Abstract

In the present work nonlinear dynamics of axially loaded energy harvester which is modeled as a fixed-guided beam with tip mass having piezoelectric patch has been investigated. The system is subjected to a base excitation and periodic axial load. The governing equation of motion of the system is developed using Lagrange principle. A closed form solution has been developed to find out the response and generated voltage using method of multiple scales (MMS). Frequency response of the system is obtained for superharmonic and subharmonic resonance conditions and parametric study has been carried out in order to investigate the variation of output voltage with different system parameters.

Keywords: Nonlinear dynamics; piezoelectric energy harvester; periodic axial load; Method of multiple scales.

1. Introduction

To make devices smart, self sufficient and battery less, available ambient energy can be utilized to power such devices by using embedded smart materials [1]. One such ambient energy is vibration energy. Resonance plays dominant role in case of energy harvesters based on vibration energy. In case of energy harvesters based on linear vibration, only primary resonance with short frequency bandwidth is available for energy conversion [2,3]. In order to extract vibration energy over a wide range of frequency bandwidth there has been attempt made by researchers to exploit the nonlinear behavior of vibration [4-6, 11]. Nonlinear dynamics of axially loaded beam structure as an energy harvester [4] and as a manipulator [12] have been investigated by few researchers. By considering the fact

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Peer-review under responsibility of the organizing committee of ICOVP 2015
that apart from primary resonance the phenomenon of secondary resonance conditions exists in systems having nonlinear vibrations may help to enhance the performance of energy harvester over wide range of excitation frequencies.

In the present work a fixed-guided beam based energy harvester embedded with piezoelectric patch is proposed. The harvester is having tip mass at its guided end. It is subjected to harmonic base and axial excitation as shown in Fig. 1. Euler-Bernoulli beam theory is applied for slender fixed-guided beam. The governing equations of motion of the system have been derived using Lagrange and Galerkin’s principles. Nonlinear governing temporal equation of motion is solved using the method of multiple scales.

2. Modelling

The proposed energy harvester consists of a uniform inverted fixed-guided beam of length $L_s$, tip mass $M_t$ with piezoelectric patches of thickness $h_p$, width $b_p$ and length $L_p$. It is subjected to a harmonic base excitation and an axial load at its guided end as shown in Fig. (1). The top end of beam is constrained to move in axial direction where tip mass is placed. Lagrange principle is used to derive the governing equations of motion and Galerkin’s method is used to discretize the space-time domain. The study is limited to the pre-buckling regime only.

Fig. 1. Axially loaded inverted fixed-guided beam configuration with tip mass, harmonic base excitation and piezoelectric patches

2.1. Governing equation of motion

The Kinetic energy $T$ and the potential energy $U$ of the system in terms of curvature $k(s,t)$ and displacements $u$ and $v$ can be expressed as [5]

$$T = \frac{1}{2} \rho A \int_0^L \left\{ \left( \ddot{v}(s,t) \right)^2 + \left( \dot{u}(s,t) \right)^2 \right\} ds + \frac{1}{2} M_t \left\{ \dot{u}^2(L,t) \right\}$$

$$U = \frac{1}{2} EI \int_0^L \left( k(s,t) \right)^2 ds - \rho A g \int_0^L u(s,t) ds - M_t g u(L,t)$$
Here \( u \) and \( v \) are the transverse and axial displacement of any point at a distance \( 's' \) from the base respectively. \( \frac{d}{dt} \) represents differentiation with respect to time \( t \). Also \( \rho, A, E, I, M_t \), and \( g \) are density, area of cross section, Young’s modulus, area moment of inertia of the beam, tip mass and acceleration due to gravity, respectively. The work done \( (W) \) by the moment \( M(s,t) \) (produced by the piezoelectric patches in electric displacement of charges about the beam neutral axis) along with work done due to axial load and base excitation is

\[
W = \int_{0}^{L} M(s,t)k(s)ds - \bar{P}(t)u(L,t) + a_{b} \int_{0}^{L} m(s)v(s,t)ds
\]

(3)

\[
\bar{P}(t) = \bar{P}_{0} + \bar{P}_{1} \sin(\Omega_{s} t) + \bar{P}_{2} \cos(2\Omega_{s} t)
\]

(4)

Here \( m(s), a_{b} \), and \( \bar{P}(t) \) are the total mass of the beam, base acceleration and axial force respectively. Geometrical nonlinearity is considered due to large oscillation of the beam so the curvature-displacement relation at a distance \( 's' \) from base becomes

\[
k(s,t) = \frac{d\phi}{ds} = \phi' = \frac{v'}{\sqrt{1-v'^{2}}} \approx v' \left( 1 + \frac{1}{2} v'^{2} \right)
\]

(5)

Here \( ()' \) and \( \phi \) denotes the differentiation with respect to \( 's' \) and rotation of the beam at a distance \( 's' \) from the base centre. Inextensibility condition [10] is used to eliminate the dynamics in longitudinal direction by transverse direction as follows,

\[
u' = 1 - \sqrt{1-v'^{2}} \approx \frac{1}{2} v'^{2}
\]

(6)

The moment \( M(s,t) \) produced by voltage \( (V) \) about the beam neutral axis across the piezoelectric layers can be given by [3]

\[
M(s,t) = \gamma_{c}V(t);
\]

\[
\gamma_{c} = E d_{31} b_{p}(h+h_{p})
\]

(7)

Here \( d_{31} \) is piezoelectric coefficient and \( \gamma_{c} \) depends on the geometry, configuration and piezoelectric material. Lagrangé’s method is used to derive the governing equations of motion. Using single mode approximation in Galerkin’s method i.e. taking \( v(s,t) = \bar{r} \psi(s)q(t) \) one may obtain the following nondimensional temporal equations of motion. Here \( \psi(s) \) is the shape function [4] which is given in the appendix A.

\[
\ddot{q} + \varepsilon \dot{q} + \alpha_{q} + \varepsilon \dot{\alpha}_{q} + \alpha_{\psi} \dot{q}^{3} + \varepsilon \dot{\psi} + \varepsilon \psi q^{2} = F(\tau) + \varepsilon q \bar{P} \sin(\Omega_{s} \tau) - \varepsilon q \bar{P}_{2} \cos(2\Omega_{s} \tau)
\]

(8)

\[
V + \dot{\psi} + \alpha_{\psi} V + \varepsilon \dot{\psi} \psi q^{2} = 0
\]

(9)

Here \( V \) denotes the open circuit voltage. The equation of motion contains inertial, cubic and coupled electromechanical nonlinear terms. Following nondimensional parameters have been used for further analysis.

\[
t = \frac{\tau}{\omega_{l}}, \quad \omega_{l} = \sqrt{\left( \frac{E I N_{s} - N_{s} \rho A g - N_{s} (M_{t} g - \bar{P}_{0})}{\rho A N_{l}} \right)}
\]

\[
\bar{r} = 0.01, \quad \bar{V} = \frac{\theta \bar{P}}{C_{p}}V, \quad \Omega_{s} = \frac{\Omega_{s}}{\omega_{l}}, \quad \Omega_{s}^{*} = \frac{\Omega_{s}^{*}}{\omega_{l}},
\]

\[
C_{p} = \int_{0}^{L} C(s)ds = \int_{0}^{L} \left(H_{t_{c}} - H_{t_{c}} \right) \frac{\varepsilon A_{p}}{h_{p}^{2}}, \quad A_{p} = b_{p}h_{p}, \quad F(\tau) = F \cos(\Omega_{s} \tau), \quad F = \frac{m(s)z_{c} \Omega_{s}^{2}}{\rho A N_{l} F \omega_{l}^{2}}
\]
Here $\bar{r}, \omega_1, \Omega_1, \Omega_2, \beta, (H_{lT} - H_{l_2}), C_p$ are length scale, first modal frequency, frequency of base excitation, frequency of axial load, permittivity, Heaviside function and internal capacitance of the bimorph [8] respectively. Other coefficients are given in the appendix A. One can observe that the first modal frequency $\omega_1$ is a function of tip mass $M_t$, axial load $P_0$ apart from geometry and material properties.

2.2. Response analysis using Method of Multiple Scales

By using the method of multiple scales a uniform first order approximate solution of Eq. (8) and (9) is obtained. The time dependence can be expressed in to multiple time scales $(T_0, T_1, T_2, \ldots) \text{ as } [7]$. Expanding the solution of system $q(\tau)$ and $V(\tau)$ in the following series form

$$q(\tau) = q_0(T_0, T_1) + \varepsilon q_1(T_0, T_1) + O(\varepsilon^2); \quad V(\tau) = V_0(T_0, T_1) + \varepsilon V_1(T_0, T_1) + O(\varepsilon^2)$$

(10)

Here $\varepsilon$ is a book keeping parameter, after substituting Eq. (10) into Eq. (8) and (9) then further equating the coefficient of $\varepsilon^0$ and $\varepsilon^1$ equal to zero, yields the following set of equations.

$$O(\varepsilon^0): \quad D_0^2 q_0 + \alpha_0^2 q_0 = F \cos(\Omega_1^2 \tau); \quad D_0 V_0 + \alpha_0 V_0 = D_0 q_0$$

(11)

$$O(\varepsilon^1): \quad D_0^2 q_1 + \alpha_0^2 q_1 = \left\{ -2D_0D_0 q_0 - \alpha_0 \left( q_0^2 D_0^2 q_0 + q_0 (D_0 q_0)^2 \right) - \alpha_0 D_0 q_0 - \alpha_2 q_0^3 \right\}
\quad + \alpha_4 V_0 + q_0 P_1 \sin(\Omega_1^2 \tau) - q_0 P_2 \cos(2\Omega_1^2 \tau)$$

(12)

Assuming the solution of zero order $O(\varepsilon^0)$ differential equations as

$$q_0 = A(T_1) e^{i\omega_0 T_0} + \Lambda e^{i\Omega_1^2 T_0} + \text{cc} \quad \text{and} \quad V_0 = ZA(T_1) e^{i\omega_0 T_0} + \text{cc}.$$  

(13)

Here

$$Z = \frac{\omega_0^2 + i\omega_0 \alpha_0}{\omega_0^2 + \alpha_0^2}, \quad \Lambda = \frac{F}{2(\omega_0^2 - \Omega_1^2)}$$

(14)

and $A(T_1)$ is a complex valued function. By substituting Eq. (13) and (14) in to Eq. (12) we get the expression

$$D_0^2 q_1 + \alpha_0^2 q_1 = \left\{ \left( (2A' + \alpha_0 A) i\omega_0 + 3\alpha_2 A(2\lambda^2 + 2\bar{A}A) \right) + \left( 2A\alpha_3 \left( \omega_0^2 + \Omega_1^2 \right) + \lambda \alpha_3 \bar{A} \alpha_0^2 \right) + \Lambda \right\} e^{i\omega_0 T_0}
\quad + \left\{ 2\alpha_3 \Omega_1^2 \lambda + 2\bar{A}A \lambda \alpha_0^2 + 2\bar{A}A \lambda \alpha_0^2 \Omega_1^2 - \lambda \left( i\alpha_0 \omega_0 \Omega_1^2 + \alpha_0 \lambda^2 + 6\alpha_0 \bar{A}A \right) \right\} e^{i\Omega_1^2 T_0}
\quad + \left( 2\alpha_2 \varepsilon_0^2 \alpha_0 - \alpha_2 \right) e^{i\omega_0 T_0} + \Lambda \left( 2\alpha_2 \varepsilon_0^2 \Omega_1^2 - \alpha_2 \right)e^{i\Omega_1^2 T_0} + \frac{\Lambda}{2} iP_1 \left( e^{i\omega_0 (\Omega_1^2 - \Omega_2^2)} - e^{i\omega_0 (\Omega_2^2 - \Omega_1^2)} - e^{i\omega_0 (\Omega_1^2 + \Omega_2^2)} \right)
\quad - \frac{\Lambda}{2} P_2 \left( e^{i\omega_0 (\Omega_1^2 + 2\Omega_2^2)} + e^{i\omega_0 (2\Omega_1^2 - \Omega_2^2)} + e^{i\omega_0 (2\Omega_2^2 - \Omega_1^2)} \right) - 0.5iP_1 \left( e^{i\omega_0 (\Omega_1^2 - \Omega_2^2)} - e^{i\omega_0 (\Omega_2^2 - \Omega_1^2)} \right)
\quad - 0.5P_2 \left( A e^{i\omega_0 (\Omega_1^2 + 2\Omega_2^2)} + \bar{A} e^{i\omega_0 (-\Omega_1^2 + 2\Omega_2^2)} \right) + A^2 \Lambda \left( 3\alpha_2 \varepsilon_0^2 \Gamma_1^2 - 3\alpha_2 \varepsilon_0^2 \Gamma_2^2 \right) e^{i\omega_0 (\Omega_1^2 - \Omega_2^2)} + \Lambda \left( 3\alpha_2 \varepsilon_0^2 \Gamma_1^2 + 3\alpha_2 \varepsilon_0^2 \Gamma_2^2 \right) e^{i\omega_0 (\Omega_1^2 + \Omega_2^2)}
\quad + 3\alpha_2 \bar{A} \Lambda \left( 3\alpha_2 \varepsilon_0^2 \Gamma_1^2 - 3\alpha_2 \varepsilon_0^2 \Gamma_2^2 \right) e^{i\omega_0 (\Omega_1^2 - 2\Omega_2^2)} + 3\alpha_2 \bar{A} \Lambda \left( 3\alpha_2 \varepsilon_0^2 \Gamma_1^2 + 3\alpha_2 \varepsilon_0^2 \Gamma_2^2 \right) e^{i\omega_0 (\Omega_1^2 + 2\Omega_2^2)}
\quad - 3\alpha_2 \bar{A} \Lambda \left( 3\alpha_2 \varepsilon_0^2 \Gamma_1^2 - 3\alpha_2 \varepsilon_0^2 \Gamma_2^2 \right) e^{i\omega_0 (\Omega_1^2 - \Omega_2^2)} + \text{cc}$$

(15)
From Eq. (15) one can observe several resonance conditions such as

(i) Primary Resonance: \( \Omega_1^* \approx \omega_0; \ \Omega_2^* \approx \omega_0 \)

(ii) Superharmonic Resonance: \( \Omega_1^* \approx \omega_0/3 \)

(iii) Subharmonic Resonance: \( \Omega_1^* \approx 3\omega_0 \)

(iv) Simultaneous Resonance: \( \Omega_1^* - \Omega_2^* \approx \omega_0; \ \Omega_1^* - \Omega_2^* \approx \omega_0; \ \Omega_1^* - 2\Omega_2^* \approx \omega_0; \ 2\Omega_1^* - \Omega_2^* \approx \omega_0 \)

In the following section only superharmonic and subharmonic resonance conditions are discussed briefly.

2.2.1 Superharmonic Resonance: \( 3\Omega_2^* = \omega_0 + \varepsilon \sigma \)

The response of the system is analyzed when frequency of periodic excitation \( \Omega_2^* \) coincides with one third times the systems natural frequency \( \omega_0 \) and \( \Omega_1^* \) is away from \( \omega_0 \). Here \( \sigma \) is detuning parameter, by using this condition into Eq. (15) the following secular term is obtained

\[
\left\{-\left(2A' + \alpha_s A\right)i\omega_0 + 3A\alpha_s \left(2\Lambda^2 + A\bar{A}\right)\right\} + 2A\alpha_s \left\{\Lambda^2 \left(\omega_0^2 + \Omega_i^*\right) + A\bar{A}\omega_0^2 \right\} + Z\alpha_s + \left(2\alpha_s\Omega_i^* - \alpha_s\right)\Lambda^3 e^{i\varepsilon T} = 0
\]  

(16)

Expressing \( A(T_i) \) in to the polar form as

\[
A(T_i) = \frac{1}{2} a(T_i) e^{i\varepsilon T_i}
\]

(17)

By using Eq. (17) into Eq. (16) and separating real and imaginary parts one can obtain the reduced or modulation equations. The frequency response equation at steady state becomes

\[
\sigma = \pm \sqrt{\left\{\frac{2\alpha_s\Omega_i^* - \alpha_s}{a\omega_0}\right\}^2 - \left\{-\frac{1}{2} \alpha_0 + \frac{\alpha_s}{2\left(a_s^2 + \omega_0^2\right)} \right\}^2 - \frac{1}{a\omega_0} \left\{-a\alpha_s \left(3\Lambda^2 + \frac{a^2}{8}\right) + \left(\frac{1}{4} a^2 + a\Lambda^2\right)\alpha_s \omega_0^2 + \frac{a\alpha_s}{2\left(a_s^2 + \omega_0^2\right)} + a\Lambda^2 \alpha_s \Omega_i^* \right\}}
\]

(18)

The appearance of excitation force term in Eq. (11) is due to the assumption of hard excitation. If soft excitation force is present then the primary resonance condition when \( \Omega_i^* = \omega_0 \) can be observed as well.

2.2.2 Subharmonic Resonance: \( \Omega_1^* = 3\omega_0 + \varepsilon \sigma \)

Response of the system is obtained when frequency of base excitation \( \Omega_1^* \) coincides with three times the systems natural frequency \( \omega_0 \) and assuming that the frequency of axial excitation \( \Omega_2^* \) is away from \( \omega_0 \). By applying the subharmonic condition into Eq. (15) and using Eq. (17) and separating real and imaginary parts we obtained the following secular term and frequency-response equation, denoted by Eq. (19) and (20) respectively.
\[ \begin{aligned}
&\left\{ 2i\omega_0^2 + i\omega_0 \alpha_1 A + 3A\alpha_3 \left( 2\Lambda^2 + \Lambda \Delta \right) \right\} + 2A\alpha_3 \left( \Lambda^2 \alpha_0^2 + \Lambda^2 \Omega_1^2 + \Lambda \Delta \omega_0^2 \right) \\
&+ ZA\alpha_3 + \left( 3\Lambda^2 \alpha_0 \omega_0^2 - 2\Lambda^2 \alpha_0 \omega_0 \Omega_1^2 + 3\alpha_2 \omega_0^2 - 3\alpha_2 \Omega_1^2 \right) \Lambda e^{i\omega t} = 0
\end{aligned} \tag{19} \]

\[ \sigma = \pm \sqrt{\left\{ \left( \frac{3\omega_0^2 - 2\omega_0 \Omega_1^2 + \Omega_1^2}{4\omega_0} \right) \alpha_3 - 3\alpha_2 \right\} a^2 \Lambda^2} - \left\{ \frac{1}{2} \frac{\alpha_0 \alpha_1}{2\left( \alpha_0^2 + \omega_0^2 \right)} \right\} \]

\[ - \frac{1}{a\omega_0} \left\{ -a\alpha_2 (3\Lambda^2 + \frac{3}{8} \alpha_1^2) + \left( \frac{1}{4} \alpha_1^2 + a\Lambda^2 \right) \alpha_3 \omega_0^2 + \frac{a\alpha_2 \omega_0^2}{2\left( \alpha_0^2 + \omega_0^2 \right)} + a\Lambda^2 \alpha_3 \Omega_1^2 \right\} \tag{20} \]

3. Results and discussion

The following geometric and material properties of beam (both substrate and piezoelectric patch) are considered for dynamic analysis for superharmonic and subharmonic resonance conditions.

<table>
<thead>
<tr>
<th>Property</th>
<th>Substrate</th>
<th>Piezo-patch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length, m</td>
<td>( L_r )</td>
<td>0.1575</td>
</tr>
<tr>
<td>Young’s Modulus, GPa</td>
<td>( E_i )</td>
<td>190</td>
</tr>
<tr>
<td>Width, m</td>
<td>( b_i )</td>
<td>0.0146</td>
</tr>
<tr>
<td>Height, m</td>
<td>( h_i )</td>
<td>0.0005</td>
</tr>
<tr>
<td>Density, kg/m³</td>
<td>( \rho_i )</td>
<td>8100</td>
</tr>
</tbody>
</table>

3.1 Superharmonic Resonance case: \( 3\Omega_1^2 = \omega_0 + \varepsilon \sigma \)

Figure 2 shows the frequency response plot for superharmonic resonance condition when the excitation frequency is close to one third of the natural frequency of the system. System exhibits the hardening type behaviour. Maximum voltage of 0.14 V is achieved here for the forcing parameter \( \Lambda = 1 \). The amplitude of both displacement and voltage is increased with increase in amplitude of excitation force. Substantial increment in the frequency bandwidth is also observed which help to maintain specific voltage over wide range of frequency variation.

![Figure 2](image-url)

Fig. 2. (a) Amplitude and (b) Voltage frequency response of the system for superharmonic resonance for \( \Lambda = 1, \Lambda = 1.1 \) and \( \Lambda = 1.2 \)
3.2 Subharmonic Resonance case:  \[ \Omega'_1 = 3\omega_0 + \varepsilon \sigma \]

In Fig. 3 the frequency response is shown for subharmonic resonance when excitation frequency is close to three times the natural frequency of the system. Large amplitude and voltage is observed here as compared to superharmonic case over a wide bandwidth of frequency. Slight change in the amplitude of both displacement and voltage is observed as response curve shifted toward right with increase in magnitude of excitation force. The frequency bandwidth remains unchanged here but slightly shifted towards right.

![Fig. 3. (a) Amplitude and (b) Voltage frequency response of the system for subharmonic resonance for \( \Lambda=1, \Lambda=1.1 \) and \( \Lambda=1.2 \)](image)

The static natural frequency of the system decreases with increase in tip mass and axial load (Fig. 4). Here the dynamic analysis is limited to pre-buckling regime. Systems natural frequency can be adjusted according to the excitation frequency available by changing either tip mass and axial load or both. This helps in tuning the systems natural frequency near to the excitation frequency.

3. Conclusions

In the present work nonlinear dynamics of fixed-guided beam with piezoelectric patch and tip mass has been investigated for energy harvesting purpose. It has been observed that by adjusting either mass or axial load the frequency of the system can be tuned according to the available excitation frequency. For wide range of frequency bandwidth harvester is productive due to existence of multiple resonance conditions like superharmonic and subharmonic. For super-harmonic resonance a substantial increment in the frequency bandwidth is observed which help to maintain specific minimum voltage over a wide range of frequency variation. The amplitude of both displacement and voltage response increases with increase in amplitude of base excitation. Occurrence of subharmonic resonance leads to large amplitude and voltage as compared to superharmonic case. This work can be further extended to study the energy harvester under primary and simultaneous resonance conditions.

![Fig. 4. (a) Variation of natural frequency of the system with tip mass (when axial load \( P_0 = 1N \)) and; (b) axial load (when tip mass \( M_t = 0.03\,\text{kg} \))](image)
The proposed energy harvester may find application in case of low frequency of excitation due to its ability to tune by changing load or tip mass. Also the existence of multiple resonance conditions makes the harvester capable of being productive for over a wide range of external excitation frequencies.

Appendix A.

The mode shapes for fixed-guided beam configuration is given by [4]

\[ \psi_n(s) = (\cosh \beta_s s - \cos \beta_s s) + (\sin \beta_s s - \sinh \beta_s s) \frac{\cos \beta_s L_s - \cosh \beta_s L_s}{\sin \beta_s L_s - \sinh \beta_s L_s}, \]

where \( \beta_s \) are the roots of the following characteristic equation

\[ 1 - \cosh(\beta_s L_s) \cos(\beta_s L_s) = 0. \]

Other coefficients are as follows

\[
\alpha_1 = 1, \quad \alpha_2 = \frac{2EI\bar{r}^2}{\rho AN_1 \omega_1^2}, \quad \alpha_3 = \frac{\left(\rho AN_1 + M_n N_s^2\right)\bar{r}^2}{\varepsilon^2 \rho AN_1 C_n \omega_1^2}, \quad \alpha_4 = \frac{\theta_1^2}{\varepsilon^2 \rho AN_1 C_n \omega_1^2}, \quad \alpha_5 = \frac{\theta_1 \theta_2 \bar{r}^2}{\varepsilon^2 \rho AN_1 C_n \omega_1^2}, \quad \alpha_6 = \frac{1}{C_n R \omega_1^2},
\]

\[
\alpha_7 = \frac{\theta_2 \bar{r}^2}{\varepsilon^2 \omega_1^2}, \quad P_1 = \frac{\bar{P} N_4}{\rho AN_1 \omega_1^2}, \quad P_2 = \frac{P_n N_4}{\rho AN_1 \omega_1^2}, \quad N_1 = r^2 \int_0^L \psi \psi^2 \, ds, \quad N_2 = r^2 \int_0^L \psi \psi \, ds, \quad N_3 = r^2 \int_0^L \left( \int_0^\tau \psi^2 \, d\zeta \right)^2 \, ds,
\]

\[
\mu = \frac{CN_4}{\rho AN_1 \omega_1^2}, \quad N_4 = r^2 \int_0^L \psi^2 \, ds, \quad N_5 = r^2 \int_0^L \psi \psi^2 \, ds, \quad N_6 = r^2 \int_0^L \psi^2 \psi^2 \, ds, \quad N_7 = r^2 \int_0^L \psi \psi^2 \psi^2 \, ds, \quad a_o = \dot{z}(t),
\]

\[
N_8 = r^2 \int_0^L \left( \int_0^\tau \psi \psi^2 \, d\zeta \right) \, ds, \quad \theta_1 = r \gamma, \quad \theta_2 = \frac{3}{2} r^2 \gamma \int_0^L \psi \psi \psi \psi \, ds, \quad F = \frac{m(s)z \Omega^2}{\rho AN_1 \bar{r} \omega_1^2}, \quad F(\tau) = F(\cos(\Omega \tau))
\]

References