Maximum Principles for Elliptic Systems

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Submitted by Murray H. Protter

Received October 2, 1989

1. INTRODUCTION

Positive definite solutions $B$ of the matrix equation $C^*B + BC = -E$ ($E > 0$) have been successfully used to construct Liapunov functions, and then to prove the stability of some ordinary differential systems $du/dx = Cu$ (cf. [11, 24]). This method usually is called Liapunov's Second Method. In 1974, Chow and Dunninger [2] applied this method to the study of $n$-metaharmonic functions, and obtained a generalized maximum principle for some classes of $n$-metaharmonic functions.

In this paper, we transfer the idea of Liapunov's second method to the study of weakly coupled second-order elliptic systems

$$Lu + Cu = 0 \text{ or } f \quad \text{in } D \subset \mathbb{R}^n.$$ 

Here

$$L = a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + a_i(x) \frac{\partial}{\partial x_i}, \quad a_{ij} = a_{ji}$$

is a second-order real elliptic operator, and $f = (f_1, ..., f_m)^T$, $u = (u_1, ..., u_m)^T$, and the $m \times m$ matrix $C$ all have entries which are complex-valued functions.

We establish the following generalized maximum principle for a certain class of the homogeneous system (Theorem 1):

If there exists a complex constant matrix $B > 0$ such that $C^*(x)B + BC(x) \leq 0$ in $D$, then for all $C^2(D) \cap C(\overline{D})$ solutions $u$ of $Lu + Cu = 0$,

$$\|u\|_{0,D} \leq K \|u\|_{0,\partial D}.$$
Here $K = (\beta_m/\beta_1)^{1/2}$, where $\beta_1$ and $\beta_m$ are the smallest and biggest eigenvalues of $B$, respectively.

We also find a simple sufficient condition for the classical maximum principle ($K = 1$ in the above inequality) holding, which is $C^*(x) + C(x) \leq 0$ in $D$ (Theorem 3). These results extend the result of Winter and Wong [23] for $C$ being negative semidefinite to a more general form of $C$. Generalized maximum principles for weakly coupled second-order elliptic systems have also been obtained by Dow [3], Hile and Protter [8], Szeptycki [21], and Wasowski [22] under different conditions on the coefficients.

We further show how our maximum principles may be used to prove the uniqueness of various boundary value problems of some classes of elliptic systems over bounded or unbounded domain $D \subset \mathbb{R}^n$. By using a recent result of Hile and Yeh [10], we even obtain uniqueness for a boundary value problem with an exceptional boundary set $\Gamma \subset \partial D$ such that the Hausdorff dimension of $\Gamma$ is less than $n - 1$.

An estimate of the best possible $K$ in our maximum principle inequality is given when $C$ is a 2 by 2 real matrix with a double eigenvalue. The condition for the classical maximum principle ($K = 1$) holding can be written as

$$\text{Re}\{a\} \leq 0, \quad \text{Re}\{d\} \leq 0, \quad |b + \xi|^2 \leq 4\text{Re}\{a\} \text{Re}\{d\}$$

when $C = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$ is any 2 by 2 complex-valued matrix function.

We also study the nonhomogeneous system. Miranda [13] has studied the weakly coupled real elliptic system

$$Lu + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial k} + Cu = f \quad \text{in } D \subset \mathbb{R}^n,$$

giving a rather complicated condition which implies the bound for solutions $u$,

$$\|u\|_{0,D} \leq \|u\|_{0,\partial D} + c_0^{-1} \|f\|_{0,D} \quad (c_0 > 0).$$

When all the matrices $B_k = 0$, $k = 1, \ldots, n$, the condition required by Miranda reduces to $\xi^T C \xi \leq -c_0 |\xi|^2$ for any $\xi \in \mathbb{R}^m$. We will extend this result to $C$ being a complex matrix function such that $C^* + C \leq -2c_0 I$ (Corollary 11). Moreover, we prove the following (Theorem 9):

If there exist $B > 0$ and $E > 0$ such that $C^*(x) B + BC(x) \leq -E$ in $D$, then for all $C^2(D) \cap C(\bar{D})$ solutions $u$ of $Lu + Cu = f$,

$$\|u\|_{0,D} \leq K_1 \|u\|_{0,\partial D} + K_2 \|f\|_{0,D}.$$
Here $K_1 = (\beta_m/\beta_1)^{1/2}$ and $K_2 = (2/\mu_1)(\beta_m/\beta_1)^{1/2}$, where $\mu_1$ is the smallest eigenvalue of $EB^{-1}$.

Results for systems are later used to yield maximum principles and bounds for some higher order elliptic homogeneous and nonhomogeneous equations. Our maximum principles include those of Chow and Dunninger [2, 6] for real metaharmonic functions as a special case. Various maximum principles for higher order elliptic equations were also studied in the papers of Agmon [1], Duffin [4, 5], Fichera [7], Payne [14], Scheaffer and Walter [16, 17], and the books of Miranda [13] and Sperb [19].

2. Notation and a Liapunov Stability Theorem

Unless otherwise stated, all matrices considered in this paper will be over the complex field. Let $X$ be any $m \times n$ matrix. Its transpose, complex conjugate, and adjoint will be denoted by $X^T$, $\bar{X}$, and $X^*$ ($X^* = \bar{X}^T$), respectively. For the sake of brevity, both Hermitian positive definite and real symmetric positive definite matrices will be named positive. Similar abbreviations hold for semipositive, negative, and seminegative definite matrices. The notations $B > 0$, $B \geq 0$, $B < 0$, and $B \leq 0$ mean that the square matrix $B$ is positive, semipositive, negative, and seminegative, respectively. The norm $\| \cdot \|_{0,D}$ means the sup norm over $D$; thus for complex-valued vector functions $u = (u_1, u_2, \ldots, u_m)$,

$$\|u\|_{0,D} := \sup_{x \in D} |u(x)| = \sup_{x \in D} (|u_1(x)|^2 + \cdots + |u_m(x)|^2)^{1/2}.$$

The following well-known result in Liapunov stability theory will be applied several times in this paper. We state this result as a lemma.

Liapunov Lemma. Let $C$ be an $m \times m$ complex or real matrix.

(a) Assume that no eigenvalue of $C$ has positive real part, and moreover that the elementary divisors of $C$ corresponding to eigenvalues with vanishing real part are linear. Then there exist matrices $B > 0$ and $E \geq 0$ such that $C^*B + BC = -E$.

(b) If each eigenvalue of $C$ has negative real part, then for any $E > 0$, there exists a unique $B > 0$ such that $C^*B + BC = -E$.

The proof of this lemma, both for real and complex versions, can be found in many papers, such as [11,18,24].
3. Maximum Principles

Consider a second-order operator

\[ L = a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + a_i(x) \frac{\partial}{\partial x_i} \]

in a bounded domain \( D \) in \( \mathbb{R}^n \). Here the summation convention is employed. We assume that \( L \) is elliptic in \( D \); i.e., for all \( x \in D \) and all \( y = (y_1, y_2, \ldots, y_n) \) in \( \mathbb{R}^n \setminus \{0\} \) the inequality

\[ a_{ij}(x) y_i y_j > 0 \]

holds. We also suppose that the coefficients \( a_{ij} \) and \( a_i \) are bounded and real-valued functions in \( D \).

Now consider the following weakly coupled second-order elliptic system,

\[ L u_s(x) + c_{sk}(x) u_k(x) = 0, \quad s = 1, 2, \ldots, m, \quad \text{in } D, \]

or, in more brief matrix form,

\[ Lu(x) + C(x) u(x) = 0 \quad \text{in } D. \]

Here \( C(x) = (c_{sk}(x)) \) is an \( m \times m \) complex matrix function and \( u \) is a \( C^2 \) \( m \times 1 \) complex vector function. Associated with (3) is the following characteristic equation of \( C \),

\[ h(\lambda) \equiv |\lambda I - C| = 0. \]

**Theorem 1.** Assume that there exists a constant matrix \( B > 0 \) such that

\[ C^*(x) B + B C(x) \leq 0, \quad x \in D. \]

Then for all \( C^2(D) \cap C(\bar{D}) \) solutions \( u \) of (3), there exists a positive constant \( K \) such that

\[ \|u\|_{0, D} \leq K \|u\|_{0, \partial D}. \]

Here \( K = (\beta_m/\beta_1)^{1/2} \), where \( \beta_1 \) and \( \beta_m \) are the smallest and biggest eigenvalues of \( B \), respectively.

**Proof.** Define

\[ v = u^* B u = u \cdot B u = B u \cdot u = b_{ks} \bar{u}_k u_s, \]

where "\cdot" is the dot product in \( \mathbb{C}^n \) defined by \( x \cdot y = y^* x = \sum_{k=1}^{n} x_k \bar{y}_k. \)
Then \( v \) is a nonnegative function and,

\[
v_{,i} \equiv \frac{\partial v}{\partial x_i} = b_{k,s} u_{k,i} u_s + b_{k,s} \bar{u}_{k} u_{s,i},
\]

\[
v_{,ij} \equiv \frac{\partial^2 v}{\partial x_i \partial x_j} = b_{k,s} \bar{u}_{k,ij} u_s + b_{k,s} \bar{u}_{k} u_{s,ij} + 2 \text{Re} \{ b_{k,s} \bar{u}_{k,i} u_{s,j} \},
\]

where \( u_{k,i} = \partial u_k / \partial x_i \), \( u_{k,ij} = \partial^2 u_k / \partial x_i \partial x_j \), etc.; and,

\[
Lv = a_{ij} v_{,ij} + a_i v_{,i}
\]

\[
= b_{k,s} a_{ij} \bar{u}_{k,ij} u_s + b_{k,s} \bar{u}_{k} a_{ij} u_{s,ij} + 2 b_{k,s} a_{ij} \bar{u}_{k,i} u_{s,j} + b_{k,s} a_{ij} \bar{u}_{k,i} u_{s,j}
\]

\[
= b_{k,s} (L u_k) u_s + b_{k,s} \bar{u}_{k} (L u_s) + 2 a_{ij} u_{s,ij} + b_{k,s} u_{s,ij}.
\]

Thus,

\[
Lv = -u^* (C*B + BC) u + 2 a_{ij} B^{1/2} u_{s,ij} \cdot B^{1/2} u_{s,j} \geq 0,
\]

since \( C*B + BC \leq 0 \) and \( a_{ij} v_{i} \cdot v_{j} \geq 0 \) for any vectors \( v_1, v_2, ..., v_n \). Therefore, by the maximum principle for the elliptic operator \( L \), we have

\[
v(x) \leq \max_{y \in \partial D} v(y) \quad \text{for all} \quad x \in D.
\]

(8)

Suppose that \( \{ \beta_i \}_{i=1}^m \) are the eigenvalues of \( B \) with \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \). Since \( B > 0 \), we know that \( \beta_1 > 0 \) and

\[
\beta_1 |u(x)|^2 \leq v(x) = u(x)^* B u(x) \leq \beta_m |u(x)|^2.
\]

Hence, from (8),

\[
|u(x)|^2 \leq \frac{\beta_m}{\beta_1} \max_{y \in \partial D} |u(y)|^2,
\]

and consequently, \( \| u \|_{0,D} \leq K \| u \|_{0,\partial D} \), where \( K = (\beta_m / \beta_1)^{1/2} \).

The Lyapunov Lemma yields the following corollary.

**Corollary 2.** Let \( C(x) = r(x) I + \tilde{C} \) in (3), where \( r(x) \leq 0 \) in \( D \) and \( \tilde{C} \) is a constant matrix over the complex field. Assume that none of the eigenvalues of \( \tilde{C} \) has a positive real part, and moreover that the elementary divisors of \( \tilde{C} \) corresponding to eigenvalues with vanishing real part are linear.
Then there exists a positive constant $K$ such that for all $C^2(D) \cap C(\overline{D})$ solutions $u$ of (3),

$$\|u\|_{0,D} \leq K \|u\|_{0,\partial D}. \quad (6)$$

**Proof.** By the Liapunov Lemma in Section 2, there exist matrices $B > 0$ and $E \geq 0$ such that $\bar{C}^*B + BC = -E \leq 0$. Since $r \leq 0$ in $D$, we get

$$C^*(x)B + BC(x) = 2r(x)B + \bar{C}^*B + BC \leq 0.$$

Now the result of this corollary follows from Theorem 1. □

**Remark.** By the Liapunov Lemma, we know that at least one positive definite $B$, satisfying $\bar{C}^*B + BC = -E \leq 0$, exists if the matrix $\bar{C}$ meets the assumption of Corollary 2. In fact, if $\bar{C}$ satisfies the condition of Corollary 2 and has at least one eigenvalue with vanishing real part, then there is an infinite number of positive definite $B$ such that $\bar{C}^*B + BC = -E \leq 0$ (see [11, 18]).

Theorem 1 and Corollary 2 are generalized maximum principles since the value of $K$ in (6) may be larger than 1. The best conceivable value of $K$ in (6), for any matrix $C$, is $K=1$, which corresponds to the classical maximum principle.

**THEOREM 3 (The Classical Maximum Principle).** (a) A sufficient condition that

$$\|u\|_{0,D} \leq \|u\|_{0,\partial D} \quad (6)_1$$

holds, for all $C^2(D) \cap C(\overline{D})$ solutions $u$ of (3), is

$$C^*(x) + C(x) \leq 0. \quad (9)$$

(b) Assume that the variable matrix $C = C(x)$ in (3) is normal (i.e., $C^*(x)C(x) = C(x)C^*(x)$, $x \in D$), and all its eigenvalues have nonpositive real parts for all $x \in D$. Then (6)$_1$ holds for all $C^2(D) \cap C(\overline{D})$ solutions $u$ of (3).

**Proof.** (a) By choosing $B = I$ in Theorem 1, (6) with $K=1$ (i.e., (6)$_1$) follows from the condition (9).

(b) Suppose

$$\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$$
are all the eigenvalues of \( C(x) \). Since \( C(x) \) is normal, there exists a unitary matrix \( U(x) \) such that

\[
U^*(x) C(x) U(x) = \begin{bmatrix}
\lambda_1(x) \\
\vdots \\
\lambda_m(x)
\end{bmatrix}.
\]

Therefore, by the assumption,

\[
U^*(x) (C^*(x) + C(x)) U(x) = \begin{bmatrix}
2 \text{ Re } \lambda_1(x) \\
\vdots \\
2 \text{ Re } \lambda_m(x)
\end{bmatrix} = A(x) \leq 0.
\]

Hence \( C^* + C = UAU^* \leq 0 \); and then (6), follows from (a).

Remarks. (1) The condition (9) is also a "necessary" condition for the proof of the classical maximum principle by the method imposed here. In fact, if, in Theorem 1, (6) holds with \( K = 1 \), then \( \beta_1 = \beta_m \), and so there exists a \( B > 0 \), with an \( m \)-multiple eigenvalue \( B > 0 \), such that \( C^* B + B C < 0 \); hence \( B = \beta I \), and then \( C^* + C \leq 0 \).

(2) Theorem 3 contains the result of Winter and Wong [23] for real negative semidefinite \( C = C(x, u, \nabla u) \) as a special case; one may view, for given \( u \), \( C(x, u(x), \nabla u(x)) \) as a matrix function \( C_1(x) \).

EXAMPLE 1. For \( n = 2 \), consider

\[
Lu + \begin{bmatrix} a & b \\ c & d \end{bmatrix} u = 0, \quad a, b, c, d \in \mathbb{R}.
\]

The associated characteristic equation,

\[
\lambda^2 - (a + d) \lambda + (ad - bc) = 0,
\]

has roots

\[
\lambda_{\pm} = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}
\]

\[
= \frac{(a + d) \pm \sqrt{(a - d)^2 + 4bc}}{2}.
\]
Hence, by Corollary 2, the inequality (6) is valid provided one of the following conditions is satisfied:

(i) \( a + d < 0, (a - d)^2 + 4bc \leq 0; \)

(ii) \( a + d < 0, (a - d)^2 + 4bc > 0, ad - bc \geq 0; \)

(iii) \( a + d = 0, ad - bc > 0. \)

The inequality (6) is not valid for the general case, when \( a + d > 0 \) or \( a + d = 0, ad - bc \leq 0. \) In fact, \( u = \sin x \sin y \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) solves the systems

\[
\Delta u + \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix} u = 0
\]

and

\[
\Delta u + \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix} u = 0
\]

in \( D = (0, \pi) \times (0, \pi) \) and vanishes on \( \partial D, \) but (6) does not hold.

Theorem 1 gives a sufficient condition for which (6) holds. It raises some open questions as to whether Theorem 1 can be extended to a more general system (3) with weaker restrictions on the matrix \( C, \) and as to whether necessary conditions can be determined so that (6) holds.

Following from the inequality

\[
L(u^*Bu) \geq -u^*(C^*B + BC)u + 2a_{ij}B^{1/2}u_{,i} \cdot B^{1/2}u_{,j} \geq 0,
\]

in the proof of Theorem 1, and from Protter and Weinberger's book [15], are the following two maximum principles for system (3).

**Corollary 4.** If \( u \in C^2(D) \cap C(\overline{D}) \) is a solution of (3), and if \( u^*Bu \) attains a maximum in \( D \) for some positive definite matrix \( B \) such that \( C^*(x)B + BC(x) \) is negative semidefinite in \( D, \) then \( u \) is a complex constant vector in \( D. \) Moreover, if \( C^*(x)B + BC(x) \) is negative definite at some \( x \in D, \) or, if \( C(x) \) is invertible for some \( x \in D, \) then \( u \equiv 0 \) in \( D. \)

**Proof.** Under the assumption of this corollary, by the proof of Theorem 1, inequality (7) holds. Thus, by the maximum principle of the second-order elliptic equation (see [15]), \( u^*Bu \equiv \text{constant}. \) Hence, from (7) again, we have

\[
0 = L(u^*Bu) = -u^*(C^*B + BC)u + 2a_{ij}B^{1/2}u_{,i} \cdot B^{1/2}u_{,j} \geq 0 \quad \text{in } D;
\]

which implies that \( u^*(C^*B + BC)u = 0 \) and \( a_{ij}B^{1/2}u_{,i} \cdot B^{1/2}u_{,j} = 0, \) and then,
Thus \( u \) is a complex constant vector in \( D \). Moreover, if \( C^*(x) B + BC(x) < 0 \) at some \( x \in D \), then, from 
\[
\begin{align*}
\ast \left((C^*(x) B + BC(x)) u = 0 \right, \text{ we have } u = 0 \text{ in } D; \text{ and if } C(x) \text{ is invertible for some } x \in D, \text{ then, from the system (3), we still have } u = 0 \text{ in } D.
\end{align*}
\]

Note that Corollary 4 actually holds even if \( D \) is unbounded.

**Corollary 5.** Let \( u \in C^2(D) \cap C(\overline{D}) \) be a solution of (3). Suppose that 
\( u^* Bu \leq M \) in \( D \) and that \( u^* Bu = M \) at a point \( P \in \partial D \) for some positive definite \( B \) such that \( C^* B + BC \) is negative semidefinite. Here \( M \) is a non-negative constant. Assume that \( P \) lies on the boundary of a ball in \( D \), and that the outward directional derivative \( \partial u / \partial v \) exists at \( P \). Then 
\[
\frac{\partial (u^* Bu)}{\partial v} = 2 \text{Re} \left[ u^* B \frac{\partial u}{\partial v} \right] = 2 \text{Re} \left[ (B^{1/2} u)^* \frac{\partial (B^{1/2} u)}{\partial v} \right] > 0 \quad \text{at } P
\]

unless \( u \) is a complex constant vector such that \( u^* Bu \equiv M \); equivalently,
\[
\frac{\partial |B^{1/2} u|}{\partial v} > 0 \quad \text{at } P
\]

unless \( u \) is constant and \( |B^{1/2} u| \equiv M^{1/2} \).

4. **Uniqueness Theorems for Some Boundary Value Problems**

As applications of the maximum principles in Section 3, we can prove uniqueness theorems for various boundary value problems.

As an example, consider the first boundary value problem for the elliptic system (3). By Theorem 1, the problem
\[
Lu(x) + C(x) u(x) = f(x), \quad x \in D, \quad (3)_N
\]
\[
u = g(x), \quad (10)
\]

where \( C \) satisfies the assumption of Theorem 1, has at most one solution.

As a second example, we have uniqueness for the following mixed boundary problem:
\[
Lu(x) + C(x) u(x) = f(x) \quad \text{in } D \subset \mathbb{R}^n, \quad (3)_N
\]
\[
\begin{align*}
\begin{cases}
\begin{align*}
u(x) &= g_1(x) \quad \text{on } \Gamma_1, \\
\frac{\partial u(x)}{\partial v} + \alpha(x) u &= g_2(x) \quad \text{on } \Gamma_2,
\end{align*}
\end{cases}
\end{align*}
\]

where \( \nu = \nu(x) \) is a given outward direction on \( \Gamma_2 \), and \( \Gamma_2 = \partial D \setminus \Gamma_1 \).
Corollary 6. Suppose \( u^1 \) and \( u^2 \) satisfy (3)\(_N\) and (11) in a bounded domain \( D \subset \mathbb{R}^n \) and \( C \) satisfies the assumption of Theorem 1. Assume that each point of \( \Gamma_2 \) lies on the boundary of a ball in \( D \). If \( L \) is elliptic as defined by (1) and \( \alpha(x) \geq 0 \) on \( \Gamma_2 \), then \( u^1 \equiv u^2 \), except when \( \alpha \equiv 0 \), \( \Gamma_1 \) is vacuous and \( C(x) \) is singular for all \( x \in D \), in which case \( u^1 - u^2 \) is a complex constant vector.

Proof. Define \( u = u^1 - u^2 \). Then \( u \) satisfies

\[
Lu + Cu = 0 \quad \text{in } D,
\]

\[
\begin{cases}
u = 0 & \text{on } \Gamma_1, \\
\frac{\partial u}{\partial v} + \alpha u = 0 & \text{on } \Gamma_2.
\end{cases}
\]

Choose positive definite \( B \), as in the proof of Theorem 1, such that \( C^*B + BC \) is negative semidefinite, and let \( v = u^*Bu \); then (7) holds. If \( v \) is not a constant, by Corollary 4, a nonzero maximum of \( v \) must occur at a point \( P \) on \( \Gamma_2 \); and by Corollary 5,

\[
\Re\left[ u^*B \frac{\partial u}{\partial v} \right] > 0 \quad \text{at } P.
\]

Hence,

\[
\Re\left[ u^*B \left( \frac{\partial u}{\partial v} + \alpha u \right) \right] > 0 \quad \text{at } P \in \Gamma_2,
\]

which contradicts the boundary condition (12). Thus \( v = u^*Bu \) must be a constant. From (7), by the proof of Corollary 4, \( u \) must be a complex constant vector. Then, from (3) and (12), we know that \( u \equiv 0 \), i.e., \( u^1 \equiv u^2 \) in \( D \), except when \( \alpha \equiv 0 \), \( \Gamma_1 \) is vacuous, and \( C(x) \) is singular for all \( x \in D \).

Besides having applications to some boundary value problems with bounded domain \( D \), the maximum principle we obtained in Section 3 can also be used to establish uniqueness theorems for various boundary value problems with unbounded domain \( D \subset \mathbb{R}^n \). In this case, the point \( \infty \) is called the exceptional boundary point; and an appropriate growth restriction on the solution at \( \infty \) is required. Moreover, by using a recent result of Hile and Yeh [10], we even obtain uniqueness for the boundary value problem with an exceptional boundary set \( \Gamma \) such that the Hausdorff dimension of \( \Gamma \) is less than \( n - 1 \).
As a third example of this section, we have uniqueness for the following boundary value problem with unbounded domain $D \subset \mathbb{R}^n$:

$$Lu(x) + C(x)u(x) = f(x) \quad \text{in } D,
\begin{align*}
\begin{cases}
u(x) = g(x) & \text{on } \partial D, \\
|u(x)| \text{ is bounded in } D.
\end{cases}
\end{align*}
$$

**COROLLARY 7.** Let $D$ be an unbounded domain contained inside a cone, let $L$ be given in $D$ by (1), with

$$a_i(x) = o(r^{-1}) \quad \text{as } x \to \infty \text{ in } D, i = 1, \ldots, n, \ (r = |x|),$$

and with the uniform ellipticity condition

$$\delta |y|^2 \leq a_{ij}(x)y_i y_j \leq \Lambda |y|^2, \quad x \in D, y \in \mathbb{R}^n,$$

holding for some positive constants $\delta$ and $\Lambda$. Suppose $u^1$ and $u^2$ satisfy (3)$_N$, (13) in $D$ and matrix $C$ satisfies the assumption of Theorem 1. Then $u^1 \equiv u^2$ in $D$.

**Proof.** Let $u = u^1 - u^2$ and $v = u^*Bu$, where $B$, as in Theorem 1, is positive definite such that $C^*(x)B + BC(x)$ is negative semidefinite for all $x \in D$. Then $v$ is nonnegative, and the proof of Theorem 1 gives

$$\begin{cases}
Lv = -u^*(C^*B + BC)u + 2a_{ij}B^{1/2}u \cdot B^{1/2} \geq 0 & \text{in } D \\
v(x) = 0 & \text{on } \partial D, \\
v(x) \text{ is bounded in } D.
\end{cases}$$

Hence, by the Phragmen–Lindelof Principle [9, Corollary 2],

$$\lim_{r \to \infty} v(x) = 0, \quad \text{as } x \to \infty \text{ in } D.$$ 

Therefore, the maximum principle (Theorem 1) implies that $v \equiv 0$ in $D$; i.e., $u^1 \equiv u^2$ in $D$.

**Remark.** When $L \equiv \Delta$ (Laplace operator) in (3)$_N$, the growth restriction of $|u|$ being bounded in Corollary 7 can be replaced by the weak restriction

$$\lim_{r \to \infty} \frac{\max_{|x| = r} |u(x)|^2}{\log r} = 0 \quad \text{if } n = 2,$$

or

$$\lim_{r \to \infty} \frac{\sup_{|x| = r} |u(x)|^2}{r^{n-2}} = 0 \quad \text{if } n \geq 3$$

(see [15]).
As a last example, we prove uniqueness for the solution of the following boundary problem without knowing the data on an exceptional boundary set $I \subset \partial D$ (domain $D$ can be unbounded):

$$\left( \Delta + a_i(x) \frac{\partial}{\partial x_i} \right) u(x) + C(x) u(x) = f(x) \quad \text{in } D, \quad (3)''$$

$$\begin{cases}
  u(x) = g(x) & \text{on } \partial D \setminus \Gamma, \\
  |u(x)| \text{ is bounded in } D. \quad (14)
\end{cases}$$

**Corollary 8.** Suppose $u^1$ and $u^2$ are two $C^2$ solutions of the problem $(3)''$, $(14)$; and assume that $|a_i(x)|$, $1 \leq i \leq n$, are bounded in $D$ and the matrix $C(x)$ satisfies the assumption of Theorem 1. Let $\Gamma$ be a subset of $\partial D$ such that for each $y$ on $\Gamma$, $D$ is contained on one side of an $(n-1)$-dimensional hyperplane passing through $y$. Suppose also that the Hausdorff dimension of $\Gamma$ is less than $n-1$. Then $u^1 \equiv u^2$ in $D$.

**Proof.** Let $u = u^1 - u^2$ and $v = u^* Bu$ where $B$, as in Theorem 1, is positive definite such that $C^*(x) B + BC(x)$ is negative semidefinite over $D$. Then $v$ is nonnegative, and the proof of Theorem 1 implies that

$$\begin{cases}
  \left( \Delta + a_i(x) \frac{\partial}{\partial x_i} \right) v(x) = -u^*(C^*B + BC) u + 2a_{ij} B^{1/2} u \cdot B^{1/2} u \geq 0 & \text{in } D, \\
  v(x) = 0 & \text{on } \partial D \setminus \Gamma, \\
  v(x) \leq M & \text{in } D, \text{ for some } M > 0.
\end{cases}$$

Hence, by a result of Hile and Yeh [10, Corollary 2],

$$v = 0 \quad \text{on } \Gamma.$$ 

Therefore,

$$v \equiv 0 \quad \text{on } \partial D.$$ 

Thus, by the maximum principle (if $D$ is bounded) or the Phragmen-Lindelof principle (if $D$ is unbounded) of the second order elliptic equation, we have $v \equiv 0$ in $D$. This gives $u^1 \equiv u^2$ in $D$. 

5. **Estimate of $K$**

In Theorem 1, the constant $K = (\beta_m/\beta_1)^{1/2}$ in (6) depends on $C$; in fact, $\beta_1$ and $\beta_m$ are the smallest and largest eigenvalues of $B$, where $B$ is chosen so that

$$C^*(x) B + BC(x) \leq 0.$$
The value of $K$ in (6) is important in applications, such as in numerical computation and estimation. The best conceivable value of $K$, for any $C$, is $K = 1$, which corresponds to the classical maximum principle; and Theorem 3 gives a sufficient condition (9) to guarantee the classical maximum principle ($K = 1$) for solutions of (3). However, for a general matrix $C$, the best possible value of $K$ can be larger than 1.

**Example 2.** For the system

$$Lu(x) + \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} u(x) = 0,$$

where $a, b, c,$ and $d$ are complex-valued functions, by Theorem 3, the classical maximum principle (6) holds if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 \text{Re} \{a\} & b + \bar{c} \\ (b + \bar{c}) & 2 \text{Re} \{d\} \end{bmatrix} \leq 0,$$

which is equivalent to

$$\text{Re} \{a\} \leq 0, \quad \text{Re} \{d\} \leq 0; \quad \text{and} \quad |b + \bar{c}|^2 \leq 4 \text{Re} \{a\} \text{Re} \{d\}. \quad (15)$$

The following example can be used to compute the best choice of $K = (\beta_m/\beta_1)^{1/2}$ when $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a real constant matrix with a double eigenvalue $\lambda = \lambda_+ = \lambda_- = (1/2)(a + d) < 0$.

**Example 3.** Consider

$$Lu + \begin{bmatrix} -1 & \varepsilon \\ 0 & -1 \end{bmatrix} u = 0, \quad \varepsilon \in \mathbb{R}. \quad (16)$$

In order to determine the best $K$, we choose a positive definite $B := \begin{bmatrix} \beta & \gamma \\ \beta & \gamma \end{bmatrix}$ which minimizes

$$K = \left( \frac{\beta_2}{\beta_1} \right)^{1/2} = \left( \frac{\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4 |\beta|^2}}{(\alpha + \gamma) - \sqrt{(\alpha - \gamma)^2 + 4 |\beta|^2}} \right)^{1/2}, \quad (17)$$

and satisfies

$$C^*B + BC = \begin{bmatrix} -2\alpha & \alpha\varepsilon - 2\beta \\ \alpha\varepsilon - 2\beta & \varepsilon(\beta + \bar{\beta}) - 2\gamma \end{bmatrix} \leq 0.$$

Without loss of generality, we can assume that

$$\alpha + \gamma = 1. \quad (18)$$

Then the problem can be reduced to the following equivalent problem.
Find \((\alpha, \beta) \in \mathbb{R} \times \mathbb{C}\) which minimizes \((2\alpha - 1)^2 + 4|\beta|^2\), subject to: \(\alpha > \alpha^2 + |\beta|^2\),
\[\varepsilon(\beta + \bar{\beta}) < 2,\]
\[\alpha^2(\varepsilon^2 + 4) + 4\beta^2 - 4\alpha \leq 0.\]

Obviously, \(\beta = 0\) is the best. Thus the problem reduces to minimize \((2\alpha - 1)^2\), subject to: \(\alpha > \alpha^2\),
\[\alpha^2(\varepsilon^2 + 4) - 4\alpha \leq 0.\]

It is easy to conclude that
\[(\alpha, \beta) = \begin{cases} 
\left(\frac{1}{2}, 0\right), & \text{if } |\varepsilon| \leq 2, \\
\left(\frac{4}{4 + \varepsilon^2}, 0\right), & \text{if } |\varepsilon| > 2.
\end{cases}\]

Hence by (17) and (18), the best value of \(K = (\beta_m/\beta_1)^{1/2}\) for the system (16) is
\[K = \begin{cases} 
1, & \text{if } |\varepsilon| \leq 2, \\
\frac{|\varepsilon|}{2}, & \text{if } |\varepsilon| > 2.
\end{cases}\]

For a general real matrix \(C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) with a double eigenvalue \(\lambda = (1/2)(a + d) < 0\), by matrix theory (see [20, Chap. 5]), there exist an orthogonal matrix \(P := \begin{bmatrix} p & q \end{bmatrix}\) and a real number \(\varepsilon_0\) such that
\[
\begin{bmatrix} \lambda & \varepsilon_0 \\ 0 & \lambda \end{bmatrix} = P^T C P = \begin{bmatrix} p^2 a + p r c + p r b + r^2 d & p q a + r q c + s p b + s r d \\ p q a + p s c + r q b + r s d & q^2 a + q s c + q s b + s^2 d \end{bmatrix}.
\]

From \(0 = p q a + p s c + r q b + r s d\), we have
\[\varepsilon_0 = p q a + r q c + s p b + s r d = (p s - r q)(b - c) = \det P \cdot (b - c) = \pm (b - c).\]

Let \(B = P B_1 P^T\). Then
\[
C^T B + BC = P \begin{bmatrix} \lambda & \varepsilon_0 \\ 0 & \lambda \end{bmatrix}^T B_1 + B_1 \begin{bmatrix} \lambda & \varepsilon_0 \\ 0 & \lambda \end{bmatrix} P^T \leq 0
\]
is equivalent to
\[
\begin{bmatrix}
\lambda & \varepsilon_0 \\
0 & \lambda
\end{bmatrix}^T B_1 + B_1 \begin{bmatrix}
\lambda & \varepsilon_0 \\
0 & \lambda
\end{bmatrix} \leq 0,
\]
or
\[
\begin{bmatrix}
-1 & -\varepsilon_0 \\
0 & -1
\end{bmatrix}^T B_1 + B_1 \begin{bmatrix}
-1 & -\varepsilon_0 \\
0 & -1
\end{bmatrix} \leq 0.
\]

By letting \( \varepsilon = -\varepsilon_0/\lambda = \pm 2(b-c)/(a+d) \), from the Example 3 we have the following:

**Conclusion.** The best value of \( K = (p_1/p_2) R^2 \) for \( C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) in (3) with a double eigenvalue \( \lambda = (1/2)(a+d) < 0 \) is

\[
K = \begin{cases}
1 & \text{if } |b-c| \leq -(a+d); \\
\frac{|b-c|}{a+d} & \text{if } |b-c| > -(a+d).
\end{cases}
\]

Since \( \lambda = \lambda_+ = \lambda_- = (1/2)(a+d) < 0 \) implies that \( (a-d)^2 + 4bc = 0 \). The condition of \( |b-c| \leq -(a+d) \) is equivalent to the condition

\[
(b+c)^2 \leq 4ad \quad (\text{which is the same as (15)}).
\]

**Remark.** When \( C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) has two different eigenvalues \( \lambda_+ < \lambda_- \leq 0 \), by the procedure above the problem of finding the best choice of \( K = (\beta_m/\beta_1)^{1/2} \) in (6) reduces to the problem of minimizing \( (2\alpha - 1)^2 + 4|\beta|^2 \) subject to some constraints. By Largange's method of multipliers a formula for the best choice of \( K \) can be obtained. Also, an upper bound for the best possible \( K \) can be found when \( C \in \mathbb{R}^{2 \times 2} \) has two complex conjugate eigenvalues with a nonpositive real part. Since the procedure of finding these formulas involves some very complicated computation, we omit the details here.

6. **Bounds for Solutions of the Nonhomogeneous Elliptic System**

In Section 3 we obtained maximum principles for some homogeneous elliptic systems. Now we extend these results to the nonhomogeneous elliptic system

\[
Lu(x) + C(x) u(x) = f(x), \quad x \in D.
\]

(3)
Under the restriction of \( C \) being an real matrix function such that 
\[ \xi^T C \xi \leq -c_0 |\xi|^2 \] in \( D \), for some \( c_0 > 0 \) and for any \( \xi \in \mathbb{R}^m \), Miranda [13] obtained a bound for solutions of the elliptic system \((3)_N\):

\[
\|u\|_{0,D} \leq \|u\|_{0,\partial D} + c_0^{-1} \|f\|_{0,D}.
\]

In this section, we obtain a similar result for more general complex matrix functions \( C \) in the elliptic system \((3)_N\).

**Theorem 9.** Suppose, for \( C = C(x) \), there exist two complex constant matrices \( B > 0, E > 0 \) such that
\[ C^*(x) B + BC(x) \leq -E. \]
Then for all \( C^2(D) \cap C(D) \) solutions \( u \) of \((3)_N\), there exist positive constants \( K_1 \) and \( K_2 \) such that

\[
\|u\|_{0,D} \leq K_1 \|u\|_{0,\partial D} + K_2 \|f\|_{0,D}.
\]  

(19)

Here \( K_1 = \left( \frac{\beta_m/\beta_1}{\beta_1} \right)^{1/2} \) and \( K_2 = 2\beta_m^{1/2}/\mu_1 \beta_1^{1/2} \), where \( \beta_1 \) and \( \beta_m \) are the smallest and biggest eigenvalues of \( B \), respectively, and \( \mu_1 \) is the smallest eigenvalue of \( EB^{-1} \).

**Proof.** Define \( v = B^{1/2} u; \) i.e., \( u = B^{-1/2} v \). First we will prove that (for \( v \)),

\[
\|v\|_{0,D} \leq \|v\|_{0,\partial D} + K \|f\|_{0,D}, \quad K > 0.
\]

(20)

It is sufficient to show that at an internal relative maximum of \( |v| \) (or of \( |v|^2 = |B^{1/2} u|^2 = Bu \cdot u \)),

\[
|v(x)| \leq K |f(x)|.
\]

(21)

We assume \( v(x) \neq 0 \) at such a maximum; otherwise the inequality is trivial.

At a relative maximum of \( |v|^2 = Bu \cdot u \), we have

\[
\frac{\partial}{\partial x_k} |v|^2 = 2 \text{Re} \left( Bu \cdot u_k \right) = 0,
\]

and the matrix of second derivatives is negative semidefinite; i.e.,

\[
\left[ \frac{\partial^2 |v|^2}{\partial x_i \partial x_k} \right]_{n \times n} = \left[ 2 \text{Re} \left( v_i \cdot v_k + v \cdot v_{ik} \right) \right]_{n \times n} \leq 0.
\]

Hence,

\[
\text{Re} \left( Bu \cdot u_k \right) = \text{Re} \left( v \cdot v_k \right) = 0,
\]

\[
\left[ \text{Re} \left( v_i \cdot v_k + v \cdot v_{ik} \right) \right]_{n \times n} \leq 0.
\]

Since \( (a_{ik}) > 0 \), we have that

\[
a_{ik} (v_i \cdot v_k + \text{Re} \left( v \cdot v_{ik} \right)) \leq 0.
\]
Into this inequality we substitute the system (3)N and Re \((Bu \cdot u_k) = 0\) to get

\[
0 \geq a_{ik} v_{i,k} v_{k} + \text{Re} (Bu \cdot a_{ik} u_{ik}) = a_{ik} v_{i,k} v_{k} + \text{Re} [Bu \cdot (-a_i u_{i} - Cu + f)] = a_{ik} v_{i,k} v_{k} - \frac{1}{2}(C^*B + BC) u \cdot u + \text{Re} (Bu \cdot f).
\]

Therefore,

\[
\beta_{m}^{1/2} |v| |f| \geq \text{Re} (Bu \cdot f) = a_{ik} v_{i,k} v_{k} - \frac{1}{2}(C^*B + BC) u \cdot u \geq \frac{1}{2} Eu \cdot u = \frac{1}{2}(B^{-1/2}EB^{-1/2}) v \cdot v.
\]

Since \(B^{-1/2}EB^{-1/2}\) and \(EB^{-1}\) have the same eigenvalues, it follows that

\[
\beta_{m}^{1/2} |v| |f| \geq \frac{\mu_1}{2} |v|^2,
\]

and then (21) holds with

\[
K = \frac{2 \beta_{m}^{1/2}}{\mu_1}.
\]

Hence we have proved (20).

By using (20) and substituting \(v = B^{1/2}u\), we have

\[
\beta_{1}^{1/2} \|u\|_{0,D} \leq \beta_{m}^{1/2} \|u\|_{0,\partial D} + K \|f\|_{0,D},
\]

so (19) holds with

\[
K_1 = \left(\frac{\beta_{m}}{\beta_{1}}\right)^{1/2} \quad \text{and} \quad K_2 = \frac{K}{\beta_{1}^{1/2}} = \frac{2 \beta_{m}^{1/2}}{\mu_1 \beta_{1}^{1/2}}.
\]

From the Liapunov Lemma, the following corollary is easily obtained:

**Corollary 10.** Assume that \(C(x) = r(x) I + \bar{C}\) in (3)N, where \(r \leq 0\) in \(D\) and each eigenvalue of the complex constant matrix \(\bar{C}\) has negative real part. Then for all \(C^2(D) \cap \mathcal{C}(\bar{D})\) solutions \(u\) of (3)N, there exist positive constants \(K_1\) and \(K_2\) such that (19) holds.

**Corollary 11.** (a) A sufficient condition that

\[
\|u\|_{0,D} \leq \|u\|_{0,\partial D} + c_0^{-1} \|f\|_{0,D}
\]

holds, for all \(C^2(D) \cap \mathcal{C}(\bar{D})\) solutions \(u\) of (3)N, is

\[
C^*(x) + C(x) \leq -2c_0 I < 0, \quad \text{for some} \quad c_0 \in \mathbb{R}.
\]
(b) Suppose that the variable matrix $C = C(x)$ is normal, and all its eigenvalues $\{\lambda_i(x)\}_{i=1}^m$ have uniformly negative real parts in $D$; i.e., $\text{Re} \lambda_i(x) \leq -c_0 < 0$ in $D$, for $1 \leq i \leq m$. Then (22) holds for all $C^2(D) \cap C(D)$ solutions $u$ of (3)$_N$.

Proof. (a) By choosing $B = I$ in Theorem 9, the inequality (22) follows from the condition (23).

(b) Since $C(x)$ is normal, there exists a unitary matrix $U(x)$ such that

$$U^*(x)[C^*(x) + C(x)] U(x) = \begin{bmatrix} 2 \text{Re} \lambda_1(x) & \cdots & \cdots & 2 \text{Re} \lambda_m(x) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 2 \text{Re} \lambda_m(x) \end{bmatrix} \leq -2c_0 I < 0$$

Hence $C^*(x) + C(x) \leq -2c_0 I$; and then (22) follows from (a).

7. APPLICATIONS TO HIGHER ORDER EQUATIONS

Note that the results we obtain are valid for complex systems which may be considered generalizations of some higher order complex equations. For example, consider a $2m$-order homogeneous equation of the form

$$\mathcal{P} u = L^m u + a_{m-1} L^{m-1} u + \cdots + a_1 L u + a_0 u = 0, \quad (24)$$

and the nonhomogeneous equation of the same form.

$$\mathcal{P} u = F. \quad (24)_N$$

Here $a_0, a_1, \ldots, a_{m-1}$ and $F$ are complex-valued functions, $\mathcal{L} := L + r(x)$ where $r \leq 0$ in $D$ and $L$ is the elliptic operator defined by (1); and $\mathcal{L}^m \equiv \mathcal{L}(\mathcal{L}^{m-1})$, $\mathcal{L}^0 \equiv I$. Let $u_1 = u$, $u_2 = \mathcal{L} u$, $\ldots$, $u_m = \mathcal{L}^{m-1} u$; then the equations (24) and (24)$_N$ reduce to the equivalent systems (3) and (3)$_N$, respectively, where

$$C = r I + \begin{bmatrix} 0 & -1 & & \\ 0 & 0 & -1 & \\ & \ddots & \ddots & \ddots \\ a_0 & a_1 & \cdots & a_{m-1} \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

(25)
Associated with each system is the characteristic equation
\[ h(\lambda) = (-\lambda)^m + a_{m-1}(-\lambda)^{m-1} + \cdots + a_1(-\lambda) + a_0 = 0. \]  
(26)

Hence, under the same assumptions on \( C \) in (25), Theorem 1 and Theorem 3 hold, but (6) changes to
\[ \sup_{x \in D} \left( \sum_{j=0}^{m-1} |\mathcal{L}^j u(x)|^2 \right)^{1/2} \leq K \sup_{x \in \partial D} \left( \sum_{j=0}^{m-1} |\mathcal{L}^j u(x)|^2 \right)^{1/2}, \quad (K \geq 1); \]  
(27)
also, under the same assumptions on \( C \) in (25), Theorem 9 holds, but (19) changes to
\[ \sup_{x \in D} \left( \sum_{j=0}^{m-1} |\mathcal{L}^j u(x)|^2 \right)^{1/2} \leq K_1 \sup_{x \in \partial D} \left( \sum_{j=0}^{m-1} |\mathcal{L}^j u(x)|^2 \right)^{1/2} + K_2 \|F\|_{0,D}. \]  
(27)_N

In the case of \( m = 2 \) and \( r = 0 \) in \( D \), the matrix in (25) is
\[ C(x) := \begin{bmatrix} 0 & -1 \\ a_0(x) & a_1(x) \end{bmatrix} \]  
with eigenvalues \( \lambda_{\pm} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2} \).

Hence the requirement for Theorem 1 holding is that there exists \( B := \begin{bmatrix} p & 0 \\ q & s \end{bmatrix} > 0 \) such that
\[ C^*B + BC = \begin{bmatrix} 0 & \overline{a_0(x)} \\ -1 & a_1(x) \end{bmatrix} \begin{bmatrix} p & s \\ \overline{s} & q \end{bmatrix} + \begin{bmatrix} p & s \\ \overline{s} & q \end{bmatrix} \begin{bmatrix} 0 & -1 \\ a_0(x) & a_1(x) \end{bmatrix} = \begin{bmatrix} 2 \text{ Re} (sa_0) & qa_0 + sa_1 - p \\ qa_0 + sa_1 - p & 2 \text{ Re}(qa_1 - s) \end{bmatrix} \leq 0, \]
which is equivalent to the existence of \( p, q \in \mathbb{R}, s \in \mathbb{C} \) such that
\[ \begin{cases} p > 0 \\ pq - |s|^2 > 0 \end{cases} \quad \text{and} \quad \begin{cases} \text{Re} \ (sa_0) \leq 0, & \text{Re} \ (qa_1 - s) \leq 0, \\ 4 \text{ Re} \ (sa_0) \cdot \text{Re} \ (qa_1 - s) \geq |qa_0 + sa_1 - p|^2. \end{cases} \]  
(28)

If we choose \( s = 0 \), then
\[ B = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}, \]
\[ K^2 = \frac{\max(p, q)}{\min(p, q)}, \]
and (28) becomes

\[ \begin{cases} p > 0 \\ q > 0 \end{cases} \quad \text{and} \quad \begin{cases} \Re a_1(x) < 0 \\ a_0 = \frac{p}{q} > 0. \end{cases} \]

So for the 4th-order equations,

\[ \mathcal{L}^2 u(x) + a_1(x) \mathcal{L} u(x) + a_0 u(x) = 0 \quad \text{in } D, \quad (29) \]

and

\[ \mathcal{L}^2 u(x) + a_1(x) \mathcal{L} u(x) + a_0 u(x) = F \quad \text{in } D, \quad (29)_N \]

we have

**Corollary 12.** Assume that \( \Re \{a_1(x)\} \leq 0 \) in \( D \) and \( a_0 > 0 \), then

(a) for all complex-valued \( C^4(D) \cap C^2(\overline{D}) \) solutions \( u \) of (29),

\[ \sup_D (|u|^2 + |\mathcal{L} u|^2)^{1/2} \leq \sqrt{\max(a_0, a_0^{-1})} \sup_{\partial D} (|u|^2 + |\mathcal{L} u|^2)^{1/2}; \]

(b) if \( r(x) \leq -r_0 < 0 \) in \( D \), then for all complex-valued \( C^4(D) \cap C^2(\overline{D}) \) solutions \( u \) of (29),

\[ \sup_D (|u|^2 + |\mathcal{L} u|^2)^{1/2} \]

\[ \leq \sqrt{\max(a_0, a_0^{-1})} \sup_{\partial D} (|u|^2 + |\mathcal{L} u|^2)^{1/2} + r_0^{-1} \| F \|_{0,D}]. \]

When \( \{a_j\}_{j=0}^{m-1} \) are complex constants, by applying Corollary 2 and Corollary 10 we have the following corollary for the \( 2m \)-order equations (24) and (24).

**Corollary 13.** Suppose the roots \( \{\lambda_i\}_{i=1}^n \) of (26) satisfy both of the conditions,

(i) \( \Re \lambda_i \leq 0 \);  
(ii) if \( \Re \lambda_i = 0 \), then the corresponding elementary divisor is linear.

Then,

(a) there exists a positive constant \( K \) such that (27) holds for all complex-valued regular solutions of (24);  
(b) if \( \Re \lambda_i < 0, 1 \leq i \leq m \); or \( r(x) \leq -r_0 < 0 \) in \( D \), there exist positive constants \( K_1 \) and \( K_2 \) such that (27) holds for all complex-valued regular solutions of (24).
Remark. Corollary 13 contains the result of Chow and Dunninger [2, 6] for real metaharmonic functions as a special case. The first part of the Corollary 13 was obtained in [2] for the case of $L = \Delta$ (Laplace operator) and $a_0, a_1, \ldots, a_{m-1}$ being real constants.

ACKNOWLEDGMENT

The author expresses his appreciation to Professor Gerald N. Hile for his helpful guidance and suggestions.

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