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On the Lebesgue constant of Leja sequences for the unit disk and its applications to multivariate interpolation

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Abstract

We estimate the growth of the Lebesgue constant of any Leja sequence for the unit disk. The main application is the construction of new multivariate interpolation points in a polydisk (and in the Cartesian product of many plane compact sets) whose Lebesgue constant grows (at most) like a polynomial.

Keywords: Lagrange interpolation; Lebesgue constants; Leja sequences

1. Introduction

Let A be a set of $d+1$ pairwise distinct points in a compact set K in the complex plane. Given a function f defined on A, the unique polynomial of degree at most d which coincides with f on A is the *Lagrange interpolation polynomial* of f and is denoted by $\mathbf{L}[A; f]$. We have

$$\mathbf{L}[A; f] = \sum_{a \in A} f(a)\ell(A, a; \cdot), \tag{1}$$

where $\ell(A, a; \cdot)$ is the fundamental Lagrange interpolation polynomial (*FLIP*) corresponding to a, that is, the unique polynomial of degree at most d such that $\ell(A, a; a) = 1$ and $\ell(A, a; \cdot) = 0$

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on $A \setminus \{a\}$,

$$\ell(A, a; z) = \prod_{b\in A, b\neq a} \frac{z-b}{a-b} = \frac{w_A(z)}{w'_A(a)(z-a)} \quad \text{where } w_A(z) = \prod_{a\in A}(z-a),\ z \in \mathbb{C}. \tag{2}$$

The *Lebesgue constant* $\Delta(A)$ is the norm of the continuous operator $\mathbf{L}[A] : f \in C(K) \mapsto \mathbf{L}[A; f] \in C(K)$. As is well known, $\Delta(A)$ is the uniform norm on K of the Lebesgue function $\delta(A, \cdot) := \sum_{a\in A} |\ell(A, a; \cdot)|$, that is,

$$\Delta(A) = \left\| \sum_{a\in A} |\ell(A, a; \cdot)| \right\|_K . \tag{3}$$

In the multivariate case ($A \subset \mathbb{C}^N$), Lagrange interpolation polynomials and Lebesgue constants are defined in a similar way, see Section 6, with the (fundamental) differences that not every set A (with a good cardinality) can be taken as an interpolation set and, even when interpolation is possible, in general, the FLIPs no longer have a simple expression—a fact which makes theoretical studies of multivariate Lebesgue constants difficult.

The Lebesgue constant is a basic object of interpolation theory because it controls the stability of interpolation at A as well as the approximation capabilities of the interpolation polynomial via the Lebesgue inequality

$$\|f - \mathbf{L}[A; f]\|_K \leq (1 + \Delta(A))\mathrm{dist}_K(f, \mathcal{P}_d),$$

where d is the degree of interpolation, $\mathcal{P}_d$ denotes the space of polynomials of degree at most d and $\mathrm{dist}_K(f, \mathcal{P}_d)$ the uniform distance on K between f and $\mathcal{P}_d$. A large part of classical Lagrange interpolation theory is devoted to the study of Lebesgue constants of natural interpolation points, such as, for instance, the roots of standard orthogonal polynomials. Unlike the classical cases, which, it seems, always deal with arrays of points (when we go from degree $d-1$ to degree d, we take $d+1$ new points), in this note we exhibit *sequences* $(e_k : k \in \mathbb{N})$ in the unit disk $D := \{|z| \leq 1\}$ such that $\Delta(\{e_0, \ldots, e_k\})$ grows at most like $k \ln k$. These sequences are Leja sequences for the unit disk. They are defined by a simple extremal metric property (Section 2). Using classical works of Alper, we may then construct sequences whose Lebesgue constant grows polynomially not only for a disk but also for a large class of plane compact sets (Section 5). The main application (and motivation) of our study is the construction of explicit multivariate interpolation sets (in the Cartesian products on many plane compact sets) having a Lebesgue constant that grows polynomially (Section 6). Very few such examples are currently available. We mention the beautiful Padua points recently discovered in the square of $\mathbb{R}^2$ and which have a Lebesgue constant that grows like $\ln^2 d$ where d is the degree, but their construction seems to be hardly generalizable to the higher dimensional cases; see [5]. The other striking properties of our multivariate interpolation points are that they are nested in the sense that the points used for the degree $d-1$ are still used for the degree d.

Let us finally point out that we may define Leja sequences for every non-empty compact subset of the plane and, in a recent interesting paper, Taylor and Totik [14] showed that the Lebesgue constant of Leja sequences for many plane compact sets has a sub-exponential growth. These sequences took their name from Franciszek Leja which used them in a classical paper on the approximation of exterior conformal mappings [13] but they were first considered by Albert Edrei in 1939; see [10].

2. Leja sequences

2.1. Definition and structure

A k-tuple $E_k = (e_0, \dots, e_{k-1}) \in D^k$ with $e_0 = 1$ is a *k-Leja section* for the unit disk D if, for $j = 1, \dots, k-1$, the $(j+1)$th entry e_j maximizes the product of the distances to the j previous points, that is

$$\prod_{m=0}^{j-1} |e_j - e_m| = \max_{z \in D} \prod_{m=0}^{j-1} |z - e_m|, \quad j = 1, \dots, k-1.$$

The maximum principle implies that all the e_i's actually lie on the unit circle ∂D. A sequence $E = (e_k : k \in \mathbb{N})$ for which $E_k := (e_0, \dots, e_{k-1})$ is a k-Leja section for every $k \in \mathbb{N}$ is called a *Leja sequence* for D.

The first purpose of this note is to estimate the Lebesgue constant $\Delta(E_k)$. As recalled in (3), it is given by

$$\Delta(E_k) = \left\| \sum_{j=0}^{k-1} \left| \ell(E_k, e_j; \cdot) \right| \right\|_D = \left\| \sum_{j=0}^{k-1} \left| \ell(E_k, e_j; \cdot) \right| \right\|_{\partial D}. \tag{4}$$

The second equality is perhaps not obvious. It follows for instance from the maximum principle applied to the Lebesgue function $\delta(E_k, \cdot)$ which is subharmonic on $\mathbb{C}$.

It is not difficult to describe the structure of a Leja sequence for D. The following theorem is proved in [3]. If A is the r-tuple $(a_0, \dots, a_{r-1})$ and B is the s-tuple $(b_0, \dots, b_{s-1})$ we denote by (A, B) the $r+s$-tuple $(a_0, \dots, a_{r-1}, b_0, \dots, b_{s-1})$.

Theorem 1. *The underlying set of a 2^n-Leja section for D is formed of the 2^nth roots of unity. If $E_{2^{n+1}}$ is a 2^{n+1}-Leja section then there exist a 2^n-root ρ of -1 and a 2^n-Leja section U_{2^n} such that $E_{2^{n+1}} = (E_{2^n}, \rho U_{2^n})$.*

Repeated applications of this theorem shows that if E_k is a k-Leja section with $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}, n_1 > n_2 > \cdots > n_s \geq 0$, then the underlying set of E_k is formed of the union of images under certain rotations of the complete sets of roots of unity of order 2^{n_j}, $j = 1, \dots, s$. Also, if $2^n + 1 \leq k \leq 2^{n+1}$, we have $E_k = (E_{2^n}, \rho U_{k-2^n})$.

The structure of a Leja sequence suggests that the binary expansion of k plays a role in the behavior of $\Delta(E_k)$. We also may expect to use a classical result of Gronwall [11,7] showing that the Lebesgue constant for complete sets of roots of unity grows like the logarithm of the degree. Indeed, since E_{2^n} is a complete set of roots of unity of degree 2^n, Gronwall theorem ensures that $\Delta(E_{2^n}) = O(n), n \to \infty$; see below.

Most of our results strongly rely on Theorem 1 and are obtained by successive reductions of the lengths of the Leja sections considered. This will be generally indicated by the use of a laconic expression like ‘by continuing in this way’. Further specific consequences of Theorem 1 are given in Section 2.3.

2.2. An example

Given a sequence of complex numbers $\eta = (\eta_s : s \in \mathbb{N})$ such that $\eta_s^{2^s} = -1$, we may define a Leja sequence $E = E(\eta)$ such that $E_{2^{n+1}} = (E_{2^n}, \eta_n E_{2^n}), n \in \mathbb{N}$. Such a Leja sequence will be said to be *simple*. There are $2^{n(n+1)/2}$ simple 2^{n+1}-Leja sections. The elements of a simple Leja sequence are readily expressed in terms of η.

Lemma 2. *If* $E(\eta) = (a_n : n \in \mathbb{N})$ *and* $k = 2^n + \sum_{j=0}^{n-1} \varepsilon_j 2^j$, $\varepsilon_j \in \{0, 1\}$, *then*

$$a_k = \eta_n \eta_{n-1}^{\varepsilon_{n-1}} \eta_{n-2}^{\varepsilon_{n-2}} \cdots \eta_1^{\varepsilon_1} \eta_0^{\varepsilon_0}, \quad k \geq 1. \tag{5}$$

Proof. Since

$$E_{2^{n+1}} = (E_{2^n}, \eta_n E_{2^n}) = (a_0, \ldots, a_{2^n-1}, \eta_n a_0, \eta_n a_1, \ldots \eta_n a_{2^n-1}),$$

we have $a_k = \eta_n a_{k-2^n}$. Now if i is the biggest index for which $\varepsilon_i = 1$ then $k - 2^n = 2^i + \sum_{j=0}^{i-1} \varepsilon_j 2^j$ and the same reasoning as above gives $a_{k-2^n} = \eta_i a_{k-2^n-2^i}$ so that $a_k = \eta_n \eta_i a_{k-2^n-2^i}$. Continuing in this way, we obtain Eq. (5). □

2.3. *Consequences of Theorem 1*

The first lemma uses the structure theorem to compute the polynomial w_{E_k} (as defined in (2)) for a Leja section E_k and the second one computes the sup norm of this polynomial on D.

Lemma 3. *Let E be a Leja sequence for D. If* $k = 2^{n_1} + \cdots + 2^{n_s}$ *with* $n_1 > n_2 > \cdots > n_s \geq 0$, *then, for every* $z \in \mathbb{C}$, *we have*

$$\prod_{m=0}^{k-1} (z - e_m) = c\left[z^{2^{n_1}} - 1\right] \cdot \left[(z\rho_1^{-1})^{2^{n_2}} - 1\right] \cdot \left[(z\rho_1^{-1}\rho_2^{-1})^{2^{n_3}} - 1\right] \cdots \left[(z\rho_1^{-1} \ldots \rho_{s-1}^{-1})^{2^{n_s}} - 1\right], \tag{6}$$

where $|c| = 1$ *and* $\rho_j^{2^{n_j}} = -1$ *for every* $1 \leq j \leq s - 1$.

Proof. Theorem 1 tells us that $E_k = (E_{2^{n_1}}, \rho_1 U_{k-2^{n_1}})$ with $\rho_1^{2^{n_1}} = -1$ and $U_{k-2^{n_1}}$ a $(k - 2^{n_1})$-Leja section, say $U_{k-2^{n_1}} = (u_0 = 1, u_1, \ldots, u_{k-2^{n_1}-1})$. Since the 2^{n_1} first elements of E form a complete set of roots of unity of degree 2^{n_1}, we have

$$\prod_{m=0}^{k-1} (z - e_m) = \prod_{m=0}^{2^{n_1}-1} (z - e_m) \cdot \prod_{m=2^{n_1}}^{k-1} (z - e_m) \tag{7}$$

$$= [z^{2^{n_1}} - 1] \cdot \prod_{m=0}^{k-2^{n_1}-1} (z - \rho_1 u_m) \tag{8}$$

$$= \rho_1^{k-2^{n_1}} \cdot [z^{2^{n_1}} - 1] \cdot \prod_{m=0}^{k-2^{n_1}-1} (\rho_1^{-1} z - u_m). \tag{9}$$

Now, since $U_{k-2^{n_1}}$ is itself a Leja section and $k - 2^{n_1} = 2^{n_2} + \cdots + 2^{n_s}$ we may factorize the third factor in the same fashion and continuing in this way we arrive at the required expression with $c = \rho_1^{k-2^{n_1}} \rho_2^{k-2^{n_1}-2^{n_2}} \cdots \rho_{s-1}^{2^{n_s}}$. □

Lemma 4. *Under the same assumptions as in Lemma 3, we have*

$$\prod_{m=0}^{k-1} |e_k - e_m| = 2^s. \tag{10}$$

Proof. Eq. (9) in the proof of the previous lemma yields

$$\prod_{m=0}^{k-1} |e_k - e_m| = |e_k^{2^{n_1}} - 1| \cdot \prod_{m=0}^{k-2^{n_1}-1} |\rho_1^{-1} e_k - u_m|.$$

However, as above, in view of Theorem 1, $e_k = \rho_1 u_{k-2^{n_1}}$. Hence, since $u_{k-2^{n_1}}$ is a 2^{n_1}th root of unity and $\rho_1^{2^{n_1}} = -1$, we have

$$\prod_{m=0}^{k-1} |e_k - e_m| = 2 \cdot \prod_{m=0}^{k-2^{n_1}-1} |u_{k-2^{n_1}} - u_m|.$$

Likewise, using the fact that $U_{k-2^{n_1}}$ is a Leja section and $k - 2^{n_1} = 2^{n_2} + \cdots + 2^{n_s}$, we may apply the same idea to $\prod_{m=0}^{k-2^{n_1}-1} |u_{k-2^{n_1}} - u_m|$ to obtain

$$\prod_{m=0}^{k-2^{n_1}-1} |u_{k-2^{n_1}} - u_m| = 2 \prod_{m=0}^{k-2^{n_1}-2^{n_2}-1} |v_{k-2^{n_1}-2^{n_2}} - v_m|,$$

where the v_m are the points of a certain Leja section. Continuing in this way we arrive to (10). □

We now give another consequence of Theorem 1 regarding the form of the FLIPs for Leja points.

Lemma 5. *Let $2^n + 1 \le k \le 2^{n+1} - 1$ and let $E_k = (E_{2^n}, \rho U_{k-2^n})$ be a k-Leja section for D.*
(1) *If $0 \le j \le 2^n - 1$ then*

$$\ell(E_k, e_j; z) = \ell(E_{2^n}, e_j; z) \cdot \prod_{m=2^n}^{k-1} (z - e_m)/(e_j - e_m), \quad z \in \mathbb{C}.$$

(2) *If $2^n \le j \le k - 1$ then*

$$\ell(E_k, e_j; z) = \ell(U_{k-2^n}, u_{j-2^n}; \rho^{-1} z) \cdot (1 - z^{2^n})/2, \quad z \in \mathbb{C}.$$

Proof. We easily check that the polynomials on the right-hand sides are polynomials of degree $k - 1$ that vanish at e_s for $s \ne j$ but take the value 1 at e_j. In the second case, we need again to use that, when $2^n \le j \le k - 1 (\le 2^{n+1} - 2)$, $e_j^{2^n} = -1$ or, equivalently, $\rho^{2^n} = -1$. □

3. The estimates on the Lebesgue constants

3.1. Upper bound

Here is the key estimate from which the more general statements presented in the last two sections are derived.

Theorem 6. *Let $2^n + 1 \le k \le 2^{n+1} - 1$. If $E_k = (E_{2^n}, \rho U_{k-2^n})$ is a k-Leja section for D then*

$$\Delta(E_k) \le 2^n \Delta(E_{2^n}) + \Delta(U_{k-2^n}). \tag{11}$$

The proof of this result is given in Section 4. Our result on the asymptotic behavior of the Lebesgue constant of a Leja sequence is an immediate consequence of Theorem 6.

Corollary 7. *Let E be a Leja sequence for D. As $k \to \infty$, $\Delta(E_k) = O(k \ln k)$.*

The constant involved in the notation O does not depend on E.

Proof. First recall that since E_{2^n} is a complete set of 2^n-roots of unity, the theorem of Gronwall cited above gives $\Delta(E_{2^n}) = O(\ln(2^n)) = O(n)$. Hence, in view of inequality (11), we have

$$\Delta(E_k) = O\left(n2^n\right) + \Delta(U_{k-2^n}), \quad 2^n + 1 \le k \le 2^{n+1} - 1.$$

Since U_{k-2^n} itself is a Leja section, we may bound its Lebesgue constant in the same fashion. Continuing in this way, if $k = 2^n + \sum_{j=0}^{n-1} \varepsilon_j 2^j$, $\varepsilon_j \in \{0, 1\}$, we arrive at

$$\Delta(E_k) = O\left(n2^n + \sum_{j=0}^{n-1} j2^j \varepsilon_j\right) = O(k \ln k). \quad \square \tag{12}$$

3.2. Lower bound

As shown by the next result, the Lebesgue constant of any Leja sequence cannot grow slower than k. We conjecture that $\Delta(E_k) \le k$ for every k.

Theorem 8. *For every Leja sequence E and every $n \in \mathbb{N}^\star$ we have $\Delta(E_{2^n-1}) = 2^n - 1$.*

Proof. We know that E_{2^n-1} is formed of the 2^nth roots of unity with only one missing. The property is therefore a consequence of Theorem 9. $\square$

Theorem 9. *Let $a_k = \exp(2ik\pi/n)$, $R = \{a_k : k = 0, \dots, n-1\}$ and $R^j = R \setminus \{a_j\}$ with $0 \le j \le n-1$. Then we have $\Delta(R^j) = n - 1$.*

Lemma 10. *Let a be an nth root of unity, $n \ge 3$, $a \ne 1$. We have*

$$\left|\frac{z^n - 1}{z - 1}\right| \cdot |a - 1| \cdot \left[\frac{1}{|z - a|} + \frac{1}{|z - \overline{a}|}\right] \le 2n, \quad |z| = 1. \tag{13}$$

Proof. We call $F(z, a)$ the left-hand side of (13). Since F is invariant by conjugation, we may assume that $\arg(z) \in [0, \pi]$ and $\arg(a) \in]0, \pi]$. Setting $z = \exp(i\theta)$ and $a = \exp(i\phi)$, a simple calculation shows that

$$\frac{F(z, a)}{|a - 1||z^n - 1|} = \frac{1}{4}\left[\left|\frac{1}{\sin(\theta/2)\sin((\theta - \phi)/2)}\right| + \left|\frac{1}{\sin(\theta/2)\sin((\theta + \phi)/2)}\right|\right].$$

(A) We assume $0 < \phi \le \theta \le \pi$. In that case, $\sin(\theta/2)$, $\sin((\theta - \phi)/2)$ and $\sin((\theta + \phi)/2)$ are nonnegative, hence

$$\frac{F(z, a)}{|a - 1||z^n - 1|} = \frac{1}{4}\left[\frac{1}{\sin(\theta/2)\sin((\theta - \phi)/2)} + \frac{1}{\sin(\theta/2)\sin((\theta + \phi)/2)}\right].$$

Returning to z and a, using $|1 - a| = |1 - \overline{a}|$ and an easily checked expansion, we find

$$\begin{aligned} F(z, a) &= |a - 1|\left|a\frac{z^n - 1}{(z - 1)(z - a)} + \frac{z^n - 1}{(z - 1)(z - \overline{a})}\right| \\ &= \left|\frac{(a - 1)(z^n - 1)}{(z - 1)(z - a)} + \frac{(1 - \overline{a})(z^n - 1)}{(z - 1)(z - \overline{a})}\right| \\ &= \left|\sum_{k=0}^{n-2} z^k(a^{n-k-1} - \overline{a}^{n-k-1})\right| \le \sum_{k=0}^{n-2} |2\sin((k + 1)\phi)| \le 2(n - 1) < 2n. \end{aligned} \tag{14}$$

(B) We now assume $0 \le \theta \le \phi \le \pi$. In that case, $\sin(\theta/2), \sin((\theta+\phi)/2)$ and $-\sin((\theta-\phi)/2)$ are nonnegative. Thus

$$\frac{F(z,a)}{|a-1||z^n-1|} = \frac{1}{4}\left[\frac{-1}{\sin(\theta/2)\sin((\theta-\phi)/2)} + \frac{1}{\sin(\theta/2)\sin((\theta+\phi)/2)}\right],$$

and working as in the previous case, we get

$$\begin{aligned}
F(z,a) &= |a-1|\left|-a\frac{z^n-1}{(z-1)(z-a)} + \frac{z^n-1}{(z-1)(z-\overline{a})}\right| \\
&= \left|\frac{(1-a)(z^n-1)}{(z-1)(z-a)} + \frac{(1-\overline{a})(z^n-1)}{(z-1)(z-\overline{a})}\right| \\
&= \left|\sum_{k=0}^{n-2} z^k\Big(2 - a^{n-k-1} - \overline{a}^{n-k-1}\Big)\right| = \left|\sum_{k=0}^{n-2} z^k\Big(2 - 2\cos\big((k+1)\phi\big)\Big)\right| \\
&\le \sum_{k=0}^{n-2}\Big(2 - 2\cos\big((k+1)\phi\big)\Big) = \sum_{k=1}^{n-1}\Big(2-2\cos(k\phi)\Big) \\
&= 2(n-1) - 2\sum_{k=1}^{n-1}\cos(k\phi) = 2n. \quad \square
\end{aligned}$$

Proof of Theorem 9. Since Lebesgue constants are invariant under rotation, we may assume that $j = 0$ so that the missing point a_j equals 1.

We first prove $\Delta(R^0) \ge n-1$. In view of (2), with $w_{R^0}(z) = w(z) = (z^n-1)/(z-1)$, the FLIPs for R^0 are given by

$$\ell(R^0, a; z) = \frac{w(z)}{w'(a)(z-a)} = \frac{(a-1)}{na^{n-1}}\frac{z^n-1}{(z-1)(z-a)}.$$

We have $w(1) = n$ and it follows that

$$\Delta(R^0) \ge \sum_{a^n=1, a\neq 1} |\ell(R^0, a; 1)| = \sum_{a^n=1, a\neq 1} 1 = n-1.$$

This shows that $\Delta(R^0) \ge n-1$.

To prove the converse, we first assume that n is odd so that the interpolation points can be written as

$$R^0 = \{1\} \cup \bigcup_{a\in B}\{a, \overline{a}\},$$

with $\sharp B = (n-1)/2$. Then

$$\begin{aligned}
\sum_{a^n=1, a\neq 1} |\ell(R^0, a; z)| &= \sum_{a\in B}\big(|\ell(R^0, a; z)| + |\ell(R^0, \overline{a}; z)|\big) \qquad (15)\\
&= \frac{1}{n}\sum_{a\in B}\left|\frac{z^n-1}{z-1}\right| \cdot |a-1| \cdot \left[\frac{1}{|z-a|} + \frac{1}{|z-\overline{a}|}\right] \\
&\le \frac{n-1}{2}\frac{2n}{n} = n-1, \quad |z| = 1, \qquad (16)
\end{aligned}$$

where we used the estimate given in Lemma 10. This shows that $\Delta(R^0) \le n-1$. (Recall that, according to (4), it suffices to bound the Lebesgue function on the unit circle.) When n is even, the proof is similar with the sole difference that $a=-1$ must be treated separately. □

4. Proof of Theorem 6

The starting point is Eq. (4) which gives

$$\Delta(E_k) \le \left\| \sum_{j=0}^{2^n-1} \left|\ell(E_k, e_j; \cdot)\right| \right\|_D + \left\| \sum_{j=2^n}^{k-1} \left|\ell(E_k, e_j; \cdot)\right| \right\|_D, \quad 2^n+1 \le k \le 2^{n+1}-1. \tag{17}$$

The estimate of the first sum is the difficult part. Our bound is based on the following lemma.

Lemma 11. *Let $E_k = (e_0, \ldots, e_{k-1})$ be a k-Leja section with $0 < k \le 2^n - 1$. If a is a 2^n-root of -1 then $\prod_{m=0}^{k-1} |e_k - e_m| \le 2^n \prod_{m=0}^{k-1} |a - e_m|$.*

To prove this lemma we need the following classical inequalities that we state as a lemma.

Lemma 12. (1) *If $0 \le \alpha \le \pi/2$ then $\sin\alpha \ge 2\alpha/\pi$.*
(2) *If $m \in \mathbb{N}^\star$ and $\alpha \in \mathbb{R}$ then $2^m|\sin\alpha| \ge 2^m|\sin\alpha\cos\alpha| \ge |\sin 2^m\alpha|$.*

Proof. The second inequality follows at once from repeated applications of

$$|\sin\alpha| \ge |\sin\alpha\cos\alpha| = |\sin 2\alpha|/2. \quad \square$$

Proof of Lemma 11. We assume that

$$k = 2^{n_1} + \cdots + 2^{n_s} \quad \text{with } n-1 \ge n_1 > \cdots > n_s \ge 0, \tag{18}$$

and use the same notation as in Lemma 3. In particular $\rho_j^{2^{n_j}} = -1$ so that for some $t_j \in \mathbb{N}$,

$$\theta_j := \arg(\rho_j^{-1}) = (2t_j+1)\pi/2^{n_j}, \quad 1 \le j \le s-1. \tag{19}$$

Eq. (6) yields

$$\prod_{m=0}^{k-1} |a - e_m| = 2^s \prod_{j=0}^{s-1} |\sin 2^{n_{j+1}-1}(\theta_0 + \cdots + \theta_j)|,$$

where $\arg a = \theta_0 = (2t_0+1)\pi/2^n$. Thus, in view of (10), the lemma will be proved if we show that

$$\prod_{j=0}^{s-1} \left|\sin 2^{n_{j+1}-1}(\theta_0 + \cdots + \theta_j)\right| \ge 1/2^n. \tag{20}$$

We first treat the case $s=1$, that is, $k = 2^{n_1}$. Here we just need to prove that

$$\left|\sin 2^{n_1-1}\theta_0\right| \ge 1/2^n.$$

Since

$$2^{n_1-1}\theta_0 = \pi/2^{n-n_1+1} + 2t_0\pi/2^{n-n_1+1},$$

we have

$$\left|\sin\left(2^{n_1-1}\theta_0\right)\right| \geq \sin\left(\pi/2^{n-n_1+1}\right) \geq (2/\pi)\pi/2^{n-n_1+1} \geq 1/2^n,$$

where we use Lemma 12(1).

We now assume $s \geq 2$ in (18). We first look at the factor corresponding to $j = s-1$ in (20). Applying Lemma 12(2) with $m = n_{s-1} - n_s$ and $\alpha = 2^{n_s-1}(\theta_0 + \cdots + \theta_{s-1})$, we obtain

$$\left|\sin 2^{n_s-1}(\theta_0 + \cdots + \theta_{s-1})\right| \geq 2^{n_s-n_{s-1}} \left|\sin 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-1})\right|. \tag{21}$$

But, in view of (19),

$$2^{n_{s-1}-1}\theta_{s-1} = 2^{n_{s-1}-1}(2t_{s-1}+1)\pi/2^{n_{s-1}} = \pi/2 + t_{s-1}\pi$$

which gives

$$\left|\sin 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-1})\right| = \left|\cos 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-2})\right|. \tag{22}$$

Hence, (21) becomes

$$\left|\sin 2^{n_s-1}(\theta_0 + \cdots + \theta_{s-1})\right| \geq 2^{n_s-n_{s-1}} \left|\cos 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-2})\right|. \tag{23}$$

We now concentrate on the last two factors of (20), those corresponding to $j = s-1$ and $j = s-2$. Thanks to (23), we have

$$\prod_{j=s-2}^{s-1} \left|\sin 2^{n_{j+1}-1}(\theta_0 + \cdots + \theta_j)\right| \geq 2^{n_s-n_{s-1}} \left|\sin 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-2}) \cdot \cos 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-2})\right|. \tag{24}$$

Another use of Lemma 12(2) with $m = n_{s-2} - n_{s-1}$ yields

$$\left|\sin 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-2}) \cdot \cos 2^{n_{s-1}-1}(\theta_0 + \cdots + \theta_{s-2})\right| \geq 2^{n_{s-1}-n_{s-2}} \left|\sin 2^{n_{s-2}-1}(\theta_0 + \cdots + \theta_{s-2})\right| \tag{25}$$

and, again, since

$$2^{n_{s-2}-1}\theta_{s-2} = 2^{n_{s-2}-1}(2t_{s-2}+1)\pi/2^{n_{s-2}} = \pi/2 + t_{s-2}\pi,$$

the absolute value of the sine on the right-hand side of (25) actually equals $|\cos 2^{n_{s-2}-1}(\theta_0 + \cdots + \theta_{s-3})|$. Thus, at this point, taking into account (24) and (25), we have

$$\prod_{j=s-3}^{s-1} \left|\sin 2^{n_{j+1}-1}(\theta_0 + \cdots + \theta_j)\right| \geq 2^{n_s-n_{s-2}} \left|\sin 2^{n_{s-2}-1}(\theta_0 + \cdots + \theta_{s-3}) \cdot \cos 2^{n_{s-2}-1}(\theta_0 + \cdots + \theta_{s-3})\right|. \tag{26}$$

Continuing in this fashion, we finally arrive at

$$\prod_{j=0}^{s-1} \left|\sin 2^{n_{j+1}-1}(\theta_0 + \cdots + \theta_j)\right| \geq 2^{n_s-n_1} \left|\sin 2^{n_1-1}\theta_0 \cdot \cos 2^{n_1-1}\theta_0\right| = 2^{n_s-n_1-1}|\sin 2^{n_1}\theta_0|.$$

Now, working as in the case $s = 1$ above, we obtain

$$2^{n_s-n_1-1}|\sin 2^{n_1}\theta_0| \geq 2^{n_s-n_1-1}(2/\pi)\pi/2^{n-n_1} = 2^{n_s-n} \geq 2^{-n}.$$

This completes the proof of the lemma. □

Conclusion of the proof of Theorem 6. We assume that $E_{2^{n+1}} = (E_{2^n}, \rho U_{2^n})$ and $2^n + 1 \leq k \leq 2^{n+1} - 1$. As indicated above, starting from (17), we want to estimate

$$\left\| \sum_{j=0}^{2^n-1} |\ell(E_k, e_j; \cdot)| \right\|_D \quad \text{and} \quad \left\| \sum_{j=2^n}^{k-1} |\ell(E_k, e_j; \cdot)| \right\|_D .$$

(A) Thanks to Lemma 5(2), we have

$$\left\| \sum_{j=2^n}^{k-1} \left|\ell(E_k, e_j; \cdot)\right| \right\|_D \leq \left\| \frac{|1 - z^{2^n}|}{2} \right\|_D \left\| \sum_{j=2^n}^{k-1} \left|\ell(U_{k-2^n}, u_{j-2^n}; \rho^{-1}\cdot)\right| \right\|_D = \Delta(U_{k-2^n}).$$

(B) On the other hand, in view of Lemma 5(1), for every z in D,

$$\sum_{j=0}^{2^n-1} |\ell(E_k, e_j; z)| = \sum_{j=0}^{2^n-1} \left\{ \left|\ell(E_{2^n}, e_j; z)\right| \prod_{m=2^n}^{k-1} |z - e_m|/|e_j - e_m| \right\}$$
$$\leq \Delta(E_{2^n}) \max_{j=0,\ldots,2^n-1} \prod_{m=2^n}^{k-1} |z - e_m|/|e_j - e_m|.$$

Hence, to prove the theorem, it suffices to show that

$$\left\| \prod_{m=2^n}^{k-1} \frac{|z - e_m|}{|e_j - e_m|} \right\|_D \leq 2^n, \quad 0 \leq j \leq 2^n - 1.$$

To see this, we observe that for $0 \leq j \leq 2^n - 1$ and $z \in D$, we have

$$\prod_{m=2^n}^{k-1} |z - e_m| = \prod_{m=0}^{k-1-2^n} |\rho^{-1}z - u_m| \leq \prod_{m=0}^{k-1-2^n} |u_{k-2^n} - u_m|$$
$$\leq 2^n \prod_{m=0}^{k-1-2^n} |\rho^{-1}e_j - u_m| = 2^n \prod_{m=2^n}^{k-1} |e_j - e_m|, \tag{27}$$

where the equalities come from the relation $E_{2^{n+1}} = (E_{2^n}, \rho U_{2^n})$; the first inequality follows from the fact that U_{2^n} is a Leja section and the second inequality is given by Lemma 11. The use of this lemma is permitted since $(\rho^{-1}e_j)^{2^n} = -1$. Indeed $\rho^{2^n} = -1$ and, since $0 \leq j \leq 2^n - 1$, $e_j^{2^n} = 1$. □

5. Alper-smooth Jordan curves

Using classical works of Alper [1,2] as in [3], we now show that if K is a compact set whose boundary is an Alper-smooth Jordan curve (see below) and ϕ denotes the exterior conformal mapping from $\overline{\mathbb{C}} \setminus D$ onto $\overline{\mathbb{C}} \setminus K$ then the image under ϕ of every Leja sequence for the disk (which lies in ∂K) still has a Lebesgue constant (with respect to K) that grows (at most) like a polynomial.

Let Γ be a smooth Jordan curve. The angle between the tangent at $\Gamma(s)$ and the positive real axis is denoted by $\theta(s)$ where s is the arc-length parameter. Following a terminology used by

Kövari and Pommerenke [12], we say that Γ is *Alper-smooth* if the modulus of continuity ω of θ satisfies

$$\int_0^h \frac{\omega(x)}{x}|\ln x|\mathrm{d}x < \infty.$$

Twice continuously differentiable Jordan curves are Alper-smooth.

An important property of the exterior conformal mapping ϕ is the following (see [1, Sections 1 and 2] or [2, Eq. (3) p. 45]). There exist positive constants $M_1 < M_2$ such that

$$0 < M_1 \leq \left|\frac{\phi(z)-\phi(w)}{z-w}\right| \leq M_2 < \infty, \quad z, w \in \partial D,\ z \neq w. \tag{28}$$

Theorem 13. *Assume that K is a compact set whose boundary is an Alper-smooth Jordan curve. We denote by ϕ the conformal mapping of the exterior to the unit disk onto the exterior of K. If $E = (e_k : k \in \mathbb{N})$ is a Leja sequence for D then the Lebesgue constant $\Delta(\phi(E_k))$ grows at most like a polynomial in k as $k \to \infty$. Here $\phi(E_k) := (\phi(e_0), \ldots, \phi(e_{k-1}))$.*

We need the following lemma.

Lemma 14. *Under the same assumptions as in the theorem, for any w on the unit circle, $w \neq e_i,\ i = 0, \ldots, k-1$, we have*

$$C^k(K)\frac{1}{c_k} \leq \prod_{l=0}^{k-1} \frac{|\phi(w)-\phi(e_l)|}{|w-e_l|} \leq C^k(K)c_k, \tag{29}$$

where $C(K)$ is the logarithmic capacity of K,

$$c_k = \exp\left(A\sum_{j=0}^{s}\epsilon_j\right), \quad k = \sum_{j=0}^{s}\epsilon_j 2^j,\ \epsilon_j \in \{0, 1\},$$

and A is a positive constant depending only on K.

Proof. The proof can be found in [3, Lemma 3]. It is an adaptation of a method due to Alper. □

Proof of Theorem 13. As in (4), we have

$$\Delta(\phi(E_k)) = \left\|\sum_{j=0}^{k-1}\left|\ell(\phi(E_k), \phi(e_j); \cdot)\right|\right\|_{\partial K}.$$

Thus, since $\partial K = \phi(\partial D)$, we just need to consider terms of the form $|\ell(\phi(E_k), \phi(e_j); \phi(w))|$ with $|w| = 1$.

Now, since, for $w \neq e_l,\ l = 0, \ldots, k-1$,

$$\prod_{l=0, l\neq m}^{k-1} \frac{|\phi(w)-\phi(e_l)|}{|w-e_l|} = \prod_{l=0}^{k-1} \frac{|\phi(w)-\phi(e_l)|}{|w-e_l|} \times \frac{|w-e_m|}{|\phi(w)-\phi(e_m)|}, \quad 0 \leq m \leq k-1.$$

Eqs. (28) and (29) give

$$\frac{C^k(K)}{M_2 c_k} \leq \prod_{l=0, l\neq m}^{k-1} \frac{|\phi(w)-\phi(e_l)|}{|w-e_l|} \leq \frac{C^k(K)c_k}{M_1}.$$

By continuity, this inequality remains true for $w = e_m$. Next, dividing the estimates for w and for e_m, we obtain for $w \neq e_l,\ l \neq m$,

$$\left|\ell\big(\phi(E_k), \phi(e_m), \phi(w)\big)\right| \leq \frac{M_2}{M_1} c_k^2 \cdot \left|\ell\big(E_k, e_m, w\big)\right|.$$

Again by continuity, the above inequality holds for every w on the unit circle. Now applying the inequality for every point e_m we get

$$\Delta(\phi(E_k)) \leq \frac{M_2}{M_1} c_k^2 \Delta(E_k),$$

from which the conclusion readily follows since, in view of Lemma 14, $c_k = O(k^{A/\ln(2)})$. □

As shown by the proof $\Delta(\phi(E_k))$ grows (at most) like $\Delta(E_k)$ apart from the factor $c_k^2 = O(k^{2A/\ln(2)})$. A precise definition of A is given in [3, Proof of Lemma 3, Eq. (44)]. The way this number depends on the geometry of K does not seem to be simple. It would be interesting to have estimates on A using as little information as possible on the Jordan curve that defines K.

6. Multivariate interpolation sets

6.1. *Intertwining of block-unisolvent arrays*

The dimension of the space $\mathcal{P}_n(\mathbb{C}^N)$ of complex polynomials of (total) degree at most n in N complex variables is $\binom{n+N}{n}$. A finite set A formed of $\binom{n+N}{n}$ distinct points is said to be *unisolvent* of degree n if Lagrange interpolation at the points of A by polynomials of degree at most n is well defined. The condition is satisfied if and only if A is not included in an algebraic hypersurface of degree $\leq n$. In that case, the Lagrange interpolation polynomial of a function f, still denoted by $\mathbf{L}[A; f]$, is given by (1) but the FLIPs $\ell(A, a; \cdot)$ no longer have a simple expression. If K is a compact subset in $\mathbb{C}^N$ containing A, the Lebesgue constant $\Delta(A)$ or $\Delta(A \mid K)$ is still defined as the operator norm on $C(K)$ of the interpolation operator and is given by the multivariate form of (3). For basic definitions and facts on multivariate Lagrange interpolation from the complex analysis point of view, the reader may consult [4].

It is useful to label the elements of a unisolvent set with multi-indexes. The length $\sum_{i=1}^N \alpha_i$ of an N-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ is denoted by $|\alpha|$. The indexes are ordered according to the graded lexicographic order $\prec$. Recall that $\alpha \prec \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and the leftmost non-zero entry of $\alpha - \beta$ is negative.

We say that an $\binom{n+N}{n}$-tuple $A = (x_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_N}) : |\alpha| \leq n)$ is *block-unisolvent* of degree n if for every $i,\ i \in \{0, 1, \ldots, n\}$, (the underlying set of) the ith block $\mathcal{B}_i(A) := (x_\alpha : |\alpha| \leq i)$ is unisolvent of degree i. Note that when A is a tuple of unidimensional interpolation points, $\mathcal{B}_i(A)$ is simply formed of the first $i + 1$ entries of A.

Given two *block-unisolvent* families of degree n,

$$A = (x_{\alpha^1} = (x_{\alpha_1^1}, \ldots, x_{\alpha_{N_1}^1}) : |\alpha^1| \leq n) \quad \text{in } \mathbb{C}^{N_1} \quad \text{and}$$
$$B = (y_{\alpha^2} = (y_{\alpha_1^2}, \ldots, y_{\alpha_{N_2}^2}) : |\alpha^2| \leq n) \quad \text{in } \mathbb{C}^{N_2},$$

the *intertwining* of A and B is

$$A \oplus B = \Big((x_{\alpha^1}, y_{\alpha^2}) : \left|(\alpha^1, \alpha^2)\right| \leq n\Big).$$

It is known [8, Theorem 3.1] that $A \oplus B$ is itself block-unisolvent of degree n with blocks

$$\mathcal{B}_i(A \oplus B) = \mathcal{B}_i(A) \oplus \mathcal{B}_i(B), \quad 0 \le i \le n. \tag{30}$$

Note that, by repeating the process, we may construct the intertwining of any finite number of block-unisolvent families of same degree. In particular, iterations of (30) give

$$\mathcal{B}_i(A_1 \oplus A_2 \oplus \cdots \oplus A_N) = \mathcal{B}_i(A_1) \oplus \mathcal{B}_i(A_2) \oplus \cdots \oplus \mathcal{B}_i(A_N). \tag{31}$$

6.2. The Lebesgue constant of the intertwining of two block-unisolvent family

Our main tool is the following result.

Theorem 15. *Let K be a compact set in $\mathbb{C}^{N_1+N_2}$ containing $A \oplus B$. We denote by K_1 (resp., K_2) the projection of K on $\mathbb{C}^{N_1}$ (resp., $\mathbb{C}^{N_2}$). We have*

$$\Delta(A \oplus B|K) \le 4 \binom{n + N_1 + N_2}{n} \sum_{i+j \le n} \Delta\left(\mathcal{B}_i(A)|K_1\right) \cdot \Delta\left(\mathcal{B}_j(B)|K_2\right).$$

Proof. See [8, Theorem 4.4]. □

In the notation for the Lebesgue constant, we indicate the compact set (containing the interpolation points) with respect to which the Lebesgue constant is computed. In order to estimate the Lebesgue constant of $A \oplus B$, we must therefore use a bound for all the blocks of A and B. The above theorem certainly greatly overestimates the Lebesgue constant but it is sufficient to prove our main application in the following subsection.

6.3. Intertwining of Leja sections and related families

For $i = 1, \dots, N$, let $E^{(i)} = (e_n^{(i)} : n \in \mathbb{N})$ denote a Leja sequence for D and K_i a plane compact set whose boundary is an Alper-smooth Jordan curve (with conformal exterior mapping $\phi_i : \overline{\mathbb{C}} \setminus D \to \overline{\mathbb{C}} \setminus K_i$). For every $n \in \mathbb{N}$, we define a family $\mathbf{P}_{N,n}$ as

$$\mathbf{P}_{N,n} = \phi_1(E_{n+1}^{(1)}) \oplus \cdots \oplus \phi_N(E_{n+1}^{(N)}).$$

The $\binom{n+N}{n}$ points of $\mathbf{P}_{N,n}$ lie in $K := K_1 \times K_2 \times \cdots \times K_N \subset \mathbb{C}^N$ and are given by the relation

$$\mathbf{P}_{N,n} = \left(\mathbf{p}_\alpha = \left(\phi_1(e_{\alpha_1}^{(1)}), \dots, \phi_N(e_{\alpha_N}^{(N)})\right) : |\alpha| \le n\right).$$

The family $\mathbf{P}_{N,n}$ is block-unisolvent of degree n in $\mathbb{C}^N$. It is obtained by induction via the relations $\mathbf{P}_{1,n} = \phi(E_{n+1}^{(1)})$ and

$$\mathbf{P}_{d+1,n} = \mathbf{P}_{d,n} \oplus \phi_{d+1}\left(E_{n+1}^{(d+1)}\right), \quad 1 \le d \le N-1. \tag{32}$$

We obtain the following theorem as a consequence of Theorems 13 and 15. The proof relies on the fact that $\mathbf{P}_{N,n}$ is a sub-family of $\mathbf{P}_{N,n+1}$.

Theorem 16. *The Lebesgue constant $\Delta(\mathbf{P}_{N,n})$ grows at most like a polynomial in n as $n \to \infty$.*

Here the Lebesgue constant is computed with respect to K, the Cartesian product of the K_i's.

Proof. The proof is by induction on N.

The case $N = 1$ is given by Theorem 13. We assume that the estimate holds true up to N and prove it for $N + 1$. In view of (32) and Theorem 15, we have

$$\Delta(\mathbf{P}_{N+1,n}) \leq 4 \binom{n+N+1}{n} \sum_{i+j\leq n} \Delta\left(\mathcal{B}_i(\mathbf{P}_{N,n})\right) \cdot \Delta\left(\mathcal{B}_j(\phi(E^{(N+1)}_{n+1}))\right), \tag{33}$$

where we use the previous notation for the blocks of both factors. Now, the important point is that

$$\mathcal{B}_i(\mathbf{P}_{N,n}) = \mathbf{P}_{N,i} \quad \text{and} \quad \mathcal{B}_j(\phi(E^{(N+1)}_{n+1})) = \phi(E^{(N+1)}_{j+1}),$$

where we use (31) for the first equality. Eq. (33) thus becomes

$$\Delta(\mathbf{P}_{N+1,n}) \leq 4 \binom{n+N+1}{n} \sum_{i+j\leq n} \Delta\left(\mathbf{P}_{N,i}\right) \cdot \Delta\left(\phi(E^{(N+1)}_{j+1})\right). \tag{34}$$

Now, in view of Theorem 13, for some constant C_{N+1} we have $\Delta\left(\phi(E^{(N+1)}_{j+1})\right) = O(j^{C_{N+1}})$ and the claim now readily follows from (34) and the induction hypothesis. □

In the case of an intertwining of Leja sequences, inequality (34) together with Corollary 7 yields the (almost certainly pessimistic) bound

$$\Delta\left(E^{(1)}_{n+1} \oplus E^{(2)}_{n+1} \oplus \cdots \oplus E^{(N)}_{n+1}\right) = O\left(n^{(N^2+7N-6)/2}(\ln n)^N\right), \quad n \to \infty.$$

The proof also shows that the intertwining of sequences having a Lebesgue constant growing sub-exponentially also has a Lebesgue constant that grows sub-exponentially. Thus starting from Leja sequences for compact K_i of the kind considered in [14], we obtain sets of interpolation points whose Lebesgue constant grows at most sub-exponentially.

6.4. Application to the construction of weakly admissible meshes

It will be shown in a forthcoming paper that the projections on the real axis of Leja sequences for D still have a Lebesgue constant that grows polynomially. Here, we conclude with a few words on the connection with a topic of recent interest. Let Ω be a compact set in $\mathbb{C}^N$ and for $n \in \mathbb{N}$ a finite subset A_n of Ω. We say that $(A_n : n \in \mathbb{N})$ is a weakly admissible mesh for Ω if the following two conditions are satisfied.

(1) The cardinality of A_n grows sub-exponentially (i.e. $(\sharp A_n)^{1/n} \to 1$ as $n \to \infty$).

(2) There exists a sequence M_n growing sub-exponentially such that

$$\|p\|_\Omega \leq M_n \|p\|_{A_n}, \quad p \in \mathcal{P}_n(\mathbb{C}^N).$$

Admissible meshes are good evaluation points for approximation by discrete least squares polynomials [9]. They also contain good points for Lagrange interpolation that, in principles, can be numerically retrieved [6]. However, for computational reasons, it is desirable to have both M_n and the cardinality of the A_n as small as possible; see [6]. Both conditions however compete with each other and such meshes are not easy to produce. An acceptable compromise is obtained with meshes having both the cardinality of A_n and the constant M_n growing at most polynomially. Now, it is readily seen that (new) examples of such meshes are given by finite unions of images of sets of the form $\mathbf{P}_{N,n}$ under affine mappings (for the union of the corresponding compact sets).

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