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Modules with many homomorphisms

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Abstract

It is proved that if R is a right FBN ring then a non-zero right R-module M has the property that $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule N of M if and only if $\mathrm{Hom}_R(M, R/P) \neq 0$ for every associated prime ideal P of M. One consequence is that over a commutative Noetherian ring R, $\mathrm{Hom}_R(X, Y) \neq 0$ for every non-zero projective R-module X and every non-zero submodule Y of X. In case R is a left Noetherian right FBN ring, then a non-zero finitely generated right R-module M has the property that $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule N of M if and only if the right (R/P)-module M/MP is not torsion for every associated prime ideal P of M. Finally, if R is a commutative Noetherian ring and M is an R-module such that $\mathrm{Hom}_R(M, R) \neq 0$ then $\mathrm{Hom}_R(M, M') \neq 0$ for every non-zero R-module M'. It is shown that this result does not extend to prime Noetherian PI rings.

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1. Slightly compressible modules

Throughout this note all rings are associative and all modules are unital right modules. Let R be a ring. An R-module M is called *compressible* if for every non-zero submodule N of M there exists a monomorphism from M to N. In [21], Zelmanowitz calls a module M *weakly compressible* if, for every non-zero submodule N of M, there exists a non-zero homomorphism $f : M \to N$ such that f^2 is non-zero. We shall call an R-module

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M slightly compressible if, for every non-zero submodule N of M, there exists a non-zero homomorphism from M to N.

Note that compressible modules are weakly compressible and weakly compressible modules are slightly compressible. Clearly any semisimple module is weakly compressible because every submodule is a direct summand. However, a non-zero semisimple module is compressible if and only if it is simple. Note next that the R-module R, and more generally any free R-module, is slightly compressible. However, if R is a commutative ring then the R-module R is weakly compressible if and only if R is a semiprime ring. For, if R is not semiprime then there exists a non-zero element a in R such that $a^2 = 0$. It is easy to check that every homomorphism $f : R \to aR$ satisfies $f^2 = 0$. Thus the R-module R is not weakly compressible. Conversely, if the ring R is semiprime then every endomorphism g of R satisfies $g^2 \neq 0$, so that the R-module R is weakly compressible.

Let I be any proper ideal of an arbitrary ring R. Then the right R-module R/I is slightly compressible. For, any non-zero submodule of R/I has the form E/I for some right ideal E of R properly containing I. Let $a \in E \backslash I$. Then the mapping $f : R/I \to E/I$ defined by $f(r + I) = ar + I$ $(r \in R)$ is a non-zero homomorphism. In this note we shall study slightly compressible modules and investigate in particular when projective modules are slightly compressible. The first lemma gives circumstances under which a right ideal is not slightly compressible.

Lemma 1.1. *Let A be a non-zero idempotent right ideal of a ring R such that A contains a non-zero right ideal B of R such that $BA = 0$. Then the right R-module A is not slightly compressible.*

Proof. Suppose that $f : A \to B$ is a homomorphism. Then

$$f(A) = f(A^2) = f(A)A \subseteq BA = 0.$$

It follows that A is not slightly compressible. □

Using Lemma 1.1 we next give an example of a projective right ideal A of a ring R such that A is not a slightly compressible module.

Example 1.2. Let S be any (non-zero) ring and let R denote the ring of 2×2 upper triangular matrices over S. Let A be the right ideal of R consisting of all matrices in R with bottom row zero. Then A is a cyclic projective right R-module which is not slightly compressible.

Proof. Note first that $A = eR$ where e is the idempotent element of R with 1 as its (1,1) entry and all other entries 0. Thus A is cyclic, idempotent and projective. Let B denote the right ideal of R consisting of all matrices with all entries 0 except for possibly the (1,2) entry. Then B is contained in A and $BA = 0$. By Lemma 1.1, A is not slightly compressible. □

Note that in Example 1.2 the ring R is not semiprime. For semiprime rings we have the following result.

Proposition 1.3. *Let R be a semiprime ring. Then every right ideal of R is a slightly compressible R-module.*

Proof. Let B be any non-zero right ideal of R such that B is contained in A. Then B^2 is non-zero so that BA is non-zero. There exists an element b of B such that bA is non-zero. Define a non-zero homomorphism $f : A \to B$ by $f(a) = ba$ for all a in A. □

Let R be a ring. In general the class of slightly compressible R-modules is not closed under taking submodules, factor modules or extensions. In Example 1.2 the right R-module R is slightly compressible but the submodule A is not. Moreover, A is also a homomorphic image of R and is not slightly compressible. Next, in Example 1.2 let S denote a field. Then the right ideal B in Example 1.2 is a simple R-module and the factor module A/B is also simple. Clearly B and A/B are both (slightly) compressible but A is not slightly compressible. The situation for direct sums is much better, however, as the next result shows.

Proposition 1.4. *Let R be any ring. Then any direct sum of slightly compressible R-modules is slightly compressible.*

Proof. Let an R-module $M = \bigoplus_{i\in I} M_i$ be a direct sum of slightly compressible submodules M_i $(i \in I)$. Let N be any non-zero submodule of M. Let m be any non-zero element of N. There exists a finite subset I' of I such that m belongs to $\bigoplus_{i\in I'} M_i$. If there exists a non-zero homomorphism $f : \bigoplus_{i\in I'} M_i \to mR$ and $p : M \to \bigoplus_{i\in I'} M_i$ is the canonical projection then $fp : M \to mR$ is a non-zero homomorphism and hence $\mathrm{Hom}_R(M, N)$ is non-zero. Thus without loss of generality we can suppose that I is a finite set.

By induction it is sufficient to prove the result when $I = \{1, 2\}$. Let $M = M_1 \oplus M_2$ be a direct sum of slightly compressible submodules M_1 and M_2. Let L be a non-zero submodule of M. If $L \cap M_1$ is non-zero then there exists a non-zero homomorphism from M_1 to $L \cap M_1$ and hence there exists a non-zero homomorphism from M to L. If $L \cap M_1$ is zero then L is isomorphic to a submodule K of M_2. There exists a non-zero homomorphism from M_2 to K and hence there exists a non-zero homomorphism from M to N. □

For some rings every projective module is slightly compressible. A ring R is called a *right V-ring* if every simple right R-module is injective. Kaplansky proved that a commutative ring R is a V-ring if and only if R is von Neumann regular (see [16]). Cozzens [5] and Koifman [12] give examples off right Noetherian domains which are right V-rings but are not division rings.

Theorem 1.5. *Let R be a right V-ring. Then every projective right R-module is slightly compressible.*

Proof. Let X be any non-zero projective R-module. Let x be any non-zero element of X. Let Y be any maximal submodule of xR. Then xR/Y is injective and hence is a direct summand of X/Y. It follows that X has a submodule Z such that X/Z is isomorphic to xR/Y. Hence there exists a non-zero homomorphism $f : X \to xR/Y$. Because X is a projective module, f can be lifted to a non-zero homomorphism $g : X \to xR$. It follows that X is slightly compressible. □

For any ring R and R-module M, we shall denote the injective hull of M by $\mathrm{E}(M)$. Following [9, p. 71], we say that the module M has *finite rank* provided $\mathrm{E}(M)$ is a finite direct sum of indecomposable modules. A non-zero R-module U is called *uniform* provided the intersection of any two non-zero submodules of U is non-zero. In [10, Proposition 4.11] it is proved that a non-zero module M has finite rank if and only if there exists an essential submodule L of M such that L is a finite direct sum of uniform submodules.

In contrast to the situation for projective modules, it is unusual for an injective module to be slightly compressible, especially in case it is non-singular. Recall that if R is any ring then a right R-module M is *non-singular* if $mE \neq 0$ for every non-zero element m of M and essential right ideal E of R.

Lemma 1.6. *Let R be any ring and let M be a right R-module such that every non-zero submodule contains a non-zero direct summand of M. Then M is slightly compressible.*

Proof. Clear. $\square$

Corollary 1.7. *Let R be any ring and let M be a non-singular injective right R-module. Then M is slightly compressible if and only if every non-zero submodule contains a non-zero direct summand of M.*

Proof. The sufficiency follows by Lemma 1.6. Conversely, suppose that M is slightly compressible. Let N be any non-zero submodule of M. By hypothesis, there exists a non-zero homomorphism $f : M \to N$. Because $M/\ker f$ is non-singular, being isomorphic to a submodule of N, $\ker f$ is injective and hence is a direct summand of M. It follows that $f(E)$ is injective. Thus $f(E)$ is a non-zero direct summand of M and $f(E) \subseteq N$. $\square$

Recall that a ring R is called *right hereditary* if every right ideal is projective, equivalently if every submodule of every projective right R-module is projective. A second consequence of Lemma 1.6 is the following result.

Corollary 1.8. *Let R be a right hereditary ring and let M be an injective right R-module. Then M is slightly compressible if and only if every non-zero submodule contains a non-zero direct summand of M.*

Proof. Note that every homomorphic image of M is injective by [1, p. 215, Example 10]. The result follows by the proof of Corollary 1.7. $\square$

Corollary 1.9. *Let R be any ring and let M be a slightly compressible injective right R-module. Then any non-singular uniform submodule of M is simple and injective.*

Proof. Let U be a non-singular uniform submodule of M. Let V be any non-zero submodule of U. By adapting the proof of Corollary 1.7, there exists a non-zero injective submodule W of V. Then $W = V = U$. Thus U is simple injective. $\square$

Corollary 1.10. *Let R be a right hereditary ring and let M be a slightly compressible injective right R-module. Then any uniform submodule of M is simple and injective.*

Proof. By Corollary 1.8 and the proof of Corollary 1.9. □

Let R be any ring and let M be any R-module. For an element m of M we set $\mathbf{r}(m) = \{r \in R : mr = 0\}$. Note that $\mathbf{r}(m)$ is a right ideal of R and $R/\mathbf{r}(m) \cong mR$ for all $m \in M$.

Theorem 1.11. *Let R be any ring and let M be a non-singular injective right R-module such that mR has finite rank for every element m of M. Then M is slightly compressible if and only if M is semisimple.*

Proof. The sufficiency is clear. Conversely, suppose that M is slightly compressible. Let $m \in M$. Then $R/\mathbf{r}(m)$ is a non-singular R-module which has finite rank. By [6, Section 5.10] R satisfies the ascending chain condition on right ideals of the form $\mathbf{r}(m)$ where $m \in M$. Now [6, Section 8.2] gives that M is a direct sum of uniform submodules. Finally, by Corollary 1.9 M is semisimple. □

A ring R is called *right non-singular* if the right R-module R, which we shall denote by R_R, is non-singular. Theorem 1.11 has the following consequence.

Corollary 1.12. *Let R be a right non-singular right Goldie ring. Then $E(R_R)$ is a slightly compressible right R-module if and only if R is a semiprime Artinian ring. In this case, every right R-module is slightly compressible.*

Proof. The sufficiency is clear. Conversely, suppose that $E(R_R)$ is a slightly compressible R-module. Note that $E(R_R)$ is non-singular and has finite rank. By Theorem 1.11 the R-module $E(R_R)$ is semisimple and hence the R-module R is semisimple, as required. The last part is clear. □

Corollary 1.12 shows that, in contrast to Theorem 1.5, even for right V-rings R not every non-zero R-module is slightly compressible. In particular, if R is one of the rings produced by Cozzens and Koifman the R-module $E(R_R)$ is not slightly compressible. Corollary 1.12 has the following further consequence.

Corollary 1.13. *Let R be a right Noetherian ring such that every non-zero right R-module is slightly compressible. Then R is right Artinian.*

Proof. Let P be any prime ideal of R. Let X denote the R/P-module which is the injective hull of the (R/P)-module R/P. Then X is an R-module. By hypothesis, X is slightly compressible as an R-module and hence also as an (R/P)-module. Now Corollary 1.12 gives that R/P is a right Artinian ring. Thus R/P is a right Artinian ring for every prime ideal P of R. Because R is right Noetherian, there exist a positive integer n and prime ideals P_i $(1 \leqslant i \leqslant n)$ of R such that $P_1 \cdots P_n = 0$. By considering the chain $R \supseteq P_1 \supseteq P_1P_2 \supseteq \cdots \supseteq P_1 \cdots P_n = 0$, we conclude that the ring R is right Artinian. □

Compare Corollaries 1.12 and 1.13 with the following result.

Proposition 1.14. *Let R be a right and left Noetherian ring such that there exists a non-zero R-homomorphism $f : E(R_R) \to R$. Then R has non-zero right socle.*

Proof. By [4, Corollary 2.4]. □

In view of the above results we note the following fact for prime rings.

Proposition 1.15. *Let R be any prime ring. Then the right R-module $E(R_R)$ is slightly compressible if and only if* $\mathrm{Hom}_R(E(R_R), R) \neq 0$.

Proof. The necessity is clear. Conversely, suppose that there exists a non-zero homomorphism $f : E \to R$, where $E = E(R_R)$. Let N be any non-zero submodule of E. Then $N \cap R$ and $f(E)$ are both non-zero right ideals of R. Because R is prime, we have $(N \cap R)f(E) \neq 0$ and hence $af(E) \neq 0$ for some $a \in N \cap R$. Define a mapping $g : E \to N$ by $g(e) = af(e)$ for all $e \in E$. Clearly g is a non-zero homomorphism. □

Given the above discussion, the next result is not too unexpected!

Theorem 1.16. *Let R be a right hereditary ring and let M be an injective right R-module such that R satisfies the ascending chain condition on right ideals of the form* $\mathbf{r}(m)$ *where* $m \in M$. *Then M is slightly compressible if and only if M is semisimple.*

Proof. By Corollary 1.10 and the proof of Theorem 1.11. □

Note that if R is a right hereditary ring then R is right non-singular so that Corollary 1.12 applies to right hereditary right Noetherian rings. Next we show that for right self-injective rings, non-singular modules are slightly compressible.

Proposition 1.17. *Let R be any right self-injective ring. Then any non-singular right R-module is slightly compressible.*

Proof. Let M be any non-zero non-singular R-module. Let $0 \neq m \in M$. Then $R/\mathbf{r}(m) \cong mR$ and hence $R/\mathbf{r}(m)$ is non-singular. Since R is an injective R-module it follows that $\mathbf{r}(m)$ is an injective R-module and hence so too is mR. Thus every cyclic submodule of M is injective. By Lemma 1.6 M is slightly compressible. □

A ring R is called *right semi-artinian* provided every non-zero right R-module has non-zero socle. Clearly right Artinian rings are right semi-artinian. Many examples of right semi-artinian right V-rings are given in [7]. Note the following result which should be compared with Theorem 1.5.

Proposition 1.18. *Let R be a right semi-artinian right V-ring. Then every right R-module is slightly compressible.*

Proof. Let M be any non-zero right R-module. Let N be any non-zero submodule of M. Let U be a simple submodule of N. Then U is injective so that U is a direct summand of M. Clearly there exists a non-zero homomorphism $f : M \to U$ and hence $\mathrm{Hom}_R(M, N) \neq 0$. □

Note that the converse of Proposition 1.18 is false in general (see Corollary 3.6).

2. Modules over Noetherian rings

We begin this section with two elementary results.

Lemma 2.1. *Let I be an ideal of an arbitrary ring R and let M be a right R-module. Then* $\mathrm{Hom}_R(M, R/I) \neq 0$ *if and only if* $\mathrm{Hom}_{R/I}(M/MI, R/I) \neq 0$.

Proof. Suppose first that there exists a non-zero homomorphism $f : M \to R/I$. Then $f(MI) = f(M)I \subseteq (R/I)I = 0$, so that f induces a non-zero (R/I)-homomorphism $\overline{f} : M/MI \to R/I$ defined by $\overline{f}(m + MI) = f(m)$ for all $m \in M$. Conversely, suppose that there exists a non-zero (R/I)-homomorphism $g : M/MI \to R/I$. Then g is an R-homomorphism. Let $p : M \to M/MI$ denote the canonical epimorphism. Then $gp : M \to R/I$ is a non-zero R-homomorphism. □

Lemma 2.2. *Let R be a right Noetherian ring. Then the following statements are equivalent for a right R-module M.*

(i) *M is slightly compressible.*
(ii) $\mathrm{Hom}_R(M, U)$ *is non-zero for every uniform submodule U of M.*
(iii) $\mathrm{Hom}_R(M, U)$ *is non-zero for every cyclic uniform submodule U of M.*

Proof. (i) ⇒ (ii) ⇒ (iii) Clear.

(iii) ⇒ (i) Let *N* be any non-zero submodule of *M*. Let *m* be any non-zero element of *N*. By hypothesis, the module *mR* is Noetherian and hence *mR* contains a uniform submodule *U*. Let *u* be any non-zero element of *U*. Then *uR* is a cyclic uniform submodule of *M*. By (iii) $\mathrm{Hom}_R(M, uR)$ is non-zero and hence $\mathrm{Hom}_R(M, N)$ is non-zero. □

Let *R* be any ring and let *X* be an *R*-module. The *annihilator* of *X* in *R* will be denoted by $\mathrm{ann}_R(X)$, i.e., $\mathrm{ann}_R(X) = \{r \in R : Xr = 0\}$. Let *M* be a non-zero *R*-module. Following [10, p.32], by an *associated prime ideal* of *M* we mean a prime ideal *P* of *R* such that, for some non-zero submodule *N* of *M*, $P = \mathrm{ann}_R(L)$ for every non-zero submodule *L* of *N*. If *R* is a ring which satisfies the ascending chain condition on ideals and *P* is maximal in $\{\mathrm{ann}_R(N)) : N$ is a non-zero submodule of $M\}$ then *P* is an associated prime ideal of *M* (see [10, Proposition 2.12]). The set of associated prime ideals of *M* will be denoted by $\mathrm{Ass}(M)$. In case *R* is a right Noetherian ring and *U* is a uniform right *R*-module then there exists a unique associated prime ideal *P* of *U* such that $\mathrm{ann}_R(L)$ is contained in *P* for every non-zero submodule *L* of *U* (see [10, Lemma 4.22]) and in this case *P* is called the *assassinator* of *U*. The assassinator of *U* will be denoted by $\mathrm{ass}U$. Non-zero right *R*-modules *M* and M' will be said to be *related* if $\mathrm{Ass}(M) \cap \mathrm{Ass}(M')$ is non-empty.

Lemma 2.3. *Let R be a right Noetherian ring and let N be a non-zero submodule of a non-zero right R-module M. Then* $\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M)$. *In particular M and N are related.*

Proof. Clear. □

Theorem 2.4. *Let R be a right Noetherian ring and let M be a non-zero right R-module such that* $\mathrm{Hom}_R(M, R/P) \neq 0$ *for every associated prime ideal P of M. Then* $\mathrm{Hom}_R(M, M') \neq 0$ *for every right R-module* M' *which is related to M. In particular, M is slightly compressible.*

Proof. Let M' be any non-zero right R-module which is related to M. Let P be any prime ideal of R which belongs to $\mathrm{Ass}(M) \cap \mathrm{Ass}(M')$. There exists a non-zero submodule N' of M' such that $P = \mathrm{ann}_R(L')$ for every non-zero submodule L' of N'. By hypothesis, there exists a non-zero homomorphism $f : M \to R/P$. Then $f(M) = E/P$ for some right ideal E of R properly containing P. Clearly $N'E \neq 0$ and hence there exists $0 \neq x' \in N'$ such that $x'E \neq 0$.

Define a mapping $g : E/P \to M'$ by $g(e + P) = x'e$ $(e \in E)$. Since $x'P = 0$ it follows that g is well defined. Clearly g is an R-homomorphism. Finally $gf : M \to M'$ is a non-zero homomorphism because $gf(M) = x'E \neq 0$. The last part follows by Lemma 2.3. □

Note that in Theorem 2.4 we do not need that R is right Noetherian but merely that the module M has associated prime ideals. Note further that the converse of Theorem 2.4 is false in general because there exist many right Noetherian rings R and non-zero slightly compressible right R-modules M such that $\mathrm{Hom}_R(M, R/P) = 0$ for some associated prime ideal P of M. For example, let R be a simple right Noetherian ring which is not Artinian. Let M be any non-zero semisimple R-module. Clearly 0 is the only associated prime ideal of M. Thus M is slightly compressible (in fact, weakly compressible) but $\mathrm{Hom}_R(M, R) = 0$. However, there are right Noetherian rings R such that $\mathrm{Hom}_R(M, R/P) \neq 0$ for every slightly compressible right R-module M and associated prime ideal P of M and we investigate this next.

A ring R is called *right bounded* if every essential right ideal contains an ideal which is essential as a right ideal. Next a ring R is called *fully right bounded* provided every prime factor ring of R is right bounded. By a *right FBN ring* we shall mean a fully right bounded right Noetherian ring. For example, any module-finite algebra over a commutative Noetherian ring is a right(and left) FBN ring (see [10, Proposition 8.1]) and more generally so too is any right Noetherian PI ring (i.e. ring satisfying a polynomial identity) by [13, Corollary 13.6.6]. For any ring R, a right R-module M will be called *bounded* if R/P is a right bounded ring for every associated prime ideal P of M. There exist right and left Noetherian rings which are not fully right (nor left) bounded such that every unfaithful right R-module is bounded (see [10, Exercise 8B]).

An element c of a general ring R is called *regular* if, given $r \in R$, $rc = 0$ or $cr = 0$ implies $r = 0$. An R-module M will be called *torsion-free* provided, for any given $m \in M$, $mc = 0$ for some regular element c in R implies that $m = 0$. On the other hand, M is called *torsion* if for each x in M there exists a regular element d of R such that $xd = 0$. Note that if R is a semiprime right Goldie ring then a right R-module M is torsion-free if and only if M is non-singular (see, for example, [10, Proposition 6.9]).

Lemma 2.5. *Let P be a prime ideal of a ring R such that the ring R/P is a prime right bounded right Goldie ring. Let U be a uniform right R-module with assassinator P. Then there exist a non-zero submodule V of U and an R-monomorphism* $f : V \to R/P$.

Proof. There exists a non-zero submodule N of U such that $P = \mathrm{ann}_R(L)$ for every non-zero submodule L of N. Consider the right (R/P)-module N. Let $u \in N$ such that $u(c+P)=0$ for some $c \in R$ with the property that $c+P$ is a regular element of the ring R/P. By [10, Lemma 5.5], $(cR+P)/P$, is an essential right ideal of R/P and hence there exists an ideal A of R properly containing P such that $A \subseteq cR+P$. It follows that $uA=0$ and hence $A \subseteq \mathrm{ann}_R(uR)$. But $P = \mathrm{ass}(U)$, so that $u=0$. Thus N is not a torsion right (R/P)-module. By [10, Lemma 6.17] there exist a non-zero submodule V of N and an (R/P)-monomorphism $f : V \to R/P$. Finally, note that f is an R-monomorphism. □

Theorem 2.6. *Let R be a right Noetherian ring. Then the following statements are equivalent for a non-zero bounded right R-module M.*

(i) *M is slightly compressible.*
(ii) $\mathrm{Hom}_R(M, R/P) \neq 0$ *for every associated prime ideal P of M.*
(iii) $\mathrm{Hom}_{R/P}(M/MP, R/P) \neq 0$ *for every associated prime ideal P of M.*
(iv) $\mathrm{Hom}_R(M, M') \neq 0$ *for every non-zero right R-module M' which is related to M.*

Proof. (ii) $\Rightarrow$ (iv) By Theorem 2.4.
(ii) $\Leftrightarrow$ (iii) By Lemma 2.1.
(iv) $\Rightarrow$ (i) By Lemma 2.3.
(i) $\Rightarrow$ (ii) Suppose that M is slightly compressible. Let P be any associated prime ideal of M. There exists a non-zero submodule N of M such that $P = \mathrm{ann}_R(L)$ for every non-zero submodule L of N. Let U be any uniform submodule of N. Clearly $P = \mathrm{ass}(U)$. By Lemma 2.5 there exists a non-zero submodule V of U and a monomorphism $f : V \to R/P$. By hypothesis there exists a non-zero homomorphism $g : M \to V$. Then $fg : M \to R/P$ is a non-zero homomorphism. □

Corollary 2.7. *Let R be a right FBN ring. Then the following statements are equivalent for a non-zero right R-module M.*

(i) *M is slightly compressible.*
(ii) $\mathrm{Hom}_R(M, R/P) \neq 0$ *for every associated prime ideal P of M.*
(iii) $\mathrm{Hom}_{R/P}(M, R/P) \neq 0$ *for every associated prime ideal P of M.*
(iv) $\mathrm{Hom}_R(M, N') \neq 0$ *for every non-zero right R-module M' which is related to M.*

Proof. By Theorem 2.6. □

Corollary 2.8. *Let R be a right Noetherian ring and let M be a non-zero slightly compressible bounded right R-module. Then $M \neq MP$ for every associated prime ideal P of M.*

Proof. Clear by Theorem 2.6. □

Corollary 2.8 can be improved in the case of projective modules, as we see in the next result (see also Corollary 2.12).

Corollary 2.9. *Let R be a right Noetherian ring. Then any non-zero bounded projective right R-module X is slightly compressible if and only if $X \neq XP$ for every associated prime ideal P of X.*

Proof. The necessity is clear by Corollary 2.8. Conversely, suppose that $X \neq XP$ for any associated prime ideal P of X. Let Q be any associated prime ideal of X. By hypothesis, X/XQ is a non-zero projective (R/Q)-module and hence $\mathrm{Hom}_{R/Q}(X/XQ, R/Q)$ is non-zero. By Theorem 2.6 it follows that X is slightly compressible. □

Let I be any (two-sided) ideal of a ring R. As usual we set $I^0 = R$. For any ordinal $\alpha \geqslant 0$ we define I^α as follows: $I^{\alpha+1} = I^\alpha I$ and $I^\alpha = \cap_{0 \leqslant \beta < \alpha} I^\beta$ if α is a limit ordinal. In this way we obtain a descending chain of ideals:

$$I \supseteq I^2 \supseteq I^3 \supseteq \cdots \supseteq I^w = \bigcap_{n=1}^{\infty} I^n \supseteq I^{w+1} \supseteq \cdots \supseteq I^\alpha \supseteq I^{\alpha+1} \supseteq \cdots,$$

and there exists an ordinal γ such that $I^\gamma = I^{\gamma+1} = \cdots$. We shall denote I^γ by $\kappa(I)$. We set $1 - I = \{1 - a : a \in I\}$ and say that $1 - I$ is a *right Ore set* if for all $a \in I, r \in R$ there exist $a' \in I, r' \in R$ such that $(1-a)r' = r(1-a')$. Note that if the ideal I is contained in the Jacobson radical of R then $1 - I$ is a right Ore set because in this case, given $a \in I, r \in R$, we have $(1-a)\{(1-a)^{-1}r(1-a)\} = r(1-a)$. Also if an ideal I of a right Noetherian ring has the right *AR* property then $1 - I$ is a right Ore set by [13, Proposition 4.2.9]. Recall that an ideal I of a ring R has the *right AR property* if, for every right ideal E of R, there exists a positive integer n such that $E \cap I^n \subseteq EI$. A consequence of [8, Lemma 5.1] is that every ideal of a commutative Noetherian ring has the (right) AR property.

Lemma 2.10. *Let I be an ideal of a right Noetherian ring R such that $1 - I$ is a right Ore set. Then*

(i) $\kappa(I) = \{r \in R : r(1-a) = 0$ *for some* $a \in I\}$,
(ii) *given* $r_i \in R, a_i \in I\,(1 \leqslant i \leqslant n)$ *such that* $r_i(1-a_i) = 0$ *for all* $1 \leqslant i \leqslant n$, *for some positive integer* n, *then there exists* $a \in I$ *such that* $r_i(1-a) = 0\ (1 \leqslant i \leqslant n)$.

Proof. (i) Let $T = \{1 - b : b \in I\}$. Let $A = \{r \in R : rt = 0$ for some $t \in T\}$. Then A is an ideal of R by [18, Lemma 1.1] and $\overline{T} = \{t + A : t \in T\}$ is a set of regular elements of the ring $\overline{R} = R/A$ by [18, Lemma 1.2]. Moreover, $\overline{T}$ is a right Ore set of $\overline{R}$ and we can form, in the usual way, the partial right (classical) quotient ring of $\overline{R}$ with respect to $\overline{T}$ and we denote this ring by R_T. Note that if $IR_T = \{(r + A)(t + A)^{-1} : r \in I, t \in T\}$ then IR_T is contained in the Jacobson radical J of R_T. By [19, Lemma 2.2],

$$(\kappa(I) + A)/A \subseteq \kappa(I/A) \subseteq \kappa(IR_T) \subseteq \kappa(J).$$

But R_T is a right Noetherian ring and hence $\kappa(J) = 0$ by [11, Theorem 11]. Thus $\kappa(I) \subseteq A$.

(ii) By induction on n, there exists $c \in I$ such that $r_i(1-c) = 0\ (1 \leqslant i \leqslant n-1)$. Also there exists $d \in I$ such that $(1-a_n)(1-d) = (1-c)s$ for some $s \in R$. Let $a = 1-(1-a_n)(1-d) \in I$. Then $r_i(1-a) = 0\ (1 \leqslant i \leqslant n)$. □

Theorem 2.11. *Let R be a right Noetherian ring and let M be a non-zero bounded projective right R-module such that $1 - P$ is a right Ore set for every associated prime ideal P of M. Then M is slightly compressible.*

Proof. Let P be any associated prime ideal of M. There exists a non-zero submodule N of M such that $P = \mathrm{ann}_R(N)$. Suppose that $M = MP$. By [19, Corollary 4.3], $M = M\kappa(P)$. Let $x \in N$. Then $x = m_1r_1 + \cdots + m_nr_n$ for some positive integer n and elements $m_i \in M$, $r_i \in \kappa(P)$ $(1 \leqslant i \leqslant n)$. By Lemma 2.10 there exists $p \in P$ such that $x(1-p) = 0$. But $xp = 0$, so that $x = 0$. It follows that $N = 0$, a contradiction. Then $M \neq MP$. By Corollary 2.9, M is slightly compressible. □

Corollary 2.12. *Let R be a commutative Noetherian ring. Then every non-zero projective R-module is slightly compressible.*

Proof. By Theorem 2.11. □

Next we aim to add to Corollary 2.8 in the case of finitely generated modules. First we prove the following elementary result.

Lemma 2.13. *Let R be a semiprime right Goldie ring and let M be a torsion right R-module. Then* $\mathrm{Hom}_R(M, R) = 0$.

Proof. Let $f : M \to R$ be a homomorphism. For any $m \in M$ there exists a regular element c of R such that $mc = 0$ and hence $f(m)c = f(mc) = 0$ which implies that $f(m) = 0$. Thus $f = 0$. □

Corollary 2.14. *Let R be a right Noetherian ring and let M be a non-zero bounded slightly compressible right R-module. Then the right (R/P)-module M/MP is not torsion for every associated prime ideal P of R.*

Proof. Let P be an associated prime ideal of M. By Theorem 2.6, $\mathrm{Hom}_{R/P}(M/MP, R/P) \neq 0$. Apply Lemma 2.13. □

Let R be a semiprime right Goldie ring and let M be any right R-module. Let $Z(M) = \{m \in M : mc = 0$ for some regular element c of $R\}$. By [10, Proposition 6.9], $Z(M)$ is a torsion module. Moreover, by [10, Proposition 3.29(a)], the R-module $M/Z(M)$ is torsion-free. Lemma 2.13 has a second consequence which will be useful later.

Corollary 2.15. *Let R be a prime right and left Goldie ring and let M be a finitely generated right R-module. Then M is not torsion if and only if* $\mathrm{Hom}_R(M, R) \neq 0$.

Proof. The sufficiency follows by Lemma 2.13. Conversely, suppose that the R-module M is not torsion. Then $M \neq Z(M)$. Let $p : M \to M/Z(M)$ denote the canonical epimorphism. Note that $M/Z(M)$ is a finitely generated torsion-free R-module. By [10, Proposition 6.19] there exists a free module F and a monomorphism $f : M/Z(M) \to F$. It follows that $fp : M \to F$ is a non-zero homomorphism and hence $\mathrm{Hom}_R(M, R) \neq 0$. □

Theorem 2.16. *Let R be a right Noetherian ring and let M be a non-zero finitely generated bounded right R-module such that the ring R/P is left Goldie for every associated prime* *ideal P of M. Then M is slightly compressible if and only if M/MP is not a torsion (R/P)-module for every associated prime ideal P of M.*

Proof. The necessity follows by Corollary 2.14. Conversely, suppose that M/MP is not a torsion (R/P)-module for every associated prime ideal P of M. By Corollary 2.15 $\mathrm{Hom}_{R/P}(M/MP, R/P) \neq 0$ for every associated prime ideal P of M. Finally, M is slightly compressible by Theorem 2.6. □

Corollary 2.17. *Let R be a left Noetherian right FBN ring. Then a non-zero finitely generated R-module is slightly compressible if and only if M/MP is not a torsion (R/P)-module for every associated prime ideal P of M.*

Proof. By Theorem 2.16. □

Corollary 2.18. *Let R be a right Noetherian PI ring. Then a non-zero finitely generated right R-module is slightly compressible if and only if M/MP is not a torsion (R/P)-module for every associated prime ideal P of M.*

Proof. By Theorem 2.16 and [13, 13.6.6(i)]. □

3. Generators, self generators and weak generators

Let R be any ring. A right R-module M is called a *generator* for Mod-R, the category of right R-modules, provided for each right R-module X there exist an index set I and an epimorphism $\phi : M^{(I)} \to X$. It is well known that a module M is a generator for Mod-R if and only if the right R-module R is isomorphic to a direct summand of $M^{(n)}$ for some positive integer n (see [20, Section 13.7]). An R-module M is called a *self-generator* if for each submodule N of M there exist an index set J and an epimorphism $\theta : M^{(J)} \to N$. Clearly every generator for Mod-R is a self-generator. Moreover it is not difficult to see that every self-generator is slightly compressible. Note that every simple R-module is a self-generator but need not be a generator for Mod-R.

Let R be a (commutative) integral domain with field of fractions Q. For any R-submodule X of Q, let $X^* = \{q \in Q : qX \subseteq R\}$. It is well known (and easy to prove) that a mapping $f : X \to R$ is an R-homomorphism if and only if there exists $q \in X^*$ such that $f(x) = qx (x \in X)$. The R-submodule X is called *invertible* if $X^*X = R$. The following result is presumably well known but is included for completeness. One consequence is that in general not every slightly compressible module is a self-generator.

Proposition 3.1. *Let R be an integral domain with field of fractions Q and let X be a non-zero R-submodule of Q. Then*

(i) *X is slightly compressible if and only if $X^* \neq 0$.*
(ii) *The following statements are equivalent*:
(a) *X is a generator for Mod-R,*
(b) *X is a self-generator,*
(c) *X is invertible.*

Proof. (i) Suppose first that X is slightly compressible. Because $X \cap R \neq 0$, there exists a non-zero homomorphism $f : X \to X \cap R$. By the above remark there exists $q \in X^*$ such that $f(x) = qx \ (x \in X)$. Clearly $q \neq 0$.

Conversely, suppose that $X^* \neq 0$. Let $0 \neq q' \in X^*$. Let Y be any non-zero submodule of X. Then $Y \cap R \neq 0$. Let $0 \neq y \in Y \cap R$. Define a mapping $g : X \to Y$ by $g(x) = q'xy \ (x \in X)$. Then g is a non-zero homomorphism. It follows that X is slightly compressible.

(ii) (a) $\Rightarrow$ (b) Clear.

(b) $\Rightarrow$ (c) Let $0 \neq x \in X$. Then there exist a positive integer n and an epimorphism from $X^{(n)}$ to Rx. Since $Rx \cong R$ it follows that there exists an epimorphism $h : X^{(n)} \to R$. Clearly $1 = h_1(x_1) + \cdots + h_n(x_n)$ for some $x_i \in X$, $h_i \in \mathrm{Hom}_R(X, R)$ $(1 \leqslant i \leqslant n)$. For each $1 \leqslant i \leqslant n$, there exists $q_i \in X^*$ such that $h_i(u) = q_i u \ (u \in X)$. Hence $1 = q_1x_1 + \cdots + q_nx_n \in X^*X$ and we conclude that $X^*X = R$.

(c) $\Rightarrow$ (a) As above, $1 = q_1x_1 + \cdots + q_nx_n$ for some positive integer n and elements $x_i \in X$, $q_i \in X^* (1 \leqslant i \leqslant n)$. Define a mapping $p : X^{(n)} \to R$ by $p(v_1, \ldots, v_n) = q_1v_1 + \cdots + q_nv_n$ for all $v_i \in X (1 \leqslant i \leqslant n)$. Then p is an epimorphism and hence the R-module R is isomorphic to a direct summand of $X^{(n)}$. This proves (a). □

Given any ring R, we shall call a non-zero right R-module M a *weak generator* for Mod-R if, for each non-zero right R-module X, there exists a non-zero homomorphism $f : M \to X$. Note that M is a weak generator for Mod-R if and only if, for each non-zero R-module X, there exist an index set I and a non-zero homomorphism $g : M^{(I)} \to X$. Clearly every generator for Mod-R is a weak generator for Mod-R and every weak generator for Mod-R is slightly compressible.

Theorem 3.2. *Let R be a ring which has a unique simple right R-module* (*up to isomorphism*). *Then every non-zero projective right R-module is a weak generator for Mod-R.*

Proof. Let M be any non-zero projective right R-module and let M' be any non-zero right R-module. Let $0 \neq m' \in M'$ and let N' be a maximal submodule of $m'R$. By [1, Proposition 17.14], M contains a maximal submodule N. By hypothesis, $M/N \cong (m'R)/N'$. Thus there exists a non-zero homomorphism $f : M \to M'/N'$. Since M is projective it follows that f can be lifted to a non-zero homomorphism from M to M'. □

Corollary 3.3. *Let R be a ring with Jacobson radical J such that R/J is a simple Artinian ring. Then every non-zero projective right R-module is a weak generator for Mod-R.*

Proof. By Theorem 3.2. □

Lemma 3.4. *Let R be any ring and let a right R-module M be a weak generator for Mod-R. Then M has a maximal submodule.*

Proof. Let U be any simple right R-module. There exists an epimorphism $f : M \to U$. Note that $\ker f$ is a maximal submodule of M. □

We saw above in Proposition 1.18 that if a ring R is a right semi-artinian right V-ring then every right R-module is slightly compressible. Note that right V-rings have zero Jacobson radical by [14, Theorem 2.1]. Now we determine which rings have the property that every module is a weak generator. Following [9] we shall call a ring R a *right max ring* if every non-zero right R-module contains a maximal submodule.

Theorem 3.5. *Let R be a ring. Then every non-zero right R-module is a weak generator for Mod-R if and only if R is a right semi-artinian right max ring which has a unique simple right R-module* (*up to isomorphism*).

Proof. Suppose first that every non-zero right R-module is a weak generator. By Lemma 3.4, R is a right max ring. Let U be a simple right R-module. If M is any non-zero right R-module then there exists a non-zero homomorphism $f : U \to M$ which must be a monomorphism. Thus every non-zero right R-module M has non-zero socle and hence R is right semi-artinian. Also if V is any simple right R-module then there exists a non-zero homomorphism $g : U \to V$ and hence $U \cong V$.

Conversely, suppose that R is a right semi-artinian right max ring which has a unique simple right R-module. Let X and X' be any two non-zero right R-modules. By hypothesis, X contains a maximal submodule Y and X' contains a simple submodule U'. But $X/Y \cong U'$, so that the mapping $X \xrightarrow{p} X/Y \to U' \xrightarrow{i} X'$ is a non-zero homomorphism, where p is the canonical epimorphism and i denotes inclusion. Thus every non-zero right R-module is a weak generator for Mod-R. □

Recall that an ideal A of a ring R is called *right T-nilpotent* if for any sequence $a_1, a_2, a_3, \ldots$ of elements of A there exists a positive integer n such that $a_n \cdots a_1 = 0$. Compare the next result with Corollary 3.3.

Corollary 3.6. *Let R be a right semi-artinian ring with Jacobson radical J such that R/J is simple Artinian and J is right T-nilpotent. Then every non-zero right R-module is a weak generator for Mod-R.*

Proof. By Theorem 3.5 and [1, Proposition 13.5 and Lemma 28.3]. □

We note the following variation of Corollary 3.6. It is proved in [15, Proposition 3.2] that if J is the Jacobson radical of a left semi-artinian ring R then J is right T-nilpotent. Thus if R is a right and left semi-artinian ring with Jacobson radical J such that the ring R/J is simple Artinian then every non-zero right R-module is a weak generator for Mod-R. Examples of right semi-artinian rings which are not left semi-artinian are given in [3]. The next result gives a characterization of weak generators over arbitrary right Noetherian rings.

Theorem 3.7. *Let R be any right Noetherian ring. Then a right R-module M is a weak generator for Mod-R if and only if $\mathrm{Hom}_R(M, R/P) \neq 0$ for every prime ideal P of R.*

Proof. The necessity is clear. Conversely, suppose that $\mathrm{Hom}_R(M, R/P) \neq 0$ for every prime ideal P of R. Adapting the proof of Theorem 2.4, we conclude that $\mathrm{Hom}_R(M, M') \neq 0$ for every non-zero R-module M', as required. □

Theorem 3.7 shows that for a right Noetherian ring R, to check if a non-zero right R-module is a weak generator for Mod-R it is sufficient to consider $\mathrm{Hom}_R(M, R/P)$ for every prime ideal P of R. In particular, if R is a simple ring, M is a weak generator for Mod-R if and only if $\mathrm{Hom}_R(M, R) \neq 0$. In fact, the ring R need not be right Noetherian as the next result shows.

Proposition 3.8. *Let R be a simple ring. Then the following statements are equivalent for a right R-module M.*

(i) *M is a generator for Mod-R.*
(ii) *M is a weak generator for Mod-R.*
(iii) $\mathrm{Hom}_R(M, R) \neq 0$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Let $f : M \to R$ be a non-zero homomorphism. Let $A = f(M)$. Then A is a non-zero right ideal of R so that $R = RA$. There exist a positive integer n and elements $r_i \in R, a_i \in A (1 \leqslant i \leqslant n)$ such that $1 = r_1 a_1 + \cdots + r_n a_n$. Define a mapping $g : M^{(n)} \to R$ by $g(m_1, \ldots, m_n) = r_1 f(m_1) + \cdots + r_n f(m_n)$ for all $m_i \in M (1 \leqslant i \leqslant n)$. Clearly g is an epimorphism and hence R is isomorphic to a direct summand of $M^{(n)}$. It follows that M is a generator for Mod-R by [20, Section 13.7]. □

There is another class of rings R for which right R-modules are weak generators for Mod-R if and only if $\mathrm{Hom}_R(M, R) \neq 0$, namely the class of commutative Noetherian domains.

Theorem 3.9. *Let R be a commutative Noetherian domain. Then an R-module M is a weak generator for Mod-R if and only if* $\mathrm{Hom}_R(M, R) \neq 0$.

Proof. The necessity is clear. Conversely, suppose that there exists a non-zero homomorphism $f : M \to R$. Let $A = f(M)$. Then A is a non-zero ideal of R. Let P be any prime ideal of R. Suppose that A/AP is a torsion (R/P)-module. It follows that for each $a \in A$ there exists $c \in R \backslash P$ such that $ac \in AP$. Let R_P denote the localization of the ring R at the prime ideal P, where we think of R as a subring of R_P. Then AR_P is a finitely generated ideal of R_P, PR_P is the Jacobson radical of R_P and $AR_P = (AR_P)(PR_P)$. By Nakayama's Lemma, $AR_P = 0$ and hence $A = 0$, a contradiction. Thus A/AP is not a torsion (R/P)-module. By Corollary 2.15, there exists a non-zero homomorphism $g : A/AP \to R/P$. But this yields a non-zero homomorphism $M \to A \to A/AP \to R/P$. Thus $\mathrm{Hom}_R(M, R/P) \neq 0$ for every prime ideal P of R. By Theorem 3.7, M is a weak generator for Mod-R. □

Corollary 3.10. *Let R be a commutative Noetherian ring with minimal prime ideals $P_i (1 \leqslant i \leqslant n)$, for some positive integer n. Then a non-zero R-module M is a weak generator for Mod-R if and only if* $\mathrm{Hom}_R(M, R/P_i) \neq 0$ *for all* $1 \leqslant i \leqslant n$.

Proof. The necessity is clear. Conversely, suppose that $\mathrm{Hom}_R(M, R/P_i) \neq 0$ for all $1 \leqslant i \leqslant n$. Let P be any prime ideal of R. There exists $1 \leqslant i \leqslant n$ such that $P_i \subseteq P$. By hypothesis, $\mathrm{Hom}_R(M, R/P_i) \neq 0$ and hence, by Lemma 2.1, $\mathrm{Hom}_{R/P_i}(M/MP_i, R/P_i) \neq 0$. By Theorem 3.9, there exists a non-zero (R/P_i)-homomorphism $f : M/MP_i \to R/P$. Note that f is an R-homomorphism. If $p : M \to M/MP_i$ is the canonical epimorphism then $fp : M \to R/P$ is a non-zero homomorphism. Thus $\mathrm{Hom}_R(M, R/P) \neq 0$ for every prime ideal P of R. By Theorem 3.7, M is a weak generator for Mod-R. $\square$

Note that in Corollary 3.10 it is not sufficient to suppose that $\mathrm{Hom}_R(M, R)$ is non-zero. For example, let R be a commutative Noetherian semiprime ring which is not a domain and let P be a minimal prime ideal of R. Clearly there exists $c \in R \backslash P$ such that $Pc = 0$. Let $f : P \to R/P$ be an R-homomorphism. Then $0 = f(Pc) = f(P)c$ and $f(P) \subseteq R/P$, so that $f(P) = 0$. Thus $\mathrm{Hom}_R(P, R/P) = 0$. It follows that $\mathrm{Hom}_R(P, R) \neq 0$ but P is not a weak generator for Mod-R.

Finally, we shall show that Theorem 3.9 fails to be true for some integral domains which are not Noetherian and also for some prime Noetherian *PI* rings. To do so we first prove the following result.

Proposition 3.11. *Let P be a maximal ideal of a commutative ring R such that $P^n \neq P^{n+1}$ for some positive integer n. Then P^n is a weak generator for Mod-R.*

Proof. Let M be any non-zero R-module. Let $0 \neq m \in M$. Suppose first that $mP^n \neq 0$. Define $f : P^n \to M$ by $f(a) = ma$ $(a \in P^n)$. Then f is a non-zero homomorphism. Now suppose that $mP^n = 0$. Clearly there exists $0 \neq u \in mR$ such that $uP = 0$. Then P^n/P^{n+1} and uR are non-zero vector spaces over the field R/P so that there exists a non-zero homomorphism $g : P^n/P^{n+1} \to uR$. If $p : P^n \to P^n/P^{n+1}$ is the canonical epimorphism and $i : uR \to M$ is inclusion then $igp : P^n \to M$ is a non-zero homomorphism. $\square$

Lemma 3.12. *Let A be an idempotent ideal of a general ring R. Then the right R-module A is not a weak generator for Mod-R.*

Proof. Let $f : A \to R/A$ be any homomorphism. Then $f(A) = f(A^2) = f(A)A \subseteq (R/A)A = 0$. Thus $\mathrm{Hom}_R(A, R/A) = 0$. $\square$

Proposition 3.11 and Lemma 3.12 can be combined to show that Theorem 3.9 fails spectacularly for commutative domains which are not Noetherian. Let n be any positive integer. In [2] an example is given of a commutative domain R and a maximal ideal P of R such that $P \supset P^2 \supset \cdots \supset P^n = P^{n+1}$. In this case the R-module P^i is a weak generator for Mod-R for all $1 \leqslant i \leqslant n-1$ but P^n is not a weak generator for Mod-R, by Proposition 3.11 and Lemma 3.12.

Finally, we give an example due to J.C. Robson (see [17, Example 5.2.28]) of a prime Noetherian PI ring R which has an idempotent maximal ideal. In view of Lemma 3.12, this shows that Theorem 3.9 cannot be extended to prime Noetherian PI rings in general.

Example 3.13. Let S be a commutative Noetherian local domain with unique maximal ideal J and let R be the subring of the ring of 2×2 matrices with entries in S consisting of all matrices with $(1, 2)$ entry in J. Let P denote the ideal of R consisting of all matrices in R with (1,1) entry also in J. Then R is a prime Noetherian *PI* ring and P is an idempotent maximal ideal of R.

Proof. See [17, Example 5.2.28]. $\square$

Note that the ideal P of Example 3.13 is slightly compressible but not a weak generator for Mod-R. In fact, $\mathrm{Hom}_R(P, X) \neq 0$ for every R-module X such that $XP \neq 0$.

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