left(x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1\right)^{p_{m+3}-1} x_1^{p_{m+1}} y_1 x_2^{-1} \\
& \times A_{m+4}^{-1} x_2^{p_m+5} y_2 \left(A_{m+4}^{-p_{m+4}} x_2^{p_{m+5}} y_2\right)^{p_{m+6}-1} A_{m+4}^{-p_{m+4}} x_2 A_{m+4}^{p_{m+4}} \\
= {} & \left|\begin{array}{c|c} A_m^{-p_m} & x_1^{-1} A_m^{p_m} \\ \hline {\scriptstyle c_m z_m} & {\scriptstyle x_1^{-1} z_m^{-1}} \end{array}\right| A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}} \\
& \times A_m^{p_m} x_2^{p_{m+5}} y_2 \left(A_{m+4}^{-p_{m+4}} x_2^{p_{m+5}} y_2\right)^{p_{m+6}-1} A_{m+4}^{-p_{m+4}} x_2 A_{m+4}^{p_{m+4}}.
\end{aligned}
$$

(b) If $1 \leqslant i < n-1$, then

$$\left(x_i^{-1}x_{i+1}\right)^{\phi_K} = \left|\begin{array}{c|c} x_i^{-\phi_K} & x_{i+1}^{\phi_K} \\ \hline y_i x_{i+1}^{-1} & x_{i+2} y_{i+1}^{-1} \end{array}\right|.$$

(c) Similarly to (10)(a) we get

$$\left(x_{n-1}^{-1}x_n\right)^{\phi_K} = \left|\begin{array}{c} A_{m+4n-8}^{-p_{m+4n-8}} \\ \hline y_{n-3}x_{n-2}^{-1} \end{array}\right| \cdot \left|\begin{array}{c} x_{n-1}^{-1} A_{m+4n-8}^{p_{m+4n-8}} \\ \hline x_{n-1}^{-1}x_{n-2} \end{array}\right| A_{m+4n-6}^{-p_{m+4n-6}-1} **.$$

(11)(a) If $1 < i < n-1$, then

$$\begin{aligned}
&(x_{i+1}x_i)^{\phi_K} \\
&\quad = A_{m+4i+4}^{-q_8+1} x_{i+2} y_{i+1}^{-1} x_{i+1}^{-q_5} \left(x_{i+1}^{-1} A_{m+4i}^{q_4} A_{m+4i+2}^{-q_6+1} y_{i+1}^{-1} x_{i+1}^{-q_5}\right)^{q_7-1} A_{m+4i}^{q_4} \\
&\qquad \times A_{m+4i}^{-q_4+1} x_{i+1} y_i^{-1} x_i^{-q_1} \left(x_i^{-1} A_{m+4i-4}^{q_0} A_{m+4i-2}^{-q_2+1} y_i^{-1} x_i^{-q_1}\right)^{q_3-1} A_{m+4i-4}^{q_0} \\
&\quad = A_{m+4i+4}^{-q_8+1} x_{i+2} y_{i+1}^{-1} x_{i+1}^{-q_5} \left(x_{i+1}^{-1} A_{m+4i}^{q_4} A_{m+4i+2}^{-q_6+1} y_{i+1}^{-1} x_{i+1}^{-q_5}\right)^{q_7-1} A_{m+4i} \\
&\qquad \times x_{i+1} y_i^{-1} x_i^{-q_1} \left(x_i^{-1} A_{m+4i-4}^{q_0} A_{m+4i-2}^{-q_2+1} y_i^{-1} x_i^{-q_1}\right)^{q_3-1} A_{m+4i-4}^{q_0} \\
&\quad = A_{m+4i+4}^{-q_8+1} x_{i+2} y_{i+1}^{-1} x_{i+1}^{-q_5} \left(x_{i+1}^{-1} A_{m+4i}^{q_4} A_{m+4i+2}^{-q_6+1} y_{i+1}^{-1} x_{i+1}^{-q_5}\right)^{q_7-1} \\
&\qquad \times A_{m+4i-4}^{-q_0} x_i^{q_1} y_i A_{m+4i-2}^{q_2-1} \left|\begin{array}{c|c} A_{m+4i-4}^{-q_0} x_i & A_{m+4i-4}^{q_0} \\ \hline x_{i-1}^{-1} x_i & x_{i-1} y_{i-2}^{-1} \end{array}\right|.
\end{aligned}$$

(b) If $n > 2$, then

$$(x_2x_1)^{\phi_K} = ** \left|\begin{array}{c|c} A_m^{-q_0} x_1 & A_m^{q_0} \\ \hline z_m x_1 & z_m^{-1} c_m^{-1} \end{array}\right|.$$

(c)
$$\begin{aligned}
(x_n x_{n-1})^{\phi_K} &= A_{m+4n-2}^{q_6} A_{m+4n-4}^{-q_4} x_n A_{m+4n-4}^{q_4} \cdot A_{m+4n-4}^{-q_4+1} \\
&\quad \times x_n y_{n-1}^{-1} x_{n-1}^{-q_1} \left(x_{n-1}^{-1} A_{m+4n-8}^{q_0} A_{m+4n-6}^{-q_2+1} y_{n-1}^{-1} x_{n-1}^{-q_1}\right)^{q_3-1} A_{m+4n-8}^{q_0} \\
&= ** \left|\begin{array}{c} A_{m+4n-8}^{-q_0} x_{n-1} \\ \hline x_{n-2}^{-1} x_{n-1} \end{array}\right| \cdot \left|\begin{array}{c} A_{m+4n-8}^{q_0} \\ \hline x_{n-2} y_{n-3}^{-1} \end{array}\right|.
\end{aligned}$$

(d) Similarly, if $n = 2$, then

$$(x_2x_1)^{\phi_K} = ** \left|\begin{array}{c|c} A_m^{-p_m} x_1 & A_m^{p_m} \\ \hline z_m x_1 & z_m^{-1} c_m^{-1} \end{array}\right|.$$

This proves the lemma. □

Notation. Denote by Y the following set of words:

(1) if $n \neq 0$ and for $n = 1$, $m \neq 1$, then

$$Y = \{x_i, y_i, c_j^{z_j} \mid i = 1, \dots, n,\ j = 1, \dots, m\};$$

(2) if $n = 0$, denote the element $c_1^{z_1} \dots c_m^{z_m} \in F(X \cup C_S)$ by a new letter d, then

$$Y = \{c_1^{z_1}, \dots, c_{m-1}^{z_{m-1}}, d\};$$

a reduced word in this alphabet is a word that does not contain subwords $(c_1^{-z_1} d)^{\pm 1}$ and $(d c_m^{-z_m})^{\pm 1}$;

(3) if $n = 1$, $m = 1$, then

$$Y = \{A_1, x_1, y_1\};$$

a reduced word in this alphabet is a word that does not contain subwords $(A_1 x_1)^{\pm 1}$.

Lemma 49. *Let $m \geqslant 3$, $n = 0$, $K = K(m, 0)$. Let $p = (p_1, \dots, p_K)$ be a 3-large tuple, $\phi_K = \gamma_K^{p_K} \dots \gamma_1^{p_1}$, and $X^{\pm\phi_K} = \{x^{\phi_K} \mid x \in X^{\pm 1}\}$. Then the following holds*:

(1) *Every element from X^{ϕ_K} can be uniquely presented as a reduced product of elements and their inverses from the set*

$$X \cup \{c_1, \dots, c_{m-1}, d\}.$$

Moreover:

- *all elements $z_i^{\phi_K}$, $i \neq m$ have the form $z_i^{\phi_K} = c_i z_i \hat{z}_i$, where $\hat{z}_i$ is a reduced word in the alphabet Y,*
- *$z_m^{\phi_K} = z_m \hat{z}_m$, where $\hat{z}_m$ is a reduced word in the alphabet Y.*

When viewing elements from X^{ϕ_K} as elements in

$$F\big(X \cup \{c_1, \dots, c_{m-1}, d\}\big),$$

the following holds:

(2) $$\mathrm{Sub}_2\left(X^{\pm\phi_K}\right) = \left\{\begin{array}{ll} c_j z_j, & 1 \leqslant j \leqslant m, \\ z_j^{-1} c_j, z_j z_{j+1}^{-1}, & 1 \leqslant j \leqslant m-1, \\ z_2 d, d z_{m-1}^{-1} & \end{array}\right\}^{\pm 1}.$$

Moreover:

- *the word $z_m z_{m-1}^{-1}$ occurs only in the beginning of $z_m^{\phi_K}$ as a part of the subword $z_m z_{m-1}^{-1} c_{m-1}^{-1} z_{m-1}$,*

- *the words $z_2 d$, dz_{m-1}^{-1} occur only as parts of subwords*

$$\left(c_1^{z_1}c_2^{z_2}\right)^2 dz_{m-1}^{-1}c_{m-1}^{-1}z_{m-1}c_{m-1}$$

and $(c_1^{z_1}c_2^{z_2})^2 d$.

(3) $$\mathrm{Sub}_3\left(X^{\pm\phi_K}\right) = \left\{ \begin{array}{ll} z_j^{-1}c_j z_j,\, c_j z_j z_{j+1}^{-1},\, z_j z_{j+1}^{-1}c_{j+1}^{-1}, & 1 \leqslant j \leqslant m-1, \\ z_j z_{j+1}^{-1}c_{j+1}, & 1 \leqslant j \leqslant m-2, \\ c_2 z_2 d,\, z_2 d z_{m-1}^{-1},\, d z_{m-1}^{-1}c_{m-1}^{-1} & \end{array} \right\}^{\pm 1}.$$

Proof. The lemma follows from Lemmas 44 and 48 by replacing all the products $c_1^{z_1}\dots c_m^{z_m}$ in subwords of $X^{\pm\phi_K}$ by the letter d. □

Notation. Let $m \neq 0$, $K = K(m,n)$, $p = (p_1,\dots,p_K)$ be a 3-large tuple, and $\phi_K = \gamma_K^{p_K}\dots\gamma_1^{p_1}$. Let $\mathcal{W}$ be the set of words in $F(X \cup C_S)$ with the following properties:

(1) if $v \in W$ then $\mathrm{Sub}_3(v) \subseteq \mathrm{Sub}_3(X^{\pm\phi_K})$, $\mathrm{Sub}_2(v) \subseteq \mathrm{Sub}_2(X^{\pm\phi_K})$;
(2) every subword $x_i^{\pm 2}$ of $v \in W$ is contained in a subword $x_i^{\pm 3}$;
(3) every subword $c_1^{\pm z_1}$ of $v \in W$ is contained in $(c_1^{z_1}c_2^{z_2})^{\pm 3}$ when $m \geqslant 2$ or in $(c_1^{z_1}x_1^{-1})^{\pm 3}$ when $m = 1$;
(4) every subword $c_m^{\pm z_m}$ $(m \geqslant 3)$ is contained in $(\prod_{i=1}^m c_i^{z_i})^{\pm 1}$;
(5) every subword $c_2^{\pm z_2}$ of $v \in W$ is contained either in $(c_1^{z_1}c_2^{z_2})^{\pm 3}$ or as the central occurrence of $c_2^{\pm z_2}$ in $(c_2^{-z_2}c_1^{-z_1})^3 c_2^{\pm z_2}(c_1^{z_1}c_2^{z_2})^3$ or in $(c_1 z_1 c_2^{z_2}(c_1^{z_1}c_2^{z_2})^3)^{\pm 1}$.

Definition 30. The following words are called *elementary periods*:

$$x_i, \quad c_1^{z_1}c_2^{z_2} \quad (\text{if } m \geqslant 2), \quad c_1^{z_1}x_1^{-1} \quad (\text{if } m = 1).$$

We call the squares (cubes) of elementary periods or their inverses elementary squares (cubes).

Notation.

(1) Denote by $\mathcal{W}_\Gamma$ the set of all subwords of words in $\mathcal{W}$.
(2) Denote by $\bar{\mathcal{W}}_\Gamma$ the set of all words $v \in \mathcal{W}_\Gamma$ that are freely reduced forms of products of elements from $Y^{\pm 1}$. In this case we say that these elements v are (group) words in the alphabet Y.

If U is a set of words in alphabet Y we denote by $\mathrm{Sub}_{n,Y}(U)$ the set of subwords of length n of words from U in alphabet Y.

Lemma 50. *Let $v \in \mathcal{W}_\Gamma$. Then the following holds*:

(1) *If v begins and ends with an elementary square and contains no elementary cube, then v belongs to the following set*:

$$\left\{\begin{array}{ll}
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i x_{i-1} y_{i-2}^{-1} x_{i-2}^{-2},\ x_i^2 y_i x_i y_{i-1}^{-1} x_{i-1}^{-2}, & m \geqslant 2,\ n \neq 0,\\
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i^2,\ x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i y_{i-1}^{-1} x_{i-1}^{-2}, & \\
(c_2^{-z_2} c_1^{-z_1})^2 c_2^{z_2} (c_1^{z_2} c_2^{z_2})^2, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_i^{z_i} c_{i-1}^{-z_{i-1}} \dots c_3^{-z_3} (c_2^{-z_2} c_1^{-z_1})^2,\ i \geqslant 3, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_m^{z_m} x_1 c_m^{-z_m} \dots c_3^{-z_3} (c_2^{-z_2} c_1^{-z_1})^2, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_m^{z_m} x_1^2, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_m^{z_m} x_1^{-1} y_1^{-1} x_1^{-2}, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_m^{z_m} x_1^{-1} x_2^{-1} x_1 c_m^{-z_m} \dots c_3^{-z_3} (c_2^{-z_2} c_1^{-z_1})^2, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_m^{z_m} x_1^{-1} x_2^2, & \\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_m^{z_m} x_1^{-1} x_2 y_1^{-1} x_1^{-2}, & \\
(c_1^{z_1} c_2^{z_2})^2 d c_{m-1}^{-z_{m-1}} \dots (c_2^{-z_2} c_1^{-z_1})^2,\ z_m c_{m-1}^{-z_{m-1}} \dots (c_2^{-z_2} c_1^{-z_1})^2, & m \geqslant 3,\ n = 0,\\
(c_1^{z_2} c_2^{z_2})^2 c_3^{z_3} \dots c_i^{z_i} c_{i-1}^{-z_{i-1}} \dots c_3^{-z_3} (c_2^{-z_2} c_1^{-z_1})^2,\ i \geqslant 3, & \\
(c_2^{-z_2} c_1^{-z_1})^2 c_2^{z_2} (c_1^{z_2} c_2^{z_2})^2, & \\
x_1^2 y_1 (x_1 c_1^{-z_1})^2,\ (c_1^{z_1} x_1^{-1})^2 x_2 (x_1 c_1^{-z_1})^2, & m = 1,\ n \geqslant 2,\\
(x_1 c_1^{-z_1})^2 x_1^2,\ x_1^2 y_1 x_2^{-1} (x_1 c_1^{-z_1})^2,\ x_2^{-2} (x_1 c_1^{-z_1})^2, & \\
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i x_{i-1} y_{i-2}^{-1} x_{i-2}^{-2},\ x_i^2 y_i x_i y_{i-1}^{-1} x_{i-1}^{-2}, & m = 0,\ n > 1,\\
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i y_{i-1}^{-1} x_{i-1}^{-2},\ x_1^2 y_1 x_2^{-1} x_1^2,\ x_1^{-2} x_2^{-1} x_1^2, & \\
A_1^2,\ A_1^{-2} x_1^2,\ A_1^{-2} x_1 A_1^2,\ x_1^2 y_1 A_1^{-2}, & m = 1,\ n = 1
\end{array}\right\}^{\pm 1};$$

(2) *If v does not contain two elementary squares and begins* (*ends*) *with an elementary square, or contains no elementary squares, then v is a subword of either one of the words above or of one of the words in $\{x_1^2 y_1 x_1, x_2^2 y_2 x_2\}$ for $m = 0$.*

Proof. Straightforward verification using the description of the set $\mathrm{Sub}_3(X^{\pm\phi_K})$ from Lemma 48. □

Definition 31. Let Y be an alphabet and E a set of words of length at least 2 in Y. We say that an occurrence of a word $w \in Y \cup E$ in a word v is *maximal* relative to E if it is not contained in any other (distinct from w) occurrence of a word from E in v.

We say that a set of words W in the alphabet Y admits *Unique Factorization Property* (*UF*) with respect to E if every word $w \in W$ can be uniquely presented as a product

$$w = u_1 \dots u_k$$

where u_i are maximal occurrences of words from $Y \cup E$. In this event the decomposition above is called *irreducible*.

Lemma 51. *Let E be a set of words of length $\geqslant 2$ in an alphabet Y. Suppose that W is a set of words in the alphabet Y such that if $w_1 w_2 w_3$ is a subword of a word from W and $w_1 w_2, w_2 w_3 \in E$ then $w_1 w_2 w_3 \in E$. Then W admits* (*UF*) *with respect to E.*

Proof. Obvious. □

Definition 32. Let Y be an alphabet, E a set of words of length at least 2 in Y and W a set of words in Y which admits (UF) relative to E. An automorphism $\phi \in \operatorname{Aut} F(Y)$ satisfies the Nielsen property with respect to W with exceptions E if for any word $z \in Y \cup E$ there exists a decomposition

$$z^{\phi} = L_z \circ M_z \circ R_z, \tag{72}$$

for some words $L_z, M_z, R_z \in F(Y)$ such that for any $u_1, u_2 \in Y \cup E$ with $u_1 u_2 \in \operatorname{Sub}(W) \setminus E$ the words $L_{u_1} \circ M_{u_1}$ and $M_{u_2} \circ R_{u_2}$ occur as written in the reduced form of $u_1^{\phi} u_2^{\phi}$.

If an automorphism ϕ satisfies the Nielsen property with respect to W and E, then for each word $z \in Y \cup E$ there exists a unique decomposition (72) with maximal length of M_z. In this event we call $M_z = M_{\phi,z}$ the *middle* of z^{ϕ} with respect to ϕ.

Lemma 52. *Let W be a set of words in the alphabet Y which admits* (*UF*) *with respect to a set of words E. If an automorphism $\phi \in \operatorname{Aut} F(Y)$ satisfies the Nielsen property with respect to W with exceptions E then for every $w \in W$ if $w = u_1 \ldots u_k$ is the irreducible decomposition of w then the words M_{u_i} occur as written* (*uncancelled*) *in the reduced form of w^{ϕ}.*

Proof. Follows directly from definitions. □

Set

$$T(m,1) = \left\{ c_s^{z_s}\ (s = 1, \ldots, m),\ \prod_{i=1}^{m} c_i^{z_i} x_1 \prod_{i=m}^{1} c_i^{-z_i} \right\}^{\pm 1}, \quad m \geqslant 2.$$

$$T(m,2) = T(m,1) \cup \left\{ \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} x_2 x_1 \prod_{i=m}^{1} c_i^{-z_i},\ y_1 x_2^{-1} x_1 \prod_{i=m}^{1} c_i^{-z_i}, \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} y_1^{-1} \right\}^{\pm 1};$$

if $n \geqslant 3$ then put

$$T(m,n) = T(m,1) \cup \left\{ \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} x_2^{-1}, \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} y_1^{-1} \right\}^{\pm 1} \cup T_1(m,n),$$

where

$$T_1(m,n) = \big\{ y_{n-2} x_{n-1}^{-1} x_n x_{n-1} y_{n-2}^{-1},\ y_{r-2} x_{r-1}^{-1} x_r^{-1},\ y_{r-1} x_r^{-1} y_r^{-1},$$
$$y_{n-1} x_n^{-1} x_{n-1} y_{n-2}^{-1}\ (n > r \geqslant 2) \big\}^{\pm 1}.$$

Now, let

$$E(m,n)=\bigcup_{i\geqslant 2}\mathrm{Sub}_i\bigl(T(m,n)\bigr)\cap\bar{\mathcal{W}}_\Gamma,\qquad E(m,0)=\emptyset,\ E(1,1)=\emptyset.$$

Lemma 53. *Let $m\neq 0$, $n\neq 0$, $K=K(m,n)$, $p=(p_1,\dots,p_K)$ be a 3-large tuple. Then the following holds.*

(1) *Let $w\in X\cup E(m,n)$, $v=v(w)$ be the leading variable of w, and $j=j(v)$ (see notations at the beginning of Section 7.1). Then the period $A_j^{p_j-1}$ occurs in w^{ϕ_K} and each occurrence of A_j^2 in w^{ϕ_K} is contained in some occurrence of $A_j^{p_j-1}$. Moreover, no square A_k^2 occurs in w for $k>j$.*

(2) *The automorphism ϕ_K satisfies the Nielsen property with respect to $\bar{\mathcal{W}}_\Gamma$ with exceptions $E(m,n)$. Moreover, the following conditions hold*:

- (a) $M_{x_j}=A_{m+4r-8}^{-p_{m+4r-8}+1}x_{r-1}$, *for $j\neq n$*;
- (b) $M_{x_n}=x_n^{q_1}\circ y_n\circ A_{m+4n-2}^{q_2-1}\circ A_{m+4n-4}^{-q_0}\circ x_n$;
- (c) $M_{y_j}=y_j^{\phi_K}$, *for $j<n$*;
- (d) $M_{y_n}=\left(x_n^{q_1}y_n\left|\frac{A_{m+4n-2}^{q_2-1}}{x_ny_{n-1}^{-1}\quad x_ny_n}\right|\frac{A_{m+4n-4}^{-q_0}}{x_ny_{n-1}^{-1}\quad y_{n-2}x_{n-1}^{-1}}\right|x_n\Bigr)^{q_3}x_n^{q_1}y_n$;
- (e) $M_w=w^{\phi_K}$ *for any $w\in E(m,n)$ except for the following words*:
  - $w_1=y_{r-2}x_{r-1}^{-1}x_r^{-1}$, $3\leqslant r\leqslant n-1$, $w_2=y_{r-1}x_r^{-1}y_r^{-1}$, $2\leqslant r\leqslant n-1$,
  - $w_3=y_{n-2}x_{n-1}^{-1}x_n$, $w_4=y_{n-2}x_{n-1}^{-1}x_ny_{n-1}^{-1}$, $w_5=y_{n-2}x_{n-1}^{-1}x_nx_{n-1}^{-1}y_{n-2}^{-1}$, $w_6=y_{n-2}x_{n-1}^{-1}x_nx_{n-1}$, $w_7=y_{n-2}x_{n-1}^{-1}x_n^{-1}$, $w_8=y_{n-1}x_n^{-1}$, $w_9=x_{n-1}^{-1}x_n$, $w_{10}=x_{n-1}^{-1}x_ny_{n-1}^{-1}$, $w_{11}=x_{n-1}^{-1}x_nx_{n-1}y_{n-2}^{-1}$.
- (f) *The only letter that may occur in a word from $\mathcal{W}_\Gamma$ to the left of a subword $w\in\{w_1,\dots,w_8\}$ ending with y_i ($i=r-1$, $r-2,n-1,n-2$, $i\geqslant 1$) is x_i. The maximal number j such that L_w contains $A_j^{p_j-1}$ is $j=m+4i-2$, and $R_{w_1}=R_{w_2}=1$.*

Proof. We first exhibit the formulas for u^{ϕ_K}, where $u\in\bigcup_{i\geqslant 2}\mathrm{Sub}_i(T_1(m,n))$.

(1)(a) Let $i<n$. Then

$$\begin{aligned}
&\bigl(x_iy_{i-1}^{-1}\bigr)^{\phi_{m+4i}}\\
&\quad=\bigl(x_iy_{i-1}^{-1}\bigr)^{\phi_K}=\left|\frac{A_{m+4i}^{-q_4+1}}{x_{i+1}y_i^{-1}\quad y_{i-1}x_i^{-1}}\right|x_{i+1}\circ y_i^{-1}x_i^{-q_1}\\
&\qquad\circ\left(x_i^{-1}\left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|\frac{A_{m+4i-2}^{-q_2+1}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|y_i^{-1}x_i^{-q_1}\Bigr)^{q_3-1}\left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|\\
&\qquad\cdot A_{m+4i-4}^{-q_0+1}\circ x_i\circ\tilde{y}_{i-1}\circ x_i^{-1}\left|\frac{A_{m+4i-4}^{q_0-1}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|
\end{aligned}$$

$$= \left|\frac{A_{m+4i}^{-q_4+1}}{x_{i+1}y_i^{-1} \quad y_{i-1}x_i^{-1}}\right| x_{i+1} \circ y_i^{-1}x_i^{-q_1}$$
$$\circ \left(x_i^{-1}\left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1} \quad y_{i-1}x_i^{-1}}\right|\frac{A_{m+4i-2}^{-q_2+1}}{x_{i-1}y_{i-2}^{-1} \quad y_{i-1}x_i^{-1}}\right| y_i^{-1}x_i^{-q_1}\right)^{q_3-1} \cdot x_i^{-1}\left|\frac{A_{m+4i-4}^{q_0-1}}{x_{i-1}y_{i-2}^{-1} \quad y_{i-1}x_i^{-1}}\right|.$$

(b) Let $i = n$. Then

$$\left(x_n y_{n-1}^{-1}\right)^{\phi_{m+4n-1}} = \left(x_n y_{n-1}^{-1}\right)^{\phi_K} = \left|\frac{A_{m+4n-2}^{q_2}}{x_n y_{n-1}^{-1} \quad x_n y_n}\right|\frac{A_{m+4n-4}^{-1}}{x_n y_{n-1}^{-1} \quad y_{n-2}x_{n-1}^{-1}}\right|.$$

Here $y_{n-1}^{-\phi_K}$ is completely cancelled.

(2)(a) Let $i < n-1$. Then

$$\begin{aligned}
&\left(x_{i+1}x_i y_{i-1}^{-1}\right)^{\phi_K} \\
&\quad = \left(x_{i+1}x_i y_{i-1}^{-1}\right)^{\phi_{m+4i+4}} \\
&\quad = A_{m+4i+4}^{-q_8+1} \circ x_{i+2} \circ y_{i+1}^{-1} \circ x_{i+1}^{-q_5} \circ \left(x_{i+1}^{-1} \circ A_{m+4i}^{q_4} \circ A_{m+4i+2}^{-q_6+1} \circ y_{i+1}^{-1}x_{i+1}^{-q_5}\right)^{q_7-1} A_{m+4i-4}^{-q_0} \\
&\quad\quad \circ x_i^{q_1} y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-1}.
\end{aligned}$$

Here $(x_i y_{i-1}^{-1})^{\phi_{m+4i+4}}$ was completely cancelled.

(b) Similarly, $(x_i y_{i-1}^{-1})^{\phi_{m+4i+3}}$ is completely cancelled in $(x_{i+1}x_i y_{i-1}^{-1})^{\phi_{m+4i+3}}$ and

$$\left(x_{i+1}x_i y_{i-1}^{-1}\right)^{\phi_{m+4i+3}} = A_{m+4i+2}^{q_6} \circ A_{m+4i}^{-q_4} \circ x_{i+1} \circ A_{m+4i-4}^{-q_0} A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-1}.$$

(c) $$\left(x_n^{-1}x_{n-1}y_{n-2}^{-1}\right)^{\phi_{m+4n-1}} = A_{m+4n-4}^{-q_4} \circ x_n^{-1} \circ A_{m+4n-4}^{q_4} \circ A_{m+4n-2}^{-q_6+1} \circ y_n^{-1} \circ x_n^{-q_5}$$
$$\circ A_{m+4n-8}^{-q_0} \circ x_{n-1}^{q_1} \circ y_{n-1} \circ A_{m+4n-6}^{q_2-1} \circ A_{m+4n-8}^{-1},$$

and $(x_{n-1}y_{n-2}^{-1})^{\phi_{m+4n-1}}$ is completely cancelled.

(3)(a) $(y_i x_i y_{i-1}^{-1})^{\phi_{m+4i}} = A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-1}$, and $(x_i y_{i-1}^{-1})^{\phi_{m+4i}}$ is completely cancelled.

(b) $(y_n x_n y_{n-1}^{-1})^{\phi_K} = y_n^{\phi_K} \circ (x_n y_{n-1}^{-1})^{\phi_K}$.

(c) $(y_{n-1}x_n^{-1}x_{n-1}y_{n-2}^{-1})^{\phi_K} = A_{m+4n-4} \circ A_{m+4n-2}^{-q_6+1} \circ y_n^{-1} \circ x_n^{-q_5} \circ A_{m+4n-8}^{-q_0} \circ x_{n-1}^{q_1} \circ y_{n-1} \circ A_{m+4n-6}^{q_2-1} \circ A_{m+4n-8}^{-1}$, and $y_{n-1}^{\phi_K}$ and $(x_{n-1}y_{n-2}^{-1})^{\phi_K}$ are completely cancelled.

(4)(a) Let $n \geqslant 2$.

$$\begin{aligned}
&\left(x_1 c_m^{-z_m}\right)^{\phi_{m+4i}} \\
&\quad = \left(x_1 c_m^{-z_m}\right)^{\phi_K} \\
&\quad = \left(\left|\frac{A_{m+4}^{-q_4+1}}{x_2 y_1^{-1} \quad c_m^{z_m} x_1^{-1}}\right| x_2 \circ y_1^{-1}x_1^{-q_1} \circ \left(x_1^{-1} \circ A_m^{q_0} \circ A_{m+2}^{-q_2+1} \circ y_1^{-1} \circ x_1^{-q_1}\right)^{q_3-1} \circ A_m^{q_0}\right)
\end{aligned}$$

$$\cdot \left(A_m^{-q_0} \circ x_1^{-1} \circ A_m^{q_0-1}\right)$$
$$= A_{m+4}^{-q_4+1} \circ x_2 \circ y_1^{-1} \circ x_1^{-q_1} \circ \left(x_1^{-1} \circ A_m^{q_0} \circ A_{m+2}^{-q_2+1} \circ y_1^{-1} \circ x_1^{-q_1}\right)^{q_3-1} \circ x_1^{-1} \circ A_m^{q_0-1}.$$

Let $n = 1$.

$$\left(x_1 z_m^{-c_m}\right)^{\phi_K} = A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1 \circ A_{m+2}^{p_{m+2}-1} \circ A_m^{-1},$$
$$\left(y_1 x_1 z_m^{-c_m}\right)^{\phi_K} = y_1^{\phi_K} \circ (x_1 z_m^{-c_m})^{\phi_K}.$$

(b) $(x_1 c_m^{-z_m})^{\phi_K}$ is completely cancelled in $x_2^{\phi_K}$ and for $n > 2$

$$\left(x_2 x_1 c_m^{-z_m}\right)^{\phi_K} = A_{m+8}^{-q_8+1} \circ x_3 \circ y_2^{-1} \circ x_3^{-q_5} \circ \left(x_3^{-1} \circ A_{m+4}^{q_4} \circ A_{m+6}^{-q_6+1} \circ y_2^{-1} \circ x_3^{-q_5}\right)^{q_7-1}$$
$$\circ A_m^{-q_0} \circ x_1^{q_1} \circ y_1 \circ A_{m+2}^{q_2-1} \circ A_m^{-1},$$

and for $n = 2$

$$\left(x_2 x_1 c_m^{-z_m}\right)^{\phi_K} = A_{m+6}^{q_6} \circ A_{m+4}^{-q_4} \circ x_i \circ A_m^{-q_0} \circ x_1^{q_1} \circ y_1 \circ A_{m+2}^{q_2-1} \circ A_m^{-1}.$$

(c) The cancellation between $(x_2 x_1 c_m^{-z_m})^{\phi_K}$ and $c_{m-1}^{-z_{m-1}}$ is the same as the cancellation between A_m^{-1} and $c_{m-1}^{-z_{m-1}^{\phi_K}}$, namely,

$$A_m^{-1} c_{m-1}^{-z_{m-1}^{\phi_K}} = \left(x_1 \circ A_{m-1}^{-p_{m-1}} \circ c_m^{-z_m} \circ A_{m-1}^{p_{m-1}}\right)$$
$$\circ \left(A_{m-1}^{-p_{m-1}+1} \circ c_m^{-z_m} \circ A_{m-2}^{-p_{m-2}} \circ c_{m-1}^{-z_{m-1}} \circ A_{m-2}^{p_{m-2}} \circ c_m^{z_m} \circ A_{m-1}^{p_{m-1}-1}\right)$$
$$= x_1 A_{m-1}^{-1},$$

and $c_{m-1}^{-z_{m-1}^{\phi_K}}$ is completely cancelled.

(d) The cancellations between $(x_2 x_1 c_m^{-z_m})^{\phi_K}$ (or between $(y_1 x_1 c_m^{-z_m})^{\phi_K}$) and $\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}}$ are the same as the cancellations between A_m^{-1} and $\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}}$ namely, the product $\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}}$ is completely cancelled and

$$A_m^{-1} \prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}} = x_1 \prod_{i=m}^{1} c_i^{-z_i}.$$

Similarly one can write expressions for u^{ϕ_K} for all $u \in E(m,n)$. The first statement of the lemma now follows from these formulas.

Let us verify the second statement. Suppose $w \in E(m,n)$ is a maximal subword from $E(m,n)$ of a word u from $\mathcal{W}_\Gamma$. If w is a subword of a word in $T(m,n)$, then either u begins with w or w is the leftmost subword of a word in $T(m,n)$. All the words in $T_1(m,n)$ begin with some y_j, therefore the only possible letters in u in front of w are x_j^2.

We have

$$x_j^{\phi_K} x_j^{\phi_K} w^{\phi_K} = x_j^{\phi_K} \circ x_j^{\phi_K} \circ w^{\phi_K}$$

if w is a 2-letter word, and

$$x_j^{\phi_K} x_j^{\phi_K} w^{\phi_K} = x_j^{\phi_K} \circ x_j^{\phi_K} w^{\phi_K}$$

if w is more than a 2-letter word. In this last case there are some cancellations between $x_j^{\phi_K}$ and w^{ϕ_K}, and the middle of x_j is the non-cancelled part of x_j because x_j as a letter not belonging to $E(m,n)$ appears only in x_j^n.

We still have to consider all letters that can appear to the right of w, if w is the end of some word in $T_1(m,n)$ or $w = y_{n-1}x_n^{-1}x_{n-1}$, $w = y_{n-1}x_n^{-1}$. There are the following possibilities:

(i) w is an end of $y_{n-2}x_{n-1}^{-1}x_n x_{n-1} y_{n-2}^{-1}$;
(ii) w is an end of $y_{r-2}x_{r-1}^{-1}x_r^{-1}$, $r < i$;
(iii) w is an end of $y_{n-2}x_{n-1}^{-1}y_{n-1}^{-1}$.

Situation (i) is equivalent to the situation when w^{-1} is the beginning of the word $y_{n-2}x_{n-1}^{-1}x_n x_{n-1} y_{n-2}^{-1}$, we have considered this case already. In the situation (ii) the only possible word to the right of w will be left end of $x_{r-1}y_{r-2}^{-1}x_{r-2}^{-2}$ and

$$w^{\phi_K} x_{r-1}^{\phi_K} y_{r-2}^{-\phi_K} x_{r-2}^{-2\phi_K} = w^{\phi_K} \circ x_{r-1}^{\phi_K} y_{r-2}^{-\phi_K} \circ x_{r-2}^{-2\phi_K}, \quad \text{and} \quad w^{\phi_K} x_{r-1}^{\phi_K} = w^{\phi_K} \circ x_{r-1}^{\phi_K}.$$

In the situation (iii) the first two letters to the right of w are $x_{n-1}x_{n-1}$, and $w^{\phi_K} x_{n-1}^{\phi_K} = w^{\phi_K} \circ x_{n-1}^{\phi_K}$.

There is no cancellation in the words

$$\left(c_j^{z_j}\right)^{\phi_K} \circ \left(c_{j+1}^{\pm z_{j+1}}\right)^{\phi_K}, \qquad \left(c_m^{z_m}\right)^{\phi_K} \circ x_1^{\pm\phi_K}, \qquad x_1^{\phi_K} \circ x_1^{\phi_K}.$$

For all the other occurrences of x_i in the words from $\mathcal{W}_\Gamma$, namely for occurrences in x_i^n, $x_i^2 y_i$, we have

$$\left(x_i^2 y_i\right)^{\phi_K} = x_i^{\phi_K} \circ x_i^{\phi_K} \circ y_i^{\phi_k} \quad \text{for } i < n.$$

In the case $n = i$, the bold subword of the word

$$x_n^{\phi_K} = A_{m+4n-4}^{-q_0} \circ \left(\boldsymbol{x_n^{q_1}} \circ \boldsymbol{y_n} \circ \boldsymbol{A_{m+4n-2}^{q_2-1}} \circ \boldsymbol{A_{m+4n-4}^{-q_0}} \circ \boldsymbol{x_n}\right) \circ A_{m+4n-4}^{q_0}$$

is M_{x_n} for ϕ_K, and the bold subword in the word

$$y_n^{\phi_K} = \begin{array}{|cc|}\multicolumn{2}{|c|}{A_{m+4n-4}^{-q_0}} \\ \hline x_n y_{n-1}^{-1} & y_{n-2}x_{n-1}^{-1}\end{array}\Bigg(\boldsymbol{x_n^{q_1} y_n} \begin{array}{|cc|cc|}\multicolumn{2}{|c|}{\boldsymbol{A_{m+4n-2}^{q_2-1}}} & \multicolumn{2}{c|}{\boldsymbol{A_{m+4n-4}^{-q_0}}} \\ \hline \boldsymbol{x_n y_{n-1}^{-1}} & \boldsymbol{x_n y_n} & \boldsymbol{x_n y_{n-1}^{-1}} & \boldsymbol{y_{n-2}x_{n-1}^{-1}}\end{array} \boldsymbol{x_n}\Bigg)^{\boldsymbol{q_3}} \boldsymbol{x_n^{q_1} y_n},$$

is M_{y_n} for ϕ_K. □

Lemma 54. *The following statements hold.*

(1) *Let $u \in E(m,n)$. If B^2 occurs as a subword in u^{ϕ_K} for some cyclically reduced word B $(B \neq c_i)$ then B is a power of a cyclic permutation of a period A_j, $j = 1,\ldots,K$.*
(2) *Let $u \in \bar{\mathcal{W}}_\Gamma$. If B^2 occurs as a subword in u^{ϕ_K} for some cyclically reduced word B $(B \neq c_i)$ then B is a power of a cyclic permutation of a period A_j, $j = 1,\ldots,K$.*

Proof. (1) follows from the formulas (1)(a)–(4)(d) from Lemma 53.

(2) We may assume that w does not contain an elementary square. In this case w is a subword of a word from Lemma 50. Now the result follows from the formulas (1)(a)–(4)(d) from Lemma 53. □

Notation. (1) Denote by $\mathcal{W}_{\Gamma,L}$ the least set of words in the alphabet Y that contains $\bar{\mathcal{W}}_\Gamma$, is closed under taking subwords, and is ϕ_K-invariant.

(2) Let $\bar{\mathcal{W}}_{\Gamma,L}$ be union of $\mathcal{W}_{\Gamma,L}$ and the set of all initial subwords of $z_i^{\phi_{Kj}}$ which are of the form

$$c_i^j \circ z_i \circ w, \quad \text{where } w \in \mathcal{W}_{\Gamma,L}.$$

Notation. Denote by Exc the following set of words in the alphabet Y.

(1) If $m > 2$, $n \geqslant 2$, then

$$\mathrm{Exc} = \left\{c_1^{-z_1} c_i^{-z_i} c_{i-1}^{-z_{i-1}}, c_1^{-z_1} x_1 c_m^{-z_m}, c_1^{-z_1} x_j y_{j-1}^{-1}\right\}.$$

(2) If $m > 2$, $n = 1$, then

$$\mathrm{Exc} = \left\{c_1^{-z_1} c_i^{-z_i} c_{i-1}^{-z_{i-1}}, c_1^{-z_1} x_1 c_m^{-z_m}\right\}.$$

(3) If $m = 2$, $n \geqslant 2$, then

$$\mathrm{Exc} = \left\{c_1^{-z_1} x_1 c_m^{-z_m}, c_1^{-z_1} x_j y_{j-1}^{-1}\right\}.$$

(4) If $m = 2$, $n = 1$, then

$$\mathrm{Exc} = \left\{c_1^{-z_1} x_1 c_m^{-z_m}\right\}.$$

(5) If $m=1$, $n\geqslant 2$, then

$$\mathrm{Exc}=\{c_1^{-z_1}x_j y_{j-1}^{-1}\}.$$

(6) If $m=0$, $n\geqslant 2$, then

$$\mathrm{Exc}=\{y_1x_1x_i,\, x_1x_i y_{i-1}^{-1},\ 2\leqslant i\leqslant n\}.$$

Lemma 55. *The following holds*:

(1) $\mathrm{Sub}_{3,Y}(\mathcal{W}_{\Gamma,L})=\mathrm{Sub}_{3,Y}(X^{\pm\phi_K})\cup \mathrm{Exc}$;
(2) *Let $v\in\mathcal{W}_{\Gamma,L}$ be a word that begins and ends with an elementary square and does not contain any elementary cubes. Then either $v\in\bar{\mathcal{W}}_\Gamma$ or $v=v_1v_2$ for some words $v_1,v_2\in\bar{\mathcal{W}}_\Gamma$ described below*:
   (a) *for $m>2$, $n\geqslant 2$,*

$$v_1\in\left\{v_{11}=\left(c_1^{z_1}c_2^{z_2}\right)^2\prod_{i=3}^{m}c_i^{z_i}x_1x_2x_1\prod_{i=m}^{1}c_i^{-z_i},\, v_{12}=x_1^2y_1x_1\prod_{i=m}^{1}c_i^{-z_i}\right\}$$

   and

$$v_2\in\left\{v_{2i}=c_i^{-z_i}\dots c_3^{-z_3}\left(c_2^{-z_2}c_1^{-z_1}\right)^2,\, u_{2,1}=x_1c_m^{-z_m}\dots c_3^{-z_3}\left(c_2^{-z_1}c_1^{-z_1}\right)^2,\right.$$
$$\left. u_{2,j}=x_jy_{j-1}^{-1}x_{j-1}^2\right\};$$

   (b) *for $m=2$, $n\geqslant 2$,*

$$v_1\in\left\{v_{11}=\left(c_1^{z_1}c_2^{z_2}\right)^2x_1x_2x_1\prod_{i=m}^{1}c_i^{-z_i},\, v_{12}=x_1^2y_1x_1\prod_{i=m}^{1}c_i^{-z_i}\right\}$$

   and

$$v_2\in\left\{u_{2,1}=x_1\left(c_2^{-z_1}c_1^{-z_1}\right)^2,\, u_{2,j}=x_jy_{j-1}^{-1}x_{j-1}^2\right\};$$

   (c) *for $m>2$, $n=1$,*

$$v_1\in\left\{v_{12}=x_1^2y_1x_1\prod_{i=m}^{1}c_i^{-z_i}\right\}$$

   and

$$v_2\in\left\{v_{2i}=c_i^{-z_i}\dots c_3^{-z_3}\left(c_2^{-z_2}c_1^{-z_1}\right)^2,\, u_{2,1}=x_1c_m^{-z_m}\dots c_3^{-z_3}\left(c_2^{-z_1}c_1^{-z_1}\right)^2\right\};$$

(d) *for* $m=2$, $n=1$,

$$v_1 \in \left\{ v_{12} = x_1^2 y_1 x_1 \prod_{i=m}^{1} c_i^{-z_i} \right\}$$

and

$$v_2 \in \left\{ u_{2,1} = x_1 \left(c_2^{-z_1} c_1^{-z_1} \right)^2 \right\};$$

(e) *for* $m=1$, $n \geqslant 2$,

$$v_1 \in \left\{ v_{11} = \left(c_1^{z_1} x_1^{-1} \right)^2 x_2 x_1 c_1^{-z_1},\ v_{12} = x_1^2 y_1 x_1 c_1^{-z_1} \right\}$$

and

$$v_2 \in \left\{ u_{2,j} = x_j y_{j-1}^{-1} x_{j-1}^2 \right\};$$

(3) *If* $v \in \mathcal{W}_{\Gamma,L}$ *and either* v *does not contain two elementary squares and begins* (*ends*) *with an elementary square, or* v *contains no elementary squares, then either* v *is a subword of one of the words from* (2) *or* (*for* $m=0$) v *is a subword of one of the words* $x_1^2 y_1 x_1$, $x_2^2 y_2 x_2$;

(4) *Automorphism* ϕ_K *satisfies Nielsen property with respect* $\mathcal{W}_{\Gamma,L}$ *with exceptions* $E(m,n)$.

Proof. Let $T = Kl$. We will consider only the case $m \geqslant 2$, $n \geqslant 2$. We will prove all the statements of the lemma by simultaneous induction on l. If $l=1$, then $T=K$ and the lemma is true. Suppose now that

$$\mathrm{Sub}_{3,Y}\left(\bar{\mathcal{W}}_{\Gamma}^{\phi_{T-K}}\right) = \mathrm{Sub}_{3,Y}\left(\bar{\mathcal{W}}_{\Gamma}\right) \cup \mathrm{Exc}.$$

Formulas in the beginning of the proof of Lemma 53 show that

$$\mathrm{Sub}_{3,Y}\left(E(m,n)^{\pm\phi_K}\right) \subseteq \mathrm{Sub}_{3,Y}\left(\bar{\mathcal{W}}_{\Gamma}\right).$$

By the third statement for $\mathrm{Sub}(\bar{\mathcal{W}}_{\Gamma}^{\phi_{T-K}})$ the automorphism ϕ_K satisfies the Nielsen property with exceptions $E(m,n)$. Let us verify that new 3-letter subwords do not occur "between" u^{ϕ_K} for $u \in T_1(m,n)$ and the power of the corresponding x_i to the left and right of it. All the cases are similar to the following:

$$\left(x_n x_{n-1} y_{n-2}^{-1}\right)^{\phi_K} \cdot x_{n-2}^{\phi_K} \cdots \left|\begin{array}{c} A_{m+4n-10}^{-q+1} \\ \hline * \quad y_{n-3} x_{n-2}^{-1} \end{array}\right| \cdot x_{n-1}^{-1} \left|\begin{array}{c} A_{m+4n-8}^{q_0-1} \\ \hline x_{n-2} \quad * \end{array}\right|.$$

Words $(v_1 v_2)^{\phi_K}$ produce the subwords from Exc. Indeed, $[(x_2 x_1 \prod_{i=m}^{1} c_i^{-z_i})]^{\phi_{Kj}}$ ends with v_{12} and $v_{12}^{\phi_K}$ ends with v_{12}. Similarly, $v_{2,j}^{\phi_K}$ begins with $v_{2,j+1}$ for $j<m$ and with $u_{2,1}$ for $j=m$. And $u_{2,j}^{\phi_K}$ begins with $u_{2,j+1}$ for $j<n$ and with $u_{2,j}$ for $j=n$.

This and the second part of Lemma 48 finish the proof. □

According to the definition of $\bar{\mathcal{W}}_{\Gamma,L}$, this set contains words which are written in the alphabet $Y^{\pm 1}$ as well as extra words u of the form $(c_i^j z_i w)^{\pm 1}$ or $(z_i w)^{\pm 1}$ whose $Y^{\pm 1}$-representation is spoiled at the start or at the end of u. For those $u \in \bar{\mathcal{W}}_{\Gamma,L}$ which are written in the alphabet $Y^{\pm 1}$, Lemma 51 gives a unique representation as the product $u_1 \dots u_k$ where $u_i \in Y^{\pm 1} \cup E(m,n)$ and the occurrences of u_i are maximal. We call this representation a *canonical decomposition of* u. For $u \in \bar{\mathcal{W}}_{\Gamma,L}$ of the form $(c_i^j z_i w)^{\pm 1}$ or $(z_i w)^{\pm 1}$ we define the canonical decomposition of u as follows: $u = c_i \dots c_i z_i u_1 \dots u_k$ where $u_i \in Y^{\pm 1} \cup E(m,n)$. Clearly, we can consider the Nielsen property of automorphisms with exceptions $E(m,n)$ relative to this extended notion of canonical decomposition. Below the Nielsen property is always assumed in this sense.

Lemma 56. *The automorphism ϕ_K satisfies Nielsen property with respect to $\bar{\mathcal{W}}_{\Gamma,L}$ with exceptions $E(m,n)$. The set $\bar{\mathcal{W}}_{\Gamma,L}$ is ϕ_K-invariant.*

Proof. The first statement follows from Lemmas 53 and 55. For the second statement notice that if $c_i^j z_i w \in \bar{\mathcal{W}}_{\Gamma,L}$, then

$$c_i^{c_i^j z_i w} = w^{-1} \circ c_i^{z_i} \circ w \in \mathcal{W}_{\Gamma,L} \quad \text{and} \quad c_i^{(c_i^j z_i w)^{\phi_K}} = w^{-\phi_K} \circ c_i^{z_i^{\phi_K}} \circ w^{\phi_K} \in \mathcal{W}_{\Gamma,L},$$

therefore $c_i^j z_i^{\phi_K} \circ w^{\phi_K} \in \bar{\mathcal{W}}_{\Gamma,L}$. □

Let $W \in G[X]$. We say that a word $U \in G[X]$ *occurs* in W if $W = W_1 \circ U \circ W_2$ for some $W_1, W_2 \in G[X]$. An occurrence of U^q in W is called *maximal* with respect to a property P of words if U^q is not a part of any occurrence of U^r with $q < r$ and which satisfies P. We say that an occurrence of U^q in W is t-*stable* if $q \geqslant 1$ and $W = W_1 \circ U^t U^q U^t \circ W_2$, $t \geqslant 1$ (it follows that U is cyclically reduced). If $t = 1$ it is *stable*. Maximal stable occurrences U^q will play an important part in what follows. If $(U^{-1})^q$ is a stable occurrence of U^{-1} in W then, sometimes, we say that U^{-q} is a stable occurrence of U in W. Two given occurrences U^q and U^p in a word W are *disjoint* if they do not have a common letter as subwords of W. Observe that if integers p and q have different signs then any two occurrences of A^q and A^p are disjoint. Also, any two different maximal stable occurrences of powers of U are disjoint. To explain the main property of stable occurrences of powers of U, we need the following definition. We say that a given occurrence of U^q *occurs correctly* in a given occurrence of U^p if $|q| \leqslant |p|$ and for these occurrences U^q and U^p one has $U^p = U^{p_1} \circ U^q \circ U^{p_1}$. We say, that two given non-disjoint occurrences of U^q, U^p *overlap correctly* in W if their common subword occurs correctly in each of them.

A cyclically reduced word A from $G[X]$ which is not a proper power and does not belong to G is called *a period*.

Lemma 57. *Let A be a period in $G[X]$ and $W \in G[X]$. Then any two stable occurrences of powers of A in W are either disjoint or they overlap correctly.*

Proof. Let A^q, A^p ($q \leqslant p$) be two non-disjoint stable occurrences of powers of A in W. If they overlap incorrectly then $A^2 = u \circ A \circ v$ for some elements $u, v \in G[X]$. This implies that $A = u \circ v = v \circ u$ and hence u and v are (non-trivial) powers of some element in $G[X]$. Since A is not a proper power it follows that $u = 1$ or $v = 1$—a contradiction. This shows that A^q and A^p overlap correctly. □

Let $W \in G[X]$ and $\mathcal{O} = \mathcal{O}(W, A) = \{A^{q_1}, \ldots, A^{q_k}\}$ be a set of pair-wise disjoint stable occurrences of powers of a period A in W (listed according to their appearance in W from the left to the right). Then $\mathcal{O}$ induces an $\mathcal{O}$-decomposition of W of the following form:

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}. \tag{73}$$

For example, let P be a property of words (or just a property of occurrences in W) such that if two powers of A (two occurrences of powers of A in W) satisfy P and overlap correctly then their union also satisfies P. We refer to such P as *preserving correct overlappings*. In this event, by $\mathcal{O}_P = \mathcal{O}_P(W, A)$ we denote the uniquely defined set of all maximal stable occurrences of powers of A in W which satisfy the property P. Notice, that occurrences in $\mathcal{O}_P$ are pair-wise disjoint by Lemma 57. Thus, if P holds on every power of A then $\mathcal{O}_P(W, A) = \mathcal{O}(W, A)$ contains all maximal stable occurrences of powers of A in W. In this case, the decomposition (73) is unique and it is called the *canonical* (*stable*) A-decomposition of W.

The following example provides another property P that will be in use later. Let N be a positive integer and let P_N be the property of A^q that $|q| \geqslant N$. Obviously, P_N preserves correct overlappings. In this case the set $\mathcal{O}_{P_N}$ provides the so-called *canonical N-large A-decompositions* of W which are also uniquely defined.

Definition 33. Let

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

be the decomposition (73) of W above. Then the numbers

$$\max_A(W) = \max\{q_i \mid i = 1, \ldots, k\}, \qquad \min_A(W) = \min\{q_i \mid i = 1, \ldots, k\}$$

are called, correspondingly, the *upper* and the *lower* A-bounds of W.

Definition 34. Let A be a period in $G[X]$ and $W \in G[X]$. For a positive integer N we say that the N-large A-decomposition of W

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

has A-size (l, r) if $\min_A(W) \geqslant l$ and $\max_A(B_i) \leqslant r$ for every $i = 1, \ldots, k$.

Let $\mathcal{A} = \{A_1, A_2, \ldots\}$ be a sequence of periods from $G[X]$. We say that a word $W \in G[X]$ has $\mathcal{A}$-rank j ($\operatorname{rank}_{\mathcal{A}}(W) = j$) if W has a stable occurrence of $(A_j^{\pm 1})^q$ ($q \geqslant 1$) and

j is maximal with this property. In this event, A_j is called the *$\mathcal{A}$-leading term* (or just the *leading term*) of W (notation $LT_{\mathcal{A}}(W) = A_j$ or $LT(W) = A_j$).

We now fix an arbitrary sequence $\mathcal{A}$ of periods in the group $G[X]$. For a period $A = A_j$ one can consider canonical A_j-decompositions of a word W and define the corresponding A_j-bounds and A_j-size. In this case we, sometimes, omit A in the writings and simply write $\max_j(W)$ or $\min_j(W)$ instead of $\max_{A_j}(W)$, $\min_{A_j}(W)$.

In the case when $\mathrm{rank}_{\mathcal{A}}(W) = j$ the canonical A_j-decomposition of W is called the *canonical* $\mathcal{A}$-decomposition of W.

Now we turn to an analog of $\mathcal{O}$-decompositions of W with respect to "periods" which are not necessarily cyclically reduced words. Let $U = D^{-1} \circ A \circ D$, where A is a period. For a set $\mathcal{O} = \mathcal{O}(W, A) = \{A^{q_1}, \ldots, A^{q_k}\}$ as above consider the $\mathcal{O}$-decomposition of a word W

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}. \tag{74}$$

Now it can be rewritten in the form:

$$W = (B_1 D)\big(D^{-1} \circ A^{q_1} \circ D\big) \ldots \big(D^{-1} B_k D\big)\big(D^{-1} \circ A^{q_k} \circ D\big)\big(D^{-1} B_{k+1}\big).$$

Let $\varepsilon_i, \delta_i = \mathrm{sgn}(q_i)$. Since every occurrence of A^{q_i} above is stable, $B_1 = \bar{B}_1 \circ A^{\varepsilon_1}$, $B_i = (A^{\delta_{i-1}} \circ \bar{B}_i \circ A^{\varepsilon_i})$, $B_{k+1} = A^{\delta_k} \circ \bar{B}_{k+1}$ for suitable words $\bar{B}_i$. This shows that the decomposition above can be written as

$$\begin{aligned} W &= \big(\bar{B}_1 A^{\varepsilon_1} D\big)\big(D^{-1} A^{q_1} D\big) \ldots \big(D^{-1} A^{\delta_{i-1}} \bar{B}_i A^{\varepsilon_i} D\big) \ldots \big(D^{-1} A^{q_k} D\big)\big(D^{-1} A^{\delta_k} \bar{B}_{k+1}\big) \\ &= \big(\bar{B}_1 D\big)\big(D^{-1} A^{\varepsilon_1} D\big)\big(D^{-1} A^{q_1} D\big) \ldots \big(D^{-1} A^{\delta_{i-1}} D\big)\big(D^{-1} \bar{B}_i D\big)\big(D^{-1} A^{\varepsilon_i} D\big) \ldots \\ &\quad \big(D^{-1} A^{q_k} D\big)\big(D^{-1} A^{\delta_k} D\big)\big(D^{-1} \bar{B}_{k+1}\big) \\ &= \big(\bar{B}_1 D\big)\big(U^{\varepsilon_1}\big)\big(U^{q_1}\big) \ldots \big(U^{\delta_{k-1}}\big)\big(D^{-1} \bar{B}_k D\big)\big(U^{\varepsilon_k}\big)\big(U^{q_k}\big)\big(U^{\delta_k}\big)\big(D^{-1} \bar{B}_{k+1}\big). \end{aligned}$$

Observe, that the cancellation between parentheses in the decomposition above does not exceed the length $d = |D|$ of D. Using notation $w = u \circ_d v$ to indicate that the cancellation between u and v does not exceed the number d, we can rewrite the decomposition above in the following form:

$$W = \big(\bar{B}_1 D\big) \circ_d U^{\varepsilon_1} \circ_d U^{q_1} \circ_d U^{\delta_1} \circ_d \cdots \circ_d U^{\varepsilon_k} \circ_d U^{q_k} \circ_d U^{\delta_k} \circ_d \big(D^{-1} \bar{B}_{k+1}\big),$$

hence

$$W = D_1 \circ_d U^{q_1} \circ_d \cdots \circ_d D_k \circ_d U^{q_k} \circ_d D_{k+1}, \tag{75}$$

where $D_1 = \bar{B}_1 D$, $D_{k+1} = D^{-1} \bar{B}_{k+1}$, $D_i = D^{-1} \bar{B}_i D$ $(2 \leqslant i \leqslant k)$, and the occurrences U^{q_i} are $(1, d)$-stable. (We similarly define (t, d)-stable occurrences.) We will refer to this decomposition of W as *U-decomposition with respect to $\mathcal{O}$* (to get a rigorous definition of U-decompositions one has to replace in the definition of the $\mathcal{O}$-decomposition of W the period A by U and $\circ$ by $\circ_{|D|}$). In the case when an A-decomposition of W (with respect to $\mathcal{O}$) is unique then the corresponding U-decomposition of W is also unique, and in this event one can easily rewrite A-decompositions of W into U-decomposition and vice versa.

We summarize the discussion above in the following lemma.

Lemma 58. *Let $A \in G[X]$ be a period and $U = D^{-1} \circ A \circ D \in G[X]$. Then for a word $W \in G[X]$ if*

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

is a stable A-decomposition of W then

$$W = D_1 \circ_d U^{q_1} \circ_d \cdots \circ_d D_k \circ_d U^{q_k} \circ_d D_{k+1}$$

is a stable U-decomposition of W, where D_i are defined as in (75). *And vice versa.*

From now on we fix the following set of leading terms

$$\mathcal{A}_{L,p} = \{A_j \mid j \leqslant L,\ \phi = \phi_{L,p}\}$$

for a given multiple L of $K = K(m,n)$ and a given tuple p.

Definition 35. Let $W \in G[X]$ and N be a positive integer. A word A_s is termed the N-large leading term $LT_N(W)$ of the word W if A_s^q has a stable occurrence in W for some $q \geqslant N$, and s is maximal with this property. The number s is called the N-rank of W ($s = \mathrm{rank}_N(W)$, $s \geqslant 1$).

In Lemmas 44–47 we described precisely the leading terms A_j for $j = 1, \ldots, K$. It is not easy to describe precisely A_j for an arbitrary $j > K$. So we are not going to do it here, instead, we chose a compromise by introducing a modified version A_j^* of A_j which is not cyclically reduced, in general, but which is “more cyclically reduced” then the initial word A_j. Namely, let L be a multiple of K and $1 \leqslant j \leqslant K$. Define

$$A^*_{L+j} = A^*(\phi_{L+j}) = A_j^{\phi_L}.$$

Lemma 59. *Let L be a multiple of K and $1 \leqslant j \leqslant K$. Let $p = (p_1, \ldots, p_n)$ be $(N+3)$-large tuple. Then*

$$A^*_{L+j} = R^{-1} \circ A_{L+j} \circ R$$

for some word $R \in F(X \cup C_S)$ such that $\mathrm{rank}(R) \leqslant L - K + j + 2$ *and* $|R| < |A_{L+j}|$.

Proof. First, let $L = K$. Consider elementary periods $x_i = A_{m+4i-3}$ and $A_1 = c_1^{z_1} c_2^{z_2}$. For $i \neq n$,

$$x_i^{2\phi_K} = x_i^{\phi_k} \circ x_i^{\phi_K}.$$

For $i=n$,

$$A^*(\phi_{K+m+4n-3}) = R^{-1} \circ A_{K+m+4n-3} \circ R,$$

where $R = A_{m+4i-4}^{p_{m+4n-4}}$, therefore $\operatorname{rank}_N(R) = m+4n-4$. For the other elementary period,

$$\left(c_1^{z_1} c_2^{z_2}\right)^{2\phi_K} = \left(c_1^{z_1} c_2^{z_2}\right)^{\phi_K} \circ \left(c_1^{z_1} c_2^{z_2}\right)^{\phi_K}.$$

Any other A_j can be written in the form $A_j = u_1 \circ v_1 \circ u_2 \circ v_2 \circ u_3$, where v_1, v_2 are the first and the last elementary squares in A_j, which are parts of big powers of elementary periods. The Nielsen property of ϕ_K implies that the word R for $A^*(\phi_{K+j})$ is the word that cancels between $(v_2u_3)^{\phi_K}$ and $(u_1v_1)^{\phi_K}$. It definitely has N-large rank $\leqslant K$, because the element $(v_2u_3u_1v_1)^{\phi_K}$ has N-large rank $\leqslant K$. To give an exact bound for the rank of R we consider all possibilities for A_j:

(1) A_i begins with z_i^{-1} and ends with z_{i+1}, $i=1,\dots,m-1$;
(2) A_m begins with z_m^{-1} and ends with x_1^{-1};
(3) A_{m+4i-4} begins with $x_{i-1}y_{i-2}^{-1}x_{i-2}^{-2}$, if $i=3,\dots,n$, and ends with $x_{i-1}^2 y_{i-1} x_i^{-1}$ if $i=2,\dots,n$; if $i=2$ it begins with $x_1 \prod_{j=m}^{1} c_j^{-z_j} (c_2^{-z_2} c_1^{-z_1})^2$;
(4) A_{m+4i-2} and A_{m+4i-1} begin with $x_i y_{i-1}^{-1} x_{i-1}^{-2}$ and end with $x_i^2 y_i$ if $i=1,\dots,n$.

Therefore, $A_i^{\phi_K}$ begins with z_{i+1}^{-1} and ends with z_{i+2}, $i=1,\dots,m-2$, and is cyclically reduced. $A_{m-1}^{\phi_K}$ begins with z_m^{-1} and ends with x_1, and is cyclically reduced. $A_m^{\phi_K}$ begins with z_m^{-1} and ends with x_1^{-1} and is cyclically reduced. We have already considered $A_{m+4i-3}^{\phi_K}$.

Elements $A_{m+4i-4}^{\phi_K}, A_{m+4i-2}^{\phi_K}, A_{m+4i-1}^{\phi_K}$ are not cyclically reduced. By Lemma 53, for $A_{K+m+4i-4}^*$, one has

$$R = \left(x_{i-1}y_{i-2}^{-1}\right)^{\phi_K} \quad \left(\operatorname{rank}(R) = m+4i-4\right);$$

for $A_{K+m+4i-2}^*$ and $A_{K+m+4i-1}^*$,

$$R = \left(x_i y_{i-1}^{-1}\right)^{\phi_K} \quad \left(\operatorname{rank}(R) = m+4i\right).$$

This proves the statement of the lemma for $L=K$.

We can suppose by induction that

$$A_{L-K+j}^* = R^{-1} \circ A_{L-K+j} \circ R, \quad \text{and} \quad \operatorname{rank}(R) \leqslant L-2K+j+2.$$

The cancellations between $A_{L-K+j}^{\phi_K}$ and R^{ϕ_K} and between $A_{L-K+j}^{\phi_K}$ and $A_{L-K+j}^{\phi_K}$ correspond to cancellations in words u^{ϕ_K}, where u is a word in $\mathcal{W}_\Gamma$ between two elementary squares. These cancellations are in rank $\leqslant K$, and the statement of the lemma follows. □

Lemma 60. *Let $W \in F(X \cup C_S)$ and $A = A_j = LT_N(W)$, and $A^* = R^{-1} \circ A \circ R$. Then W can be presented in the form*

$$W = B_1 \circ_d A^{*q_1} \circ_d B_2 \circ_d \cdots \circ_d B_k \circ_d A^{*q_k} \circ_d B_{k+1} \tag{76}$$

where A^{*q_i} are maximal stable N-large occurrences of A^* in W and $d \leqslant |R|$. This presentation is unique and it is called the canonical N-large A^*-decomposition of W.

Proof. The result follows from existence and uniqueness of the canonical A-decompositions. Indeed, if

$$W = B_1 \circ A^{q_1} \circ B_2 \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

is the canonical A-decomposition of W, then

$$(B_1R)\big(R^{-1}AR\big)^{q_1}\big(R^{-1}B_2R\big)\ldots\big(R^{-1}B_kR\big)\big(R^{-1}AR\big)^{q_k}\big(R^{-1}B_{k+1}\big)$$

is the canonical A^*-decomposition of W. Indeed, since every A^{q_i} is a stable occurrence, then every B_i starts with A (if $i \neq 1$) and ends with A (if $i = k+1$). Hence $R^{-1}B_iR = R^{-1} \circ B_i \circ R$.

Conversely if

$$W = B_1A^{*q_1}B_2 \ldots B_kA^{*q_k}B_{k+1}$$

is an A^*-representation of W then

$$W = \big(B_1R^{-1}\big) \circ A^{q_1} \circ \big(RB_2R^{-1}\big) \circ \cdots \circ \big(RB_kR^{-1}\big) \circ A^{q_k} \circ (RB_{k+1})$$

is the canonical A-decomposition for W. □

Let ϕ be an automorphism of $F(X \cup C)$ which satisfies the Nielsen property with respect to a set W with exceptions E. In Definition 32, we have introduced the notation $M_{\phi,w}$ for the middle of w with respect to ϕ for $w \in Y \cup E$. We now introduce a similar notation for any $w \in \mathrm{Sub}(W)$ denoting by $\bar{M}_{\phi,w}$ the maximal non-cancelled part of w^ϕ in the words $(uwv)^\phi$ for all $uwv \in W$ with $w \neq u^{-1}, v^{-1}$. Observe that, in general, $\bar{M}_{\phi,w}$ may be empty while this cannot hold for $M_{\phi,w}$. If $\bar{M}_{\phi,w}$ is non-empty then we represent w^ϕ as

$$w^\phi = \bar{L}_{\phi,w} \circ \bar{M}_{\phi,w} \circ \bar{R}_{\phi,w}.$$

Lemma 61. *Let $L = lK$, $l > 0$, p a 3-large tuple.*

(1) *If E is closed under taking subwords then $\bar{M}_{\phi,w}$ is non-empty whenever the irreducible decomposition of w has length at least 3.*
(2) *$\bar{M}_{\phi_L,(A^2)}$ is non-empty for an elementary period A.*

(3) *The automorphism* $\phi_L = \phi_{L,p}$ *has the Nielsen property with respect to* $\bar{\mathcal{W}}_{\Gamma,L}$ *with exceptions* $E(m,n)$. *For* $w \in X \cup E(m,n)$ *and* $l > 1$, *the middle* $M_{\phi_L,w}$ *can be described in the following way. Let*

$$M_{\phi_K,w} = f \circ A^r \circ g \circ B^s \circ h$$

where A^r *and* B^s *are the first and the last maximal occurrences of elementary powers in* $M_{\phi_K,w}$. *Then* $M_{\phi_L,w}$ *contains* $\bar{M}_{\phi_{L-K},(A^r g B^s)}$ *as a subword.*

(4) *If* $i < j \leqslant L$ *then* A_j^2 *does not occur in* A_i.

Proof. To prove (1) observe that if $w = u_1u_2u_3$, $u_i \in Y \cup E$, is the irreducible decomposition of w then $\bar{M}_{\phi,w}$ should contain M_{ϕ,u_2}.

The middles $M_{\phi_K,x}$ of elements from X and from $E(m,n)$ contain big powers of some A_j, where $j = 1,\ldots,K$, and, therefore, big powers of elementary periods. Therefore, statements (2) and (3) can be proved by the simultaneous induction on l. Notice that for $l = 1$ both statements follow from Lemma 56.

The statement (4) follows from Lemmas 44–46. □

Lemma 62. *Let* $L = lK > 0$, $1 \leqslant ir \leqslant K$, $t \geqslant 2$, p *a* 3*-large tuple,*

(1) *and*

$$w = u \circ A_r^s \circ v$$

be a t*-stable occurrence of* A_r^s *in a word* $w \in \bar{\mathcal{W}}_{\Gamma,L}$. *Let* $A_{r+L}^* = R^{-1} \circ A_{r+L} \circ R$ *and* $d = |R|$. *Then*

$$w^{\phi_L} = u^{\phi_L} \circ_d \left(A_{r+L}^*\right)^s \circ_d v^{\phi_L}$$

where the occurrence of $(A_{r+L}^*)^s$ *is* $(t-2,d)$*-stable.*

(2) *Let* $W \in \bar{\mathcal{W}}_{\Gamma,L}$, *and* $A_{L+r}^* = R^{-1} \circ A_{L+r} \circ R$ *and* $d = |R|$. *If* $t \geqslant 2$ *and*

$$W = D_1 \circ A_r^{q_1} \circ D_2 \circ \cdots \circ A_r^{q_k} \circ D_{k+1}$$

is a t*-stable* A_r*-decomposition of* W *then*

$$W^{\phi_L} = D_1^{\phi_L} \circ_d (A_{L+r}^*)^{q_1} \circ_d D_2^{\phi_L} \circ_d \cdots \circ_d (A_{L+r}^*)^{q_k} \circ_d D_{k+1}^{\phi_L}$$

is a $(t-2,d)$*-stable* A_{L+r}^**-decomposition of* W^{ϕ_L}.

Proof. (1) Clearly, we can assume $t = 2$ without loss of generality. Suppose first that A_r is not an elementary period. Then the canonical decomposition of A_r is of length $\geqslant 3$ and thus $\bar{M}_{\phi_L}(A_r)$ is non-empty by Lemma 61. This implies that u^{ϕ_L} ends with $\bar{M}_{\phi_L}(A_r)\bar{R}_{\phi_L}(A_r)$, and thus the cancellation between u^{ϕ_L} and $(A_{r+L}^*)^r$ is the same as in the product $A^*_{r+L} \cdot A^*_{r+L}$. Similarly, the same is the cancellation between $(A^*_{r+L})^r$ and v^{ϕ_L} and the statement of lemma follows.

If A_r is an elementary period, a slightly more careful analysis is needed. We first consider the image of w under ϕ_K. If $r=1$ one of the images $A_1^{\pm\phi_K}$ of the periods $A_1^{\pm 1}$ in the occurrence of $A^{s+4\,\mathrm{sgn}(r)}$ in w (i.e., the first or the last one) may be completely cancelled in w^{ϕ_K}, but all the others have non-empty non-cancelled contributions in w^{ϕ_K}. Then an easy application of Lemma 61 (with L replaced with $L-K$) gives the result, and this is the case when only $(t-2,d)$-stability can be stated. If A_r is an elementary period of the form x_j, a similar argument applies but with no possibility of completely cancelled period $A_r^{\pm 1}$ under ϕ_K.

(2) follows from (1). □

Lemma 63. *Let $A_{j_1}\dots, A_{j_k}$, $k \geqslant 0$, be elementary periods, $1 \leqslant j_1, \dots, j_k \leqslant K$. If $w \in \bar{\mathcal{W}}_{\Gamma,L}$ and*

$$w^{\phi_K} = \tilde{w}_0 \circ_{d_{j_1}} A^{*q_1}_{j_1+K} \circ_{d_{j_1}} \tilde{w}_1 \circ_{d_{j_2}} \cdots \circ_{d_{j_k}} A^{*q_k}_{j_k+K} \tilde{w}_k, \tag{77}$$

where $q_i \geqslant 5$, $\tilde{w}_i$ does not contain an elementary square, and $d_{j_i} = |R_{j_i}|$, where $A^*_{j_i+K} = R_{j_i}^{-1} \circ A_{j_i+K} \circ R_{j_i}$ (see Lemma 59), $i = 1, \dots, k$, then

$$w = w_0 \circ A_{j_1}^{q_1} \circ w_1 \circ \cdots \circ A_{j_k}^{q_k} \circ w_k,$$

where $w_i^{\phi_K} = \tilde{w}_i$, $i = 0, \dots, k$.

Proof. (1) Suppose that w does not contain an elementary square.

In this case either $w \in \bar{\mathcal{W}}_\Gamma$ or $w = v_1 v_2$ for some words $v_1, v_2 \in \bar{\mathcal{W}}_\Gamma$ which are described in Lemma 55.

Claim 1. *If w^{ϕ_K} contains B^s for some cyclically reduced word $B \neq c_i$, $i = 1, \dots, m$, and $s \geqslant 2$, then B is a power of a cyclic permutation of some uniquely defined period A_i, $i = 1, \dots, K$.*

It suffices to consider the case $s=2$. Notice that for $w \in \bar{\mathcal{W}}_\Gamma$ the claim follows from Lemma 54. Now observe that if $w = v_1 v_2$ for $v_1, v_2 \in \bar{\mathcal{W}}_\Gamma$ then

$$w^{\phi_K} = v_1^{\phi_K} \circ v_2^{\phi_K}$$

and "illegal" squares do not occur on the boundary between $v_1^{\phi_K}$ and $v_2^{\phi_K}$ (direct inspection).

Claim 2. *w^{ϕ_K} does not contain $(E^{\phi_K})^2$, where E is an elementary period.*

By Claim 1, w^{ϕ_K} contains $(E^{\phi_K})^2$, where E is an elementary period, if and only if E^{ϕ_K} is a power of a cyclic permutation of some period A_i, $i = 1, \dots, K$. So it suffices to show that E^{ϕ_K} is not a power of a cyclic permutation of some period A_i, $i = 1, \dots, K$. To this end we list below all the words E^{ϕ_K}.

By Lemma 44

$$A_1^{\phi_K} = A_1^{-p_1+1} c_2^{-z_2} A_1^{p_1} A_2^{-p_2+1} c_3^{-z_3} A_1^{-p_1+1} c_2^{z_2} A_1^{p_1-1} c_3^{z_3} A_2^{p_2-1} \quad (m \geqslant 2);$$

by Lemma 47

$$x_i^{\phi_K} = \Big| \begin{array}{c} A_{m+4i-2}^{q_2} \\ \hline x_i y_{i-1}^{-1} \qquad x_i y_i \end{array} \Big| \begin{array}{c} A_{m+4i-4}^{-q_0} \\ \hline x_i y_{i-1}^{-1} \qquad y_{i-2} x_{i-1}^{-1} \end{array} \Big| x_i \Big| \begin{array}{c} A_{m+4i-4}^{q_0} \\ \hline x_{i-1} y_{i-2}^{-1} \qquad y_{i-1} x_i^{-1} \end{array} \Big| \quad (i \neq n);$$

and (direct computation from Lemmas 44 and 47)

$$\begin{aligned} \left(c_1^{z_1} x_1^{-1}\right)^{\phi_K} &= z_1 x_1^{p_2} y_1 A_3^{p_3} A_1^{-p_1} x_1 \left(A_1^{-p_1} x_1^{p_2} y_1 A_3^{p_3} A_1^{-p_1} x_1\right)^{p_4-2} x_1^{p_2} y_1 \\ &\quad \times x_2^{-1} \left(y_1^{\phi_4} x_2^{-1}\right)^{p_5-1} \quad (n > 1), \\ \left(c_1^{z_1} x_1^{-1}\right)^{\phi_K} &= z_1 x_1^{-1} A_1^{p_1} A_3^{-p_3+1} y_1^{-1} x_1^{-p_2} A_1^{p_1} \quad (n = 1). \end{aligned}$$

The claim follows by comparing the formulas above with the corresponding formulas for A_j (Lemmas 44–47).

Now Claim 2 implies the lemma since in this case the decomposition (77) for the w^{ϕ_K} is of the form $w^{\phi_K} = \tilde{w}_0$ and $w = w_0$, as required.

(2) w^{ϕ_K} contains $(E^{\phi_K})^2$, where E is an elementary period. By the case (1) w has a non-trivial decomposition of the form

$$w = w_0 \circ A_{j_1}^{q_1} \circ w_1 \circ \cdots \circ A_{j_k}^{q_k} \circ w_k,$$

where $q_i \geqslant 2$, and w_i does not have squares of elementary periods. Consider the A_r-decomposition of w where $r = \max\{j_1, \ldots, j_k\}$:

$$w = D_1 \circ A_r^{q_1} \circ \cdots \circ A_r^{q_s} \circ D_{s+1},$$

where D_i does not contain a square of an elementary period. It follows from the case (1) that this decomposition is at least 3-large canonical stable A_r-decomposition of w. Indeed, if E_1 and E_2 are two distinct elementary periods then $E_1^{s\phi_K}$ does not contain a cyclically reduced part of $E_2^{2\phi_K}$ as a subword (see the formulas above). So in the canonical 3-stable A_{r+K}^*-decomposition of w^{ϕ_K} the powers $A_{r+K}^{*q_i}$ come from the corresponding powers of A_r. By Lemma 62

$$w^{\phi_K} = D_1^{\phi_K} \circ_d A_{K+r}^{*q_1} \circ_d \cdots \circ A_{K+r}^{*q_s} \circ_d D_{s+1}^{\phi_K},$$

is the canonical stable A_{j+K}^*-decomposition of w^{ϕ_K} that contains all the occurrences of powers of A_{j+K}^* in the decomposition (77). Now by induction on the maximal rank of elementary periods which squares appear in the words D_i we can finish the proof. □

Lemma 64. *Let $L = lK > 0$, $1 \leqslant r \leqslant K$, $A_{r+L}^* = R^{-1} \circ A_{r+L} \circ R$ and $d = |R|$. Then the following holds for every $w \in \bar{\mathcal{W}}_{\Gamma,L}$.*

(1) *Suppose there is a decomposition*

$$w^{\phi_K} = \tilde{u} \circ_f \left(A^*_{r+K}\right)^s \circ_f \tilde{v},$$

where $s \geqslant 5$ *and the cancellation between* $\tilde{u}$ *and* A^*_{r+K} (*respectively, between* A^*_{r+K} *and* $\tilde{v}$) *is not more than* f *which is the maximum of the corresponding* d *and length of the part of* A^*_{r+K} *before the first stable occurrence of an elementary power* (*respectively, after the last stable occurrence of an elementary power*). *Then*

$$w = u \circ A_r^s \circ v, \qquad u^{\phi_K} = \tilde{u}, \qquad v^{\phi_K} = \tilde{v}.$$

(2) *Let*

$$W^{\phi_L} = \tilde{D}_1 \circ_d \left(A^*_{L+r}\right)^{q_1} \circ_d \tilde{D}_2 \circ_d \cdots \circ_d \left(A^*_{L+r}\right)^{q_k} \circ_d \tilde{D}_{k+1}$$

be a $(1, d)$*-stable* 3*-large* A^*_{L+r}*-decomposition of* W^{ϕ_L}. *Then* W *has a stable* A_r*-decomposition*

$$W = D_1 \circ A_r^{q_1} \circ D_2 \circ \cdots \circ A_k^{q_k} \circ D_{k+1}$$

where $D_i^{\phi_L} = \tilde{D}_i$.

Proof. (1) If A_r is an elementary period, the statement follows from Lemma 63. Otherwise represent A_r as $A_r = A_{j_1}^{q_1} \circ w_1 \circ A_{j_2}^{q_2} \circ w_2$, where $A_{j_1}^{q_1}$ and $A_{j_2}^{q_2}$ are the first and the last maximal elementary powers (each A_i begins with an elementary power).

Then

$$w^{\phi_K} = \tilde{u} \circ_d \left(A^{q_1}_{j_1+K} \circ_d w_1^{\phi_K} \circ_d A^{q_2}_{j_2+K} \circ_d w_2^{\phi_K}\right)^{s-1} \circ_d A^{q_1}_{j_1+K} \circ_d w_1^{\phi_K} \circ_d A^{q_2}_{j_2+K} \\ \circ_d w_2^{\phi_K} \circ_f \tilde{v}.$$

Since ϕ_K is a monomorphism, by Lemma 63 we obtain

$$w = u \circ \left(A_{j_1}^{q_1} \circ w_1 \circ A_{j_2}^{q_2} \circ w_2\right)^{s-1} \circ A_{j_1}^{q_1} \circ w_1 \circ A_{j_2}^{q_2} \circ w_2 v,$$

where $u^{\phi_K} = \tilde{u}$, $v^{\phi_K} = \tilde{v}$. We will show that $w_2 v = w_2 \circ v$. Indeed, w_2 is either $c_i^{z_i}$, $i \geqslant 3$, or $y_{i-1}x_i^{-1}$, or y_i. If there is a cancellation between w and v, then v must respectively begin either with $c_i^{-z_i}$, or x_i or y_i^{-1} and the image of this letter when ϕ_K is applied to v must be almost completely cancelled. It follows from Lemma 53 that this does not happen. Therefore $w = u \circ A_r^s \circ v$, and (1) is proved.

(2) For $L = K$ statement (1) implies statement (2). We now use induction on l to prove (2).

Suppose

$$w^{\phi_L} = \tilde{u} \circ_d A^*_{r+L} \circ_d \tilde{v}. \tag{78}$$

Represent A^*_{r+K} as

$$A^*_{r+K} = w_0 \circ A_{i_1}^{q_1} \circ w_1 \circ A_{i_2}^{q_2} \circ w_2,$$

where $A_{i_1}^{s_1}$ and $A_{i_2}^{s_2}$ are the first and the last maximal occurrences of elementary powers.

Then

$$w^{\phi_L} = \tilde{u} w_0^{\phi_{L-K}} \circ_d \left(A_{i_1}^{s_1 \phi_{L-K}} \circ_d w_1^{\phi_{L-K}} \circ_d A_{i_2}^{s_2 \phi_{L-K}} \circ_d (w_2 w_0)^{\phi_{L-K}}\right)^{s-1} \circ_d A_{i_1}^{s_1 \phi_{L-K}} \circ_d w_1^{\phi_{L-K}} \circ_d A_{i_2}^{s_2 \phi_{L-K}} \circ_d (w_2)^{\phi_{L-K}} \tilde{v}.$$

By the assumption of induction

$$w^{\phi_K} = \hat{u} w_0 \circ \left(A_{i_1}^{s_1} \circ w_1 \circ A_{i_2}^{s_2} \circ (w_2 w_0)\right)^{s-1} \circ A_{i_1}^{s_1} \circ w_1 \circ A_{i_2}^{s_2} \circ (w_2 \hat{v}),$$

where $\hat{u}^{\phi_{L-K}} = \tilde{u}$, $\hat{v}^{\phi_{L-K}} = \tilde{v}$. Therefore

$$w^{\phi_K} = \hat{u} \circ_f A^{*s}_{r+K} \circ_f \hat{v}.$$

By statement (1), $w = u \circ A_r^s \circ v$, where $u^{\phi_K} = \hat{u}$, $v^{\phi_K} = \hat{v}$. Therefore (78) implies that $w = u \circ A_r^s \circ v$, where $u^{\phi_L} = \tilde{u}$, $v^{\phi_L} = \tilde{v}$. This implies (2) for L. □

Corollary 10.

(1) *Let $m \neq 0$, $n \neq 0$, $K = K(m,n)$, $p = (p_1, \dots, p_K)$ be a 3-large tuple, $L = Kl$. Then for any $u \in Y \cup E(m,n)$ the element $M_{\phi_L,u}$ contains A_j^q for some $j > L - K$ and $q > p_j - 3$.*

(2) *For any $x \in X$ if $\operatorname{rank}(x^{\phi_L}) = j$ then every occurrence of A_j^2 in x^{ϕ_L} occurs inside some occurrence of A_j^{N-3}.*

Proof. (1) follows from the formulas for M_u with respect to ϕ_K in Lemmas 53 and 62. □

Corollary 11. *Let $u, v \in \mathcal{W}_{\Gamma,L}$. If the canceled subword in the product $u^{\phi_K} v^{\phi_K}$ does not contain A_j^l for some $j \leqslant K$ and $l \in \mathbb{Z}$ then the canceled subword in the product $u^{\phi_{K+L}} v^{\phi_{K+L}}$ does not contain the subword A_{L+j}^l.*

Lemma 65. *Suppose p is an $(N+3)$-large tuple, $\phi_j = \phi_{jp}$. Let L be a multiple of K. Then*:

(1) (a) *$x_i^{\phi_j}$ has a canonical N-large A_j^*-decomposition of size $(N,2)$ if either $j \equiv m + 4(i-1) \pmod K$, or $j \equiv m + 4i - 2 \pmod K$, or $j \equiv m + 4i \pmod K$. In all other cases*

$$\operatorname{rank}\left(x_i^{\phi_j}\right) < j;$$

(b) $y_i^{\phi_j}$ *has a canonical* N*-large* A_j^**-decomposition of size* $(N,2)$ *if either* $j \equiv m + 4(i-1) \pmod K$, *or* $j \equiv m + 4i - 3 \pmod K$, *or* $j \equiv m + 4i - 1 \pmod K$, *or* $j \equiv m + 4i \pmod K$. *In all other cases*

$$\operatorname{rank}\left(y_i^{\phi_j}\right) < j;$$

(c) $z_i^{\phi_j}$ *has a canonical* N*-large* A_j^**-decomposition of size* $(N,2)$ *if* $j \equiv i \pmod K$ *and either* $1 \leqslant i \leqslant m-1$ *or* $i = m$ *and* $n \neq 0$. *In all other cases*

$$\operatorname{rank}\left(z_i^{\phi_j}\right) < j;$$

(d) *if* $n = 0$ *then* $z_m^{\phi_j}$ *has a canonical* N*-large* A_j^**-decomposition of size* $(N,2)$ *if* $j \equiv m - 1 \pmod K$. *In all other cases*

$$\operatorname{rank}\left(z_m^{\phi_j}\right) < j.$$

(2) *If* $j = r + L$, $0 < r \leqslant K$, $(w_1 \dots w_k) \in \mathrm{Sub}_k(X^{\pm\gamma_K \cdots \gamma_{r+1}})$ *then either* $(w_1 \dots w_k)^{\phi_j} = (w_1 \dots w_k)^{\phi_{j-1}}$, *or* $(w_1 \dots w_k)^{\phi_j}$ *has a canonical* N*-large* A_j^**-decomposition. In any case,* $(w_1 \dots w_k)^{\phi_j}$ *has a canonical* N*-large* A_s^**-decomposition in some rank* s, $j - K + 1 \leqslant s \leqslant j$.

Proof. (1) Consider $y_i^{\phi_{L+m+4i}}$:

$$y_i^{\phi_{L+m+4i}} = \left(x_{i+1}^{\phi_L} y_i^{-\phi_{L+m+4i-1}}\right)^{q_4 - 1} x_{i+1}^{\phi_L} \left(y_i^{\phi_{L+m+4i-1}} x_{i+1}^{-\phi_L}\right)^{q_4}.$$

In this case $A^*(\phi_{L+m+4i}) = x_{i+1}^{\phi_{L+m+4i-1}} y_i^{-\phi_{L+m+4i-1}}$.

To write a formula for $x_i^{\phi_{L+m+4i}}$, denote $\tilde{y}_{i-1} = y_{i-1}^{\phi_{L+m+4i-5}}$, $\bar{x}_i = x_i^{\phi_L}$, $\bar{y}_i = y_i^{\phi_L}$. Then

$$\begin{aligned} x_i^{\phi_{L+m+4i}} = {} & \left(\bar{x}_{i+1} y_i^{-\phi_{L+m+4i-1}}\right)^{q_4-1} \bar{x}_{i+1} \\ & \left(\left(\left(\bar{x}_i \tilde{y}_{i-1}^{-1}\right)^{q_0} \bar{x}_i^{q_1} \bar{y}_i\right)^{q_2-1} \left(\bar{x}_i \tilde{y}_{i-1}^{-1}\right)^{q_0} \bar{x}_i^{q_1+1} \bar{y}_i\right)^{-q_3+1} \bar{y}_i^{-1} \bar{x}_i^{-q_1} \left(\tilde{y}_{i-1} \bar{x}_i^{-1}\right)^{q_0}. \end{aligned}$$

Similarly we consider $z_i^{\phi_{L+i}}$.

(2) If in a word $(w_1 \dots w_k)^{\phi_j}$ all the powers of $A_j^{p_j}$ are cancelled (they can only cancel completely and the process of cancellations does not depend on p) then if we consider an A_j^*-decomposition of $(w_1 \dots w_k)^{\phi_j}$, all the powers of A_j^* are also completely cancelled. By construction of the automorphisms γ_j, this implies that

$$(w_1 \dots w_k)^{\gamma_j \phi_{j-1}} = (w_1 \dots w_k)^{\phi_{j-1}}. \qquad \square$$

7.2. Generic solutions of orientable quadratic equations

Let G be a finitely generated fully residually free group and $S = 1$ a standard quadratic orientable equation over G which has a solution in G. In this section we effectively construct discriminating sets of solutions of $S = 1$ in G. The main tool in this construction is an embedding

$$\lambda : G_{R(S)} \to G(U, T)$$

of the coordinate group $G_{R(S)}$ into a group $G(U, T)$ which is obtained from G by finitely many extensions of centralizers. There is a nice set Ξ_P (see Section 2.5) of discriminating G-homomorphisms from $G(U, T)$ onto G. The restrictions of homomorphisms from Ξ_P onto the image $G^{\lambda}_{R(S)}$ of $G_{R(S)}$ in $G(U, T)$ give a discriminating set of G-homomorphisms from $G^{\lambda}_{R(S)}$ into G, i.e., solutions of $S = 1$ in G. This idea was introduced in [12] to describe the radicals of quadratic equations.

It has been shown in [12] that the coordinate groups of non-regular standard quadratic equations $S = 1$ over G are already extensions of centralizers of G, so in this case we can immediately put $G(U, T) = G_{R(S)}$ and the result follows. Hence we can assume from the beginning that $S = 1$ is regular.

Notice, that all regular quadratic equations have solutions in general position, except for the equation $[x_1, y_1][x_2, y_2] = 1$ (see Section 2.7).

For the equation $[x_1, y_1][x_2, y_2] = 1$ we do the following trick. In this case we view the coordinate group $G_{R(S)}$ as the coordinate group of the equation $[x_1, y_1] = [y_2, x_2]$ over the group of constants $G * F(x_2, y_2)$. So the commutator $[y_2, x_2] = d$ is a non-trivial constant and the new equation is of the form $[x, y] = d$, where all solutions are in general position. Therefore, we can assume that $S = 1$ is one of the following types (below d, c_i are non-trivial elements from G):

$$\prod_{i=1}^{n} [x_i, y_i] = 1, \quad n \geqslant 3, \tag{79}$$

$$\prod_{i=1}^{n} [x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n \geqslant 1,\ m \geqslant 0, \tag{80}$$

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad m \geqslant 2, \tag{81}$$

and it has a solution in G in general position.

Observe, that since $S = 1$ is regular then Nullstellensats holds for $S = 1$, so $R(S) = \mathrm{ncl}(S)$ and $G_{R(S)} = G[X]/\,\mathrm{ncl}(S) = G_S$.

For a group H and an element $u \in H$ by $H(u, t)$ we denote the extension of the centralizer $C_H(u)$ of u:

$$H(u, t) = \bigl\langle H, t \bigm| t^{-1} x t = x \ \bigl(x \in C_H(u)\bigr)\bigr\rangle.$$

If

$$G = G_1 \leqslant G_1(u_1, t_1) = G_2 \leqslant \cdots \leqslant G_n(u_n, t_n) = G_{n+1}$$

is a chain of extensions of centralizers of elements $u_i \in G_i$, then we denote the resulting group G_{n+1} by $G(U, T)$, where $U = \{u_1, \ldots, u_n\}$ and $T = \{t_1, \ldots, t_n\}$.

Let $\beta : G_{R(S)} \to G$ be a solution of the equation $S(X) = 1$ in the group G such that

$$x_i^\beta = a_i, \qquad y_i^\beta = b_i, \qquad z_i^\beta = e_i.$$

Then

$$d = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n} [a_i, b_i].$$

Hence we can rewrite the equation $S = 1$ in the following form (for appropriate m and n):

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n} [a_i, b_i]. \tag{82}$$

Proposition 4. *Let $S = 1$ be a regular quadratic equation* (82) *and $\beta : G_{R(S)} \to G$ a solution of $S = 1$ in G in a general position. Then one can effectively construct a sequence of extensions of centralizers*

$$G = G_1 \leqslant G_1(u_1, t_1) = G_2 \leqslant \cdots \leqslant G_n(u_n, t_n) = G(U, T)$$

and a G-homomorphism $\lambda_\beta : G_{R(S)} \to G(U, T)$.

Proof. By induction we define a sequence of extensions of centralizers and a sequence of group homomorphisms in the following way.

Case: $m \neq 0$, $n = 0$. In this event for each $i = 1, \ldots, m-1$ we define by induction a pair (θ_i, H_i), consisting of a group H_i and a G-homomorphism $\theta_i : G[X] \to H_i$.

Before we will go into formalities let us explain the idea that lies behind this. If $z_1 \to e_1, \ldots, z_m \to e_m$ is a solution of an equation

$$z_1^{-1} c_1 z_1 \ldots z_m^{-1} c_m z_m = d, \tag{83}$$

then transformations

$$e_i \to e_i \left(c_i^{e_i} c_{i+1}^{e_{i+1}}\right)^q, \qquad e_{i+1} \to e_{i+1} \left(c_i^{e_i} c_{i+1}^{e_{i+1}}\right)^q, \qquad e_j \to e_j \quad (j \neq i, i+1), \tag{84}$$

produce a new solution of Eq. (83) for an arbitrary integer q. This solution is composition of the automorphism γ_i^q and the solution e. To avoid collapses under cancellation of the periods $(c_i^{e_i} c_{i+1}^{e_{i+1}})^q$ (which is an important part of the construction of the discriminating set of homomorphisms Ξ_P in Section 2.5) one might want to have number q as big as possible, the best way would be to have $q = \infty$. Since there are no infinite powers in G, to realize this idea one should go outside the group G into a bigger group, for example, into an ultrapower G' of G, in which a non-standard power, say t, of the element $c_i^{e_i} c_{i+1}^{e_{i+1}}$ exists. It is not hard to see that the subgroup $\langle G, t\rangle \leqslant G'$ is an extension of the centralizer $C_G(c_i^{e_i} c_{i+1}^{e_{i+1}})$ of the element $c_i^{e_i} c_{i+1}^{e_{i+1}}$ in G. Moreover, in the group $\langle G, t\rangle$ the transformation (84) can be described as

$$e_i \to e_i t, \qquad e_{i+1} \to e_{i+1} t, \qquad e_j \to e_j \quad (j \neq i, i+1), \tag{85}$$

Now, we are going to construct formally the subgroup $\langle G, t\rangle$ and the corresponding homomorphism using (85).

Let H be an arbitrary group and $\beta : G_S \to H$ a homomorphism. Composition of the canonical projection $G[X] \to G_S$ and β gives a homomorphism $\beta_0 : G[X] \to H$. For $i = 0$ put

$$H_0 = H, \qquad \theta_0 = \beta_0.$$

Suppose now, that a group H_i and a homomorphism $\theta_i : G[X] \to H_i$ are already defined. In this event we define H_{i+1} and θ_{i+1} as follows

$$H_{i+1} = \left\langle H_i, r_{i+1} \;\middle|\; \left[C_{H_i}\left(c_{i+1}^{z_{i+1}^{\theta_i}} c_{i+2}^{z_{i+2}^{\theta_i}} \right), r_{i+1} \right] = 1 \right\rangle,$$

$$z_{i+1}^{\theta_{i+1}} = z_{i+1}^{\theta_i} r_{i+1}, \qquad z_{i+2}^{\theta_{i+1}} = z_{i+2}^{\theta_i} r_{i+1}, \qquad z_j^{\theta_{i+1}} = z_j^{\theta_i} \quad (j \neq i+1, i+2).$$

By induction we constructed a series of extensions of centralizers

$$G = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{m-1} = H_{m-1}(G)$$

and a homomorphism

$$\theta_{m-1,\beta} = \theta_{m-1} : G[X] \to H_{m-1}(G).$$

Observe, that

$$c_{i+1}^{z_{i+1}^{\theta_i}} c_{i+2}^{z_{i+2}^{\theta_i}} = c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_i}$$

so the element r_{i+1} extends the centralizer of the element $c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_i}$. In particular, the following equality holds in the group $H_{m-1}(G)$ for each $i = 0, \ldots, m-1$:

$$\left[r_{i+1}, c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_i} \right] = 1 \tag{86}$$

(where $r_0 = 1$). Observe also, that

$$z_1^{\theta_{m-1}} = e_1 r_1, \quad z_i^{\theta_{m-1}} = e_i r_{i-1} r_i, \quad z_m^{\theta_{m-1}} = e_m r_{m-1} \quad (0 < i < m). \tag{87}$$

From (86) and (87) it readily follows that

$$\left(\prod_{i=1}^{m} z_i^{-1} c_i z_i\right)^{\theta_{m-1}} = \prod_{i=1}^{m} e_i^{-1} c_i e_i, \tag{88}$$

so θ_{m-1} gives rise to a homomorphism (which we again denote by θ_{m-1} or θ_β)

$$\theta_{m-1} : G_S \to H_{m-1}(G).$$

Now we iterate the construction one more time replacing H by $H_{m-1}(G)$ and β by θ_{m-1} and put:

$$H_\beta(G) = H_{m-1}\big(H_{m-1}(G)\big), \qquad \lambda_\beta = \theta_{\theta_{m-1}} : G_S \to H_\beta(G).$$

The group $H_\beta(G)$ is union of a chain of extensions of centralizers which starts at the group H.

If $H = G$ then all the homomorphisms above are G-homomorphisms. Now we can write

$$H_\beta(G) = G(U, T)$$

where $U = \{u_1, \ldots, u_{m-1}, \bar{u}_1, \ldots, \bar{u}_{m-1}\}$, $T = \{r_1, \ldots, r_{m-1}, \bar{r}_1, \ldots, \bar{r}_{m-1}\}$ and $\bar{u}_i, \bar{r}_i$ are the corresponding elements when we iterate the construction:

$$u_{i+1} = c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_{i+2}}, \qquad \bar{u}_{i+1} = c_{i+1}^{e_{i+1} r_i r_{i+1} \bar{r}_i} c_{i+2}^{e_{i+2} r_{i+1} r_{i+2}}.$$

Case: $m = 0$, $n > 0$. In this case $S = [x_1, y_1] \ldots [x_n, y_n] d^{-1}$. Similar to the case above we start with the principal automorphisms. They consist of two Dehn's twists:

$$x \to y^p x, \qquad y \to y, \tag{89}$$

$$x \to x, \qquad y \to x^p y, \tag{90}$$

which fix the commutator $[x, y]$, and the third transformation which ties two consequent commutators $[x_i, y_i][x_{i+1}, y_{i+1}]$:

$$\begin{aligned} &x_i \to \big(y_i x_{i+1}^{-1}\big)^{-q} x_i, \qquad y_i \to \big(y_i x_{i+1}^{-1}\big)^{-q} y_i \big(y_i x_{i+1}^{-1}\big)^{q}, \\ &x_{i+1} \to \big(y_i x_{i+1}^{-1}\big)^{-q} x_{i+1} \big(y_i x_{i+1}^{-1}\big)^{q}, \qquad y_{i+1} \to \big(y_i x_{i+1}^{-1}\big)^{-q} y_{i+1}. \end{aligned} \tag{91}$$

Now we define by induction on i, for $i = 0, \ldots, 4n-1$, pairs (G_i, α_i) of groups G_i and G-homomorphisms $\alpha_i : G[X] \to G_i$. Put

$$G_0 = G, \qquad \alpha_0 = \beta.$$

For each commutator $[x_i, y_i]$ in $S = 1$ we perform consequently three Dehn's twists (90), (89), (90) (more precisely, their analogs for an extension of a centralizer) and an analog of the connecting transformation (91) provided the next commutator exists. Namely, suppose G_{4i} and α_{4i} have been already defined. Then

$$\begin{gathered}
G_{4i+1} = \langle G_{4i}, t_{4i+1} \mid [C_{G_{4i}}(x_{i+1}^{\alpha_{4i}}), t_{4i+1}] = 1\rangle, \\
y_{i+1}^{\alpha_{4i+1}} = t_{4i+1} y_{i+1}^{\alpha_{4i}}, \qquad s^{\alpha_{4i+1}} = s^{\alpha_{4i}} \quad (s \neq y_{i+1}). \\
G_{4i+2} = \langle G_{4i+1}, t_{4i+2} \mid [C_{G_{4i+1}}(y_{i+1}^{\alpha_{4i+1}}), t_{4i+2}] = 1\rangle, \\
x_{i+1}^{\alpha_{4i+2}} = t_{4i+2} x_{i+1}^{\alpha_{4i+1}}, \qquad s^{\alpha_{4i+2}} = s^{\alpha_{4i+1}} \quad (s \neq x_{i+1}), \\
G_{4i+3} = \langle G_{4i+2}, t_{4i+3} \mid [C_{G_{4i+2}}(x_{i+1}^{\alpha_{4i+2}}), t_{4i+3}] = 1\rangle, \\
y_{i+1}^{\alpha_{4i+3}} = t_{4i+3} y_{i+1}^{\alpha_{4i+2}}, \qquad s^{\alpha_{4i+3}} = s^{\alpha_{4i+2}} \quad (s \neq y_{i+1}), \\
G_{4i+4} = \langle G_{4i+3}, t_{4i+4} \mid [C_{G_{4i+3}}(y_{i+1}^{\alpha_{4i+3}} x_{i+2}^{-\alpha_{4i+3}}), t_{4i+4}] = 1\rangle, \\
x_{i+1}^{\alpha_{4i+4}} = t_{4i+4}^{-1} x_{i+1}^{\alpha_{4i+3}}, \qquad y_{i+1}^{\alpha_{4i+4}} = y_{i+1}^{\alpha_{4i+3} t_{4i+4}}, \qquad x_{i+2}^{\alpha_{4i+4}} = x_{i+2}^{\alpha_{4i+3} t_{4i+4}}, \\
y_{i+2}^{\alpha_{4i+4}} = t_{4i+4}^{-1} y_{i+2}^{\alpha_{4i+3}}, \qquad s^{\alpha_{4i+4}} = s^{\alpha_{4i+3}} \quad (s \neq x_{i+1}, y_{i+1}, x_{i+2}, y_{i+2}).
\end{gathered}$$

Thus we have defined groups G_i and mappings α_i for all $i = 0, \ldots, 4n-1$. As above, the straightforward verification shows that the mapping α_{4n-1} gives rise to a G-homomorphism $\alpha_{4n-1} : G_S \to G_{4n-1}$. We repeat now the above construction once more time with G_{4n-1} in the place of G_0, α_{4n-1} in the place of β, and $\bar{t}_j$ in the place of t_j. We denote the corresponding groups and homomorphisms by $\bar{G}_i$ and $\bar{\alpha}_i : G_S \to \bar{G}_i$.

Put

$$G(U, T) = \bar{G}_{4n-1}, \qquad \lambda_\beta = \bar{\alpha}_{4n-1},$$

By induction we have constructed a G-homomorphism

$$\lambda_\beta : G_S \to G(U, T).$$

Case: $m > 0$, $n > 0$. In this case we combine the two previous cases together. To this end we take the group H_{m-1} and the homomorphism $\theta_{m-1} : G[X] \to H_{m-1}$ constructed in the first case and put them as the input for the construction in the second case. Namely, put

$$G_0 = \langle H_{m-1}, r_m \mid [C_{H_{m-1}}(c_m^{z_m^{\theta_{m-1}}} x_1^{-\theta_{m-1}}), r_m] = 1\rangle,$$

and define the homomorphism α_0 as follows

$$\begin{gathered}
z_m^{\alpha_0} = z_m^{\theta_{m-1}} r_m, \qquad x_1^{\alpha_0} = a_1^{r_m}, \qquad y_1^{\alpha_0} = r_m^{-1} b_1, \\
s^{\alpha_0} = s^{\theta_{m-1}} \quad (s \in X,\ s \neq z_m, x_1, y_1).
\end{gathered}$$

Now we apply the construction from the second case. Thus we have defined groups G_i and mappings $\alpha_i : G[X] \to G_i$ for all $i = 0, \dots, 4n-1$. As above, the straightforward verification shows that the mapping α_{4n-1} gives rise to a G-homomorphism $\alpha_{4n-1} : G_S \to G_{4n-1}$.

We repeat now the above construction once more time with G_{4n-1} in place of G_0 and α_{4n-1} in place of β. This results in a group $\bar{G}_{4n-1}$ and a homomorphism $\bar{\alpha}_{4n-1} : G_S \to \bar{G}_{4n-1}$.

Put

$$G(U,T) = \bar{G}_{4n-1}, \qquad \lambda_\beta = \bar{\alpha}_{4n-1}.$$

We have constructed a G-homomorphism

$$\lambda_\beta : G_S \to G(U,T).$$

We proved the proposition for all three types of Eqs. (79)–(81), as required. □

Proposition 5. *Let $S = 1$ be a regular quadratic equation* (2) *and $\beta : G_{R(S)} \to G$ a solution of $S = 1$ in G in a general position. Then the homomorphism $\lambda_\beta : G_{R(S)} \to G(U,T)$ is a monomorphism.*

Proof. In the proof of this proposition we use induction on the atomic rank of the equation in the same way as in the proof of Theorem 1 in [12].

Since all the intermediate groups are also fully residually free by induction it suffices to prove the following:

(1) $n = 1$, $m = 0$; prove that $\psi = \alpha_3$ is an embedding of G_S into G_3;
(2) $n = 2$, $m = 0$; prove that $\psi = \alpha_4$ is a monomorphism on $H = \langle G, x_1, y_1 \rangle$;
(3) $n = 1$, $m = 1$; prove that $\psi = \alpha_3 \bar{\alpha}_0$ is a monomorphism on $H = \langle G, z_1 \rangle$;
(4) $n = 0$, $m \geqslant 3$; prove that $\theta_2 \bar{\theta}_2$ is an embedding of G_S into $\bar{H}_2$.

Now we consider all these cases one by one.

(1) Choose an arbitrary non-trivial element $h \in G_S$. It can be written in the form

$$h = g_1 v_1(x_1, y_1) g_2 v_2(x_1, y_1) g_3 \dots v_n(x_1, y_1) g_{n+1},$$

where $1 \neq v_i(x_1, y_1) \in F(x_1, y_1)$ are words in x_1, y_1, not belonging to the subgroup $\langle [x_1, y_1] \rangle$, and $1 \neq g_i \in G$, $g_i \notin \langle [a, b] \rangle$ (with the exception of g_1 and g_{n+1}, they could be trivial). Then

$$h^\psi = g_1 v_1(t_3 t_1 a, t_2 b) g_2 v_2(t_3 t_1 a, t_2 b) g_3 \dots v_n(t_3 t_1 a, t_2 b) g_{n+1}. \tag{92}$$

The group $G(U,T)$ is obtained from G by three HNN extensions (extensions of centralizers), so every element in $G(U,T)$ can be rewritten to its reduced form by making finitely many pinches. It is easy to see that the leftmost occurrence of either t_3 or t_1 in the product (92) occurs in the reduced form of h^ψ uncancelled.

(2) $x_1 \to t_4^{-1}t_2a_1,\ y_1 \to t_4^{-1}t_3t_1b_1t_4,\ x_2 \to t_4^{-1}a_2t_4,\ y_2 \to t_4^{-1}b_2$. Choose an arbitrary non-trivial element $h \in H = G * F(x_1, y_1)$. It can be written in the form

$$h = g_1v_1(x_1, y_1)g_2v_2(x_1, y_1)g_3 \dots v_n(x_1, y_1)g_{n+1},$$

where $1 \neq v_i(x_1, y_1) \in F(x_1, y_1)$ are words in x_1, y_1, and $1 \neq g_i \in G$ (with the exception of g_1 and g_{n+1}, they could be trivial). Then

$$h^\psi = g_1v_1\bigl(t_4^{-1}t_2a, (t_3t_1b)^{t_4}\bigr)g_2v_2\bigl(t_4^{-1}t_2a, (t_3t_1b)^{t_4}\bigr)g_3 \dots v_n\bigl(t_4^{-1}t_2a, (t_3t_1b)^{t_4}\bigr)g_{n+1}. \quad (93)$$

The group $G(U, T)$ is obtained from G by four HNN extensions (extensions of centralizers), so every element in $G(U, T)$ can be rewritten to its reduced form by making finitely many pinches. It is easy to see that the leftmost occurrence of either t_4 or t_1 in the product (93) occurs in the reduced form of h^ψ uncancelled.

(3) We have an equation

$$c^z[x, y] = c[a, b], \quad z \to zr_1\bar{r}_1, \quad x \to \bigl(t_2a^{r_1}\bigr)^{\bar{r}_1}, \quad y \to \bar{r}_1^{-1}t_3t_1r_1^{-1}b, \quad \text{and}$$
$$\bigl[r_1, ca^{-1}\bigr] = 1, \qquad \bigl[\bar{r}_1, \bigl(c^{r_1}a^{-r_1}t_2^{-1}\bigr)\bigr] = 1.$$

Here we can always suppose, that $[c, a] \neq 1$, by changing a solution, hence $[r_1, \bar{r}_1] \neq 1$. The proof for this case is a repetition of the proof of Proposition 11 in [12].

(4) We will consider the case when $m = 3$; the general case can be considered similarly. We have an equation $c_1^{z_1}c_2^{z_2}c_3^{z_3} = c_1c_2c_3$, and can suppose $[c_i, c_{i+1}] \neq 1$.

We will prove that $\psi = \theta_2\bar{\theta}_1$ is an embedding. The images of z_1, z_2, z_3 under $\theta_2\bar{\theta}_1$ are the following:

$$z_1 \to c_1r_1\bar{r}_1, \qquad z_2 \to c_2r_1r_2\bar{r}_1, \qquad z_3 \to c_3r_2,$$

where

$$[r_1, c_1c_2] = 1, \qquad \bigl[r_2, c_2^{r_1}c_3\bigr] = 1, \qquad \bigl[\bar{r}_1, c_1^{r_1}c_2^{r_1r_2}\bigr] = 1.$$

Let w be a reduced word in $G * F(z_i, i = 1, 2, 3)$, which does not have subwords $c_1^{z_1}$. We will prove that if $w^\psi = 1$ in $\bar{H}_1$, then $w \in N$, where N is the normal closure of the element $c_1^{z_1}c_2^{z_2}c_3^{z_3}c_3^{-1}c_2^{-1}c_1^{-1}$. We use induction on the number of occurrences of $z_1^{\pm 1}$ in w. The induction basis is obvious, because homomorphism ψ is injective on the subgroup $\langle F, z_2, z_3\rangle$.

Notice, that the homomorphism ψ is also injective on the subgroup $K = \langle z_1z_2^{-1}, z_3, F\rangle$.

Consider $\bar{H}_1$ as an HNN extension by letter $\bar{r}_1$. Suppose $w^\psi = 1$ in $\bar{H}_1$. Letter $\bar{r}_1$ can disappear in two cases: (1) $w \in KN$, (2) there is a pinch between $\bar{r}_1^{-1}$ and $\bar{r}_1$ (or between $\bar{r}_1$ and $\bar{r}_1^{-1}$) in w^ψ. This pinch corresponds to some element $z_{1,2}^{-1}uz'_{1,2}$ (or $z_{1,2}u(z'_{1,2})^{-1}$), where $z_{1,2}, z'_{1,2} \in \{z_1, z_2\}$.

In the first case $w^\psi \neq 1$, because $w \in K$ and $w \notin N$.

In the second case, if the pinch happens in $(z_{1,2}u(z'_{1,2})^{-1})^\psi$, then $z_{1,2}u(z'_{1,2})^{-1} \in KN$, therefore it has to be at least one pinch that corresponds to $(z_{1,2}^{-1}uz'_{1,2})^\psi$. We can suppose, up to a cyclic shift of w, that $z_{1,2}^{-1}$ is the first letter, w does not end with some $z''_{1,2}$, and w cannot be represented as $z_{1,2}^{-1}uz'_{1,2}v_1z''_{1,2}v_2$, such that $z'_{1,2}v_1 \in KN$. A pinch can only happen if $z_{1,2}^{-1}uz'_{1,2} \in \langle c_1^{z_1}c_2^{z_2}\rangle$. Therefore, either $z_{1,2} = z_1$, or $z'_{1,2} = z_1$, and one can replace $c_1^{z_1}$ by $c_1c_2c_3c_3^{-z_3}c_2^{-z_2}$, therefore replace w by w_1 such that $w = uw_1$, where u is in the normal closure of the element $c_1^{z_1}c_2^{z_2}c_3^{z_3}c_3^{-1}c_2^{-1}c_1^{-1}$, and apply induction. □

The embedding $\lambda_\beta : G_S \to G(U,T)$ allows one to construct effectively discriminating sets of solutions in G of the equation $S = 1$. Indeed, by the construction above the group $G(U,T)$ is union of the following chain of length $2K = 2K(m,n)$ of extension of centralizers:

$$\begin{aligned} G = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{m-1} \leqslant G_0 \leqslant G_1 \leqslant \cdots \leqslant G_{4n-1} \\ = \bar{H}_0 \leqslant \bar{H}_1 \leqslant \cdots \leqslant \bar{H}_{m-1} = \bar{G}_0 \leqslant \cdots \leqslant \bar{G}_{4n-1} = G(U,T). \end{aligned}$$

Now, every $2K$-tuple $p \in \mathbb{N}^{2K}$ determines a G-homomorphism

$$\xi_p : G(U,T) \to G.$$

Namely, if Z_i is the ith term of the chain above then Z_i is an extension of the centralizer of some element $g_i \in Z_{i-1}$ by a stable letter t_i. The G-homomorphism ξ_p is defined as composition

$$\xi_p = \psi_1 \circ \cdots \circ \psi_K$$

of homomorphisms $\psi_i : Z_i \to Z_{i-1}$ which are identical on Z_{i-1} and such that $t_i^{\psi_i} = g_i^{p_i}$, where p_i is the ith component of p.

It follows (see Section 2.5) that for every unbounded set of tuples $P \subset \mathbb{N}^{2K}$ the set of homomorphisms

$$\Xi_P = \{\xi_p \mid p \in P\}$$

G-discriminates $G(U,T)$ into G. Therefore (since λ_β is monic), the family of G-homomorphisms

$$\Xi_{P,\beta} = \{\lambda_\beta \xi_p \mid \xi_p \in \Xi_P\}$$

G-discriminates G_S into G.

One can give another description of the set $\Xi_{P,\beta}$ in terms of the basic automorphisms from the basic sequence Γ. Observe first that

$$\lambda_\beta \xi_p = \phi_{2K,p}\beta,$$

therefore

$$\Xi_{P,\beta} = \{\phi_{2K,p}\beta \mid p \in P\}.$$

We summarize the discussion above as follows.

Theorem 10. *Let G be a finitely generated fully residually free group, $S = 1$ a regular standard quadratic orientable equation, and Γ its basic sequence of automorphisms. Then for any solution $\beta : G_S \to G$ in general position, any positive integer $J \geqslant 2$, and any unbounded set $P \subset \mathbb{N}^{JK}$ the set of G-homomorphisms $\Xi_{P,\beta}$ G-discriminates $G_{R(S)}$ into G. Moreover, for any fixed tuple $p' \in \mathbb{N}^{tK}$ the family*

$$\Xi_{P,\beta,p'} = \{\phi_{tK,p'}\theta \mid \theta \in \Xi_{P,\beta}\}$$

G-discriminates $G_{R(S)}$ into G.

For tuples $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_m)$ denote the tuple

$$fg = (f_1, \ldots, f_k, g_1, \ldots, g_m).$$

Similarly, for a set of tuples P put

$$fPg = \{fpg \mid p \in P\}.$$

Corollary 12. *Let G be a finitely generated fully residually free group, $S = 1$ a regular standard quadratic orientable equation, Γ the basic sequence of automorphisms of S, and $\beta : G_S \to G$ a solution of $S = 1$ in general position. Suppose $P \subseteq \mathbb{N}^{2K}$ is unbounded set, and $f \in \mathbb{N}^{Ks}$, $g \in \mathbb{N}^{Kr}$ for some $r, s \in \mathbb{N}$. Then there exists a number N such that if f is N-large and $s \geqslant 2$ then the family*

$$\Phi_{P,\beta,f,g} = \{\phi_{K(r+s+2),q}\beta \mid q \in fPg\}$$

G-discriminates $G_{R(S)}$ into G.

Proof. By Theorem 10 it suffices to show that if f is N-large for some N then $\beta_f = \phi_{2K,f}\beta$ is a solution of $S = 1$ in general position, i.e., the images of some particular finitely many non-commuting elements from $G_{R(S)}$ do not commute in G. It has been shown above that the set of solutions $\{\phi_{2K,h}\beta \mid h \in \mathbb{N}^{2K}\}$ is a discriminating set for $G_{R(S)}$. Moreover, for any finite set M of non-trivial elements from $G_{R(S)}$ there exists a number N such that for any N-large tuple $h \in \mathbb{N}^{2K}$ the solution $\phi_{2K,h}\beta$ discriminates all elements from M into G. Hence the result. $\square$

7.3. Small cancellation solutions of standard orientable equations

Let $S(X) = 1$ be a standard regular orientable quadratic equation over F written in the form (82):

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n} [a_i, b_i].$$

In this section we construct solutions in F of $S(X)=1$ which satisfy certain small cancellation conditions.

Definition 36. Let $S=1$ be a standard regular orientable quadratic equation written in the form (82). We say that a solution $\beta: F_S \to F$ of $S=1$ satisfies the small cancellation condition $(1/\lambda)$ with respect to the set $\bar{\mathcal{W}}_\Gamma$ (respectively $\bar{\mathcal{W}}_{\Gamma,L}$) if the following conditions are satisfied:

(1) β is in general position;
(2) for any 2-letter word $uv \in \bar{\mathcal{W}}_\Gamma$ (respectively $uv \in \mathcal{W}_{\Gamma,L}$) (in the alphabet Y) the cancellation in the word $u^\beta v^\beta$ does not exceed $(1/\lambda)\min\{|u^\beta|, |v^\beta|\}$ (we assume here and below that u^β, v^β are given by their reduced forms in F);
(3) the cancellation in a word $u^\beta v^\beta$ does not exceed $(1/\lambda)\min\{|u^\beta|, |v^\beta|\}$ provided u, v satisfy one of the conditions below:
   (a) $u=z_i$, $v=(z_{i-1}^{-1}c_{i-1}^{-1}z_{i-1})$,
   (b) $u=c_i$, $v=z_i$,
   (c) $u=v=c_i$

(we assume here that u^β, v^β are given by their reduced forms in F).

Notation. For a homomorphism $\beta: F[X] \to F$ by C_β we denote the set of all elements that cancel in $u^\beta v^\beta$ where u, v are as in (2), (3) from Definition 36 and the word that cancels in the product $(c_2^{z_2})^\beta \cdot (dc_{m-1}^{-z_{m-1}})^\beta$.

Lemma 66. *Let u, v be cyclically reduced elements of $G * H$ such that $|u|, |v| \geqslant 2$. If for some $m, n > 1$ elements u^m and v^n have a common initial segment of length $|u|+|v|$, then u and v are both powers of the same element $w \in G * H$. In particular, if both u and v are not proper powers then $u=v$.*

Proof. The same argument as in the case of free groups. □

Corollary 13. *If $u, v \in F$, $[u,v] \neq 1$, then for any $\lambda \geqslant 0$ there exist m_0, n_0 such that for any $m \geqslant m_0$, $n \geqslant n_0$ cancellation between u^m and v^n is less than $(1/\lambda)\max\{|u^m|, |v^n|\}$.*

Lemma 67. *Let $S(X)=1$ be a standard regular orientable quadratic equation written in the form* (82):

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n} [a_i, b_i], \quad n \geqslant 1,$$

where all c_i are cyclically reduced. Then there exists a solution β of this equation that satisfies the small cancellation condition with respect to $\bar{\mathcal{W}}_{\Gamma,L}$. Moreover, for any word $w \in \bar{\mathcal{W}}_{\Gamma,L}$ that does not contain elementary squares, the word w^β does not contain a cyclically reduced part of $A_i^{2\beta}$ for any elementary period A_i.

Proof. We will begin with a solution

$$\beta_1 : x_i \to a_i,\ y_i \to b_i,\ z_i \to e_i$$

of $S = 1$ in F in general position. We will show that for any $\lambda \in \mathbb{N}$ there are positive integers $m_i,\ n_i,\ k_i,\ q_j$ and a tuple $p = (p_1, \dots, p_m)$ such that the map $\beta : F[X] \to F$ defined by

$$x_1^\beta = \left(\tilde{b}_1^{n_1}\tilde{a}_1\right)^{[\tilde{a}_1,\tilde{b}_1]^{m_1}}, \qquad y_1^\beta = \left(\left(\tilde{b}_1^{n_1}\tilde{a}_1\right)^{k_1}\tilde{b}_1\right)^{[\tilde{a}_1,\tilde{b}_1]^{m_1}},$$

where $\tilde{a}_1 = x_1^{\phi_m\beta_1}$, $\tilde{b}_1 = y_1^{\phi_m\beta_1}$,

$$x_i^\beta = \left(b_i^{n_i}a_i\right)^{[a_i,b_i]^{m_i}}, \quad y_i^\beta = \left(\left(b_i^{n_i}a_i\right)^{k_i}b_i\right)^{[a_i,b_i]^{m_i}}, \quad i = 2, \dots, n,$$
$$z_i^\beta = c_i^{q_i} z_i^{\phi_m\beta_1}, \quad i = 1, \dots, m,$$

is a solution of $S = 1$ satisfying the small cancellation condition $(1/\lambda)$ with respect to $\bar{\mathcal{W}}_\Gamma$. Moreover, we will show that one can choose the solution β_1 such that β satisfies the small cancellation condition with respect to $\bar{\mathcal{W}}_{\Gamma,L}$.

The solution β_1 is in general position, therefore the neighboring items in the sequence

$$c_1^{e_1}, \dots, c_m^{e_m}, [a_1, b_1], \dots, [a_n, b_n]$$

do not commute. We have $[c_i^{e_i}, c_{i+1}^{e_{i+1}}] \neq 1$.

There is a homomorphism $\theta_{\beta_1} : F_S \to \bar{F} = F(\bar{U}, \bar{T})$ into the group $\bar{F}$ obtained from F by a series of extensions of centralizers, such that $\beta = \theta_{\beta_1}\psi_p$, where $\psi_p : \bar{F} \to F$. This homomorphism θ_{β_1} is a monomorphism on $F * F(z_1, \dots, z_m)$ (this follows from the proof of Theorem 4 in [12], where the same sequence of extensions of centralizers is constructed).

The set of solutions ψ_p for different tuples p and numbers m_i, n_i, k_i, q_j is a discriminating family for $\bar{F}$. We just have to show that the small cancellation condition for β is equivalent to a finite number of inequalities in the group $\bar{F}$.

We have $z_i^\beta = c_i^{q_i} z_i^{\phi_m\beta_1}$ such that $\beta_1(z_i) = e_i$, and $p = (p_1, \dots, p_m)$ is a large tuple. Denote $\bar{A}_j = A_j^{\beta_1}$, $j = 1, \dots, m$. Then it follows from Lemma 44 that

$$z_i^\beta = c_i^{q_i+1} e_i \bar{A}_{i-1}^{p_{i-1}} c_{i+1}^{e_{i+1}} \bar{A}_i^{p_i-1}, \quad \text{where } i = 2, \dots, m-1,$$
$$z_m^\beta = c_m^{q_m+1} e_m \bar{A}_{m-1}^{p_{m-1}} a_1^{-1} \bar{A}_m^{p_m-1},$$

where

$$\bar{A}_1 = c_1^{e_1} c_2^{e_2},\ \bar{A}_2 = \bar{A}_1(p_1) = \bar{A}_1^{-p_1} c_2^{e_2} \bar{A}_1^{p_1} c_3^{e_3},\ \dots,$$
$$\bar{A}_i = \bar{A}_{i-1}^{-p_{i-1}} c_i^{e_i} \bar{A}_{i-1}^{p_{i-1}} c_{i+1}^{e_{i+1}}, \quad i = 2, \dots, m-1,$$
$$\bar{A}_m = \bar{A}_{m-1}^{-p_{m-1}} c^{e_m} \bar{A}_{m-1}^{p_{m-1}} a_1^{-1}.$$

One can choose p such that $[\bar{A}_i, \bar{A}_{i+1}] \neq 1, [\bar{A}_{i-1}, c_{i+1}^{e_{i+1}}] \neq 1, [\bar{A}_{i-1}, c_i^{e_i}] \neq 1$ and $[\bar{A}_m, [a_1, b_1]] \neq 1$, because their pre-images do not commute in $\bar{F}$. We need the second and third inequality here to make sure that $\bar{A}_i$ does not end with a power of $\bar{A}_{i-1}$. Alternatively, one can prove by induction on i that p can be chosen to satisfy these inequalities.

Then $c_i^{z_i^\beta}$ and $c_{i+1}^{z_{i+1}^\beta}$ have small cancellation, and $c_m^{z_m^\beta}$ has small cancellation with $x_1^{\pm\beta}$, $y_1^{\pm\beta}$.

Let

$$x_i^\beta = \left(b_i^{n_i} a_i\right)^{[a_i, b_i]^{m_i}}, \quad y_i^\beta = \left(\left(b_i^{n_i} a_i\right)^{k_i} b_i\right)^{[a_i, b_i]^{m_i}}, \quad i = 2, \ldots, n,$$

for some positive integers m_i, n_i, k_i, s_j which values we will specify in a due course. Let $uv \in \bar{\mathcal{W}}_\Gamma$. There are several cases to consider.

(1) $uv = x_i x_i$. Then

$$u^\beta v^\beta = \left(b_i^{n_i} a_i\right)^{[a_i, b_i]^{m_i}} \left(b_i^{n_i} a_i\right)^{[a_i, b_i]^{m_i}}.$$

Observe that the cancellation between $(b_i^{n_i} a_i)$ and $(b_i^{n_i} a_i)$ is not more then $|a_i|$. Hence the cancellation in $u^\beta v^\beta$ is not more then $|[a_i, b_i]^{m_i}| + |a_i|$. We chose $n_i \gg m_i$ such that

$$\left|[a_i, b_i]^{m_i}\right| + |a_i| < \frac{1}{\lambda} \left|\left(b_i^{n_i} a_i\right)^{[a_i, b_i]^{m_i}}\right|$$

which is obviously possible. Similar arguments prove the cases $uv = x_i y_i$ and $uv = y_i x_i$.

(2) In all other cases the cancellation in $u^\beta v^\beta$ does not exceed the cancellation between $[a_i, b_i]^{m_i}$ and $[a_{i+1}, b_{i+1}]^{m_{i+1}}$, hence by Lemma 66 it is not greater than $|[a_i, b_i]| + |[a_{i+1}, b_{i+1}]|$.

Let $u = z_i^\beta$, $v = c_{i-1}^{-z_{i-1}^\beta}$. The cancellation is the same as between $\bar{A}_{2i}^{p_{2i}}$ and $\bar{A}_{i-1}^{-p_{i-1}}$ and, therefore, small.

Since c_i is cyclically reduced, there is no cancellation between c_i and z_i^β.

The first statement of the lemma is proved.

We now will prove the second statement of the lemma. We have to show that if $u = c_i^{z_i}$ or $u = x_j^{-1}$ and $v = c_1^{z_1}$, then the cancellation between u^β and v^β is less than $(1/\lambda)\min\{|u|, |v|\}$. We can choose the initial solution $e_1, \ldots, e_m, a_1, b_1, \ldots, a_n, b_n$ so that

$$\left[c_1^{e_1} c_2^{e_2}, c_3^{e_3} \ldots c_i^{e_i}\right] \neq 1 \quad (i \geqslant 3), \qquad \left[c_1^{e_1} c_2^{e_2}, [a_i, b_i]\right] \neq 1 \quad (i = 2, \ldots, n) \qquad \text{and}$$
$$\left[c_1^{e_1} c_2^{e_2}, b_1^{-1} a_1^{-1} b_1\right] \neq 1.$$

Indeed, the equations

$$\left[c_1^{z_1} c_2^{z_2}, c_3^{z_3} \ldots c_i^{z_i}\right] = 1, \qquad \left[c_1^{z_1} c_2^{z_2}, [x_i, y_i]\right] = 1 \quad (i = 2, \ldots, n) \qquad \text{and}$$
$$\left[c_1^{z_1} c_2^{z_2}, y_1^{-1} x_1^{-1} y_1\right] = 1$$

are not consequences of the equation $S = 1$, and, therefore, there is a solution of $S(X) = 1$ which does not satisfy any of these equations.

To show that $u = c_i^{z_i^\beta}$ and $v = c_1^{z_1^\beta}$ have small cancellation, we have to show that p can be chosen so that $[\bar{A}_1, \bar{A}_i] \neq 1$ (which is obvious, because the pre-images in $\bar{G}$ do not commute), and that $\bar{A}_i^{-1}$ does not begin with a power of $\bar{A}_1$. The period $\bar{A}_i^{-1}$ has form $(c_{i+1}^{-z_{i+1}} \ldots c_3^{-z_3} \bar{A}_1^{-p_2} \ldots)$. It begins with a power of $\bar{A}_1$ if and only if $[\bar{A}_1, c_3^{e_3} \ldots c_i^{e_i}] = 1$, but this equality does not hold.

Similarly one can show, that the cancellation between $u = x_j^{-\beta}$ and $v = c_1^{z_1^\beta}$ is small. □

Lemma 68. *Let $S(X) = 1$ be a standard regular orientable quadratic equation of the type* (81)

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i = c_1^{e_1} \ldots c_m^{e_m} = d,$$

where all c_i are cyclically reduced, and

$$\beta_1 : z_i \to e_i$$

a solution of $S = 1$ in F in general position. Then for any $\lambda \in \mathbb{N}$ there is a positive integer s and a tuple $p = (p_1, \ldots, p_K)$ such that the map $\beta : F[X] \to F$ defined by

$$z_i^\beta = c_i^{q_i} z_i^{\phi_K \beta_1} d^s,$$

is a solution of $S = 1$ satisfying the small cancellation condition $(1/\lambda)$ with respect to $\bar{\mathcal{W}}_{\Gamma,L}$ with one exception when $u = d$ and $v = c_{m-1}^{-z_{m-1}}$ (in this case d cancels out in v^β). Notice, however, that such word uv occurs only in the product wuv with $w = c_2^{z_2}$, in which case cancellation between w^β and dv^β is less than $\min\{|w^\beta|, |dv^\beta|\}$. Moreover, for any word $w \in \bar{\mathcal{W}}_{\Gamma,L}$ that does not contain elementary squares, the word w^β does not contain a cyclically reduced part of $A_i^{2\beta}$ for any elementary period A_i.

Proof. Solution β is chosen the same way as in the previous lemma (except for the multiplication by d^s) on the elements z_i, $i \neq m$. We do not take s very large, we just need it to avoid cancellation between z_2^β and d. Therefore the cancellation between

$$c_i^{z_i^\beta} \quad \text{and} \quad c_{i+1}^{\pm z_{i+1}^\beta}$$

is small for $i < m - 1$. Similarly, for $u = c_2^{z_2}$, $v = d$, $w = c_{m-1}^{-z_{m-1}}$, we can make the cancellation between u^β and dw^β less than $\min\{|u^\beta|, |dw^\beta|\}$. □

Lemma 69. *Let $U, V \in \mathcal{W}_{\Gamma,L}$ such that $UV = U \circ V$ and $UV \in \mathcal{W}_{\Gamma,L}$.*

(1) *Let $n \neq 0$. If u is the last letter of U and v is the first letter of V then the cancellation between U^β and V^β is equal to the cancellation between u^β and v^β.*
(2) *Let $n = 0$. If u_1u_2 are the last two letters of U and v_1, v_2 are the first two letters of V then the cancellation between U^β and V^β is equal to the cancellation between $(u_1u_2)^\beta$ and $(v_1v_2)^\beta$.*

Since β has the small cancellation property with respect to $\bar{\mathcal{W}}_{\Gamma,L}$, this implies that the cancellation in $U^\beta V^\beta$ is equal to the cancellation in $u^\beta v^\beta$, which is equal to some element in C_β. This proves the lemma.

Let $w \in \bar{\mathcal{W}}_{\Gamma,L}$, $\phi_j = \phi_{j,p}$, $W = w^{\phi_j}$, and $A = A_j$.

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1} \tag{94}$$

the canonical N-large A-representation of W for some positive integer N.

Since the occurrences A^{q_i} above are stable we have

$$B_1 = \bar{B}_1 \circ A^{\mathrm{sgn}(q_1)}, \qquad B_i = A^{\mathrm{sgn}(q_{i-1})} \circ \bar{B}_i \circ A^{\mathrm{sgn}(q_i)} \quad (2 \leqslant i \leqslant k),$$
$$B_{k+1} = A^{\mathrm{sgn}(q_k)} \circ \bar{B}_{k+1}.$$

Denote $A^\beta = c^{-1}A'c$, where A' is cyclically reduced, and $c \in C_\beta$. Then

$$B_1^\beta = \bar{B}_1^\beta c^{-1}\big(A'\big)^{\mathrm{sgn}(q_1)}c, \qquad B_i^\beta = c^{-1}\big(A'\big)^{\mathrm{sgn}(q_{i-1})}c\bar{B}_i^\beta c^{-1}\big(A'\big)^{\mathrm{sgn}(q_i)}c,$$
$$B_{k+1}^\beta = c^{-1}\big(A'\big)^{\mathrm{sgn}(q_k)}c\bar{B}_{k+1}^\beta.$$

By Lemma 69 we can assume that the cancellation in the words above is small, i.e., it does not exceed a fixed number σ which is the maximum length of words from C_β. To get an N-large canonical A'-decomposition of W^β one has to take into account stable occurrences of A'. To this end, put $\varepsilon_i = 0$ if ${A'}^{\mathrm{sgn}(q_i)}$ occurs in the reduced form of $\bar{B}_i^\beta c^{-1}(A')^{\mathrm{sgn}(q_i)}$ as written (the cancellation does not touch it), and put $\varepsilon_i = \mathrm{sgn}(q_i)$ otherwise. Similarly, put $\delta_i = 0$ if ${A'}^{\mathrm{sgn}(q_i)}$ occurs in the reduced form of $(A')^{\mathrm{sgn}(q_i)}c\bar{B}_{i+1}^\beta$ as written, and put $\delta_i = \mathrm{sgn}(q_i)$ otherwise.

Now one can rewrite W^β in the following form

$$W^\beta = E_1 \circ \big(A'\big)^{q_1-\varepsilon_1-\delta_1} \circ E_2 \circ \big(A'\big)^{q_2-\varepsilon_2-\delta_2} \circ \cdots \circ \big(A'\big)^{q_k-\varepsilon_k-\delta_k} \circ E_{k+1}, \tag{95}$$

where $E_1 = (B_1^\beta c^{-1}(A')^{\varepsilon_1})$, $E_2 = ((A')^{\delta_1}cB_2^\beta c^{-1}(A')^{\varepsilon_2})$, $E_{k+1} = ((A')^{\delta_k}cB_{k+1}^\beta)$.

Observe, that d_i and ε_i, δ_i can be effectively computed from W and β. It follows that one can effectively rewrite W^β in the form (95) and the form is unique.

The decomposition (95) of W^β induces the corresponding A^*-decomposition of W. This can be shown by an argument similar to the one in Lemmas 63 and 64, where it has been proven that A^*_{r+L}-decomposition induces the corresponding A_r-decomposition. To see that the argument works we need the last statement in Lemmas 67 ($n > 0$) and 68

($n = 0$) which ensure that the "illegal" elementary squares do not occur because of the choice of the solution β.

If the canonical N-large A^*-decomposition of W has the form:

$$D_1\big(A^*\big)^{q_1} D_2 \dots D_k\big(A^*\big)^{q_k} D_{k+1}$$

then the induced one has the form:

$$W = \big(D_1 A^{*\varepsilon_1}\big) A^{*q_1-\varepsilon_1-\delta_1}\big(A^{*\delta_1} D_2 A^{*\varepsilon_2}\big)\dots\big(A^{*\delta_{k-1}} D_k A^{*\varepsilon_k}\big) A^{*q_k-\varepsilon_k-\delta_k}\big(A^{*\delta_k} D_{k+1}\big). \quad (96)$$

We call this decomposition the *induced* A^*-decomposition of W with respect to β and write it in the form:

$$W = D_1^*\big(A^*\big)^{q_1^*} D_2^* \dots D_k^*\big(A^*\big)^{q_k^*} D_{k+1}^*, \quad (97)$$

where $D_i^* = (A^*)^{\delta_{i-1}} D_i (A^*)^{\varepsilon_i}$, $q_i^* = q_i - \varepsilon_i - \delta_i$, and, for uniformity, $\delta_1 = 0$ and $\varepsilon_{k+1} = 0$.

Lemma 70. *For given positive integers j, N and a real number $\varepsilon > 0$ there is a constant $C = C(j, \varepsilon, N) > 0$ such that if $p_{t+1} - p_t > C$ for every $t = 1, \dots, j-1$, and a word $W \in \bar{\mathcal{W}}_{\Gamma,L}$ has a canonical N-large A^*-decomposition* (97)*, then this decomposition satisfies the following conditions*:

$$\big(D_1^*\big)^\beta = E_1 \circ_\theta \big(cR^\beta\big), \qquad \big(D_i^*\big)^\beta = \big(R^{-\beta}c^{-1}\big) \circ_\theta E_i \circ_\theta \big(cR^\beta\big),$$
$$\big(D_{k+1}^*\big)^\beta = \big(R^{-\beta}c^{-1}\big) \circ_\theta E_{k+1}, \quad (98)$$

where $\theta < \varepsilon|A'|$. Moreover, this constant C can be found effectively.

Proof. Applying homomorphism β to the reduced A^*-decomposition of W (97) we can see that

$$W^\beta = \big(\big(D_1^*\big)^\beta R^\beta c\big)\big(A'\big)^{q_1^*}\big(cR^\beta\big(D_2^*\big)^\beta R^{-\beta}c^{-1}\big)\big(A'\big)^{q_2^*}\dots$$
$$\big(cR^\beta\big(D_k^*\big)^\beta R^{-\beta}c^{-1}\big)\big(A'\big)^{q_k^*}\big(cR^\beta\big(D_{k+1}^*\big)^\beta\big).$$

Observe that this decomposition has the same powers of A' as the canonical N-large A'-decomposition (95). From the uniqueness of such decompositions we deduce that

$$E_1 = \big(D_1^*\big)^\beta c^{-1}R^{-\beta}, \qquad E_i = cR^\beta\big(D_i^*\big)^\beta R^{-\beta}c^{-1}, \qquad E_{k+1} = cR^\beta\big(D_{k+1}^*\big)^\beta.$$

Put $\theta = |c| + |R^\beta|$. Rewriting the equalities above one can get

$$\big(D_1^*\big)^\beta = E_1 \circ_\theta \big(cR^\beta\big), \qquad \big(D_i^*\big)^\beta = \big(R^{-\beta}c^{-1}\big) \circ_\theta E_i \circ_\theta \big(cR^\beta\big),$$
$$\big(D_{k+1}^*\big)^\beta = \big(R^{-\beta}c^{-1}\big) \circ_\theta E_{k+1}.$$

Indeed, in the decomposition (95) every occurrence $(A')^{q_i-\varepsilon_i-\delta_i}$ is stable hence E_i starts (ends) on A'. The rank of R is at most $\operatorname{rank}(A) - K + 2$, and β has small cancellation. Taking $p_{i+1} \gg p_i$ we obtain $\varepsilon|A'| > |c| + |R^\beta|$. $\square$

Notice, that one can effectively write down the induced A^*-decomposition of W with respect to β.

We summarize the discussion above in the following statement.

Lemma 71. *For given positive integers j, N there is a constant $C = C(j, N)$ such that if $p_{t+1} - p_t > C$, for every $t = 1, \ldots, j-1$, then for any $W \in \bar{\mathcal{W}}_{\Gamma,L}$ the following conditions are equivalent*:

(1) *decomposition* (94) *is the canonical* (*the canonical N-large*) *A-decomposition of W*,
(2) *decomposition* (95) *is the canonical* (*the canonical N-large*) *A'-decomposition of W^β*,
(3) *decomposition* (96) *is the canonical* (*the canonical N-large*) *A^*-decomposition of W*.

7.4. Implicit function theorem for quadratic equations

In this section we prove Theorem 9 for orientable quadratic equations over a free group $F = F(A)$. Namely, we prove the following statement.

> *Let $S(X, A) = 1$ be a regular standard orientable quadratic equation over F. Then every equation $T(X, Y, A) = 1$ compatible with $S(X, A) = 1$ admits an effective complete S-lift.*

A special discriminating set of solutions $\mathcal{L}$ and the corresponding cut equation Π

Below we continue to use notations from the previous sections. Fix a solution β of $S(X, A) = 1$ which satisfies the cancellation condition $(1/\lambda)$ (with $\lambda > 10$) with respect to $\bar{\mathcal{W}}_\Gamma$.

Put

$$x_i^\beta = \tilde{a}_i, \qquad y_i^\beta = \tilde{b}_i, \qquad z_i^\beta = \tilde{c}_i.$$

Recall that

$$\phi_{j,p} = \gamma_j^{p_j} \ldots \gamma_1^{p_1} = \overleftarrow{\Gamma}_j^p$$

where $j \in \mathbb{N}$, $\Gamma_j = (\gamma_1, \ldots, \gamma_j)$ is the initial subsequence of length j of the sequence $\Gamma^{(\infty)}$, and $p = (p_1, \ldots, p_j) \in \mathbb{N}^j$. Denote by $\psi_{j,p}$ the following solution of $S(X) = 1$:

$$\psi_{j,p} = \phi_{j,p}\beta.$$

Sometimes we omit p in $\phi_{j,p}$, $\psi_{j,p}$ and simply write ϕ_j, ψ_j.

Below we continue to use notation:

$$A = A_j, \qquad A^* = A_j^* = A^*(\phi_j) = R_j^{-1} \circ A_j \circ R_j, \qquad d = d_j = |R_j|.$$

Recall that R_j has rank $\leqslant j-K+2$ (Lemma 59). By A' we denote the cyclically reduced form of A^β (hence of $(A^*)^\beta$). Recall that the set C_β was defined right after Definition 36.

Let

$$\Phi = \left\{\phi_{j,p} \mid j \in \mathbb{N},\ p \in \mathbb{N}^j\right\}.$$

For an arbitrary subset $\mathcal{L}$ of Φ denote

$$\mathcal{L}^\beta = \{\phi\beta \mid \phi \in \mathcal{L}\}.$$

Specifying step by step various subsets of Φ we will eventually ensure a very particular choice of a set of solutions of $S(X)=1$ in F.

Let $K=K(m,n)$ and $J\in\mathbb{N}$, $J\geqslant 3$, a sufficiently large positive integer which will be specified precisely in due course. Put $L=JK$ and define $\mathcal{P}_1=\mathbb{N}^L$,

$$\mathcal{L}_1 = \{\phi_{L,p} \mid p \in \mathcal{P}_1\}.$$

By Theorem 10 the set $\mathcal{L}_1^\beta$ is a discriminating set of solutions of $S(X)=1$ in F. In fact, one can replace the set $\mathcal{P}_1$ in the definition of $\mathcal{L}_1$ by any unbounded subset $\mathcal{P}_2\subseteq\mathcal{P}_1$, so that the new set is still discriminating. Now we construct by induction a very particular unbounded subset $\mathcal{P}_2\subseteq\mathbb{N}^L$. Let $a\in\mathbb{N}$ be a natural number and $h:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ a function. Define a tuple

$$p^{(0)} = \left(p_1^{(0)}, \dots, p_L^{(0)}\right)$$

where

$$p_1^{(0)} = a, \qquad p_{j+1}^{(0)} = p_j^{(0)} + h(0,j).$$

Similarly, if a tuple $p^{(i)}=(p_1^{(i)},\dots,p_L^{(i)})$ is defined then put $p^{(i+1)}=(p_1^{(i+1)},\dots,p_L^{(i+1)})$, where

$$p_1^{(i+1)} = p_1^{(i)} + h(i+1,0), \qquad p_{j+1}^{(i+1)} = p_j^{(i+1)} + h(i+1,j).$$

This defines by induction an infinite set

$$\mathcal{P}_{a,h} = \left\{p^{(i)} \mid i \in \mathbb{N}\right\} \subseteq \mathbb{N}^L$$

such that any infinite subset of $\mathcal{P}_{a,h}$ is also unbounded.

From now on fix a recursive monotonically increasing with respect to both variables function h (which will be specified in due course) and put

$$\mathcal{P}_2 = \mathcal{P}_{a,h}, \qquad \mathcal{L}_2 = \{\phi_{L,p} \mid p \in \mathcal{P}_2\}.$$

Proposition 6. *Let $r \geqslant 2$ and $K(r+2) \leqslant L$. Then there exists a number a_0 such that if $a \geqslant a_0$ and the function h satisfies the condition*

$$h(i+1, j) > h(i, j) \quad \text{for any } j = Kr+1, \ldots, K(r+2),\ i = 1, 2, \ldots, \tag{99}$$

then for any infinite subset $\mathcal{P} \subseteq \mathcal{P}_2$ the set of solutions

$$\mathcal{L}_{\mathcal{P}}{}^{\beta} = \{\phi_{L,p}\beta \mid p \in \mathcal{P}\}$$

is a discriminating set of solutions of $S(X, A) = 1$.

Proof. The result follows from Corollary 12. □

Let $\psi \in \mathcal{L}_2^{\beta}$. Denote by U_ψ the solution X^ψ of the equation $S(X) = 1$ in F. Since $T(X, Y) = 1$ is compatible with $S(X) = 1$ in F the equation $T(U_\psi, Y) = 1$ (in variables Y) has a solution in F, say $Y = V_\psi$. Set

$$\Lambda = \bigl\{(U_\psi, V_\psi) \bigm| \psi \in \mathcal{L}_2^{\beta}\bigr\}.$$

It follows that every pair $(U_\psi, V_\psi) \in \Lambda$ gives a solution of the system

$$R(X, Y) = \bigl(S(X) = 1 \ \wedge \ T(X, Y) = 1\bigr).$$

By Theorem 8 there exists a finite set $\mathcal{CE}(R)$ of cut equations which describes all solutions of $R(X, Y) = 1$ in F, therefore there exists a cut equation $\Pi_{\mathcal{L}_3,\Lambda} \in \mathcal{CE}(R)$ and an infinite subset $\mathcal{L}_3 \subseteq \mathcal{L}_2$ such that $\Pi_{\mathcal{L}_3,\Lambda}$ describes all solutions of the type (U_ψ, V_ψ), where $\psi \in \mathcal{L}_3$. We state the precise formulation of this result in the following proposition which, as we have mentioned already, follows from Theorem 8.

Proposition 7. *Let $\mathcal{L}_2$ and Λ be as above. Then there exists an infinite subset $\mathcal{P}_3 \subseteq \mathcal{P}_2$ and the corresponding set $\mathcal{L}_3 = \{\phi_{L,p} \mid p \in \mathcal{P}_3\} \subseteq \mathcal{L}_2$, a cut equation $\Pi_{\mathcal{L}_3,\Lambda} = (\mathcal{E}, f_X, f_M) \in \mathcal{CE}(R)$, and a tuple of words $Q(M)$ such that the following conditions hold*:

(1) $f_X(\mathcal{E}) \subset X^{\pm 1}$;
(2) *for every $\psi \in \mathcal{L}_3^{\beta}$ there exists a tuple of words $P_\psi = P_\psi(M)$ and a solution $\alpha_\psi : M \to F$ of $\Pi_{\mathcal{L}_3,\Lambda}$ with respect to $\psi : F[X] \to F$ such that*:
   - *the solution $U_\psi = X^\psi$ of $S(X) = 1$ can be presented as $U_\psi = Q(M^{\alpha_\psi})$ and the word $Q(M^{\alpha_\psi})$ is reduced as written,*
   - $V_\psi = P_\psi(M^{\alpha_\psi})$;
(3) *there exists a tuple of words P such that for any solution* (*any group solution*) (β, α) *of $\Pi_{\mathcal{L}_3,\Lambda}$ the pair (U, V), where $U = Q(M^\alpha)$ and $V = P(M^\alpha)$, is a solution of $R(X, Y) = 1$ in F.*

Put

$$\mathcal{P} = \mathcal{P}_3, \qquad \mathcal{L} = \mathcal{L}_3, \qquad \Pi_{\mathcal{L}} = \Pi_{\mathcal{L}_3,\Lambda}.$$

By Proposition 6 the set $\mathcal{L}^{\beta}$ is a discriminating set of solutions of $S(X) = 1$ in F.

The initial cut equation Π_ϕ

Now fix a tuple $p \in \mathcal{P}$ and the automorphism $\phi = \phi_{L,p} \in \mathcal{L}$. Recall, that for every $j \leqslant L$ the automorphism ϕ_j is defined by $\phi_j = \overleftarrow{\Gamma}_j^{p_j}$, where p_j is the initial subsequence of p of length j. Sometimes we use notation $\psi = \phi\beta$, $\psi_j = \phi_j\beta$.

Starting with the cut equation $\Pi_{\mathcal{L}}$ we construct a cut equation $\Pi_\phi = (\mathcal{E}, f_{\phi,X}, f_M)$ which is obtained from $\Pi_{\mathcal{L}}$ by replacing the function $f_X : \mathcal{E} \to F[X]$ by a new function $f_{\phi,X} : \mathcal{E} \to F[X]$, where $f_{\phi,X}$ is the composition of f_X and the automorphism ϕ. In other words, if an interval $e \in \mathcal{E}$ in $\Pi_{\mathcal{L}}$ has a label $x \in X^{\pm 1}$ then its label in Π_ϕ is x^ϕ.

Notice, that $\Pi_{\mathcal{L}}$ and Π_ϕ satisfy the following conditions:

(a) $\sigma^{f_X\phi\beta} = \sigma^{f_{\phi,X}\beta}$ for every $\sigma \in \mathcal{E}$;
(b) the solution of $\Pi_{\mathcal{L}}$ with respect to $\phi\beta$ is also a solution of Π_ϕ with respect to β;
(c) any solution (any group solution) of Π_ϕ with respect to β is a solution (a group solution) of $\Pi_{\mathcal{L}}$ with respect to $\phi\beta$.

The cut equation Π_ϕ has a very particular type. To deal with such cut equations we need the following definitions.

Definition 37. Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a cut equation. Then the number

$$\operatorname{length}(\Pi) = \max\bigl\{\bigl|f_M(\sigma)\bigr| \mid \sigma \in \mathcal{E}\bigr\}$$

is called the length of Π. We denote it by $\operatorname{length}(\Pi)$ or simply by N_Π.

Notice, by construction, $\operatorname{length}(\Pi_\phi) = \operatorname{length}(\Pi_{\phi'})$ for every $\phi, \phi' \in \mathcal{L}$. Denote

$$N_{\mathcal{L}} = \operatorname{length}(\Pi_\phi).$$

Definition 38. A cut equation $\Pi = (\mathcal{E}, f_X, f_M)$ is called a Γ-cut equation in *rank* j ($\operatorname{rank}(\Pi) = j$) and size l if it satisfies the following conditions.

(1) Let $W_\sigma = f_X(\sigma)$ for $\sigma \in \mathcal{E}$ and $N = (l+2)N_\Pi$. Then for every $\sigma \in \mathcal{E}$ $W_\sigma \in \bar{\mathcal{W}}_{\Gamma,L}$ and one of the following conditions holds:
  (1.1) W_σ has N-large rank j and its canonical N-large A_j-decomposition has size $(N, 2)$, i.e., W_σ has the canonical N-large A_j-decomposition
$$W_\sigma = B_1 \circ A_j^{q_1} \circ \cdots \circ B_k \circ A_j^{q_k} \circ B_{k+1}, \tag{100}$$
  with $\max_j(B_i) \leqslant 2$ and $q_i \geqslant N$;
  (1.2) W_σ has rank j and $\max_j(W_\sigma) \leqslant 2$;
  (1.3) W_σ has rank $< j$.
  Moreover, there exists at least one interval $\sigma \in \mathcal{E}$ satisfying the condition (1.1).
(2) There exists a solution $\alpha : F[M] \to F$ of the cut equation Π with respect to the homomorphism $\beta : F[X] \to F$.

Lemma 72. *Let $l \geqslant 3$. The cut equation Π_ϕ is a Γ-cut equation in rank L and size l, provided*

$$p_L \geqslant (l+2)N_{\Pi_\phi} + 3.$$

Proof. By construction the labels of intervals from Π_ϕ are precisely the words of the type x^{ϕ_L} and every such word appears as a label. Observe, that

$$\operatorname{rank}\bigl(x_i^{\phi_L}\bigr) < L \quad \text{for every } i,\ 1 \leqslant i \leqslant n$$

(Lemma 65(1)(a)). Similarly,

$$\operatorname{rank}\bigl(x_i^{\phi_L}\bigr) < L \quad \text{for every } i < n \qquad \text{and} \qquad \operatorname{rank}\bigl(y_n^{\phi_L}\bigr) = L$$

(Lemma 65(1)(b)). Also,

$$\operatorname{rank}\bigl(z_i^{\phi_L}\bigr) < L \quad \text{unless } n = 0 \text{ and } i = m,$$

in the latter case $z_m^{\phi_L} = L$ (Lemma 65(1)(c) and (1)(d)). Now consider the labels $y_n^{\phi_L}$ and $z_m^{\phi_L}$ (in the case $n = 0$) of rank L. Again, it has been shown in Lemma 65(1) that these labels have N-large A_L-decompositions of size $(N, 2)$, as required in (1.1) of the definition of a Γ-cut equation of rank L and size l. □

Agreement 1 on $\mathcal{P}$. Fix an arbitrary integer l, $l \geqslant 5$. We may assume, choosing the constant a to satisfy the condition

$$a \geqslant (l+2)N_{\Pi_\phi} + 3,$$

that all tuples in the set $\mathcal{P}$ are $((l+2)N_{\Pi_\phi} + 3)$-large. Denote $N = (l+2)N_{\Pi_\phi}$.

Now we introduce one technical restriction on the set $\mathcal{P}$, its real meaning will be clarified later.

Agreement 2 on $\mathcal{P}$. Let r be an arbitrary fixed positive integer with $Kr \leqslant L$ and q be a fixed tuple of length Kr which is an initial segment of some tuple from $\mathcal{P}$. The choice of r and q will be clarified later. We may assume (suitably choosing the function h) that all tuples from $\mathcal{P}$ have q as their initial segment. Indeed, it suffices to define $h(i,0) = 0$ and $h(i,j) = h(i+1,j)$ for all $i \in \mathbb{N}$ and $j = 1, \ldots, Kr$.

Agreement 3 on $\mathcal{P}$. Let r be the number from Agreement 2. By Proposition 6 there exists a number a_0 such that for every infinite subset of $\mathcal{P}$ the corresponding set of solutions is a discriminating set. We may assume that $a > a_0$.

Transformation T^* of Γ-cut equations

Now we describe a transformation T^* defined on Γ-cut equations and their solutions, namely, given a Γ-cut equation Π and its solution α (relative to the fixed map $\beta : F[X] \to F$ defined above) T^* transforms Π into a new Γ-cut equation $\Pi^* = T^*(\Pi)$ and α into a solution $\alpha^* = T^*(\alpha)$ of $T^*(\Pi)$ relative to β.

Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a Γ-cut equation in rank j and size l. The cut equation

$$T^*(\Pi) = \left(\mathcal{E}^*, f^*_{X^*}, f^*_{M^*}\right)$$

is defined as follows.

Definition of the set $\mathcal{E}^*$

For $\sigma \in \mathcal{E}$ we denote $W_\sigma = f_X(\sigma)$. Put

$$\mathcal{E}_{j,N} = \left\{\sigma \in \mathcal{E} \;\middle|\; W_\sigma \text{ satisfies (1.1)}\right\}.$$

Then $\mathcal{E} = \mathcal{E}_{j,N} \cup \mathcal{E}_{<j,N}$ where $\mathcal{E}_{<j,N}$ is the complement of $\mathcal{E}_{j,N}$ in $\mathcal{E}$.

Now let $\sigma \in \mathcal{E}_{j,N}$. Write the word W_σ^β in its canonical A' decomposition:

$$W_\sigma^\beta = E_1 \circ A'^{q_1} \circ E_2 \circ \cdots \circ E_k \circ A'^{q_k} \circ E_{k+1} \tag{101}$$

where $|q_i| \geqslant 1$, $E_i \neq 1$ for $2 \leqslant i \leqslant k$.

Consider the partition

$$f_M(\sigma) = \mu_1 \ldots \mu_n$$

of σ. By the condition (2) of the definition of Γ-cut equations for the solution $\beta : F[X] \to F$ there exists a solution $\alpha : F[M] \to F$ of the cut equation Π relative to β. Hence

$$W_\sigma^\beta = f_M\left(M^\alpha\right)$$

and the element

$$f_M\left(M^\alpha\right) = \mu_1^\alpha \ldots \mu_n^\alpha$$

is reduced as written. It follows that

$$W_\sigma^\beta = E_1 \circ A'^{q_1} \circ E_2 \circ \cdots \circ E_k \circ A'^{q_k} \circ E_{k+1} = \mu_1^\alpha \circ \cdots \circ \mu_n^\alpha. \tag{102}$$

We say that a variable μ_i is *long* if $A'^{\pm(l+2)}$ occurs in μ_i^α (i.e., μ_i^α contains a stable occurrence of A'^l), otherwise it is called *short*. Observe, that the definition of long (short) variables $\mu \in M$ does not depend on a choice of σ, it depends only on the given homomorphism α. The graphical equalities (102) (when σ runs over $\mathcal{E}_{j,N}$) allow one to effectively recognize long and short variables in M. Moreover, since for every $\sigma \in \mathcal{E}$ the length of the word $f_M(\sigma)$ is bounded by length$(\Pi) = N_\Pi$ and $N = (l+2)N_\Pi$, every word $f_M(\sigma)$

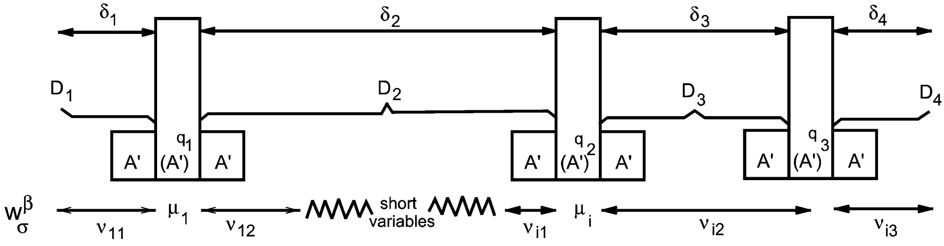


Fig. 9. Decomposition (103).

($\sigma \in \mathcal{E}_{j,N}$) contains long variables. Denote by M_{short}, M_{long} the sets of short and long variables in M. Thus, $M = M_{\text{short}} \cup M_{\text{long}}$ is a non-trivial partition of M.

Now we define the following property $P = P_{\text{long},l}$ of occurrences of powers of A' in W_σ^β: a given stable occurrence A'^q satisfies P if it occurs in μ^α for some long variable $\mu \in M_{\text{long}}$ and $q \geqslant l$. It is easy to see that P preserves correct overlappings. Consider the set of stable occurrences $\mathcal{O}_P$ which are maximal with respect to P. As we have mentioned already in Section 7.1, occurrences from $\mathcal{O}_P$ are pair-wise disjoint and this set is uniquely defined. Moreover, W_σ^β admits the unique A'-decomposition relative to the set $\mathcal{O}_P$:

$$W_\sigma^\beta = D_1 \circ \left(A'\right)^{q_1} \circ D_2 \circ \cdots \circ D_k \circ \left(A'\right)^{q_k} \circ D_{k+1}, \tag{103}$$

where $D_i \neq 1$ for $i = 2, \ldots, k$. See Fig. 9.

Denote by $k(\sigma)$ the number of non-trivial elements among $D_1, \ldots, D_{k+1}$.

According to Lemma 71 the A'-decomposition (103) gives rise to the unique associated A-decomposition of W_σ and hence the unique associated A^*-decomposition of W_σ.

Now with a given $\sigma \in \mathcal{E}_{j,N}$ we associate a finite set of new intervals E_σ (of the equation $T^*(\Pi)$):

$$E_\sigma = \{\delta_1, \ldots, \delta_{k(\sigma)}\}$$

and put

$$\mathcal{E}^* = \mathcal{E}_{<j,N} \cup \bigcup_{\sigma \in \mathcal{E}_{j,N}} E_\sigma.$$

Definition of the set M^*

Let $\mu \in M_{\text{long}}$ and

$$\mu^\alpha = u_1 \circ \left(A'\right)^{s_1} \circ u_2 \circ \cdots \circ u_t \circ \left(A'\right)^{s_t} \circ u_{t+1} \tag{104}$$

be the canonical l-large A'-decomposition of μ^α. Notice that if μ occurs in $f_M(\sigma)$ (hence μ^α occurs in W_σ^β) then this decomposition (104) is precisely the A'-decomposition of μ^α induced on μ^α (as a subword of W_σ^β) from the A'-decomposition (103) of W_σ^β relative to $\mathcal{O}_P$.

Denote by $t(\mu)$ the number of non-trivial elements among $u_1, \dots, u_{t+1}$ (clearly, $u_i \neq 1$ for $2 \leqslant i \leqslant t$).

We associate with each long variable μ a sequence of new variables (in the equation $T^*(\Pi)$) $S_\mu = \{\nu_1, \dots, \nu_{t(\mu)}\}$. Observe, since the decomposition (104) of μ^α is unique, the set S_μ is well defined (in particular, it does not depend on intervals σ).

It is convenient to define here two functions ν_{left} and ν_{right} on the set M_{long}: if $\mu \in M_{\text{long}}$ then

$$\nu_{\text{left}}(\mu) = \nu_1, \qquad \nu_{\text{right}}(\mu) = \nu_{t(\mu)}.$$

Now we define a new set of variable M^* as follows:

$$M^* = M_{\text{short}} \cup \bigcup_{\mu \in M_{\text{long}}} S_\mu.$$

Definition of the labelling function $f^*_{X^*}$

Put $X^* = X$. We define the labelling function $f^*_{X^*} : \mathcal{E}^* \to F[X]$ as follows.

Let $\delta \in \mathcal{E}^*$. If $\delta \in \mathcal{E}_{<j,N}$, then put

$$f^*_{X^*}(\delta) = f_X(\delta).$$

Let now $\delta = \delta_i \in E_\sigma$ for some $\sigma \in \mathcal{E}_{j,N}$. Then there are three cases to consider.

(a) δ corresponds to the consecutive occurrences of powers $A'^{q_{j-1}}$ and A'^{q_j} in the A'-decomposition (103) of W_σ^β relative to $\mathcal{O}_P$. Here $j = i$ or $j = i - 1$ with respect to whether $D_1 = 1$ or $D_1 \neq 1$.

As we have mentioned before, according to Lemma 71 the A'-decomposition (103) gives rise to the unique associated A^*-decomposition of W_σ:

$$W_\sigma = D_1^* \circ_d \left(A^*\right)^{q_1^*} \circ_d D_2^* \circ \dots \circ_d D_k^* \circ_d \left(A^*\right)^{q_k^*} \circ_d D_{k+1}^*. \tag{105}$$

Now put

$$f_X^*(\delta_i) = D_j^* \in F[X]$$

where $j = i$ if $D_1 = 1$ and $j = i - 1$ if $D_1 \neq 1$. See Fig. 10.

The other two cases are treated similarly to case (a).

(b) δ corresponds to the interval from the beginning of σ to the first A' power A'^{q_1} in the decomposition (103) of W_σ^β. Put

$$f_X^*(\delta) = D_1^*.$$

(c) δ corresponds to the interval from the last occurrence of a power A'^{q_k} of A' in the decomposition (103) of W_σ^β to the end of the interval. Put

$$f_X^*(\delta) = D_{k+1}^*.$$

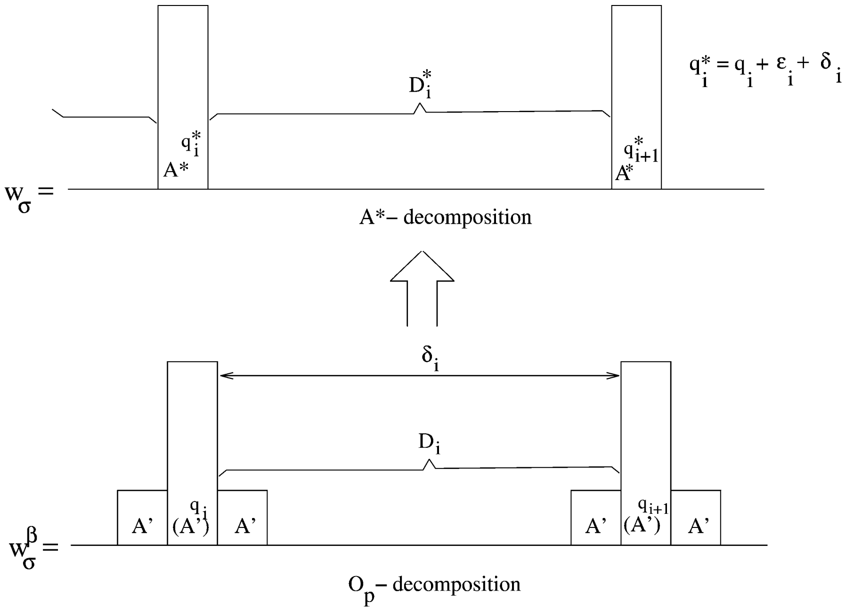


Fig. 10. Defining $f^*_{X^*}$.

Definition of the function $f^*_{M^*}$

Now we define the function $f^* : \mathcal{E}^* \to F[M^*]$.

Let $\delta \in \mathcal{E}^*$. If $\delta \in \mathcal{E}_{<j,N}$, then put

$$f^*_{M^*}(\delta) = f_M(\delta)$$

(observe that all variables in $f_M(\delta)$ are short, hence they belong to M^*).

Let $\delta = \delta_i \in E_\sigma$ for some $\sigma \in \mathcal{E}_{j,N}$. Again, there are three cases to consider.

(a) δ corresponds to the consecutive occurrences of powers A'^{q_s} and $A'^{q_{s+1}}$ in the A'-decomposition (103) of W_σ^β relative to $\mathcal{O}_P$. Let the stable occurrence A'^{q_s} occur in μ_i^α for a long variable μ_i, and the stable occurrence $A'^{q_{s+1}}$ occur in μ_j^α for a long variable μ_j.

Observe that

$$D_s = \text{right}(\mu_i) \circ \mu_{i+1}^\alpha \circ \cdots \circ \mu_{j-1}^\alpha \circ \text{left}(\mu_j),$$

for some elements right(μ_i), left$(\mu_j) \in F$.

Now put

$$f^*_{M^*}(\delta) = \nu_{i,\text{right}}\mu_{i+1} \dots \mu_{j-1}\nu_{j,\text{left}}.$$

See Fig. 11.

The other two cases are treated similarly to case (a).

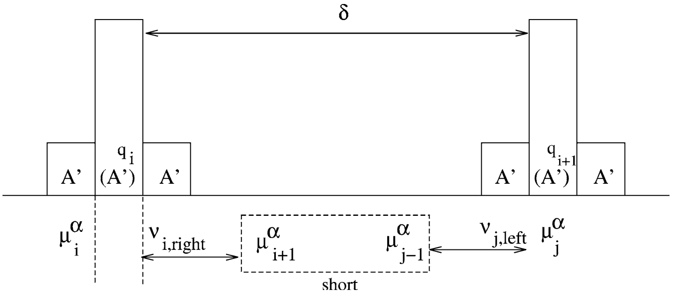


Fig. 11. Defining $f^*_{M^*}$, case (a).

(b) δ corresponds to the interval from the beginning of σ to the first A' power A'^{q_1} in the decomposition (103) of W_σ^β. Put

$$f^*_{M^*}(\delta) = \mu_1 \dots \mu_{j-1}\nu_{j,\mathrm{left}}.$$

(c) δ corresponds to the interval from the last occurrence of a power A'^{q_k} of A' in the decomposition (103) of W_σ^β to the end of the interval.

Denote $\Pi^* = (\mathcal{E}^*, f^*_{X^*}, f^*_{M^*})$.

Now we apply an auxiliary transformation T' to the cut equation Π^* as follows. The resulting cut equation will be

$$T'\big(\Pi^*\big) = \big(\tilde{\mathcal{E}}, \tilde{f}_X, \tilde{f}_M\big),$$

with the same sets X^* and M^*, and where $\tilde{f}_{X^*}, \tilde{f}_{M^*}$ are defined as follows. The transformation T' can be applied only in the following two situations.

(1) Suppose there are two intervals $\sigma, \gamma \in \mathcal{E}^*$ such that

$$f^*_{M^*}(\sigma) = \mu \in M^{*\pm 1}, \qquad f^*_{M^*}(\gamma) = u \circ \mu \in F\big[M^*\big],$$

for some $u \in F[M^*]$ and $f^*_{X^*}(\sigma) = (A^*)^k$, $f^*_{X^*}(\gamma) = w \circ (A^*)^k$. Then put

$$\tilde{f}_{X^*}(\gamma) = w, \qquad \tilde{f}_{M^*}(\gamma) = u,$$
$$\tilde{f}_{X^*}(\delta) = f^*_{X^*}(\delta), \quad \tilde{f}_{M^*}(\delta) = f^*_{M^*}(\delta) \quad (\delta \neq \gamma).$$

(2) Suppose there are two intervals $\sigma, \gamma \in \mathcal{E}^*$ such that

$$f^*_{M^*}(\sigma) = \mu \in M^{*\pm 1}, \qquad f^*_{M^*}(\gamma) = \nu \circ \mu \in F\big[M^*\big],$$

and $f^*_{X^*}(\gamma) = (A^*)^k \circ f^*_{X^*}(\sigma)$. Then put

$$\tilde{f}_{X^*}(\gamma) = \big(A^*\big)^k, \qquad \tilde{f}_{M^*}(\gamma) = \nu,$$
$$\tilde{f}_{X^*}(\delta) = f^*_{X^*}(\delta), \quad \tilde{f}_{M^*}(\delta) = f^*_{M^*}(\delta) \quad (\delta \neq \gamma).$$

We apply the transformation T' consecutively to Π^* until it is applicable. Notice, since T' decreases the length of the element $f^*_{M^*}(\gamma)$ it can only be applied a finite number of times, say s, so $(T')^s(\Pi^*) = (T')^{s+1}(\Pi^*)$. Observe also, that the resulting cut equation $(T')^s(\Pi^*)$ does not depend on a particular sequence of applications of the transformation T' to Π^*. This implies that the cut equation $T^*(\Pi) = (T')^s(\Pi^*)$ is well defined.

Claim 1. *The homomorphism* $\alpha^* : F[M^*] \to F$ *defined as* (*in the notations above*):

$$\alpha^*(\mu) = \alpha(\mu) \quad (\mu \in M_{\text{short}}),$$
$$\alpha^*(\nu_{i,\text{right}}) = R^{-\beta} c^{-1} \text{right}(\mu_i) \quad (\nu_i \in S_\mu \textit{ for } \mu \in M_{\text{long}}),$$
$$\alpha^*(\nu_{i,\text{left}}) = \text{left}(\mu_i) c R^\beta$$

is a solution of the cut equation $T^*(\Pi)$ *with respect to* $\beta : F[X] \to F$.

Proof. The statement follows directly from the construction. □

Agreement 4 on $\mathcal{P}$. We assume (by choosing the function h properly, i.e., $h(i, j) > C(L, N+3)$, see Lemma 70) that every tuple $p \in \mathcal{P}$ satisfies the conditions of Lemma 70, so Claim 1 holds for every $p \in \mathcal{P}$.

Definition 39. Let $w \in \bar{\mathcal{W}}_{\Gamma,L}$. Let $1 \leqslant i \leqslant K$. A *cut of rank* i of w is a decomposition $w = u \circ v$ where either u ends with $A_i^{\pm 2}$ or v begins with $A_i^{\pm 2}$. In this event we say that u and v are obtained by a cut (in rank i) from w.

Definition 40. Given a 3-large tuple $p \in \mathbb{N}^L$, for any $0 \leqslant j \leqslant L$ we define by induction (on $L - j$) a set of *patterns of rank* j which are certain words in $F(X \cup C)$.

(1) Patterns of rank L are precisely the letters from the alphabet $X^{\pm 1}$.
(2) Now suppose $j = Ks + r$, where $0 \leqslant r < K$ and $Ks < L$. We represent p as

$$p = p' q p'' \quad \text{where} \quad |p'| = Ks, \quad |q| = K, \quad |p''| = L - Ks - K. \tag{106}$$

Then a pattern of rank j is either a word of the form $u^{\phi_{K,q}}$ where u is a pattern of rank $Ks + K$, or a subword of $u^{\phi_{K,q}}$ formed by one or two cuts of ranks $> r$ (see Definition 39).

Remark 8. $w \in \bar{\mathcal{W}}_{\Gamma,L}$ for any pattern w of any rank $j \leqslant L$.

Claim 2. *Let* $\phi_L = \phi_{L,p}$, *where* $p \in \mathbb{N}^L$ *such that* $p_t \geqslant (l+2)N_\Pi + 3$ *for* $t = 1, \ldots, L$, *and* $l \geqslant 3$. *Denote* $\Pi_L = \Pi_{\phi_L}$.

(1) *For $j \leqslant L$ the cut equation $\Pi_{L-j} = (T^*)^j(\Pi_L)$ is well defined and it is a Γ-cut equation of rank $\leqslant L - j$ and size l. In particular, the sequence $\Sigma_{L,p}$ of Γ-cut equations*

$$\Sigma_{L,p}: \Pi_L \xrightarrow{T^*} \Pi_{L-1} \xrightarrow{T^*} \cdots \xrightarrow{T^*} \Pi_j \to \cdots \tag{107}$$

is well defined.

(2) *Let $j = Ks + r$, where $0 \leqslant r < K$, $L = K(s+i)$, and p' be from the representation* (106). *Denote $\phi_{Ks} = \phi_{Ks,p'}$. Then the following are true*:
   (a) *for any interval σ of Π_j there is a pattern w of rank j such that $f_X(\sigma) = w^{\phi_{Ks}}$;*
   (b) *if $j = Ks$ $(r = 0)$ then for every interval σ of the cut equation Π_j the pattern w, where $f_X(\sigma) = w^{\phi_{Ks}}$, does not contain N-large powers of elementary periods.*

Proof. Let $j = Ks + r$, $0 \leqslant r < K$, $L = K(s+i)$. We prove the claim by induction on i and $m = K - r$ for $i > 0$.

Case $i = 0$. In this case $j = L$, so the labels of the intervals of Π_L are of the form x^{ϕ_L}, $x \in X$, and the claim is obvious.

Case $i = 1$. We use induction on $m = 1, \ldots, K - 1$ to prove that for every interval σ from the cut equation

$$\Pi_{L-m} = \left(\mathcal{E}^{(L-m)}, f_X^{(L-m)}, f_M^{(L-m)}\right)$$

the label $f_X^{(L-m)}(\sigma)$ is of the form $u^{\phi_{L-K}}$ for some pattern $u \in \mathrm{Sub}(X^{\phi_K})$.

Let $m = 1$. In this case $j = L - 1$. For every $x \in X^{\pm 1}$ one can represent the element x^{ϕ_L} as a product of elements of the type $y^{\phi_{L-K}}$, $y \in X^{\pm 1}$ (so the element x^{ϕ_L} is a word in the alphabet $X^{\phi_{L-K}}$). Indeed,

$$x^{\phi_L} = \left(x^{\phi_K}\right)^{\phi_{L-K}} = w^{\phi_{L-K}},$$

where $w = x^{\phi_K}$ is a word in X. By Lemma 64 there is a precise correspondence between stable A_L^*-decompositions of

$$x^{\phi_L} = w^{\phi_{L-K}} = D_1^{\phi_{L-K}} \circ_d A_L^{*q_1} \circ_d D_2^{\phi_{L-K}} \circ_d \cdots \circ_d D_k^{\phi_{L-K}} \circ_d A_L^{*q_k} \circ D_{k+1}^{\phi_{L-K}}$$

and stable A_K-decompositions of w

$$w = D_1 \circ {A_K}^{q_1} \circ D_2 \circ \cdots \circ D_k \circ A_K^{q_k} \circ D_{k+1}.$$

By construction, application of the transformation T^* to Π_L removes powers

$$A_L^{*q_s} = A_K^{q_s \phi_{L-K}}$$

which are subwords of the word $w^{\phi_{L-K}}$ written in the alphabet $X^{\phi_{L-K}}$. By construction the words $D_s^{\phi_{L-K}}$ are the labels of the new intervals of the equation Π_{L-1}. Notice, that D_s are subwords of $w = x^{\phi_K}$ which obtained from w by one or two cuts in rank L. Hence D_s are patterns in rank $L-1$, as required in (2)(a).

Now we show that Π_{L-1} is a Γ-cut equation in rank $\leqslant L-1$ and size l. By (2)(a) and Remark 8, $f_X(\sigma) \in \bar{\mathcal{W}}_{\Gamma,L}$ for every interval $\sigma \in \Pi_{L-1}$. Thus the initial part of the first condition from the definition of Γ-cut equations is satisfied. To show (1) it suffices to show that (1.1) in rank L does not hold for Π_{L-1}. Let $\delta \in \mathcal{E}^{L-1}$. By the construction $(A')^{l+2}$ does not occur in μ^α for any $\mu \in M^{L-1}$. Therefore the maximal power of A' that can occur in $f_M(\delta)^\alpha$ is bounded from above by $(l+1)|f_M(\delta)|$ which is less then $(l+2)\,\mathrm{length}(\Pi_{L-1})$. Hence there are no intervals in Π_{L-1} which satisfy the condition (1.1) from the definition of Γ-cut equations. It follows that the rank of Π_{L-1} is at most $L-1$, as required. Let t be the rank of Π_{L-1}. For an interval $\delta \in \Pi_{L-1}$ if the conditions (1.1) and (1.3) for $f_X(\delta)$ and the rank t are not satisfied, then the condition (1.2) is satisfied. Indeed, it is obvious from the definition of patterns that either $f_X(\delta)$ has a non-trivial N-large decomposition in rank t or $\max_t(f_X(\delta)) \leqslant 2$. Finally, it has been shown in Claim 1 that $T^*(\Pi)$ has a solution α^* relative to β. This proves the condition (2) in the definition of the Γ-cut equation. Hence Π_{L-1} is a Γ-cut equation of rank at $t \leqslant j-1$ and size l.

Suppose now by induction on m that for an interval σ of the cut equation Π_j (for $m = L-j$)

$$f_X^{(j)}(\sigma) = u^{\phi_{L-K}} \quad \text{for some } u \in \mathrm{Sub}\big(X^{\pm\phi_K}\big).$$

Then either σ does not change under T^* or $f_X^{(j)}(\sigma)$ has a stable $(l+2)$-large A_j^*-decomposition in rank $j = r + (L-K)$ associated with long variables in $f_M^{(j)}(\sigma)$:

$$u^{\phi_{L-K}} = \bar{D}_1^{\phi_{L-K}} \circ_d A_j^{*q_1} \circ_d \bar{D}_2^{\phi_{L-K}} \circ_d \cdots \circ_d \bar{D}_k^{\phi_{L-K}} \circ_d A_j^{*q_k} \circ \bar{D}_{k+1}^{\phi_{L-K}},$$

and σ is an interval in Π_j. By Lemma 64, in this case there is a stable A_r-decomposition of u:

$$u = \bar{D}_1 \circ A_r^{q_1} \circ \bar{D}_2 \circ \cdots \circ \bar{D}_k \circ A_r^{q_k} \circ \bar{D}_{k+1}.$$

The application of the transformation T^* to Π_j removes powers

$$A_j^{*q_s} = A_r^{q_s \phi_{L-K}} \quad \big(\text{since } A_j{}^* = A_r^{\phi_{L-K}}\big)$$

which are subwords of the word $u^{\phi_{L-K}}$ written in the alphabet $X^{\phi_{L-K}}$. By construction the words $\bar{D}_s^{\phi_{L-K}}$ are the labels of the new intervals of the equation Π_{j-1}, so they have the required form. This proves statement (2)(a) for $m+1$. Statement (1) now follows from (2)(a) (the argument is the same as in rank $L-1$). By induction the claim holds for $m = K$, so the label $f_X^{(L-K)}(\sigma)$ of an interval σ in Π_{L-K} is of the form $u^{\phi_{L-K}}$, for some pattern u, where $u \in \mathrm{Sub}(X^{\pm\phi_K})$. Notice that $\mathrm{Sub}(X^{\pm\phi_K}) \subseteq \mathcal{W}_{\Gamma,L}$ which proves statement (2) (and, therefore, statement (1)) of the claim for $i = 1$.

Suppose, by induction, that labels of intervals in the cut equation Π_{L-Ki} have form $w^{\phi_{L-Ki}}$, w is a pattern in $\bar{\mathcal{W}}_{\Gamma,L}$. We can rewrite each label in the form $v^{\phi_{L-K(i+1)}}$, where $v = w^{\phi_K} \in \bar{\mathcal{W}}_{\Gamma,L}$. Similarly to case $i = 1$ we can construct the T^*-sequence

$$\Pi_{L-Ki} \to \cdots \to \Pi_{L-K(i+1)}$$

where each application of the transformation T^* removes subwords in the alphabet $X^{\phi_{L-K(i+1)}}$. The argument above shows that the labels of the new intervals in all cut equations $\Pi_{L-Ki-1)}, \ldots, \Pi_{L-K(i+1)}$ are of the required form $v^{\phi_{L-K(i+1)}}$, for patterns v where $v \in \bar{\mathcal{W}}_{\Gamma,L}$. Following the proof it is easy to see that in labels of intervals in $\Pi_{L-K(i+1)}$ the word v does not contain N-large powers of $e^{\phi_{L-K(i+1)}}$ for an elementary period e. □

Claim 3. *Let $\mathcal{P} \subseteq \mathbb{N}^L$ be an infinite set of L-tuples and for $p \in \mathcal{P}$ let*

$$\Sigma_{L,p} : \Pi_L^{(p)} \xrightarrow{T^*} \Pi_{L-1}^{(p)} \xrightarrow{T^*} \cdots \xrightarrow{T^*} \Pi_j^{(p)} \to \cdots$$

be the sequence (107) *of cut equations $\Pi_j^{(p)} = (\mathcal{E}^{j,p}, f_X^{j,p}, f_M^{j,p})$. Suppose that for a given $j > 2K$ the following $\mathcal{P}$-*uniformity *property $U(\mathcal{P}, j)$ (consisting of three conditions) holds*:

(1) *$\mathcal{E}^{j,p} = \mathcal{E}^{j,q}$ for every $p, q \in \mathcal{P}$, we denote this set by $\mathcal{E}^j$*;
(2) *$f_M^{j,p} = f_M^{j,q}$ for every $p, q \in \mathcal{P}$*;
(3) *for any $\sigma \in \mathcal{E}^j$ there exists a pattern w_σ of rank j such that for any $p \in \mathcal{P}$*

$$f_X^{j,p}(\sigma) = w_\sigma^{\phi_{Kl,p'}}$$

where p' is the initial segment of p of length Kl, where $j = Kl + r$ and $0 < r \leqslant K$.

Then there exists an infinite subset $\mathcal{P}'$ of $\mathcal{P}$ such that the $\mathcal{P}'$-uniformity condition $U(\mathcal{P}', j-1)$ holds for $j - 1$.

Proof. Follows from the construction. □

Agreement 5 on $\mathcal{P}$. We assume, in addition to all the agreements above, that for the set $\mathcal{P}$ the uniformity condition $U(\mathcal{P}, j)$ holds for every $j > 2K$. Indeed, by Claim 3 we can adjust $\mathcal{P}$ consecutively for each $j > 2K$.

Claim 4. *Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a Γ-cut equation in rank $j \geqslant 1$ from the sequence* (107). *Then for every variable $\mu \in M$ there exists a word $\mathcal{M}_\mu(M_{T(\Pi)}, X^{\phi_{j-1}}, F)$ such that the following equality holds in the group F*

$$\mu^\alpha = \mathcal{M}_\mu\bigl(M_{T(\Pi)}^{\alpha^*}, X^{\phi_{j-1}}\bigr)^\beta.$$

Moreover, there exists an infinite subset $P' \subseteq P$ such that the words $\mathcal{M}_\mu(M_{T(\Pi)}, X)$ depend only on exponents $s_1, \ldots, s_t$ of the canonical l-large decomposition (104) *of the words μ^α.*

Proof. The claim follows from the construction. Indeed, in constructing $T^*(\Pi)$ we cut out leading periods of the type $(A'_j)^s$ from μ^α (see (104)). It follows that to get μ^α back from $M^{\alpha^*}_{T(\Pi)}$ one needs to put the exponents $(A'_j)^s$ back. Notice, that

$$A_j = A(\gamma_j)^{\phi_{j-1}}.$$

Therefore,

$$(A_j)^s = A(\gamma_j)^{\phi_{j-1}\beta}.$$

Recall that A'_j is the cyclic reduced form of A_j^β, so

$$\left(A'_j\right)^s = uA(\gamma_j)^{\phi_{j-1}\beta}v$$

for some constants $u, v \in C_\beta \subseteq F$. To see existence of the subset $P' \subseteq P$ observe that the length of the words $f_M(\sigma)$ does not depend on p, so there are only finitely many ways to cut out the leading periods $(A'_j)^s$ from μ^α. This proves the claim. □

Agreement 6 on $\mathcal{P}$. We assume (replacing P with a suitable infinite subset) that every tuple $p \in \mathcal{P}$ satisfies the conditions of Claim 4. Thus, for every $\Pi = \Pi_i$ from the sequence (107) with a solution α (relative to β) the solution α^* satisfies the conclusion of Claim 4.

Definition 41. We define a new transformation T which is a modified version of T^*. Namely, T transforms cut equations and their solutions α precisely as the transformation T^*, but it also transforms the set of tuples $\mathcal{P}$ producing an infinite subset $\mathcal{P}^* \subseteq \mathcal{P}$ which satisfies Agreements 1–6.

Now we define a sequence

$$\Pi_L \xrightarrow{T} \Pi_{L-1} \xrightarrow{T} \cdots \xrightarrow{T} \Pi_1 \tag{108}$$

of N-large Γ-cut equations, where $\Pi_L = \Pi_\phi$, and $\Pi_{i-1} = T(\Pi_i)$. From now on we fix the sequence (108) and refer to it as the *T-sequence.*

Definition 42. Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a cut equation. For a positive integer n by $k_n(\Pi)$ we denote the number of intervals $\sigma \in \mathcal{E}$ such that $|f_M(\sigma)| = n$. The following finite sequence of integers

$$\mathrm{Comp}(\Pi) = \left(k_2(\Pi), k_3(\Pi), \ldots, k_{\mathrm{length}(\Pi)}(\Pi)\right)$$

is called the *complexity* of Π.

We well-order complexities of cut equations in the (right) shortlex order: if Π and Π' are two cut equations then $\mathrm{Comp}(\Pi) < \mathrm{Comp}(\Pi')$ if and only if $\mathrm{length}(\Pi) < \mathrm{length}(\Pi')$ or $\mathrm{length}(\Pi) = \mathrm{length}(\Pi')$ and there exists $1 \leqslant i \leqslant \mathrm{length}(\Pi)$ such that $k_j(\Pi) = k_j(\Pi')$ for all $j > i$ but $k_i(\Pi) < k_i(\Pi')$.

Observe that intervals $\sigma \in \mathcal{E}$ with $|f_M(\sigma)| = 1$ have no input into the complexity of a cut equation $\Pi = (\mathcal{E}, f_X, f_M)$. In particular, equations with $|f_M(\sigma)| = 1$ for every $\sigma \in \mathcal{E}$ have the minimal possible complexity among equations of a given length. We will write $\mathrm{Comp}(\Pi) = \mathbf{0}$ in the case when $k_i(\Pi) = 0$ for every $i = 2, \dots, \mathrm{length}(\Pi)$.

Claim 5. *Let* $\Pi = (\mathcal{E}, f_X, f_M)$. *Then the following holds*:

(1) $\mathrm{length}(T(\Pi)) \leqslant \mathrm{length}(\Pi)$;
(2) $\mathrm{Comp}(T(\Pi)) \leqslant \mathrm{Comp}(\Pi)$.

Proof. By straightforward verification. Indeed, if $\sigma \in \mathcal{E}_{<j}$ then $f_M(\sigma) = f^*_{M^*}(\sigma)$. If $\sigma \in \mathcal{E}_j$ and $\delta_i \in E_\sigma$ then

$$f^*_{M^*}(\delta_i) = \mu^*_{i_1}\mu_{i_1+1}\dots\mu^*_{i_1+r(i)},$$

where $\mu_{i_1}\mu_{i_1+1}\dots\mu_{i_1+r(i)}$ is a subword of $\mu_1\dots\mu_n$ and hence $|f^*_{M^*}(\delta_i)| \leqslant |f_M(\sigma)|$, as required. □

We need a few definitions related to the sequence (108). Denote by M_j the set of variables in the equation Π_j. Variables from Π_L are called *initial* variables. A variable μ from M_j is called *essential* if it occurs in some $f_{M_j}(\sigma)$ with $|f_{M_j}(\sigma)| \geqslant 2$, such occurrence of μ is called *essential*. By $n_{\mu,j}$ we denote the total number of all essential occurrences of μ in Π_j. Then

$$S(\Pi_j) = \sum_{i=2}^{N_{\Pi_j}} i k_i(\Pi_j) = \sum_{\mu \in M_j} n_{\mu,j}$$

is the total number of all essential occurrences of variables from M_j in Π_j.

Claim 6. *If* $1 \leqslant j \leqslant L$ *then* $S(\Pi_j) \leqslant 2S(\Pi_L)$.

Proof. Recall, that every variable μ in M_j either belongs to M_{j+1} or it is replaced in M_{j+1} by the set S_μ of new variables (see definition of the function $f^*_{M^*}$ above). We refer to variables from S_μ as to *children* of μ. A given occurrence of μ in some $f_{M_{j+1}}(\sigma)$, $\sigma \in \mathcal{E}_{j+1}$, is called a *side occurrence* if it is either the first variable or the last variable (or both) in $f_{M_{j+1}}(\sigma)$. Now we formulate several properties of variables from the sequence (108) which come directly from the construction. Let $\mu \in M_j$. Then the following conditions hold:

(1) Every child of μ occurs only as a side variable in Π_{j+1};
(2) Every side variable μ has at most one essential child, say μ^*. Moreover, in this event $n_{\mu^*,j+1} \leqslant n_{\mu,j}$;

(3) Every initial variable μ has at most two essential children, say μ_{left} and μ_{right}. Moreover, in this case $n_{\mu_{\text{left}},j+1} + n_{\mu_{\text{right}},j+1} \leqslant 2n_{\mu}$.

Now the claim follows from the properties listed above. Indeed, every initial variable from Π_j doubles, at most, the number of essential occurrences of its children in the next equation Π_{j+1}, but all other variables (not the initial ones) do not increase this number. □

Denote by width(Π) the *width* of Π which is defined as

$$\text{width}(\Pi) = \max_i k_i(\Pi).$$

Claim 7. *For every* $1 \leqslant j \leqslant L$ width$(\Pi_j) \leqslant 2S(\Pi_L)$.

Proof. It follows directly from Claim 6. □

Denote by $\kappa(\Pi)$ the number of all (length$(\Pi) - 1$)-tuples of non-negative integers which are bounded by $2S(\Pi_L)$.

Claim 8. $\text{Comp}(\Pi_L) = \text{Comp}(\Pi_{\mathcal{L}})$.

Proof. The complexity $\text{Comp}(\Pi_L)$ depends only on the function f_M in Π_L. Recall that $\Pi_L = \Pi_\phi$ is obtained from the cut equation $\Pi_{\mathcal{L}}$ by changing only the labelling function f_X, so $\Pi_{\mathcal{L}}$ and Π_L have the same functions f_M, hence the same complexities. □

We say that a sequence

$$\Pi_L \xrightarrow{T} \Pi_{L-1} \xrightarrow{T} \cdots$$

has $3K$*-stabilization* at $K(r+2)$, where $2 \leqslant r \leqslant L/K$, if

$$\text{Comp}(\Pi_{K(r+2)}) = \cdots = \text{Comp}(\Pi_{K(r-1)}).$$

In this event we denote

$$K_0 = K(r+2), \qquad K_1 = K(r+1), \qquad K_2 = Kr, \qquad K_3 = K(r-1).$$

For the cut equation Π_{K_1} by $M_{\text{veryshort}}$ we denote the subset of variables from $M(\Pi_{K_1})$ which occur unchanged in Π_{K_2} and are short in Π_{K_2}.

Claim 9. *For a given Γ-cut equation Π and a positive integer $r_0 \geqslant 2$ if $L \geqslant Kr_0 + \kappa(\Pi)4K$ then for some $r \geqslant r_0$ either the sequence* (108) *has $3K$-stabilization at $K(r+2)$ or* $\text{Comp}(\Pi_{K(r+1)}) = 0$.

Proof. Indeed, the claim follows by the "pigeon hole" principle from Claims 5 and 7 and the fact that there are not more than $\kappa(\Pi)$ distinct complexities which are less or equal to $\mathrm{Comp}(\Pi)$. □

Now we define a special set of solutions of the equation $S(X)=1$. Let $L=4K+\kappa(\Pi)4K$, p be a fixed N-large tuple from $\mathbb{N}^{L-4K}$, q be an arbitrary fixed N-large tuple from $\mathbb{N}^{2K}$, and p^* be an arbitrary N-large tuple from $\mathbb{N}^{2K}$. In fact, we need N-largeness of p^* and q only to formally satisfy the conditions of the claims above. Put

$$\mathcal{B}_{p,q,\beta}=\left\{\phi_{L-4K,p}\phi_{2K,p^*}\phi_{2K,q}\beta \;\middle|\; p^*\in\mathbb{N}^{2K}\right\}.$$

It follows from Theorem 10 that $\mathcal{B}_{p,q,\beta}$ is a discriminating family of solutions of $S(X)=1$. Denote $\beta_q=\phi_{2K,q}\circ\beta$. Then β_q is a solution of $S(X)=1$ in general position and

$$\mathcal{B}_{q,\beta}=\left\{\phi_{2K,p^*}\beta_q \;\middle|\; p^*\in\mathbb{N}^{2K}\right\}$$

is also a discriminating family by Theorem 10.

Let

$$\mathcal{B}=\left\{\psi_{K_1}=\phi_{K(r-2),p'}\phi_{2K,p^*}\phi_{2K,q}\beta \;\middle|\; p^*\in\mathbb{N}^{2K}\right\},$$

where p' is a beginning of p.

Proposition 8. *Let $L=2K+\kappa(\Pi)4K$ and $\phi_L\in\mathcal{B}_{p,q,\beta}$. Suppose the sequence*

$$\Pi_L\xrightarrow{T}\Pi_{L-1}\xrightarrow{T}\cdots$$

of cut equations (108) *has $3K$-stabilization at $K(r+2)$, $r\geqslant 2$. Then the set of variables M of the cut equation $\Pi_{K(r+1)}$ can be partitioned into three disjoint subsets*

$$M=M_{\mathrm{veryshort}}\cup M_{\mathrm{free}}\cup M_{\mathrm{useless}}$$

for which the following holds:

(1) *there exists a finite system of equations $\Delta(M_{\mathrm{veryshort}})=1$ over F which has a solution in F*;
(2) *for every $\mu\in M_{\mathrm{useless}}$ there exists a word $V_\mu\in F[X\cup M_{\mathrm{free}}\cup M_{\mathrm{veryshort}}]$ which does not depend on tuples p^* and q*;
(3) *for every solution $\delta\in\mathcal{B}$, for every map $\alpha_{\mathrm{free}}:M_{\mathrm{free}}\to F$, and every solution $\alpha_s:F[M_{\mathrm{veryshort}}]\to F$ of the system $\Delta(M_{\mathrm{veryshort}})=1$ the map $\alpha:F[M]\to F$ defined by*

$$\mu^\alpha=\begin{cases}\mu^{\alpha_{\mathrm{free}}}, & \text{if } \mu\in M_{\mathrm{free}},\\ \mu^{\alpha_s}, & \text{if } \mu\in M_{\mathrm{veryshort}},\\ V_\mu(X^\delta,M_{\mathrm{free}}^{\alpha_{\mathrm{free}}},M_{\mathrm{veryshort}}^{\alpha_s}), & \text{if } \mu\in M_{\mathrm{useless}}\end{cases}$$

is a group solution of $\Pi_{K(r+1)}$ with respect to β.

Proof. Below we describe (in a series of Claims 10–21) some properties of partitions of intervals of cut equations from the sequence (108):

$$\Pi_{K_1} \xrightarrow{T} \Pi_{K_1-1} \xrightarrow{T} \cdots \xrightarrow{T} \Pi_{K_2}.$$

Fix an arbitrary integer s such that $K_1 \geqslant s \geqslant K_2$.

Claim 10. *Let $f_M(\sigma) = \mu_1 \dots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then*:

(1) *the variables $\mu_2, \dots, \mu_{k-1}$ are very short*;
(2) *either μ_1 or μ_k, or both, are long variables.*

Proof. Indeed, if any of the variables $\mu_2, \dots, \mu_{k-1}$ is long then the interval σ of Π_s is replaced in $T(\Pi_s)$ by a set of intervals E_σ such that $|f_M(\delta)| < |f_M(\sigma)|$ for every $\delta \in E_\sigma$. This implies that complexity of $T(\Pi_s)$ is smaller than of Π_s—contradiction. On the other hand, since σ is a partition of rank s some variables must be long—hence the result. □

Let $f_M(\sigma) = \mu_1 \dots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then the variables μ_1 and μ_k are called *side variables*.

Claim 11. *Let $f_M(\sigma) = \mu_1 \dots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then this partition will induce a partition of the form $\mu_1'\mu_2 \dots \mu_{k-1}\mu_k'$ of some interval in rank $s-1$ in Π_{s-1} such that if μ_1 is short in rank s then $\mu_1' = \mu_1$, if μ_1 is long in Π_s then μ_1' is a new variable which does not appear in the previous ranks. Similar conditions hold for μ_k.*

Proof. Indeed, this follows from the construction of the transformation T. □

Claim 12. *Let σ_1 and σ_2 be two intervals of ranks s in Π_s such that $f_X(\sigma_1) = f_X(\sigma_2)$ and*

$$f_M(\sigma_1) = \mu_1\nu_2 \dots \nu_k, \qquad f_M(\sigma_2) = \mu_1\lambda_2 \dots \lambda_l.$$

Then for any solution α of Π_s one has

$$\nu_k^\alpha = \nu_{k-1}^{-\alpha} \dots \nu_2^{-\alpha}\lambda_2^{-\alpha} \dots \lambda_{l-1}^{-\alpha}\lambda_l^{-\alpha},$$

i.e., ν_k^α can be expressed via λ_l^α and a product of images of short variables.

Claim 13. *Let $f_M(\sigma) = \mu_1 \dots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then for any $u \in X \cup E(m,n)$ the word $\mu_2^\alpha \dots \mu_{k-1}^\alpha$ does not contain a subword of the type*

$$c_1\big(M_u^{\phi_{K_1}}\big)^\beta c_2,$$

where $c_1, c_2 \in C_\beta$, and $M_u^{\phi_{K_1}}$ is the middle of u with respect to ϕ_{K_1}.

Proof. By Corollary 10 every word $M_u^{\phi_{K_1}}$ contains a big power (greater than $(l+2)N_{\Pi_s}$) of a period in rank strictly greater than K_2. Therefore, if $(M_u^{\phi_{K_1}})^\beta$ occurs in the word

$\mu_2^{\alpha} \ldots \mu_{k-1}^{\alpha}$ then some of the variables $\mu_2, \ldots, \mu_{k-1}$ are not short in some rank greater than K_2-contradiction. □

Claim 14. *Let σ be an interval in Π_{K_1}. Then $f_X(\sigma) = W_\sigma$ can be written in the form*

$$W_\sigma = w^{\phi_{K_1}},$$

and the following holds:

(1) *the word w can be uniquely written as $w = v_1 \ldots v_e$, where $v_1, \ldots v_e \in X^{\pm 1} \cup E(m,n)^{\pm 1}$, and $v_i v_{i+1} \notin E(m,n)^{\pm 1}$;*
(2) *w is either a subword of a word from the list in Lemma* 50 *or there exists i such that $v_i = x_2 x_1 \prod_{s=m}^{1} c_s^{-z_s}$ and $v_1 \ldots v_i$, $v_{i+1} \ldots v_e$ are subwords of words from the list in Lemma* 50; *in addition, $(v_i v_{i+1})^{\phi_K} = v_i^{\phi_K} \circ v_{i+1}^{\phi_K}$;*
(3) *if w is a subword of a word from the list in Lemma* 50, *then at most for two indices i, j elements v_i, v_j belong to $E(m,n)^{\pm 1}$, and, in this case $j = i + 1$.*

Proof. The fact that W_σ can be written in such a form follows from Claim 2. Indeed, by Claim 2, $W_\sigma = w^{\phi_{K_1}}$, where $w \in \mathcal{W}_{\Gamma,L}$, therefore it is either a subword of a word from the list in Lemma 50 or contains a subword from the set Exc from statement (3) of Lemma 53. It can contain only one such subword, because two such subwords of a word from $X^{\pm \phi_L}$ are separated by big (unbounded) powers of elementary periods.

The uniqueness of w in the first statement follows from the fact that ϕ_{K_1} is an automorphism. Obviously, w does not depend on p.

Property (3) follows from the comparison of the set $E(m,n)$ with the list from Lemma 50. □

Claim 15. *Let $\Pi_{K_1} = (\mathcal{E}, f_X, f_M)$ and $\mu \in M$ be a long variable* (*in rank K_1*) *such that $f_M(\delta) \neq \mu$ for any $\delta \in \mathcal{E}$. If μ occurs as the left variable in $f_M(\sigma)$ for some $\delta \in \mathcal{E}$ then it does not occur as the right variable in $f_M(\delta)$ for any $\delta \in \mathcal{E}$* (*however, μ^{-1} can occur as the right variable*). *Similarly, If μ occurs as the right variable in $f_M(\sigma)$ then it does not occur as the right variable in any $f_M(\delta)$.*

Proof. Suppose μ is a long variable such that $f_M(\sigma) = \mu\mu_2 \ldots$ and $f_M(\delta) = \ldots \mu_s \mu$ for some intervals σ, δ from Π_{K_1}. By Claim 14, $W_\sigma = w^{\phi_{K_1}}$ for some $w = v_1 \ldots v_e$, where $v_1, \ldots v_e \in X^{\pm 1} \cup E(m,n)^{\pm 1}$, and $v_i v_{i+1} \notin E(m,n)^{\pm 1}$. We divide the proof into three cases.

(1) Let $v_1 \neq z_i, y_n^{-1}$. Then W_σ begins with a big power of some period A_j^*, $j > K_2$ (see Lemmas 44–47), therefore μ_1 begins with a big power of $A_j^{*\beta}$. It follows that in the rank j the transformation T decreases the complexity of the current cut equation. Indeed, when T transforms μ and σ it produces a new set of variables $S_\mu = \{\nu_1, \ldots, \nu_{t(\mu)}\}$ and a new set of intervals $E_\sigma = \{\sigma_1, \ldots, \sigma_{k(\sigma)}\}$ such that $f_X^*(\sigma_1) = A_j^{*k}$ for some $k \geqslant 1$ and $f_M^*(\sigma_1) = \nu_1$. Simultaneously, when T transforms δ it produces (among other things) a new set of intervals $E_\delta = \{\delta_1, \ldots, \delta_{k(\delta)}\}$ such that $f_X^*(\delta_{k(\delta)-1})$ ends on A_j^{*k} and $f_M^*(\delta_{k(\delta)-1})$

ends on ν_1. Now the transformation T' (part 1) applies to σ_1 and $\delta_{k(\delta)-1}$ and decreases the complexity of the cut equation—contradiction.

(2) Let $v_{\text{left}} = z_i$. Then μ^α begins with $z_i^\beta = c_i^{q_i} z_i^{\phi_m \beta_1}$ (see Lemma 67) for some sufficiently large q_i. This implies that $c_i^{q_i}$ occurs in $f_M(\delta)^\alpha = f_X(\delta)^\beta$ somewhere inside (since $f_M(\delta) \neq \mu$). On the other hand, $f_X(\delta) \in \bar{\mathcal{W}}_{\Gamma,L}$, so $c_i^{q_i}$ can occur only at the beginning of $f_X(\delta)^\beta$ (see Lemmas 55 and 50)—contradiction.

(3) Let $v_{\text{left}} = y_n^{-1}$. Then $W_\delta = \ldots x_n^{-1} \circ y_n^{-1}$. In this case, similar to the case (1), after application of T^* to the current cut equation in the rank $K_2 + m + 4n - 4$ one can apply the transformation T' (part 2) which decreases the complexity—contradiction.

This proves the claim. $\square$

Our next goal is to transform further the cut equation Π_{K_1} to the form where all intervals are labeled by elements $x^{\phi_{K_1}}$, $x \in (X \cup E(m,n))^{\pm 1}$. To this end we introduce several new transformations of Γ-cut equations.

Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a Γ-cut equation in rank K_1 and size l with a solution $\alpha : F[M] \to F$ relative to $\beta : F[X] \to F$. Let $\sigma \in \mathcal{E}$ and

$$W_\sigma = (v_1 \ldots v_e)^{\phi_{K_1}}, \quad e \geqslant 2,$$

be the canonical decomposition of W_σ. For i, $1 \leqslant i < e$, put

$$v_{\sigma,i,\text{left}} = v_1 \ldots v_i, \qquad v_{\sigma,i,\text{right}} = v_{i+1} \ldots v_e.$$

Let, as usual,

$$f_M(\sigma) = \mu_1 \ldots \mu_k.$$

We start with a *transformation* $T_{1,\text{left}}$. For $\sigma \in \mathcal{E}$ and $1 \leqslant i < e$ denote by θ the boundary between $v_{\sigma,i,\text{left}}^{\phi_{K_1}\beta}$ and $v_{\sigma,i,\text{right}}^{\phi_{K_1}\beta}$ in the reduced form of the product $v_{\sigma,i,\text{left}}^{\phi_{K_1}\beta} v_{\sigma,i,\text{right}}^{\phi_{K_1}\beta}$. Suppose now that there exist σ and i such that the following two conditions hold:

(C1) μ_1^α almost contains the beginning of the word $v_{\sigma,i,\text{left}}^{\phi_{K_1}\beta}$ till the boundary θ (up to a very short end of it), i.e., there are elements $u_1, u_2, u_3, u_4 \in F$ such that

$$v_{\sigma,i,\text{left}}^{\phi_{K_1}\beta} = u_1 \circ u_2 \circ u_3, \qquad v_{i+1}^{\phi_{K_1}\beta} = u_3^{-1} \circ u_4, \qquad u_1 u_2 u_4 = u_1 \circ u_2 \circ u_4,$$

and μ_1^α begins with u_1, and u_2 is very short (does not contain $A_{K_2}^{\pm l}$) or trivial.

(C2) the boundary θ does not lie inside μ_1^α.

In this event the transformation $T_{1,\text{left}}$ is applicable to Π as described below. We consider three cases with respect to the location of θ on $f_M(\sigma)$.

(1) θ is inside μ_k^α (see Fig. 12). In this case we perform the following.

(a) Replace the interval σ by two new intervals σ_1, σ_2 with the labels $v_{\sigma,i,\text{left}}^{\phi_{K_1}}$, $v_{\sigma,i,\text{right}}^{\phi_{K_1}}$.

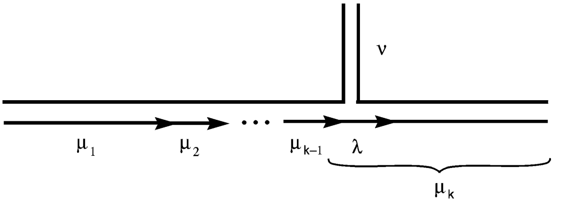


Fig. 12. T_1, case (1).

(b) Put $f_M(\sigma_1) = \mu_1 \dots \mu_{k-1}\lambda\nu$, $f_M(\sigma_2) = \nu^{-1}\mu'_k$, where λ is a new very short variable, ν is a new variable.
(c) Replace everywhere μ_k by $\lambda\mu'_k$. This finishes the description of the cut equation $T_{1,\mathrm{left}}(\Pi)$.
(d) Define a solution α^* (with respect to β) of $T_{1,\mathrm{left}}(\Pi)$ in the natural way. Namely, $\alpha^*(\mu) = \alpha(\mu)$ for all variables μ which came unchanged from Π. The values λ^{α^*}, ${\mu'}_k^{\alpha^*}$, ν^{α^*} are defined in the natural way, that is ${\mu'}_k^{\alpha^*}$ is the whole end part of μ_k^α after the boundary θ,

$$\left(\nu^{-1}\mu'_k\right)^{\alpha^*} = v_{\sigma,i,\mathrm{right}}^{\phi_{K_1}\beta}, \qquad \lambda^{\alpha^*} = \mu_k^\alpha\left({\mu'}_k^\alpha\right)^{-1}.$$

(2) θ is on the boundary between μ_j^α and μ_{j+1}^α for some j. In this case we perform the following.

(a) We split the interval σ into two new intervals σ_1 and σ_2 with labels $v_{\sigma,i,\mathrm{left}}^{\phi_{K_1}}$ and $v_{\sigma,i,\mathrm{right}}^{\phi_{K_1}}$.
(b) We introduce a new variable λ and put $f_M(\sigma_1) = \mu_1 \dots \mu_j\lambda$, $f_M(\sigma_2) = \lambda^{-1}\mu_{j+1}\dots\mu_k$.
(c) Define λ^{α^*} naturally.

(3) The boundary θ is contained inside μ_i^α for some i $(2 \leqslant i \leqslant r-1)$. In this case we do the following.

(a) We split the interval σ into two intervals σ_1 and σ_2 with labels $v_{\mathrm{left}}^{\phi_{K_1}}$ and $v_{\sigma,i,\mathrm{right}}^{\phi_{K_1}}$, respectively.
(b) Then we introduce three new variables μ'_j, μ''_j, λ, where μ'_j, μ''_j are "very short", and add equation $\mu_j = \mu'_j\mu''_j$ to the system $\Delta_{\mathrm{veryshort}}$.
(c) We define $f_M(\sigma_1) = \mu_1 \dots \mu'_j\lambda$, $f_M(\sigma_2) = \lambda^{-1}\mu''_j\mu_{i+1}\dots\mu_k$.
(d) Define values of α^* on the new variables naturally. Namely, put λ^{α^*} to be equal to the terminal segment of $v_{\mathrm{left}}^{\phi_{K_1}\beta}$ that cancels in the product $v_{\mathrm{left}}^{\phi_{K_1}\beta}v_{\sigma,i,\mathrm{right}}^{\phi_{K_1}\beta}$. Now the values ${\mu'}_j^{\alpha^*}$ and ${\mu''}_j^{\alpha^*}$ are defined to satisfy the equalities

$$f_X(\sigma_1)^\beta = f_M(\sigma_1)^{\alpha^*}, \qquad f_X(\sigma_2)^\beta = f_M(\sigma_2)^{\alpha^*}.$$

We described the transformation $T_{1,\text{left}}$. The transformation $T_{1,\text{right}}$ is defined similarly. We denote both of them by T_1.

Now we describe a *transformation* $T_{2,\text{left}}$.

Suppose again that a cut equation Π satisfies (C1). Assume in addition that for these σ and i the following condition holds:

(C3) the boundary θ lies inside μ_1^{α}.

Assume also that one of the following three conditions holds:

(C4) there are no intervals $\delta \neq \sigma$ in Π such that $f_M(\delta)$ begins with μ_1 or ends on μ_1^{-1};
(C5) $v_{\sigma,i,\text{left}} \neq x_n$ (i.e., either $i > 1$ or $i = 1$ but $v_1 \neq x_n$) and for every $\delta \in \mathcal{E}$ in Π if $f_M(\delta)$ begins with μ_1 (or ends on μ_1^{-1}) then the canonical decomposition of $f_X(\delta)$ begins with $v_{\sigma,i,\text{left}}^{\phi_{K_1}}$ (ends with $v_{\sigma,i,\text{left}}^{-\phi_{K_1}}$);
(C6) $v_{\sigma,i,\text{left}} = x_n$ ($i = 1$ and $v_1 = x_n$) and for every $\delta \in \mathcal{E}$ if $f_M(\delta)$ begins with μ_1 (ends with μ_i^{-1}) then the canonical decomposition of $f_X(\delta)$ begins with $x_n^{\phi_{K_1}}$ or with $y_n^{\phi_{K_1}}$ (ends with $x_n^{-\phi_{K_1}}$ or $y_n^{-\phi_{K_1}}$).

In this event the transformation $T_{2,\text{left}}$ is applicable to Π as described below.

(C4) Suppose the condition (C4) holds. In this case we do the following.

(a) Replace σ by two new intervals σ_1, σ_2 with the labels $v_{\sigma,i,\text{left}}^{\phi_{K_1}}, v_{\sigma,i,\text{right}}^{\phi_{K_1}}$.
(b) Replace μ_1 with two new variables μ_1', μ_1'' and put $f_M(\sigma_1) = \mu_1'$, $f_M(\sigma_2) = \mu_1''\mu_2 \ldots \mu_k$.
(c) Define $(\mu_1')^{\alpha^*}$ and $(\mu_1'')^{\alpha^*}$ such that $f_M(\sigma_1)^{\alpha^*} = v_{\sigma,i,\text{left}}^{\phi_{K_1}\beta}$ and $f_M(\sigma_2)^{\alpha^*} = v_{\sigma,i,\text{right}}^{\phi_{K_1}\beta}$.

(C5) Suppose $v_{\sigma,i,\text{left}} \neq x_n$. Then do the following.

(a) Transform σ as described in (C4).
(b) If for some interval $\delta \neq \sigma$ the word $f_M(\delta)$ begins with μ_1 then replace μ_1 in $f_M(\delta)$ by the variable μ_1'' and replace $f_X(\delta)$ by $v_{\sigma,i,\text{left}}^{-\phi_{K_1}} f_X(\delta)$. Similarly transform intervals δ that end with μ_1^{-1}.

(C6) Suppose $v_{\sigma,i,\text{left}} = x_n$. Then do the following.

(a) Transform σ as described in (C4).
(b) If for some δ the word $f_M(\delta)$ begins with μ_1 and $f_X(\delta)$ does not begin with y_n then transform δ as described in case (C5).
(c) Leave all other intervals unchanged.

We described the transformation $T_{2,\text{left}}$. The transformation $T_{2,\text{right}}$ is defined similarly. We denote both of them by T_2.

Suppose now that $\Pi = \Pi_{K_1}$. Observe that the transformations T_1 and T_2 preserve the properties described in Claims 5–8 above. Moreover, for the homomorphism $\beta : F[X] \to$

F we have constructed a solution $\alpha^*: F[M] \to F$ of $T_n(\Pi_{K_1})$ $(n = 2, 3)$ such that the initial solution α can be reconstructed from α^* and the equations Π and $T_n(\Pi)$. Notice also that the length of the elements $W_{\sigma'}$ corresponding to new intervals σ are shorter than the length of the words W_σ of the original intervals σ from which σ' were obtained. Notice also that the transformations T_1, T_2 preserves the property of intervals formulated in Claim 10.

Claim 16. *Let Π be a cut equation which satisfies the conclusion of Claim* 10. *Suppose σ is an interval in Π such that W_σ satisfies the conclusion of Claim* 14. *If for some i*

$$(v_1 \dots v_e)^{\phi_K} = (v_1 \dots v_i)^{\phi_K} \circ (v_{i+1} \dots v_e)^{\phi_K}$$

then either T_1 or T_2 is applicable to given σ and i.

Proof. By Corollary 61 the automorphism ϕ_{K_1} satisfies the Nielsen property with respect to $\bar{\mathcal{W}}_\Gamma$ with exceptions $E(m,n)$. By Corollary 12, equality

$$(v_1 \dots v_e)^{\phi_K} = (v_1 \dots v_i)^{\phi_K} \circ (v_{i+1} \dots v_e)^{\phi_K}$$

implies that the element that is cancelled between $(v_1 \dots v_i)^{\phi_K \beta}$ and $(v_{i+1} \dots v_e)^{\phi_K \beta}$ is short in rank K_2. Therefore either μ_1^α almost contains $(v_1 \dots v_i)^{\phi_K \beta}$ or μ_k^α almost contains $(v_{i+1} \dots v_e)^{\phi_K \beta}$. Suppose μ_1^α almost contains $(v_1 \dots v_i)^{\phi_K \beta}$. Either we can apply $T_{1,\text{left}}$, or the boundary θ belongs to μ_1^α. One can verify using formulas from Lemmas 44–47 and 53 directly that in this case one of the conditions (C4)–(C6) is satisfied, and, therefore $T_{2,\text{left}}$ can be applied. □

Lemma 73. *Given a cut equation Π_{K_1} one can effectively find a finite sequence of transformations $Q_1, \dots, Q_s$ where $Q_i \in \{T_1, T_2\}$ such that for every interval σ of the cut equation $\Pi'_{K_1} = Q_s \dots Q_1(\Pi_{K_1})$ the label $f_X(\sigma)$ is of the form $u^{\phi_{K_1}}$, where $u \in X^{\pm 1} \cup E(m,n)$.*

Moreover, there exists an infinite subset P' of the solution set P of Π_{K_1} such that this sequence is the same for any solution in P'.

Proof. Let σ be an interval of the equation Π_{K_1}. By Claim 14 the word W_σ can be uniquely written in the canonical decomposition form

$$W_\sigma = w^{\phi_{K_1}} = (v_1 \dots v_e)^{\phi_{K_1}},$$

so that the conditions (1)–(3) of Claim 14 are satisfied.

It follows from the construction of Π_{K_1} that either w is a subword of a word between two elementary squares $x \neq c_i$ or begins and (or) ends with some power $\geqslant 2$ of an elementary period. If u is an elementary period, $u^{2\phi_K} = u^{\phi_K} \circ u^{\phi_K}$, except $u = x_n$, when the middle is exhibited in the proof of Lemma 53. Therefore, by Claim 16, we can apply T_1 and T_2 and cut σ into subintervals σ_i such that for any i $f_X(\sigma_i)$ does not contain powers $\geqslant 2$ of elementary periods. All possible values of u^{ϕ_K} for $u \in E(m,n)^{\pm 1}$ are shown in the proof of Lemma 53. Applying T_1 and T_2 as in Claim 16 we can split intervals (and their labels) into parts with labels of the form $x^{\phi_{K_1}}$, $x \in (X \cup E(m,n))$, except for the following cases:

(1) $w = uv$, where u is x_i^2, $i < n$, $v \in E_{m,n}$, and v has at least three letters,
(2) $w = x_{n-2}^2 y_{n-2} x_{n-1}^{-1} x_n x_{n-1} y_{n-2}^{-1} x_{n-2}^2$,
(3) $w = x_{n-1}^2 y_{n-1} x_n^{-1} x_{n-1} y_{n-2}^{-1} x_{n-2}^{-2}$,
(4) $y_{r-1} x_r^{-1} y_r^{-1}$, $r < n$,
(5) $w = uv$, where $u = (c_1^{z_1} c_2^{z_2})^2$, $v \in E(m,n)$, and v is one of the following:

$$v = \prod_{t=1}^{m} c_t^{z_t} x_1^{\pm 1}, \qquad v = \prod_{t=1}^{m} c_t^{z_t} x_1^{\pm 1} \prod_{t=m}^{1} c_t^{-z_t}, \qquad v = \prod_{t=1}^{m} c_t^{z_t} x_1 \prod_{t=m}^{1} c_t^{-z_t} \left(c_1^{z_1} c_2^{z_2}\right)^{-2},$$

(6) $w = uv$, where $u = (c_1^{z_1} c_2^{z_2})^2$, $v \in E(m,n)$, and v is one of the following:

$$v = \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2^{-1} \quad \text{or} \quad v = \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} y_1^{-1}.$$

(7) $w = z_i v$.

Consider the first case. If $f_M(\sigma) = \mu_1 \ldots \mu_k$, and μ_1^α almost contains

$$x_i^{\phi_{K_1}} \left(A_{m+4i+K_2}^*\right)^{-p_{m+4i+K_2}+1} x_{i+1}^{\phi_{K_2}\beta}$$

(which is a non-cancelled initial peace of $x_i^{2\phi_{K_1}\beta}$ up to a very short part of it), then either $T_{1,\text{left}}$ or $T_{2,\text{left}}$ is applicable and we split σ into two intervals σ_1 and σ_2 with labels $x_i^{2\phi_{K_1}}$ and $v^{\phi_{K_1}}$.

Suppose μ_1^α does not contain

$$x_i^{\phi_{K_1}} \left(A_{m+4i+K_2}^*\right)^{-p_{m+4i+K_2}+1} x_{i+1}^{\phi_{K_2}\beta}$$

up to a very short part. Then μ_k^α contains the non-cancelled left end E of $v^{\phi_{K+1}\beta}$, and $\mu_k^\alpha E^{-1}$ is not very short. In this case $T_{2,\text{right}}$ is applicable.

We can similarly consider all cases (2)–(6).

(7) Letter z_i can appear only in the beginning of w (if z_i^{-1} appears at the end of w, we can replace w by w^{-1}) If $w = z_i t_1 \ldots t_s$ is the canonical decomposition, then $t_k = c_j^{\pm z_j}$ for each k. If μ_1^α is longer than the non-cancelled part of $(c_i^p z_i)^\beta$, or the difference between μ_1^α and $(c_i^p z_i)^\beta$ is very short, we can split σ into two parts, σ_1 with label $f_X(\sigma_1) = z^{\phi_{K_1}}$ and σ_2 with label $f_X(\sigma_2) = (t_1 \ldots t_s)^{\phi_{K_1}}$.

If the difference between μ_1^α and $(c_i^p z_i)^\beta$ is not very short, and μ_1^α is shorter than the non-cancelled part of $(c_i^p z_i)^\beta$, then there is no interval δ with $f(\delta) \neq f(\sigma)$ such that $f_M(\delta)$ and $f_M(\sigma)$ end with μ_k, and we can split σ into two parts using T_1, T_2 and splitting μ_k.

We have considered all possible cases. □

Denote the resulting cut equation by Π'_{K_1}.

Corollary 14. *The intervals of Π'_{K_1} are labelled by elements $u^{\phi_{K_1}}$, where for $n = 1$*

$$u \in \left\{ z_i, x_i, y_i, \prod c_s^{z_s}, x_1 \prod_{t=m}^{1} c_t^{-z_t}, \right\},$$

for $n = 2$

$$u \in \left\{ z_i, x_i, y_i, \prod c_s^{z_s}, y_1 x_1 \prod_{t=m}^{1} c_t^{-z_t}, y_1 x_1, \prod_{t=1}^{m} c_t^{z_t} x_1 \prod_{t=m}^{1} c_t^{-z_t} \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2^{\pm 1}, \right.$$
$$\prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2 x_1, \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2 x_1 \prod_{t=m}^{1} c_t^{-z_t}, x_1^{-1} x_2 x_1 \prod_{t=m}^{1} c_t^{-z_t},$$
$$\left. x_2 x_1 \prod_{t=m}^{1} c_t^{-z_t}, x_1^{-1} x_2, x_2 x_1 \right\},$$

and for $n \geqslant 3$,

$$u \in \left\{ z_i, x_i, y_i, c_s^{z_s}, y_1 x_1 \prod_{t=m}^{3} c_t^{-z_t}, \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2^{-1}, y_r x_r, x_1 \prod_{t=m}^{1} c_t^{-z_t}, y_{r-2} x_{r-1}^{-1} x_r^{-1}, \right.$$
$$y_{r-2} x_{r-1}^{-1}, x_{r-1}^{-1} x_r^{-1}, y_{r-1} x_r^{-1}, r < n; x_{n-1}^{-1} x_n x_{n-1}, y_{n-2} x_{n-1}^{-1} x_n x_{n-1} y_{n-2}^{-1},$$
$$\left. y_{n-2} x_{n-1}^{-1} x_n^{\pm 1}, x_{n-1}^{-1} x_n, x_n x_{n-1}, y_{n-1} x_n^{-1} x_{n-1} y_{n-2}^{-1}, y_{n-1} x_n^{-1}, y_{r-1} x_r^{-1} y_r^{-1} \right\}.$$

Proof. Direct inspection from Lemma 73. □

Below we suppose $n > 0$.

We still want to reduce the variety of possible labels of intervals in Π'_{K_1}. We cannot apply T_1, T_2 to some of the intervals labelled by $x^{\phi_{K_1}}$, $x \in X \cup E(m,n)$, because there are some cases when $x^{\phi_{K_1}}$ is completely cancelled in $y^{\phi_{K_1}}$, $x, y \in (X \cup E(m,n))^{\pm 1}$.

We will change the basis of $F(X \cup C_S)$, and then apply transformations T_1, T_2 to the labels written in the new basis. Replace, first, the basis $(X \cup C_S)$ by a new basis $\bar{X} \cup C_S$ obtained by replacing each variable x_s by $u_s = x_s y_{s-1}^{-1}$ for $s > 1$, and replacing x_1 by $u_1 = x_1 c_m^{-z_m}$:

$$\bar{X} = \left\{ u_1 = x_1 c_m^{-z_m}, u_2 = x_2 y_1^{-1}, \ldots, u_n = x_n y_{n-1}^{-1}, y_1, \ldots, y_n, z_1, \ldots, z_m \right\}.$$

Consider the case $n \geqslant 3$. Then the labels of the intervals will be rewritten as $u^{\phi_{K_1}}$, where

$$
\begin{aligned}
u \in \Bigg\{ & z_i, u_i y_{i-1}, y_i, \prod_s c_s^{z_s}, y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j}, u_1^{-1} y_1^{-1} u_2^{-1}, y_r u_r y_{r-1}, u_r,\ u_{r-1}^{-1} y_{r-1}^{-1} u_r^{-1}, \\
& u_r y_{r-1} u_{r-1} y_{r-2}, u_2 y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j}, r < n; \\
& y_{n-2}^{-1} u_{n-1}^{-1} u_n y_{n-1} u_{n-1} y_{n-2}, u_{n-1}^{-1} u_n y_{n-1} u_{n-1}, u_{n-1}^{-1} u_n y_{n-1}, \\
& u_{n-1}^{-1} y_{n-1}^{-1} u_n^{-1}, y_{n-2}^{-1} u_{n-1}^{-1} u_n y_{n-1}, u_n y_{n-1} u_{n-1} y_{n-2}, u_n^{-1} u_{n-1}, u_n \Bigg\}.
\end{aligned}
$$

In the cases $n = 1, 2$ some of the labels above do not appear, some coincide. Notice, that $x_n^{\phi_K} = u_n^{\phi_K} \circ y_{n-1}^{\phi_K}$, and that the first letter of $y_{n-1}^{\phi_K}$ is not cancelled in the products $(y_{n-1} x_{n-1} y_{n-2}^{-1})^{\phi_K}$, $(y_{n-1} x_{n-1})^{\phi_K}$ (see Lemma 46). Therefore, applying transformations similar to T_1 and T_2 to the cut equation Π'_{K_1} with labels written in the basis $\bar{X}$, we can split all the intervals with labels containing $(u_n y_{n-1})^{\phi_{K_1}}$ into two parts and obtain a cut equation with the same properties and intervals labelled by $u^{\phi_{K_1}}$, where

$$
\begin{aligned}
u \in \Bigg\{ & z_i, u_i y_{i-1}, y_i, \prod_s c_s^{z_s}, y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j}, u_1^{-1} y_1^{-1} u_2^{-1}, y_r u_r y_{r-1}, u_r, u_{r-1}^{-1} y_{r-1}^{-1} u_r^{-1}, \\
& u_r y_{r-1} u_{r-1} y_{r-2}, u_2 y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j}, r < n; \\
& y_{n-2}^{-1} u_{n-1}^{-1} u_n, y_{n-1} u_{n-1} y_{n-2}, u_{n-1}^{-1} u_n, y_{n-1} u_{n-1}, u_n \Bigg\}.
\end{aligned}
$$

Consider for $i < n$ the expression for

$$
(y_i u_i)^{\phi_K} = A_{m+4i}^{-p_{m+4i}+1} \circ x_{i+1} \circ A_{m+4i-4}^{-p_{m+4i-4}} \circ x^{p_{m+4i-3}} \circ y_i \circ A_{m+4i-2}^{p_{m+4i-2}-1} \circ x_i \circ \tilde{y}_{i-1}^{-1}.
$$

Formula (3)(a) from Lemma 53 shows that $u_i^{\phi_K}$ is completely cancelled in the product $y_i^{\phi_K} u_i^{\phi_K}$. This implies that $y_i^{\phi_K} = v_i^{\phi_K} \circ u_i^{-\phi_K}$.

Consider also the product

$$
\begin{aligned}
& y_{i-1}^{-\phi_K} u_i^{-\phi_K} \\
&\quad = \big(\boldsymbol{A_{m+4i-4}^{-p_{m+4i-4}+1} \circ x_i \circ \tilde{y}_{i-1} \circ x_i^{-1} A_{m+4i-4}^{p_{m+4i-4}-1}} \big) \\
&\qquad \times \big(A_{m+4i-4}^{-p_{m+4i-4}+1} x_i \circ \big(\boldsymbol{x_i^{p_{m+4i-3}} y_{i-1} \ldots *} \big)^{\boldsymbol{p_{m+4i-1}-1}} \boldsymbol{x_i^{p_{m+4i-3}} y_i x_{i+1}^{-1} A_{m+4i}^{p_{m+4i}-1}} \big),
\end{aligned}
$$

where the non-cancelled part is made bold.

Notice, that $(y_{r-1}u_{r-1})^{\phi_K} y_{r-2}^{\phi_K} = (y_{r-1}u_{r-1})^{\phi_K} \circ y_{r-2}^{\phi_K}$, because $u_{r-1}^{\phi_K}$ is completely cancelled in the product $y_i^{\phi_K} u_i^{\phi_K}$.

Therefore, we can again apply the transformations similar to T_1 and T_2 and split the intervals into the ones with labels $u^{\phi_{K_1}}$, where

$$u \in \left\{ z_s, y_i, u_i, \prod_s c_s^{z_s}, y_r u_r, y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j}, u_{n-1}^{-1} u_n = \bar{u}_n, \right.$$
$$\left. 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ 1 \leqslant r < n \right\}.$$

Now we change the basis $\bar{X}$ with a new basis $\hat{X}$ replacing $y_r, 1 < r < n$, by a new variable $v_r = y_r u_r$, $y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j}$ by v_1, and $u_{n-1}u_n$ by $\bar{u}_n$:

$$\hat{X} = \left\{ z_1, \ldots, z_m, u_1, \ldots, u_{n-1}, \bar{u}_n = u_{n-1}u_n, v_1 = y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j}, \right.$$
$$\left. v_2 = y_2 u_2, \ldots, v_n = y_n u_n, y_n \right\}.$$

Then $y_r^{\phi_K} = v_r^{\phi_K} \circ u_r^{-\phi_K}$, and $y_1^{\phi_K} = v_1^{\phi_K} \circ c_1^{z_1^{\phi_K}} \circ c_{m-1}^{z_{m-1}^{\phi_K}} \circ u_1^{-\phi_K}$ (if $n \neq 1$). Formula (2)(c) shows that $u_n^{\phi_K} = u_{n-1}^{\phi_K} \circ (u_{n-1}^{-1} u_n)^{\phi_K}$.

Apply transformations similar to T_1 and T_2 to the intervals with labels written in the new basis $\hat{X}$ and obtain intervals with labels $u^{\phi_{K_1}}$, where

$$u \in \hat{X} \cup \left\{ c_m^{z_m} \right\}.$$

Denote the resulting cut equation by $\bar{\Pi}_{K_1} = (\bar{\mathcal{E}}, f_{\bar{X}}, f_{\bar{M}})$. Let α be the corresponding solution of $\bar{\Pi}_{K_1}$ with respect to β.

Denote by $\bar{M}_{\text{side}}$ the set of long variables in $\bar{\Pi}_{K_1}$, then $\bar{M} = \bar{M}_{\text{veryshort}} \cup \bar{M}_{\text{side}}$.

Define a binary relation $\sim_{\text{left}}$ on $\bar{M}_{\text{side}}^{\pm 1}$ as follows. For $\mu_1, \mu_1' \in \bar{M}_{\text{side}}^{\pm 1}$ put $\mu_1 \sim_{\text{left}} \mu_1'$ if and only if there exist two intervals $\sigma, \sigma' \in \bar{E}$ with $f_{\bar{X}}(\sigma) = f_{\bar{X}}(\sigma')$ such that

$$f_{\bar{M}}(\sigma) = \mu_1 \mu_2 \ldots \mu_r, \qquad f_{\bar{M}}(\sigma') = \mu_1' \mu_2' \ldots \mu_{r'}'$$

and either $\mu_r = \mu_{r'}'$ or $\mu_r, \mu_{r'}' \in M_{\text{veryshort}}$. Observe that if $\mu_1 \sim_{\text{left}} \mu_1'$ then

$$\mu_1 = \mu_1' \lambda_1 \ldots \lambda_t$$

for some $\lambda_1, \ldots, \lambda_t \in M_{\text{veryshort}}^{\pm 1}$. Notice, that $\mu \sim_{\text{left}} \mu$.

Similarly, we define a binary relation $\sim_{\text{right}}$ on $\bar{M}^{\pm 1}_{\text{side}}$. For $\mu_r, \mu'_{r'} \in \bar{M}^{\pm 1}_{\text{side}}$ put $\mu_r \sim_{\text{right}} \mu'_{r'}$ if and only if there exist two intervals $\sigma, \sigma' \in \bar{E}$ with $f_{\bar{X}}(\sigma) = f_{\bar{X}}(\sigma')$ such that

$$f_{\bar{M}}(\sigma) = \mu_1 \mu_2 \dots \mu_r, \qquad f_{\bar{M}}(\sigma') = \mu'_1 \mu'_2 \dots \mu'_{r'}$$

and either $\mu_1 = \mu'_1$ or $\mu_1, \mu'_1 \in M_{\text{veryshort}}$. Again, if $\mu_r \sim_{\text{right}} \mu'_{r'}$ then

$$\mu_r = \lambda_1 \dots \lambda_t \mu'_{r'}$$

for some $\lambda_1, \dots, \lambda_t \in M^{\pm 1}_{\text{veryshort}}$.

Denote by $\sim$ the transitive closure of

$$\{(\mu, \mu') \mid \mu \sim_{\text{left}} \mu'\} \cup \{(\mu, \mu') \mid \mu \sim_{\text{right}} \mu'\} \cup \{(\mu, \mu^{-1}) \mid \mu \in \bar{M}^{\pm 1}_{\text{side}}\}.$$

Clearly, $\sim$ is an equivalence relation on $\bar{M}^{\pm 1}_{\text{side}}$. Moreover, $\mu \sim \mu'$ if and only if there exists a sequence of variables

$$\mu = \mu_0, \mu_1, \dots, \mu_k = \mu' \tag{109}$$

from $\bar{M}^{\pm 1}_{\text{side}}$ such that either $\mu_{i-1} = \mu_i$, or $\mu_{i-1} = \mu_i^{-1}$, or $\mu_{i-1} \sim_{\text{left}} \mu_i$, or $\mu_{i-1} \sim_{\text{right}} \mu_i$ for $i = 1, \dots, k$. Observe that if μ_{i-1} and μ_i from (109) are side variables of "different sides" (one is on the left, and the other is on the right) then $\mu_i = \mu_{i-1}^{-1}$. This implies that replacing in the sequence (109) some elements μ_i with their inverses one can get a new sequence

$$\mu = \nu_0, \nu_1, \dots, \nu_k = (\mu')^{\varepsilon} \tag{110}$$

for some $\varepsilon \in \{1, -1\}$ where $\nu_{i-1} \sim \nu_i$ and all the variables ν_i are of the same side. It follows that if μ is a left-side variable and $\mu \sim \mu'$ then

$$(\mu')^{\varepsilon} = \mu \lambda_1 \dots \lambda_t \tag{111}$$

for some $\lambda_j \in M^{\pm 1}_{\text{veryshort}}$.

It follows from (111) that for a variable $\nu \in \bar{M}^{\pm 1}_{\text{side}}$ all variables from the equivalence class $[\nu]$ of ν can be expressed via ν and very short variables from $M_{\text{veryshort}}$. So if we fix a system of representatives R of $\bar{M}^{\pm 1}_{\text{side}}$ relative to $\sim$ then all other variables from $\bar{M}_{\text{side}}$ can be expressed as in (111) via variables from R and very short variables.

This allows one to introduce a new transformation T_3 of cut equations. Namely, we fix a set of representatives R such that for every $\nu \in R$ the element ν^{α} has minimal length among all the variables in this class. Now, using (111) replace every variable ν in every word $f_M(\sigma)$ of a cut equation Π by its expression via the corresponding representative variable from R and a product of very short variables.

Now we repeatedly apply the transformation T_3 till the equivalence relations $\sim_{\text{left}}$ and $\sim_{\text{right}}$ become trivial. This process stops in finitely many steps since the non-trivial relations decrease the number of side variables.

Denote the resulting equation again by $\bar{\Pi}_{K_1}$.

Now we introduce an equivalence relation on partitions of $\bar{\Pi}_{K_1}$. Two partitions $f_M(\sigma)$ and $f_M(\delta)$ are equivalent ($f_M(\sigma) \sim f_M(\delta)$) if $f_X(\sigma) = f_X(\delta)$ and either the left side variables or the right side variables of $f_M(\sigma)$ and $f_M(\delta)$ are equal. Observe, that $f_X(\sigma) = f_X(\delta)$ implies $f_M(\sigma)^\alpha = f_M(\delta)^\alpha$, so in this case the partitions $f_M(\sigma)$ and $f_M(\delta)$ cannot begin with μ and μ^{-1} correspondingly. It follows that if $f_M(\sigma) \sim f_M(\delta)$ then the left side variables and, correspondingly, the right side variables of $f_M(\sigma)$ and $f_M(\delta)$ (if they exist) are equal. Therefore, the relation $\sim$ is, indeed, an equivalence relation on the set of partitions of $\bar{\Pi}_{K_1}$.

If an equivalence class of partitions contains two distinct elements $f_M(\sigma)$ and $f_M(\delta)$ then the equality

$$f_M(\sigma)^\alpha = f_M(\delta)^\alpha$$

implies the corresponding equation on the variables $\bar{M}_{\text{veryshort}}$, which is obtained by deleting all side variables (which are equal) from $f_M(\sigma)$ and $f_M(\delta)$ and equalizing the resulting words in very short variables.

Denote by $\Delta(\bar{M}_{\text{veryshort}}) = 1$ this system.

Now we describe a transformation T_4. Fix a set of representatives R_p of partitions of $\bar{\Pi}_{K_1}$ with respect to the equivalence relation $\sim$. For a given class of equivalent partitions we take as a representative an interval σ with $f_M(\sigma) = \mu_{\text{left}} \ldots \mu_{\text{right}}$.

Below we say that: a word $w \in F[X]$ is *very short* if the reduced form of w^β does not contain $(A'_j)^3$ for any $j \geqslant K_2$; a word $v \in F$ is *very short* if it does not contain $(A'_j)^3$ for any $j \geqslant K_2$; we also say that μ^α almost contains u^β for some word u in the alphabet X if μ^α contains a subword which is the reduced form of $f_1 u^\beta f_2$ for some $f_1, f_2 \in C_\beta$.

Principal variables. A long variable μ_{left} or μ_{right} for an interval σ of $\bar{\Pi}_{K_1}$ with $f_M(\sigma) = \mu_{\text{left}} \ldots \mu_{\text{right}}$ is called *principal* in σ in the following cases.

(1) Let $f_X(\sigma) = u_i$ $(i \neq n)$, where $u_i = x_i y_{i-1}^{-1}$ for $i > 1$ and $u_1 = x_1 c_m^{-z_m}$ for $m \neq 0$. Then (see Lemma 53)

$$\begin{aligned} u_i^{\phi_{K_1}} = {} & A^{*-q_4+1}_{K_2+m+4i} x_{i+1}^{\phi_{K_2}} y_i^{-\phi_{K_2}} x_i^{-q_1\phi_{K_2}} \\ & \times \left(x_i^{-\phi_{K_2}} A^{*q_0}_{K_2+m+4i-4} A^{*(-q_2+1)}_{K_2+m+4i-2} y_i^{\phi_{K_2}} x_i^{-q_1\phi_{K_2}}\right)^{q_3-1} A^{*q_0}_{K_2+m+4i-4}. \end{aligned}$$

The variable μ_{right} is *principal* in σ if and only if $\mu^\alpha_{\text{right}}$ almost contains a cyclically reduced part of

$$\begin{aligned} & \left(x_i^{-\psi_{K_2}} A^{*q_0\beta}_{K_2+m+4i-4} A^{*(-q_2+1)\beta}_{m+4i-2} y_i^{\psi_{K_2}} x_i^{-q_1\psi_{K_2}}\right)^q \\ & \quad = \left(x_i^{q_1} y_i\right)^{\psi_{K_2}} \left(A^{*\beta}_{K_2+m+4i-1}\right)^{-q} \left(y_i^{-1} x_i^{-q_1}\right)^{\psi_{K_2}}, \end{aligned}$$

for some $q > 2$. Now, the variable μ_{left} is *principal* in σ if and only if μ_{right} is not principal in σ.

(2) Let $f_X(\sigma) = v_i$, where $v_i = y_i u_i$ $(i \neq 1, n)$ and $v_1 = y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j}$. Then (see formula (3)(a) from Lemma 53)

$$v_i^{\phi_{K_1}} = A_{K_2+m+4i}^{*(-q_4+1)} x_{i+1}^{\phi_{K_2}} A_{K_2+m+4i-4}^{*(-q_0)} x_i^{q_1 \phi_{K_2}} y_i^{\phi_{K_2}} A_{K_2+m+4i-2}^{*(q_2-1)} A_{K_2+m+4i-4}^{*-1}, \quad i \neq 1,$$

and

$$v_1^{\phi_{K_1}} = A_{K_2+m+4}^{*(-q_4+1)} x_2^{\phi_{K_2}} A_{K_2+2m}^{*(-q_0)} x_1^{q_1 \phi_{K_2}} y_1^{\phi_{K_2}} A_{K_2+m+1}^{*(q_2-1)} x_1 \prod_{j=n}^{1} c_j^{-z_j}.$$

A side variable μ_{right} (μ_{left}) is *principal* in σ if and only if $\mu_{\text{right}}^{\alpha}$ (correspondingly, $\mu_{\text{left}}^{\alpha}$) almost contains $(A_{K_2+m+4i}^{\beta})^{-q}$, for some $q > 2$. In this case both variables $\mu_{\text{left}}^{\alpha}$, $\mu_{\text{right}}^{\alpha}$ could be simultaneously principal.

(3) Let $f_X(\sigma) = \bar{u}_n = u_{n-1} u_n$. Then (by formula (3)(c)) from Lemma 53)

$$\bar{u}_n^{\phi_{K_1}} = A_{K_2+m+4n-8}^{*} A_{K_2+m+4n-6}^{-q_2+1} \left(y_{n-1}^{-1} x_n^{-q_1}\right)^{\phi_{K_1}} A_{K_2+m+4n-8}^{*q_0} \left(x_n^{q_5} y_n\right)^{\phi_{K_1}} \\ \times A_{K_2+m+4n-2}^{*q_6-1} A_{K_2+m+4n-4}^{*-1}.$$

A side variable μ_{right} (μ_{left}) is *principal* in σ if $\mu_{\text{right}}^{\alpha}$ (correspondingly, $\mu_{\text{left}}^{\alpha}$) almost contains $(A_{K_2+m+4n-2}^{\beta})^{q}$, for some $q > 2$. In this case both variables $\mu_{\text{left}}^{\alpha}$, $\mu_{\text{right}}^{\alpha}$ could be simultaneously principal.

(4) Let $f_X(\sigma) = y_n$. Then (by Lemma 47)

$$y_n^{\phi_{K_1}} = A_{K_2+m+4n-4}^{*q_0 \beta} A_{K_1}^{*q_3 \beta} x_n^{q_1 \psi_{K_2}} y_1^{\psi_{K_2}}.$$

The variable μ_{right} (μ_{left}) is *principal* in σ if $\mu_{\text{right}}^{\alpha}$ (correspondingly, $\mu_{\text{left}}^{\alpha}$) almost contains $(A_{K_1}^{\beta})^{q}$, for some q such that $2q > p_{K_1} - 2$. In this case both variables $\mu_{\text{left}}^{\alpha}$, $\mu_{\text{right}}^{\alpha}$ could be simultaneously principal.

(5) Let $f_X(\sigma) = z_j$, $j = 1, \ldots, m-1$. Then (by Lemma 44)

$$z_j^{\phi_{K_1}} = c_j z_j^{\phi_{K_2}} A_{K_2+j-1}^{*\beta p_{j-1}} c_{j+1}^{z_{j+1}^{\phi_{K_2}}} A_{K_2+j}^{*\beta p_j - 1}.$$

A variable μ_{left} (μ_{right}) is *principal* if $\mu_{\text{right}}^{\alpha}$ (correspondingly, $\mu_{\text{left}}^{\alpha}$) almost contains $(A_{K_2+j}^{\beta})^{q}$, for some $|q| > 2$. Both left and right side variables can be simultaneously principal.

(6) Let $f_X(\sigma) = z_m$. Then (by Lemma 44)

$$z_m^{\phi_{K_1}} = c_m^{K_2} z_m^{\phi_{K_2}} A_{K_2+m-1}^{*p_{m-1}} x_1^{-\phi_{K_2}} A_{K_2+m}^{*p_m - 1}.$$

In this case $\mu_{\rm left}$ is *principal* in σ if and only if $\mu_{\rm left}$ is long (i.e., it is not very short), and we define $\mu_{\rm right}$ to be always non-principal. Observe that if $\mu_{\rm left}$ is very short then

$$\mu_{\rm right}^{\alpha} = f z_m^{\phi_{K_1}\beta} \quad \text{for a very short } f \in F.$$

Let $f_X(\sigma) = z_m^{-1} c_m z_m$. Then (by Lemma 44)

$$f_X(\sigma)^{\phi_{K_1}} = A_{K_2+m}^{*-p_m+1} x_1^{\phi_{K_2}} A_{K_2+m}^{*p_m}.$$

The variable $\mu_{\rm left}$ is *principal* in σ if and only if the following two conditions hold: $\mu_{\rm left}^{\alpha}$ almost contains $(A_{K_2+m}^{\beta})^q$, for some q with $|q| > 2$;

$$\mu_{\rm left}^{-1} \neq f z_m^{\phi_{K_1}\beta} \quad \text{for a very short } f \in F.$$

Similarly, the variable $\mu_{\rm right}$ is *principal* in σ if and only if the following two conditions hold: $\mu_{\rm right}^{\alpha}$ almost contains $(A_{K_2+m}^{\beta})^q$, for some q with $|q| > 2$;

$$\mu_{\rm right}^{\alpha} \neq f z_m^{\phi_{K_1}\beta} \quad \text{for a very short } f \in F.$$

Observe, that in this case the variables $\mu_{\rm left}$ and $\mu_{\rm right}$ can be simultaneously principal in σ and non-principal in σ. The latter happens if and only if

$$\mu_{\rm right}^{\alpha} = f_1 z_m^{\phi_{K_1}\beta} \quad \text{and} \quad \mu_{\rm left}^{\alpha} = z_m^{-\phi_{K_1}\beta} f_2$$

for some very short elements $f_1, f_2 \in F$. Therefore, if both $\mu_{\rm left}$ and $\mu_{\rm right}$ are non-principal then they can be expressed in terms of $z_m^{\phi_{K_1}}$ and very short variables.

Claim 17. *For every interval σ of $\bar{\Pi}_{K_1}$ its partition $f_M(\sigma)$ has at least one principal variable, unless this interval σ and its partition $f_M(\sigma)$ are of those two particular types described in case* (6).

Proof. Let $f_M(\sigma) = \mu_{\rm left}\nu_1 \dots \nu_k \mu_{\rm right}$, where $\nu_1 \dots \nu_k$ are very short variables. Suppose A_{r+K_2} is the oldest period such that $f_X(\sigma)$ has N-large A_{r+K_2}-decomposition.

If $r \neq 1$ then (see Lemmas 44–47) A_{r+K_2} contains some N-large exponent of A_{r-1+K_2}. Therefore $\nu_1^{\alpha} \dots \nu_k^{\alpha}$ does not contain A'_{r+K_2}, hence either $\mu_{\rm left}$ or $\mu_{\rm right}$ almost contains $A_{r+K_2}^{\beta q}$, where $|q| > 2$. This finishes all the cases except for the case (1). In case (1) a similar argument shows that $\nu_1^{\alpha} \dots \nu_k^{\alpha}$ does not contain A'_{r-1+K_2}, so one of the side variables is principal.

If $r = 1$, then A_{1+K_2} contains some N-large exponent of A_{2+K_3}. Again, $\nu_1^{\alpha} \dots \nu_k^{\alpha}$ does not contain A'_{1+K_2}, because the complexity of the cut equation Π_{K_1} does not changed in ranks from K_0 to K_3. Now, an argument similar to the one above finishes the proof. □

Claim 18. *If both side variables of a partition $f_M(\sigma)$ of an interval σ from $\bar{\Pi}_{K_1}$ are non-principal, then they are non-principal in every partition of an interval from $\bar{\Pi}_{K_1}$.*

Proof. It follows directly from the description of the side variables μ_{left} and μ_{right} in the case (6) of the definition of principal variables. Indeed, if μ_{left} and μ_{right} are both non-principal, then (see case (6)) each of them is either very short, or it is equal to

$$f_1 z_m^{\phi_{K_1}\beta} f_2$$

for some very short $f_1, f_2 \in F$. Clearly, neither of them could be principal in other partitions. □

Claim 19. *Let $n \neq 0$. Then a side variable can be principal only in one class of equivalent partitions of intervals from $\bar{\Pi}_{K_1}$.*

Proof. Let $f_M(\sigma) = \mu_{\text{left}}\nu_1 \ldots \nu_k\mu_{\text{right}}$, where $\nu_1 \ldots \nu_k$ are very short variables. Suppose A_{r+K_2} is the oldest period such that $f_X(\sigma)$ has N-large A_{r+K_2}-decomposition.

In every case from the definition of principal variables (except for case (1)) a principal variable in σ almost contains a cube $(A'_{r+K_2})^3$. In case (1) the principal variable almost contains $(A'_{r-1+K_2})^3$, moreover, if μ_{left} is the principal variable then $\mu_{\text{left}}^{\alpha}$ contains an N-large exponent of (A'_{r+K_2}).

We consider only the situation when the partition $f_M(\sigma)$ satisfies case (1), all other cases can be done similarly.

Clearly, if $f_X(\sigma) = u_i$ then a principal variable in σ does not appear as a principal variable in the partition of any other interval δ with $f_X(\delta) \neq u_i$, $f_X(\delta) \neq v_i$. Suppose that a principal variable in σ appears as a principal variable of the partition of δ with $f_X(\delta) = u_i$. Then partitions $f_M(\sigma)$ and $f_M(\delta)$ are equivalent, as required. Suppose now that a principal variable μ in σ appears as a principal variable of the partition of δ with $f_X(\delta) = v_i$. If $\mu = \mu_{\text{right}}$ then it cannot appear as the right principal variable, say λ_{right}, of $f_M(\delta)$. Indeed, $\mu_{\text{right}}^{\alpha}$ ends (see case (1) above) with almost all of the word $(A^{*q_0}_{K_2+m+4i-4})^{\beta}$ (except, perhaps, for a short initial segment of it). But the write principal variable λ_{right} should end (see case (2) above) with almost all of the word $A^{*-1}_{K_2+m+4i-4}$ (except, perhaps, for a short initial segment of it), so $\mu_{\text{right}} \neq \lambda_{\text{right}}$. Similarly, if the left side variable λ_{left} of $f_M(\delta)$ is principal in δ then $\mu_{\text{right}} \neq \lambda_{\text{left}}$. Suppose now that $\mu = \mu_{\text{left}}$, then μ_{right} is not principle in σ, so it is not true that μ_{right} almost contains the cube of the cyclically reduced part of

$$x_i^{-\psi_{K_2}} A^{*q_0\beta}_{K_2+m+4i-4} A^{*(-q_2+1)\beta}_{m+4i-2} y_i^{\psi_{K_2}} x_i^{-q_1\psi_{K_2}}.$$

Then μ_{left} is very long, so it is easy to see that it does not appear in the partition of δ as a principle variable. This finishes the case (1). □

For the cut equation $\bar{\Pi}_{K_1}$ we construct a finite graph $\Gamma = (V, E)$. Every vertex from V is marked by variables from $\bar{M}^{\pm 1}_{\text{side}}$ and letters from the alphabet $\{P, N\}$. Every edge from

E is colored either as red or blue. The graph Γ is constructed as follows. Every partition $f_M(\sigma) = \mu_1 \dots \mu_k$ of $\bar{\Pi}_{K_1}$ gives two vertices $v_{\sigma,\text{left}}$ and $v_{\sigma,\text{right}}$ into Γ, so

$$V = \bigcup_{\sigma} \{v_{\sigma,\text{left}}, v_{\sigma,\text{right}}\}.$$

We mark $v_{\sigma,\text{left}}$ by μ_1 and $v_{\sigma,\text{right}}$ by μ_k. Now we mark the vertex $v_{\sigma,\text{left}}$ by a letter P or letter N if μ_1 is correspondingly principal or non-principal in σ. Similarly, we mark $v_{\sigma,\text{right}}$ by P or N if μ_k is principal or non-principal in σ.

For every σ the vertices $v_{\sigma,\text{left}}$ and $v_{\sigma,\text{right}}$ are connected by a *red* edge. Also, we connect by a *blue* edge every pair of vertices which are marked by variables μ, ν provided $\mu = \nu$ or $\mu = \nu^{-1}$. This describes the graph Γ.

Below we construct a new graph Δ which is obtained from Γ by deleting some blue edges according to the following procedure. Let B be a maximal connected blue component of Γ, i.e., a connected component of the graph obtained from Γ by deleting all red edges. Notice, that B is a complete graph, so every two vertices in B are connected by a blue edge. Fix a vertex v in B and consider the star-subgraph Star_B of B generated by all edges adjacent to v. If B contains a vertex marked by P then we choose v with label P, otherwise v is an arbitrary vertex of B. Now, replace B in Γ by the graph Star_B, i.e., delete all edges in B which are not adjacent to v. Repeat this procedure for every maximal blue component B of Γ. Suppose that the blue component corresponds to long bases of case (6) that are non-principal and equal to

$$f_1 z_m^{\phi_{K_1}} f_2$$

for very short f_1, f_2. In this case, we remove all the blue edges that produce cycles if the red edge from Γ connecting non-principal μ_{left} and μ_{right} is added to the component (if such a red edge exists).

Denote the resulting graph by Δ.

In the next claim we describe connected components of the graph Δ.

Claim 20. *Let C be a connected component of Δ. Then one of the following holds.*

(1) *There is a vertex in C marked by a variable which does not occur as a principal variable in any partition of $\bar{\Pi}_{K_1}$. In particular, any component which satisfies one of the following conditions has such a vertex*:
   (a) *there is a vertex in C marked by a variable which is a short variable in some partition of $\bar{\Pi}_{K_1}$;*
   (b) *there is a red edge in C with both endpoints marked by N (it corresponds to a partition described in case* (6) *above*).

(2) *Both endpoints of every red edge in C are marked by P. In this case C is an isolated vertex.*

(3) *There is a vertex in C marked by a variable μ and N and if μ occurs as a label of an endpoint of some red edge in C then the other endpoint of this edge is marked by P.*

Proof. Let C be a connected component of Δ. Observe first, that if μ is a short variable in $\bar{\Pi}_{K_1}$ then μ is not principle in σ for any interval σ from $\bar{\Pi}_{K_1}$, so there is no vertex in C marked by both μ and P. Also, it follows from Claim 18 that if there is a red edge e in C with both endpoints marked by N, then the variables assigned to endpoints of e are non-principle in any interval σ of $\bar{\Pi}_{K_1}$. This proves the part "in particular" of (1).

Now assume that the component C does not satisfy any of the conditions (1), (2). We need to show that C has type (3). It follows that every variable which occurs as a label of a vertex in C is long and it labels, at least, one vertex in C with label P. Moreover, there are non-principle occurrences of variables in C.

We summarize some properties of C below.

- There are no blue edges in Δ between vertices with labels N and N (by construction).
- There are no blue edges between vertices labelled by P and P (Claim 19).
- There are no red edges in C between vertices labelled by N and N (otherwise (1) would hold).
- Any reduced path in Δ consists of edges of alternating color (by construction).

We claim that C is a tree. Let $p = e_1 \dots e_k$ be a simple loop in C (every vertex in p has degree 2 and the terminal vertex of e_k is equal to the starting point of e_1).

We show first that p does not have red edges with endpoints labelled by P and P. Indeed, suppose there exists such an edge in p. Taking cyclic permutation of p we may assume that e_1 is a red edge with labels P and P. Then e_2 goes from a vertex with label P to a vertex with label N. Hence the next red edge e_3 goes from N to P, etc. This shows that every blue edge along p goes from P to N. Hence the last edge e_k which must be blue goes from P to N—contradiction, since all the labels of e_1 are P.

It follows that both colors of edges and labels of vertices in p alternate. We may assume now that p starts with a vertex with label N and the first edge e_1 is red. It follows that the end point of e_1 is labelled by N and all blue edges go from N to P. Let e_i be a blue edge from v_i to v_{i+1}. Then the variable μ_i assign to the vertex v_i is principal in the partition associated with the red edge e_{i-1}, and the variable $\mu_{i+1} = \mu_i^{\pm 1}$ associated with v_{i+1} is a non-principal side variable in the partition $f_M(\sigma)$ associated with the red edge e_{i+1}. Therefore, the side variable μ_{i+2} associated with the end vertex v_{i+2} is a principal side variable in the partition $f_M(\sigma)$ associated with e_{i+1}. It follows from the definition of principal variables that the length of μ_{i+2}^{α} is much longer than the length of μ_{i+1}^{α}, unless the variable μ_i is described in the case (1). However, in the latter case the variable μ_{i+2} cannot occur in any other partition $f_M(\delta)$ for $\delta \neq \sigma$. This shows that there no blue edges in Δ with endpoints labelled by such μ_{i+2}. This implies that v_{i+2} has degree one in Δ—contradiction wit the choice of p. This shows that there are no vertices labelled by such variables described in case (1). Notice also, that the length of variables (under α) is preserved along blue edges: $|\mu_{i+1}^{\alpha}| = |(\mu_i^{\pm 1})^{\alpha}| = |\mu_i^{\alpha}|$. Therefore,

$$\left|\mu_i^{\alpha}\right| = \left|\mu_{i+1}^{\alpha}\right| < \left|\mu_{i+2}^{\alpha}\right| \quad \text{for every } i.$$

It follows that going along p the length of $|\mu_i^{\alpha}|$ increases, so p cannot be a loop. This implies that C is a tree.

Now we are ready to show that the component C has type (3) from Claim 20. Let μ_1 be a variable assigned to some vertex v_1 in C with label N. If μ_1 satisfies the condition (3) from Claim 20 then we are done. Otherwise, μ_1 occurs as a label of one of P-endpoints, say v_2 of a red edge e_2 in C such that the other endpoint of e_2, say v_3 is non-principal. Let μ_3 be the label of v_3. Thus v_1 is connected to v_2 by a blue edge and v_2 is connected to v_3 by a red edge. If μ_3 does not satisfy the condition (3) from Claim 20 then we can repeat the process (with μ_3 in place of μ_1). The graph C is finite, so in finitely many steps either we will find a variable that satisfies (3) or we will construct a closed reduced path in C. Since C is a tree the latter does not happen, therefore C satisfies (3), as required. □

Claim 21. *The graph Δ is a forest, i.e., it is union of trees.*

Proof. Let C be a connected component of Δ. If C has type (3) then it is a tree, as has been shown in Claim 20. If C of the type (2) then by Claim 20 C is an isolated vertex—hence a tree. If C is of the type (1) then C is a tree because each interval corresponding to this component has exactly one principal variable (except some particular intervals of type (6) that do not have principal variables at all and do not produce cycles), and the same long variable cannot be principal in two different intervals. Although the same argument as in (3) also works here. □

Now we define the sets $\bar{M}_{\text{useless}}$, $\bar{M}_{\text{free}}$ and assign values to variables from $\bar{M} = \bar{M}_{\text{useless}} \cup \bar{M}_{\text{free}} \cup \bar{M}_{\text{veryshort}}$. To do this we use the structure of connected components of Δ. Observe first, that all occurrences of a given variable from $\bar{M}_{\text{sides}}$ are located in the same connected component.

Denote by $\bar{M}_{\text{free}}$ subset of $\bar{M}$ which consists of variables of the following types:

(1) variables which do not occur as principal in any partition of $(\bar{\Pi}_{K_1})$;
(2) one (but not the other) of the variables μ and ν if they are both principal side variables of a partition of the type (20) and such that $\nu \neq \mu^{-1}$.

Denote by $\bar{M}_{\text{useless}} = \bar{M}_{\text{side}} - \bar{M}_{\text{free}}$.

Claim 22. *For every $\mu \in \bar{M}_{\text{useless}}$ there exists a word $V_\mu \in F[X \cup \bar{M}_{\text{free}} \cup \bar{M}_{\text{veryshort}}]$ such that for every map $\alpha_{\text{free}} : \bar{M}_{\text{free}} \to F$, and every solution $\alpha_s : F[\bar{M}_{\text{veryshort}}] \to F$ of the system $\Delta(\bar{M}_{\text{veryshort}}) = 1$ the map $\alpha : F[\bar{M}] \to F$ defined by*

$$\mu^\alpha = \begin{cases} \mu^{\alpha_{\text{free}}}, & \text{if } \mu \in \bar{M}_{\text{free}}, \\ \mu^{\alpha_s}, & \text{if } \mu \in \bar{M}_{\text{veryshort}}, \\ \bar{V}_\mu(X^\delta, \bar{M}_{\text{free}}^{\alpha_{\text{free}}}, \bar{M}_{\text{veryshort}}^{\alpha_s}), & \text{if } \mu \in \bar{M}_{\text{useless}} \end{cases}$$

is a group solution of $\bar{\Pi}_{K_1}$ with respect to β.

Proof. The claim follows from Claims 20 and 21. Indeed, take as values of short variables an arbitrary solution α_s of the system $\Delta(\bar{M}_{\text{veryshort}}) = 1$. This system is obviously consistent, and we fix its solution. Consider connected components of type (1) in Claim 20. If μ

is a principal variable for some σ in such a component, we express μ^α in terms of values of very short variables $\bar{M}_{\text{veryshort}}$ and elements $t^{\psi_{K_1}}$, $t \in X$, that correspond to labels of the intervals. This expression does not depend on α_s, β and tuples q, p^*. For connected components of Δ of types (2) and (3) we express values μ^α for $\mu \in M_{\text{useless}}$ in terms of values ν^α, $\nu \in M_{\text{free}}$ and $t^{\psi_{K_1}}$ corresponding to the labels of the intervals. □

We can now finish the proof of Proposition 8. Observe, that $M_{\text{veryshort}} \subseteq \bar{M}_{\text{veryshort}}$. If λ is an additional very short variable from $M^*_{\text{veryshort}}$ that appears when transformation T_1 or T_2 is performed, λ^α can be expressed in terms $M^\alpha_{\text{veryshort}}$. Also, if a variable λ belongs to $\bar{M}_{\text{free}}$ and does not belong to M, then there exists a variable $\mu \in M$, such that $\mu^\alpha = u^{\psi_{K_1}}\lambda^\alpha$, where $u \in F(X, C_S)$, and we can place μ into M_{free}.

Observe, that the argument above is based only on the tuple p, it does not depend on the tuples p^* and q. Hence the words V_μ do not depend on p^* and q.

The proposition is proved for $n \neq 0$. If $n = 0$, partitions of the intervals with labels $z_{n-1}^{\phi_{K_1}}$ and $z_n^{\phi_{K_1}}$ can have equivalent principal right variables, but in this case the left variables will be different and do not appear in other non-equivalent partitions. The connected component of Δ containing these partitions will have only four vertices one blue edge.

In the case $n = 0$ we transform equation Π_{K_1} applying transformation T_1 to the form when the intervals are labelled by $u^{\phi_{K_1}}$, where

$$u \in \left\{z_1, \ldots, z_m, c_{m-1}^{z_{m-1}}, z_m c_{m-1}^{-z_{m-1}}\right\}.$$

If μ_{left} is very short for the interval δ labelled by $(z_m c_{m-1}^{-z_{m-1}})^{\phi_{K_1}}$, we can apply T_2 to δ, and split it into intervals with labels

$$z_m^{\phi_{K_1}} \quad \text{and} \quad c_{m-1}^{-z_{m-1}^{\phi_{K_1}}}.$$

Indeed, even if we had to replace μ_{right} by the product of two variables, the first of them would be very short.

If μ_{left} is not very short for the interval δ labelled by

$$\left(z_m c_{m-1}^{-z_{m-1}}\right)^{\phi_{K_1}} = c_m z_m^{\phi_{K_2}} A_{m-1}^{*p_{m-1}-1},$$

we do not split the interval, and μ_{left} will be considered as the principal variable for it. If μ_{left} is not very short for the interval δ labelled by

$$z_m^{\phi_{K_1}} = z_m^{\phi_{K_2}} A_{m-1}^{*p_{m-1}},$$

it is a principal variable, otherwise μ_{right} is principal.

If an interval δ is labelled by

$$\left(c_{m-1}^{z_{m-1}}\right)^{\phi_{K_1}} = A_{m-1}^{*-p_{m-1}+1} c_m^{-z_m^{\phi_{K_2}}} A_{m-1}^{*p_{m-1}},$$

we consider μ_{right} principal if $\mu^{\alpha}_{\text{right}}$ ends with

$$\left(c_m^{-z_m^{\phi_{K_2}}} A_{m-1}^{*p_{m-2}}\right)^{\beta},$$

and the difference is not very short. If $\mu^{\alpha}_{\text{left}}$ is almost $z_m^{-\phi_k\beta}$ and $\mu^{\alpha}_{\text{right}}$ is almost $z_m^{\phi_k\beta}$, we do not call any of the side variables principal. In all other cases μ_{left} is principal.

Definition of the principal variable in the interval with label $z_i^{\phi_{K_1}}$, $i = 1, \dots, m-2$, is the same as in (5) for $n \neq 0$.

A variable can be principal only in one class of equivalent partitions. All the rest of the proof is the same as for $n > 0$. □

Now we continue the proof of Theorem 9. Let $L = 2K + \kappa(\Pi)4K$ and

$$\Pi_\phi = \Pi_L \to \Pi_{L-1} \to \cdots$$

be the sequence of Γ-cut equations (108). For a Γ-cut equation Π_j from (108) by M_j and α_j we denote the corresponding set of variables and the solution relative to β.

By Claim 9 in the sequence (108) either there is $3K$-stabilization at $K(r+2)$ or $\mathrm{Comp}(\Pi_{K(r+1)}) = 0$.

Case 1. Suppose there is $3K$-stabilization at $K(r+2)$ in the sequence (108).

By Proposition 8 the set of variables $M_{K(r+1)}$ of the cut equation $\Pi_{K(r+1)}$ can be partitioned into three subsets

$$M_{K(r+1)} = M_{\text{veryshort}} \cup M_{\text{free}} \cup M_{\text{useless}}$$

such that there exists a finite consistent system of equations $\Delta(M_{\text{veryshort}}) = 1$ over F and words $V_\mu \in F[X, M_{\text{free}}, M_{\text{veryshort}}]$, where $\mu \in M_{\text{useless}}$, such that for every solution $\delta \in \mathcal{B}$, for every map $\alpha_{\text{free}} : M_{\text{free}} \to F$, and every solution $\alpha_{\text{short}} : F[M_{\text{veryshort}}] \to F$ of the system $\Delta(M_{\text{veryshort}}) = 1$ the map $\alpha_{K(r+1)} : F[M] \to F$ defined by

$$\mu^{\alpha_{K(r+1)}} = \begin{cases} \mu^{\alpha_{\text{free}}}, & \text{if } \mu \in M_{\text{free}}, \\ \mu^{\alpha_{\text{short}}}, & \text{if } \mu \in M_{\text{veryshort}}, \\ V_\mu(X^\delta, M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_s}), & \text{if } \mu \in M_{\text{useless}} \end{cases}$$

is a group solution of $\Pi_{K(r+1)}$ with respect to β. Moreover, the words V_μ do not depend on tuples p^* and q.

By Claim 4 if $\Pi = (\mathcal{E}, f_X, f_M)$ is a Γ-cut equation and $\mu \in M$ then there exists a word $\mathcal{M}_\mu(M_{T(\Pi)}, X)$ in the free group $F[M_{T(\Pi)} \cup X]$ such that

$$\mu^{\alpha_\Pi} = \mathcal{M}_\mu\left(M_{T(\Pi)}^{\alpha_{T(\Pi)}}, X^{\phi_{K(r+1)}}\right)^{\beta},$$

where α_Π and $\alpha_{T(\Pi)}$ are the corresponding solutions of Π and $T(\Pi)$ relative to β.

Now, going along the sequence (108) from $\Pi_{K(r+1)}$ back to the cut equation Π_L and using repeatedly the remark above for each $\mu \in M_L$ we obtain a word

$$\mathcal{M}'_{\mu,L}\big(M_{K(r+1)}, X^{\phi_{K(r+1)}}\big) = \mathcal{M}'_{\mu,L}\big(M_{\text{useless}}, M_{\text{free}}, M_{\text{veryshort}}, X^{\phi_{K(r+1)}}\big)$$

such that

$$\mu^{\alpha_L} = \mathcal{M}'_{\mu,L}\big(M_{K(r+1)}^{\alpha_{K(r+1)}}, X^{\phi_{K(r+1)}}\big)^{\beta}.$$

Let $\delta = \phi_{K(r+1)} \in \mathcal{B}$ and put

$$\mathcal{M}_{\mu,L}\big(X^{\phi_{K(r+1)}}\big) = \mathcal{M}'_{\mu,L}\big(V_\mu\big(X^{\phi_{K(r+1)}}, M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_{\text{short}}}\big), M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_{\text{short}}}, X^{\phi_{K(r+1)}}\big).$$

Then for every $\mu \in M_L$

$$\mu^{\alpha_L} = \mathcal{M}_{\mu,L}\big(X^{\phi_{K(r+1)}}\big)^{\beta}.$$

If we denote by $\mathcal{M}_L(X)$ a tuple of words

$$\mathcal{M}_L(X) = \big(\mathcal{M}_{\mu_1,L}(X), \ldots, \mathcal{M}_{\mu_{|M_L|},L}(X)\big),$$

where $\mu_1, \ldots, \mu_{|M_L|}$ is some fixed ordering of M_L then

$$M_L^{\alpha_L} = \mathcal{M}_L\big(X^{\phi_{K(r+1)}}\big)^{\beta}.$$

Observe, that the words $\mathcal{M}_{\mu,L}(X)$, hence $\mathcal{M}_L(X)$ (where $X^{\phi_{K(r+1)}}$ is replaced by X) are the same for every $\phi_L \in \mathcal{B}_{p,q}$.

It follows from property (c) of the cut equation Π_ϕ that the solution α_L of Π_ϕ with respect to β gives rise to a group solution of the original cut equation $\Pi_{\mathcal{L}}$ with respect to $\phi_L \circ \beta$.

Now, property (c) of the initial cut equation $\Pi_{\mathcal{L}} = (\mathcal{E}, f_X, f_{M_L})$ insures that for every $\phi_L \in \mathcal{B}_{p,q}$ the pair $(U_{\phi_L\beta}, V_{\phi_L\beta})$ defined by

$$U_{\phi_L\beta} = Q\big(M_L^{\alpha_L}\big) = Q\big(\mathcal{M}_L\big(X^{\phi_{K(r+1)}}\big)\big)^{\beta},$$
$$V_{\phi_L\beta} = P\big(M_L^{\alpha_L}\big) = P\big(\mathcal{M}_L\big(X^{\phi_{K(r+1)}}\big)\big)^{\beta}$$

is a solution of the system $S(X) = 1 \wedge T(X, Y) = 1$.

We claim that

$$Y(X) = P\big(\mathcal{M}_L(X)\big)$$

is a solution of the equation $T(X, Y) = 1$ in $F_{R(S)}$. By Theorem 10 $\mathcal{B}_{p,q,\beta}$ is a discriminating family of solutions for the group $F_{R(S)}$. Since

$$T\big(X, Y(X)\big)^{\phi\beta} = T\big(X^{\phi\beta}, Y\big(X^{\phi\beta}\big)\big) = T\big(X^{\phi\beta}, \mathcal{M}_L\big(X^{\phi\beta}\big)\big) = T(U_{\phi_L\beta}, V_{\phi_L\beta}) = 1$$

for any $\phi\beta \in \mathcal{B}_{p,q,\beta}$ we deduce that $T(X, Y_{p,q}(X)) = 1$ in $F_{R(S)}$.

Now we need to show that $T(X,Y)=1$ admits a complete S-lift. Let $W(X,Y)\neq 1$ be an inequality such that $T(X,Y)=1\wedge W(X,Y)\neq 1$ is compatible with $S(X)=1$. In this event, one may assume (repeating the argument from the beginning of this section) that the set

$$\Lambda=\left\{(U_\psi,V_\psi)\;\middle|\;\psi\in\mathcal{L}_2\right\}$$

is such that every pair $(U_\psi,V_\psi)\in\Lambda$ satisfies the formula $T(X,Y)=1\wedge W(X,Y)\neq 1$. In this case, $W(X,Y_{p,q}(X))\neq 1$ in $F_{R(S)}$, because its image in F is non-trivial:

$$W\bigl(X,Y_{p,q}(X)\bigr)^{\phi\beta}=W(U_\psi,V_\psi)\neq 1.$$

Hence $T(X,Y)=1$ admits a complete lift into generic point of $S(X)=1$.

Case 2. A similar argument applies when $\mathrm{Comp}(\Pi_{K(r+2)})=0$. Indeed, in this case for every $\sigma\in\mathcal{E}_{K(r+2)}$ the word $f_{M_{K(r+1)}}(\sigma)$ has length one, so $f_{M_{K(r+1)}}(\sigma)=\mu$ for some $\mu\in M_{K(r+2)}$. Now one can replace the word $V_\mu\in F[X\cup M_{\mathrm{free}}\cup M_{\mathrm{veryshort}}]$ by the label $f_{X_{K(r+1)}}(\sigma)$ where $f_{M_{K(r+1)}}(\sigma)=\mu$ and then repeat the argument. □

7.5. Non-orientable quadratic equations

Consider now the equation

$$\prod_{i=1}^{m} z_i^{-1}c_i z_i\prod_{i=1}^{n}x_i^2=c_1\dots c_m\prod_{i=1}^{n}a_i^2, \tag{112}$$

where a_i, c_j give a solution in general position (in all the cases when it exists). We will now prove Theorem 9 for a regular standard non-orientable quadratic equation over F.

Let $S(X,A)=1$ be a regular standard non-orientable quadratic equation over F. Then every equation $T(X,Y,A)=1$ compatible with $S(X,A)=1$ admits an complete S-lift.

The proof of the theorem is similar to the proof in the orientable case, but the basic sequence of automorphisms is different. We will give a sketch of the proof in this section.

It is more convenient to consider a non-orientable equation in the form

$$S=\prod_{i=1}^{m} z_i^{-1}c_i z_i\prod_{i=1}^{n}[x_i,y_i]x_{n+1}^2=c_1\dots c_m\prod_{i=1}^{n}[a_i,b_i]a_{n+1}^2, \tag{113}$$

or

$$S=\prod_{i=1}^{m} z_i^{-1}c_i z_i\prod_{i=1}^{n}[x_i,y_i]x_{n+1}^2x_{n+2}^2=c_1\dots c_m\prod_{i=1}^{n}[a_i,b_i]a_{n+1}^2a_{n+2}^2. \tag{114}$$

Without loss of generality we consider Eq. (114). We define a basic sequence

$$\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{K(m,n)})$$

of G-automorphisms of the free G-group $G[X]$ fixing the left side of Eq. (114).

We assume that each $\gamma \in \Gamma$ acts identically on all the generators from X that are not mentioned in the description of γ.

Automorphisms γ_i, $i = 1, \ldots, m + 4n - 1$, are the same as in the orientable case.

Let $n = 0$. In this case $K = K(m, 0) = m + 2$. Put

$$\gamma_m : z_m \to z_m\left(c_m^{z_m} x_1^2\right),\ x_1 \to x_1^{(c_m^{z_m} x_1^2)},$$
$$\gamma_{m+1} : x_1 \to x_1(x_1x_2),\ x_2 \to (x_1x_2)^{-1}x_2,$$
$$\gamma_{m+2} : x_1 \to x_1^{(x_1^2x_2^2)},\ x_2 \to x_2^{(x_1^2x_2^2)}.$$

Let $n \geqslant 1$. In this case $K = K(m, n) = m + 4n + 2$. Put

$$\gamma_{m+4n} : x_n \to (y_n x_{n+1}^2)^{-1} x_n,\ y_n \to y_n^{(y_n x_{n+1}^2)},\ x_{n+1} \to x_{n+1}^{(y_n x_{n+1}^2)},$$
$$\gamma_{m+4n+1} : x_{n+1} \to x_{n+1}(x_{n+1}x_{n+2}),\ x_{n+2} \to (x_{n+1}x_{n+2})^{-1}x_{n+2},$$
$$\gamma_{m+4n+2} : x_{n+1} \to x_{n+1}^{(x_{n+1}^2 x_{n+2}^2)},\ x_{n+2} \to x_{n+2}^{(x_{n+1}^2 x_{n+2}^2)}.$$

These automorphisms induce automorphisms on G_S which we denote by the same letters.

Let $\Gamma = (\gamma_1, \ldots, \gamma_K)$ be the basic sequence of automorphisms for $S = 1$. Denote by Γ_∞ the infinite periodic sequence with period Γ, i.e., $\Gamma_\infty = \{\gamma_i\}_{i \geqslant 1}$ with $\gamma_{i+K} = \gamma_i$. For $j \in \mathbf{N}$ denote by Γ_j the initial segment of Γ_∞ of length j. Then for a given j and $p \in \mathbf{N}^j$ put

$$\phi_{j,p} = \overleftarrow{\Gamma}_j^{\overleftarrow{p}}.$$

Let

$$\Gamma_P = \{\phi_{j,p} \mid p \in P\}.$$

We can prove the analogue of Theorem 10, namely, that a family of homomorphisms $\Gamma_P \beta$ from $G_S = G_{R(S)}$ onto G, where β is a solution in general position, and P is unbounded, is a discriminating family.

The rest of the proof is the same as in the orientable case.

7.6. *Implicit function theorem: NTQ systems*

Definition 43. Let G be a group with a generating set A. A system of equations $S = 1$ is called *triangular quasiquadratic* (shortly, TQ) if it can be partitioned into the following subsystems

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1, \\ S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1 \end{aligned}$$

where for each i one of the following holds:

(1) S_i is quadratic in variables X_i;
(2) $S_i = \{[y, z] = 1, [y, u] = 1 \mid y, z \in X_i\}$ where u is a group word in $X_{i+1} \cup \cdots \cup X_n \cup A$ such that its canonical image in G_{i+1} defined below is not a proper power; in this case we say that $S_i = 1$ corresponds to an extension of a centralizer;
(3) $S_i = \{[y, z] = 1 \mid y, z \in X_i\}$;
(4) S_i is the empty equation.

Define $G_i = G_{R(S_i,\ldots,S_n)}$ for $i = 1, \ldots, n$ and put $G_{n+1} = G$. The TQ system $S = 1$ is called *non-degenerate* (shortly, NTQ) if each system $S_i = 1$, where $X_{i+1}, \ldots, X_n$ are viewed as the corresponding constants from G_{i+1} (under the canonical maps $X_j \to G_{i+1}$, $j = i + 1, \ldots, n$, has a solution in G_{i+1}. The coordinate group of an NTQ system is called an *NTQ group*.

An NTQ system $S = 1$ is called *regular* if for each i the system $S_i = 1$ is either of the type (1) or (4), and in the former case the quadratic equation $S_i = 1$ is in standard form and regular (see Definition 6).

One of the results to be proved in this section is the following.

Theorem 11. *Let $U(X, A) = 1$ be a regular NTQ-system. Every equation $V(X, Y, A) = 1$ compatible with $U = 1$ admits a complete U-lift.*

Proof. We use induction on the number n of levels in the system $U = 1$. We construct a solution tree $T_{\text{sol}}(V(X, Y, A) \wedge U(X, Y))$ with parameters $X = X_1 \cup \cdots \cup X_n$. In the terminal vertices of the tree there are generalized equations $\Omega_{v_1}, \ldots, \Omega_{v_k}$ which are equivalent to cut equations $\Pi_{v_1}, \ldots, \Pi_{v_k}$.

If $S_1(X_1, \ldots, X_n) = 1$ is an empty equation, we can take Merzljakov's words (see Theorem 4) as values of variables from X_1, express Y as functions in X_1 and a solution of some $W(Y_1, X_2, \ldots, X_n) = 1$ such that for any solution of the system

$$\begin{aligned} S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1 \end{aligned}$$

equation $W = 1$ has a solution.

Suppose, now that $S_1(X_1, \ldots, X_n) = 1$ is a regular quadratic equation. Let Γ be a basic sequence of automorphisms for the equation $S_1(X_1, \ldots, X_n, A) = 1$. Recall that

$$\phi_{j,p} = \gamma_j^{p_j} \ldots \gamma_1^{p_1} = \overleftarrow{\Gamma}_j^p,$$

where $j \in \mathbb{N}$, $\Gamma_j = (\gamma_1, \ldots, \gamma_j)$ is the initial subsequence of length j of the sequence $\Gamma^{(\infty)}$, and $p = (p_1, \ldots, p_j) \in \mathbb{N}^j$. Denote by $\psi_{j,p}$ the following solution of $S_1(X_1) = 1$:

$$\psi_{j,p} = \phi_{j,p}\alpha,$$

where α is a composition of a solution of $S_1 = 1$ in G_2 and a solution from a generic family of solutions of the system

$$\begin{aligned} S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1 \end{aligned}$$

in $F(A)$. We can always suppose that α satisfies a small cancellation condition with respect to Γ.

Set

$$\Phi = \left\{\phi_{j,p} \mid j \in \mathbb{N},\ p \in \mathbb{N}^j\right\}$$

and let $\mathcal{L}^\alpha$ be an infinite subset of Φ^α satisfying one of the cut equations above. Without loss of generality we can suppose it satisfies Π_1. By Proposition 8 we can express variables from Y as functions of the set of Γ-words in X_1, coefficients, variables M_{free} and variables $M_{\text{veryshort}}$, satisfying the system of equations $\Delta(M_{\text{veryshort}})$ The system $\Delta(M_{\text{veryshort}})$ can be turned into a generalized equation with parameters $X_2 \cup \cdots \cup X_n$, such that for any solution of the system

$$\begin{aligned} S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1 \end{aligned}$$

the system $\Delta(M_{\text{veryshort}})$ has a solution. Therefore, by induction, variables $(M_{\text{veryshort}})$ can be found as elements of G_2, and variables Y as elements of G_1. $\square$

Lemma 74. *All stabilizing automorphisms* (*see* [9]) *of the left side of the equation*

$$c_1^{z_1} c_2^{z_2} (c_1 c_2)^{-1} = 1 \tag{115}$$

have the form $z_1^\phi = c_1^k z_1(c_1^{z_1}c_2^{z_2})^n$, $z_2^\phi = c_2^m z_2(c_1^{z_1}c_2^{z_2})^n$. *All stabilizing automorphisms of the left side of the equation*

$$x^2c^z\left(a^2c\right)^{-1} = 1 \tag{116}$$

have the form $x^\phi = x^{(x^2c^z)^n}$, $z^\phi = c^k z(x^2c^z)^n$. *All stabilizing automorphisms of the left side of the equation*

$$x_1^2x_2^2\left(a_1^2a_2^2\right)^{-1} = 1 \tag{117}$$

have the form $x_1^\phi = (x_1(x_1x_2)^m)^{(x_1^2x_2^2)^n}$, $x_2^\phi = ((x_1x_2)^{-m}x_2)^{(x_1^2x_2^2)^n}$.

Proof. The computation of the automorphisms can be done by utilizing the Magnus software system. □

If a quadratic equation $S(X) = 1$ has only commutative solutions then the radical $R(S)$ of $S(X)$ can be described (up to a linear change of variables) as follows (see [12]):

$$\mathrm{Rad}(S) = \mathrm{ncl}\left\{[x_i, x_j], [x_i, b] \mid i, j = 1, \dots, k\right\},$$

where b is an element (perhaps, trivial) from F. Observe, that if b is not trivial then b is not a proper power in F. This shows that $S(X) = 1$ is equivalent to the system

$$U_{\mathrm{com}}(X) = \left\{[x_i, x_j] = 1, [x_i, b] = 1 \mid i, j = 1, \dots, k\right\}. \tag{118}$$

The system $U_{\mathrm{com}}(X) = 1$ is equivalent to a single equation, which we also denote by $U_{\mathrm{com}}(X) = 1$. The coordinate group $H = F_{R(U_{\mathrm{com}})}$ of the system $U_{\mathrm{com}} = 1$, as well as of the corresponding equation, is F-isomorphic to the free extension of the centralizer $C_F(b)$ of rank n. We need the following notation to deal with H. For a set X and $b \in F$ by $A(X)$ and $A(X, b)$ we denote free abelian groups with basis X and $X \cup \{b\}$, correspondingly. Now, $H \simeq F *_{b=b} A(X, b)$. In particular, in the case when $b = 1$ we have $H = F * A(X)$.

Lemma 75. *Let* $F = F(A)$ *be a non-abelian free group and* $V(X, Y, A) = 1$, $W(X, Y, A) = 1$ *be equations over* F. *If a formula*

$$\Phi = \forall X\left(U_{\mathrm{com}}(X) = 1 \;\to\; \exists Y\left(V(X, Y, A) = 1 \;\wedge\; W(X, Y, A) \neq 1\right)\right)$$

is true in F *then there exists a finite number of* $\langle F\rangle$*-embeddings* $\phi_k : F *_{b=b} A(X, b) \to F *_{b=b} A(X, b)$ $(k \in K)$ *such that*:

(1) *every formula*

$$\Phi_k = \exists Y\left(V\left(X^{\phi_k}, Y, A\right) = 1 \;\wedge\; W\left(X^{\phi_k}, Y, A\right) \neq 1\right)$$

holds in the coordinate group $H = F *_{b=b} A(X, b)$;

(2) *for any solution* $\lambda : H \to F$ *there exists an* F*-homomorphism* $\lambda^* : H \to F$ *such that* $\lambda = \phi_k \lambda^*$ *for some* $k \in K$.

Proof. We construct a set of initial parameterized generalized equations $\mathcal{G}E(S) = \{\Omega_1, \ldots, \Omega_r\}$ for $V(X, Y, A) = 1$ with respect to the set of parameters X. For each $\Omega \in \mathcal{G}E(S)$ in Section 5.6 we constructed the finite tree $T_{\rm sol}(\Omega)$ with respect to parameters X. Observe, that non-active part $[j_{v_0}, \rho_{v_0}]$ in the root equation $\Omega = \Omega_{v_0}$ of the tree $T_{\rm sol}(\Omega)$ is partitioned into a disjoint union of closed sections corresponding to X-bases and constant bases (this follows from the construction of the initial equations in the set $\mathcal{G}E(S)$). We label every closed section σ corresponding to a variable $x \in X^{\pm 1}$ by x, and every constant section corresponding to a constant a by a. Due to our construction of the tree $T_{\rm sol}(\Omega)$ moving along a brunch B from the initial vertex v_0 to a terminal vertex v we transfer all the bases from the non-parametric part into parametric part until, eventually, in Ω_v the whole interval consists of the parametric part. For a terminal vertex v in $T_{\rm sol}(\Omega)$ equation Ω_v is periodized (see Section 5.4). We can consider the correspondent periodic structure $\mathcal{P}$ and the subgroup $\tilde{Z}_2$. Denote the cycles generating this subgroup by $z_1, \ldots, z_m$. Let $x_i = b^{k_i}$ and $z_i = b^{s_i}$. All x_i's are cycles, therefore the corresponding system of equations can be written as a system of linear equations with integer coefficients in variables $\{k_1, \ldots, k_n\}$ and variables $\{s_1, \ldots, s_m\}$:

$$k_i = \sum_{j=1}^{m} \alpha_{ij} s_j + \beta_i, \quad i = 1, \ldots, n. \tag{119}$$

We can always suppose $m \leqslant n$ and at least for one equation Ω_v $m = n$, because otherwise the solution set of the irreducible system $U_{\rm com} = 1$ would be represented as a union of a finite number of proper subvarieties.

We will show now that all the tuples $(k_1, \ldots, k_n)$ that correspond to some system (119) with $m < n$ (the dimension of the subgroup H_v generated by $\bar{k} - \bar{\beta} = k_1 - \beta_1, \ldots, k_n - \beta_n$ in this case is less than n), appear also in the union of systems (119) with $m = n$. Such systems have form $\bar{k} - \bar{\beta}_q \in H_q$, q runs through some finite set Q, and where H_q is a subgroup of finite index in $Z^n = \langle s_1 \rangle \times \cdots \times \langle s_n \rangle$. We use induction on n. If for some terminal vertex v, the system (119) has $m < n$, we can suppose without loss of generality that the set of tuples H satisfying this system is defined by the equations $k_r = \cdots = k_n = 0$. Consider just the case $k_n = 0$. We will show that all the tuples $\bar{k}_0 = (k_1, \ldots, k_{n-1}, 0)$ appear in the systems (119) constructed for the other terminal vertices with $n = m$. First, if N_q is the index of the subgroup H_q, $N_q \bar{k} \in H_q$ for each tuple $\bar{k}$. Let N be the least common multiple of $N_1, \ldots, N_Q$. If a tuple $(k_1, \ldots, k_{n-1}, tN)$ for some t belongs to $\bar{\beta}_q + H_q$ for some q, then $(k_1, \ldots, k_{n-1}, 0) \in \bar{\beta}_q + H_q$, because $(0, \ldots, 0, tN) \in H_q$. Consider the set K of all tuples $(k_1, \ldots, k_{n-1}, 0)$ such that $(k_1, \ldots, k_{n-1}, tN) \notin \bar{\beta}_q + H_q$ for any $q = 1, \ldots, Q$ and $t \in \mathbb{Z}$. The set $\{(k_1, \ldots, k_{n-1}, tN) \mid (k_1, \ldots, k_{n-1}, 0) \in K, t \in \mathbb{Z}\}$ cannot be a discriminating set for $U_{\rm com} = 1$. Therefore it satisfies some proper equation. Changing variables $k_1, \ldots, k_{n-1}$ we can suppose that for an irreducible component the equation has form $k_{n-1} = 0$. The contradiction arises from the fact that we cannot obtain a discriminating set for $U_{\rm com} = 1$ which does not belong to $\bar{\beta}_q + H_q$ for any $q = 1, \ldots, Q$.

Embeddings ϕ_k are given by the systems (119) with $n = m$ for generalized equations Ω_v for all terminal vertices v. $\square$

There are two more important generalizations of the implicit function theorem, one—for arbitrary NTQ-systems, and another—for arbitrary systems. We need a few more definitions to explain this. Let $U(X_1, \ldots, X_n, A) = 1$ be an NTQ-system:

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1, \\ S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1 \end{aligned}$$

and $G_i = G_{R(S_i, \ldots, S_n)}$, $G_{n+1} = F(A)$.

A G_{i+1}-automorphism σ of G_i is called a *canonical automorphism* if the following holds:

(1) if S_i is quadratic in variables X_i then σ is induced by a G_{i+1}-automorphism of the group $G_{i+1}[X_i]$ which fixes S_i;
(2) if $S_i = \{[y, z] = 1, [y, u] = 1 \mid y, z \in X_i\}$ where u is a group word in $X_{i+1} \cup \cdots \cup X_n \cup A$, then $G_i = G_{i+1} *_{u=u} \mathrm{Ab}(X_i \cup \{u\})$, where $\mathrm{Ab}(X_i \cup \{u\})$ is a free abelian group with basis $X_i \cup \{u\}$, and in this event σ extends an automorphism of $\mathrm{Ab}(X_i \cup \{u\})$ (which fixes u);
(3) if $S_i = \{[y, z] = 1 \mid y, z \in X_i\}$ then $G_i = G_{i+1} * \mathrm{Ab}(X_i)$, and in this event σ extends an automorphism of $\mathrm{Ab}(X_i)$;
(4) if S_i is the empty equation then $G_i = G_{i+1}[X_i]$, and in this case σ is just the identity automorphism of G_i.

Let π_i be a fixed $G_{i+1}[Y_i]$-homomorphism

$$\pi_i : G_i[Y_i] \to G_{i+1}[Y_{i+1}],$$

where $\emptyset = Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_n \subseteq Y_{n+1}$ is an ascending chain of finite sets of parameters, and $G_{n+1} = F(A)$. Since the system $U = 1$ is non-degenerate such homomorphisms π_i exist. We assume also that if $S_i(X_i) = 1$ is a standard quadratic equation (the case (1)) above) which has a non-commutative solution in G_{i+1}, then X^{π_i} is a non-commutative solution of $S_i(X_i) = 1$ in $G_{i+1}[Y_{i+1}]$.

A *fundamental sequence* (or a *fundamental set*) of solutions of the system $U(X_1, \ldots, X_n, A) = 1$ in $F(A)$ with respect to the fixed homomorphisms $\pi_1, \ldots, \pi_n$ is a set of all solutions of $U = 1$ in $F(A)$ of the form

$$\sigma_1 \pi_1 \cdots \sigma_n \pi_n \tau,$$

where σ_i is Y_i-automorphism of $G_i[Y_i]$ induced by a canonical automorphism of G_i, and τ is an $F(A)$-homomorphism $\tau : F(A \cup Y_{n+1}) \to F(A)$. Solutions from a given fundamental set of U are called *fundamental* solutions.

Below we describe two useful constructions. The first one is a *normalization* construction which allows one to rewrite effectively an NTQ-system $U(X) = 1$ into a normalized NTQ-system $U^* = 1$. Suppose we have an NTQ-system $U(X) = 1$ together with a fundamental sequence of solutions which we denote $\bar{V}(U)$.

Starting from the bottom we replace each non-regular quadratic equation $S_i = 1$ which has a non-commutative solution by a system of equations effectively constructed as follows.

(1) If $S_i = 1$ is in the form

$$c_1^{x_{i1}} c_2^{x_{i2}} = c_1 c_2,$$

where $[c_1, c_2] \neq 1$, then we replace it by a system

$$\{x_{i1} = z_1 c_1 z_3, x_{i2} = z_2 c_2 z_3, [z_1, c_1] = 1, [z_2, c_2] = 1, [z_3, c_1 c_2] = 1\}.$$

(2) If $S_i = 1$ is in the form

$$x_{i1}^2 c^{x_{i2}} = a^2 c,$$

where $[a, c] \neq 1$, we replace it by a system

$$\{x_{i1} = a^{z_1}, x_{i2} = z_2 c z_1, [z_2, c] = 1, [z_1, a^2 c] = 1\}.$$

(3) If $S_i = 1$ is in the form

$$x_{i1}^2 x_{i2}^2 = a_1^2 a_2^2$$

then we replace it by the system

$$\{x_{i1} = (a_1 z_1)^{z_2}, x_{i2} = (z_1^{-1} a_2)^{z_2}, [z_1, a_1 a_2] = 1, [z_2, a_1^2 a_2^2] = 1\}.$$

The normalization construction effectively provides an NTQ-system $U^* = 1$ such that each fundamental can be obtained from a solution of $U^* = 1$. We refer to this system as to the normalized system of U corresponding to the fundamental sequence. Similarly, the coordinate group of the normalized system is called the *normalized* coordinate group of $U = 1$.

Lemma 76. *Let $U(X) = 1$ be an NTQ-system, and $U^* = 1$ be the normalized system corresponding to the fundamental sequence $\bar{V}(U)$. Then the following holds*:

(1) *The coordinate group $F_{R(U)}$ canonically embeds into $F_{R(U^*)}$*;

(2) *The system* $U^* = 1$ *is an NTQ-system of the type*

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1, \\ S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1 \end{aligned}$$

in which every $S_i = 1$ *is either a regular quadratic equation or an empty equation or a system of the type*

$$U_{\mathrm{com}}(X, b) = \{[x_i, x_j] = 1, [x_i, b] = 1 \mid i, j = 1, \ldots, k\}$$

where $b \in G_{i+1}$.

(3) *Every solution* X_0 *of* $U(X) = 1$ *that belongs to the fundamental sequence* $\bar{V}(U)$ *can be obtained from a solution of the system* $U^* = 1$.

Proof. Statement (1) follows from the normal forms of elements in free constructions or from the fact that applying standard automorphisms ϕ_L to a non-commuting solution (in particular, to a basic one) one obtains a discriminating set of solutions (see Section 7.2). Statements (2) and (3) are obvious from the normalization construction. □

Definition 44. A family of solutions Ψ of a regular NTQ-system $U(X, A) = 1$ is called *generic* if for any equation $V(X, Y, A) = 1$ the following is true: if for any solution from Ψ there exists a solution of $V(X^\psi, Y, A) = 1$, then $V = 1$ admits a complete U-lift.

A family of solutions Θ of a regular quadratic equation $S(X) = 1$ over a group G is called *generic* if for any equation $V(X, Y, A) = 1$ with coefficients in G the following is true: if for any solution $\theta \in \Theta$ there exists a solution of $V(X^\theta, Y, A) = 1$ in G, then $V = 1$ admits a complete S-lift.

A family of solutions Ψ of an NTQ-system $U(X, A) = 1$ is called *generic* if $\Psi = \Psi_1 \ldots \Psi_n$, where Ψ_i is a generic family of solutions of $S_i = 1$ over G_{i+1} if $S_i = 1$ is a regular quadratic system, and Ψ_i is a discriminating family for $S_i = 1$ if it is a system of the type U_{com}.

The second construction is a *correcting extension of centralizers* of a normalized NTQ-system $U(X) = 1$ relative to an equation $W(X, Y, A) = 1$, where Y is a tuple of new variables. Let $U(X) = 1$ be an NTQ-system in the normalized form:

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1, \\ S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1. \end{aligned}$$

So every $S_i = 1$ is either a regular quadratic equation or an empty equation or a system of the type

$$U_{\mathrm{com}}(X,b) = \bigl\{[x_i, x_j] = 1, [x_i, b] = 1 \bigm| i, j = 1, \dots, k\bigr\}$$

where $b \in G_{i+1}$. Again, starting from the bottom we find the first equation $S_i(X_i) = 1$ which is in the form $U_{\mathrm{com}}(X) = 1$ and replace it with a new centralizer extending system $\bar{U}_{\mathrm{com}}(X) = 1$ as follows.

We construct T_{sol} for the system $W(X,Y) = 1 \wedge U(X) = 1$ with parameters $X_i, \dots, X_n$. We obtain generalized equations corresponding to final vertices. Each of them consists of a periodic structure on X_i and generalized equation on $X_{i+1} \dots X_n$. We can suppose that for the periodic structure the set of cycles $C^{(2)}$ is empty. Some of the generalized equations have a solution over the extension of the group G_i. This extension is given by the relations $\bar{U}_{\mathrm{com}}(X_i) = 1$, $S_{i+1}(X_{i+1}, \dots, X_n) = 1, \dots, S_n(X_n) = 1$, so that there is an embedding $\phi_k : A(X,b) \to A(X,b)$. The others provide a proper (abelian) equation $E_j(X_i) = 1$ on X_i. The argument above shows that replacing each centralizer extending system $S_i(X_i) = 1$ which is in the form $U_{\mathrm{com}}(X_i) = 1$ by a new system of the type $\bar{U}_{\mathrm{com}}(X_i) = 1$ we eventually rewrite the system $U(X) = 1$ into finitely many new ones $\bar{U}_1(X) = 1, \dots, \bar{U}_m(X) = 1$. We denote this set of NTQ-systems by $\mathcal{C}_W(U)$. For every NTQ-system $\bar{U}_m(X) = 1 \in \mathcal{C}_W(U)$ the embeddings ϕ_k described above give rise to embeddings $\bar{\phi} : F_{R(U)} \to F_{R(\bar{U})}$. Finally, combining normalization and correcting extension of centralizers (relative to $W = 1$) starting with an NTQ-system $U = 1$ and a fundamental sequence of its solutions $\bar{V}(U)$ we can obtain a finite set

$$\mathcal{N}\mathcal{C}_W(U) = \mathcal{C}_W\bigl(U^*\bigr)$$

which comes equipped with a finite set of embeddings $\theta_i : F_{R(U)} \to F_{R(\bar{U}_i)}$ for each $\bar{U}_i \in \mathcal{N}\mathcal{C}_W(U)$. These embeddings are called *correcting normalizing embeddings*. The construction implies the following result.

Theorem 12 (Parametrization theorem). *Let $U(X,A) = 1$ be an NTQ-system with a fundamental sequence of solutions $\bar{V}(U)$. Suppose a formula*

$$\Phi = \forall X\bigl(U(X) = 1 \ \to\ \exists Y\bigl(W(X,Y,A) = 1 \ \wedge\ W_1(X,Y,A) \neq 1\bigr)\bigr)$$

is true in F. Then for every $\bar{U}_i \in \mathcal{N}\mathcal{C}_W(U)$ the formula

$$\exists Y\bigl(W\bigl(X^{\theta_i}, Y, A\bigr) = 1 \ \wedge\ W_1\bigl(X^{\theta_i}, Y, A\bigr) \neq 1\bigr)$$

is true in the group $G_{R(\bar{U}_i)}$ for every correcting normalizing embedding $\theta_i : F_{R(U)} \to F_{R(\bar{U}_i)}$. This formula can be effectively verified and solution Y can be effectively found.

Furthermore, for every fundamental solution $\phi : F_{R(U)} \to F$ there exists a fundamental solution ψ of one of the systems $\bar{U}_i = 1$, where $\bar{U}_i \in \mathcal{N}\mathcal{C}_W(U)$ such that $\phi = \theta_i \psi$.

As a corollary of this theorem and results from Section 5 we obtain the following theorems.

Theorem 13. *Let $U(X,A)=1$ be an NTQ-system and $\bar{V}(U)$ a fundamental set of solutions of $U=1$ in $F=F(A)$. If a formula*

$$\Phi = \forall X\big(U(X)=1 \ \rightarrow \ \exists Y\big(W(X,Y,A)=1 \ \wedge \ W_1(X,Y,A)\neq 1\big)\big)$$

is true in F then one can effectively find finitely many NTQ systems $U_1=1,\dots,U_k=1$ and embeddings $\theta_i: F_{R(U)} \rightarrow F_{R(U_i)}$ such that the formula

$$\exists Y\big(W\big(X^{\theta_i},Y,A\big)=1 \ \wedge \ W_1\big(X^{\theta_i},Y,A\big)\neq 1\big)$$

is true in each group $F_{R(U_i)}$. Furthermore, for every solution $\phi: F_{R(U)} \rightarrow F$ of $U=1$ from $\bar{V}(U)$ there exists $i\in\{1,\dots,k\}$ and a fundamental solution $\psi: F_{R(U_i)} \rightarrow F$ such that $\phi=\theta_i\psi$.

Theorem 14. *Let $S(X)=1$ be an arbitrary system of equations over F. If a formula*

$$\Phi = \forall X\exists Y\big(S(X)=1 \ \rightarrow \ \big(W(X,Y,A)=1 \ \wedge \ W_1(X,Y,A)\neq 1\big)\big)$$

is true in F then one can effectively find finitely many NTQ systems $U_1=1,\dots,U_k=1$ and F-homomorphisms $\theta_i: F_{R(S)} \rightarrow F_{R(U_i)}$ such that the formula

$$\exists Y\big(W\big(X^{\theta_i},Y,A\big)=1 \ \wedge \ W_1\big(X^{\theta_i},Y,A\big)\neq 1\big)$$

is true in each group $F_{R(U_i)}$. Furthermore, for every solution $\phi: F_{R(S)} \rightarrow F$ of $S=1$ there exists $i\in\{1,\dots,k\}$ and a fundamental solution $\psi: F_{R(U_i)} \rightarrow F$ such that $\phi=\theta_i\psi$.

Acknowledgment

We thank Igor Lysenok who carefully read the manuscript and whose numerous remarks and suggestions have substantially improved the exposition.

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