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THE DEVELOPMENT OF FRACTIONAL CALCULUS 1695 - 1900

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SUMMARIES

This paper describes an example of mathematical growth from scholarly curiousity to application. The contributions of Liouville, Riemann, and Laurent to the field of fractional operators are discussed. Motivation for the writing of this paper is based on the statement by Harold T. Davis [1927]: "The great elegance that can be secured by the proper use of fractional operators and the power they have in simplifying the solution of complicated functional equations should more than justify a more general recognition and use."

Dieser Artikel beschreibt ein Beispiel mathematischer Weiterentwicklung, das aus wissenschaftlicher Kuriosität bis zu deren Anwendung hervorgegangen ist. Man bespricht darin die Beiträge von Liouville, Riemann und Laurent auf dem Gebiet der gebrochenen Operatoren. Anregung zur aktuellen Arbeit findet man bei diesen Zitat von H. T Davis [1927]: "Der Gewinn an Eleganz, den man sich bei der richtigen Anwendung gebrochener Operatoren verschaffen kann, sowohl wie ihre Wirksamkeit, die Lösung komplizierte Funktionalgleichungen zu vereinfachen, sollten um so mehr eine breitere Anerkennung und Gebrauch rechtfertigen."

Cette étude donne l'exemple d'un accroissement mathématique partant d'une curiosité savante pour atteindre finalement le domaine des applications pratiques. Les contributions de Liouville, Riemann et Laurent au calcul des dérivées à indices quelconques y sont traitées. L'inspiration de cet article vient de la citation de H. T. Davis [1927]: "La grande élégance qu'apporte l'emploi correct des opérateurs fractionnaires et leur puissance qu'ils détiennent à la résolution des équations fonctionelles compliquées ferait plus que justifier une réconnaissance plus répandue et un emploi plus frequent de ces opérateurs."

THE ORIGIN OF FRACTIONAL CALCULUS

Fractional calculus has its origin in the question of the extension of meaning. A well-known example is the extension of the meaning of factorials of positive integers to factorials of complex numbers. The original question which led to the name fractional calculus was: Can the meaning of a derivative of integer order $d^n y/dx^n$ be extended to have meaning when n is a fraction? Later the question became: Can n be any number--fractional, irrational or complex? Because the latter question was answered affirmatively, the name fractional calculus has become a misnomer and is better called integration and differentiation of arbitrary order.

Leibniz invented the notation $d^n y/dx^n$. Perhaps it was naive play with symbols that prompted L'Hospital in 1695 to ask Leibniz, "What if n be 1/2?" Leibniz [1695a] replied: "You can see by that, sir, that one can express by an infinite series a quantity such as $d^{\frac{1}{2}}\,\overline{xy}$ or $d^{1:2}\,\overline{xy}$. Although infinite series and geometry are distant relations, infinite series admits only the use of exponents which are positive and negative integers, and does not, as yet, know the use of fractional exponents." Later, in the same letter, Leibniz continues prophetically: "Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx:x}$. This is an apparent paradox from which, one day, useful consequences will be drawn."

In his correspondence with Johann Bernoulli, Leibniz [1695b] mentions derivatives of "general order." In Leibniz's correspondence with John Wallis, in which Wallis's infinite product for $\pi/2$ is discussed, Leibniz [1697] states that differential calculus might have been used to achieve this result. He uses the notation $d^{\frac{1}{2}}y$ to denote the derivative of order $\frac{1}{2}$.

The subject of fractional calculus did not escape Euler's attention. In 1730 he wrote "When n is a positive integer, and if p should be a function of x, the ratio $d^n p$ to dx^n can always be expressed algebraically, so that if $n = 2$ and $p = x^3$, then $d^2(x^3)$ to $d(x^2)$ is $6x$ to 1. Now it is asked what kind of ratio can then be made if n be a fraction. The difficulty in this case can easily be understood. For if n is a positive integer d^n can be found by continued differentiation. Such a way, however, is not evident if n is a fraction. But yet with the help of interpolation which I have already explained in this dissertation, one may be able to expedite the matter." [Euler 1738]

J. L. Lagrange [1849] contributed to fractional calculus indirectly. In 1772 he developed the law of exponents (indices) for differential operators of integer order and wrote:

$$\frac{d^m}{dx^m} \cdot \frac{d^n}{dx^n}\, y = \frac{d^{m+n}}{dx^{m+n}}\, y \ .$$

In modern notation the dot is omitted, for it is not a multiplication. Later, when the theory of fractional calculus developed, mathematicians were interested in knowing what restrictions had to be imposed upon $y(x)$ so that an analogous rule held true for m and n arbitrary.

In 1812 P. S. Laplace [1820 vol. 3, 85 and 186] defined a fractional derivative by means of an integral, and in 1819 the first mention of a derivative of arbitrary order appears in a text. S. F. Lacroix [1819, 409-410] devoted less than two pages of his 700 page text to this topic. He developed a mere mathematical exercise generalizing from a case of integer order. Starting with $y = x^m$, m a positive integer, Lacroix easily develops the nth derivative:

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \qquad m \geqq n. \tag{1}$$

Using Legendre's symbol Γ for the generalized factorial, he gets

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

He then gives the example for $y = x$ and $n = \frac{1}{2}$, and obtains

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{\frac{1}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}} \tag{2}$$

because $\Gamma(2) = 1$ and $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$. It is interesting to note that the result obtained by Lacroix, in the manner typical of the classical formalists of this period, is the same as that yielded by the present-day Riemann-Liouville definition of a fractional derivative. Lacroix's method offered no clue for a possible application for a derivative of arbitrary order.

Joseph B. J. Fourier [1822] was the next to mention derivatives of arbitrary order. His definition of fractional operations was obtained from his integral representation of $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha)\, d\alpha \int_{-\infty}^{+\infty} \cos\ [p(x-\alpha)+\frac{n\pi}{2}]\ dp.$$

Now, $\frac{d^n}{dx^n} \cos p(x-\alpha) = p^n \cos[p(x-\alpha) + \frac{n\pi}{2}]$ for integral values of n. Formally replacing n with u, u arbitrary, he obtains the generalization

$$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha)\, d(\alpha) \int_{-\infty}^{+\infty} p^u \cos[p(x-\alpha) + \frac{u\pi}{2}]\, dp.$$

Fourier states, "The number u which appears in the above will be regarded as any quantity whatsoever, positive or negative."

THE CONTRIBUTIONS OF ABEL AND LIOUVILLE

Leibniz, Euler, Laplace, Lacroix, and Fourier made mention of derivatives or arbitrary order, but the first use of fractional operations was by Niels Henrik Abel in 1823 [Abel 1881]. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone (isochrone) problem. The formulation of Abel's integral equation can be found in many texts. In this problem, the time of slide is a known constant k such that

$$k = \int_0^x (x-t)^{-\frac{1}{2}} f(t)\,dt. \tag{3}$$

The integral on the right above, except for a multiplicative constant $1/\Gamma(\frac{1}{2})$ is a particular case of a definite integral that defines fractional integration of order $\frac{1}{2}$. In integral equations, such as (3), the function f in the integrand is unknown and is to be determined. Abel wrote the right side of (3) as

$\sqrt{\pi}\,\dfrac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} f(x)$. Then he operated on both sides with $\dfrac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ to obtain

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\, k = \sqrt{\pi}\, f(x) \tag{4}$$

because these fractional operators, with suitable conditions on f, have the extended semigroup property $D^{\frac{1}{2}}D^{-\frac{1}{2}} f = D^0 f = f$. Thus when the fractional derivative of order $\frac{1}{2}$ of the constant k in (4) is computed, $f(x)$ is determined. This is the remarkable achievement of Abel in the fractional calculus. It is important to note that the fractional derivative of a constant is not always equal to zero unless, perchance, the constant is zero. (See equation (7) below.) It is this curious fact that lies at the center of a mathematical controversy to be discussed shortly.

The topic of fractional calculus lay dormant for almost a decade until the works of Joseph Liouville appeared. P. Kelland later remarked, "Our astonishment is great, when we reflect on the time of its first announcement to (Liouville's) applications." But it was in 1974 that the first text [Oldham and Spanier] solely devoted to this topic was published, and in the same year the first conference was held [Ross 1974].

Mathematicians have described Abel's solution as "elegant." Perhaps it was Fourier's integral formula and Abel's solution which had attracted the attention of Liouville, who made the first major study of fractional calculus. He published three large memoirs in 1832 and several more publications in rapid succession [Liouville

1832]. Liouville was successful in applying his definitions to problems in potential theory.

The starting point for his theoretical development was the known result for derivatives of integral order:

$$D^m e^{ax} = a^m e^{ax},$$

which he extended in a natural way to derivatives of arbitrary order

$$D^\nu e^{ax} = a^\nu e^{ax}.$$

He assumes that the arbitrary derivative of a function $f(x)$ which can be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$$

is

$$D^\nu f(x) = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n x} \tag{5}$$

The above formula is known as Liouville's first formula for a fractional derivative. It generalizes, in a natural way, a derivative of arbitrary order where ν is any number--rational, irrational or complex. But it has the obvious disadvantage that ν must be restricted to those values for which the series converges. Perhaps Liouville was quite aware of these restrictions, for he formulated a second definition.

To obtain his second definition he started with a definite integral related to the gamma integral of Euler:

$$I = \int_0^\infty u^{a-1} e^{-xu} du, \quad a>0,\ u>0.$$

The change of variables $xu = t$ yields

$$I = \int_0^\infty (t^{a-1} e^{-t}) dt/x^a = \Gamma(a)/x^a,$$

or

$$x^{-a} = \frac{1}{\Gamma(a)} I.$$

Then he operates on both sides of the above with D^ν:

$$D^\nu x^{-a} = \frac{1}{\Gamma(a)} D^\nu \int_0^\infty u^{a-1} e^{-xu} du.$$

The arbitrary derivative with respect to x according to Liouville's basic assumption gives

$$D^\nu x^{-a} = \frac{(-1)^\nu}{\Gamma(a)} \int_0^\infty u^{a+\nu-1} e^{-xu} du.$$

Make the same transformation as before: $xu = t$. The transformed integral is recognized as the familiar gamma integral of Euler, which has the value $\Gamma(a+\nu)$. Thus, Liouville obtains his second definition for a fractional derivative:

$$D^{\nu}x^{-a} = \frac{(-1)^{\nu}\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}. \tag{6}$$

But Liouville's definitions were too narrow to last. The first definition is restricted to certain values of ν, and the second definition, useful for functions of the type x^{-a}, is not suitable to be applied to a wide class of functions.

Liouville was the first to attempt solving differential equations by means of fractional operators. A complementary function, familiar to those who have studied differential equations, was the object of some of his investigations. In one of his memoirs [1834], to justify the existence of a complementary function, he wrote, "The ordinary differential equation $\frac{d^n y}{dx^n} = 0$ has the complementary solution $y_c = c_o + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$. Thus $\frac{d^u y}{dx^u} = 0$ (u arbitrary) should have a corresponding complementary solution."

Liouville did publish his version of the complementary solution. Further mention of it is made later in this paper, for it played a role in planting the seeds of distrust in the general theory of fractional operations. George Peacock [1833], and S. S. Greatheed [1839] published papers which, in part, dealt with the complementary function. Greatheed was the first to call attention to the indeterminate nature of the complementary function.

A LONGSTANDING CONTROVERSY

Two essentially different definitions of fractional operations have been given which have different domains of usefulness. One definition was the generalization of a case of integral order used by Lacroix and Abel for functions of the type x^a, later called functions of the Riemann class. Peacock supported this version and spoke of Liouville's definitions as being erroneous in many points. P. Kelland, who published two works on this topic in 1839 and 1846, supported Liouville's definitions useful for functions of the type x^{-a}, later called functions of the Liouville class. William Center [1850] observed that the fractional derivative of a constant, according to the Lacroix-Peacock method, is unequal to zero. Using x^0 to denote unity, Center finds the fractional derivative of unity of order $\frac{1}{2}$, by means of (2), in the following manner:

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x^0 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} = 1/\sqrt{\pi x}. \tag{7}$$

But, as Center points out, according to Liouville's "system," referring to Liouville's second definition given above in formula

(6), the fractional derivative of unity equals zero because $\Gamma(0)=\infty$. He continues, "The whole question is plainly reduced to what is $d^u(x^0)/dx^u$. For when this is determined we shall determine at the same time which is the correct system."

Augustus De Morgan [1840] devoted three pages to fractional calculus. He comments on the two versions of a fractional derivative: "Both these systems may very possibly be parts of a more general system, but at present I incline (in deference to supporters of both systems) to the conclusion that neither system has any claim to be considered as giving the form $D^n x^n$, though either may be a form."

The state of affairs complained about by De Morgan and Center is now thoroughly cleared up. De Morgan's judgment proved to be correct, for the two systems which Center thought led to irreconcilable results have now been incorporated into a more general system. It is only fair to state that mathematicians at that time were aiming for a plausible definition of generalized integration and differentiation without attempting to examine the consequences of their definitions in the complex plane.

RIEMANN'S CONTRIBUTION, ERRORS BY NOTED MATHEMATICIANS

G. F. Berhard Riemann developed his theory of fractional integration in his student days, but he withheld publication. It was published posthumously in his *Gesammelte Werke* [1876]. He sought a generalization of a Taylor series and derived

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\,dt + \psi(x). \tag{8}$$

Because of the ambiguity in the lower limit of integration c, Riemann saw fit to add to his definition a complementary function $\psi(x)$. This complementary function is essentially an attempt to provide a measure of the deviation from the law of exponents. For example, this law, as mentioned later, is

${}_cD_x^{-\mu}\ {}_cD_x^{-\nu} f(x) = {}_cD_x^{-\mu-\nu} f(x)$ and is valid when the lower terminals c are equal. Riemann was concerned with a measure of deviation for the case ${}_cD_x^{-\mu}\ {}_{c'}D_x^{-\nu} f(x)$.

A. Cayley [1880] remarked, "The greatest difficulty in Riemann's theory, it appears to me, is the question of the meaning of a complementary function containing an infinity of arbitrary constants." Any satisfactory definition of a fractional operation will demand that this difficulty be removed. Indeed, the present-day definition of fractional integration is (8) without the complementary function,

The question of the existence of a complementary function caused considerable confusion. Liouville made an error when he gave an explicit evaluation of his own interpretation of a complementary function. He did not consider the special case for $x = 0$ which led to a contradiction [Davis 1936, 71]. Peacock made two errors in the topic of fractional calculus. These errors involved the misapplication of the *Principle of the permanence of equivalent forms*. Although this principle is stated for algebra, Peacock assumed this principle valid for all symbolic operations. He considered the existence of a complementary function and developed an expansion for $D^{-m} x$, m a positive integer. He erred when he naively concluded that he could formally replace m with a fraction as did Lacroix when Lacroix let $m = \frac{1}{2}$ as in (2). Peacock made another error of the same kind when he developed the expansion for the derivative of integer order $D^{m}(ax+b)^{n}$ and then sought to extend his result to the general case [Davis 1936, 71].

In addition to the errors of Liouville and Peacock, there was the long dispute as to whether the Lacroix-Peacock version or the Liouville version of a fractional derivative was the correct definition. Later, Cayley noted, as already mentioned, that Riemann was hopelessly entangled in his version of a complementary function. Thus, I suggest that when Oliver Heaviside published his work in the last decade of the nineteenth century, he was met with disdain and haughty silence not only because he exacerbated the situation with his hilarious jibes at mathematicians, but also because mathematicians had a general distrust of the theory of fractional operators.

THE MID-NINETEENTH CENTURY

C. J. Hargreave [1848] appears to be the first to write on the generalization of Leibniz's nth derivative of a product. In modern form it is

$$D^{\nu} f(x)g(x) = \sum_{n=0}^{\infty} \binom{\nu}{n} D^{(n)} f(x)\, D^{(\nu-n)} g(x)$$

where $D^{(n)}$ is ordinary differentiation, $D^{(\nu-n)}$ is a fractional operation and $\binom{\nu}{n}$ is the generalized binomial coefficient $\Gamma(\nu+1)/n!\Gamma(\nu-n+1)$. The generalized Leibniz Rule can be found in many modern applications [Ross 1974, 32]. H. R. Greer [1858] wrote on finite differences of fractional order. Surprisingly the most recent access to a fractional derivative is by means of finite differences [Mikolâs 1974]. Mention should also be made of a paper by W. Zachartchenxo [1861]. He improves on the work of Greer and he ends his paper with an amusing note, which no modern mathematician would admit, concerning his research on a topic: "I know that Liouville, Peacock and Kelland have written on this topic, but I have had no opportunity to read their works." H. Holmgren [1868] wrote a 58-page monograph on the application of fractional calculus to the solution of certain ordinary differential equations. In the introduction to this work, he asserts that his predecessors Liouville and Spitzer had obtained results which were too restrictive. Holmgren, taking Liouville's work as his point of departure, states that his aim in this paper is to find a complete solution not subject to the restrictions on the independent variable which his predecessors had made. He proceeds along formal lines. For example, the index law is used:

$$D^{\nu} y'' = D^{\nu}D^2 y = D^{\nu+2} y.$$

Although this rule is valid for ν a positive integer, modern mathematicians would seek to justify this rule when ν is arbitrary.

THE ORIGINS OF THE RIEMANN-LIOUVILLE DEFINITION

The earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Ya. Sonin [1869] entitled "On differentiation with arbitrary index." His starting point was Cauchy's integral formula. A. V. Letnikov wrote four papers on this topic from 1868 to 1872. His paper "An explanation of the fundamental concepts of the theory of differentiation of arbitrary order" [1872] is an extension of Sonin's paper. Sonin and Letnikov, in their attempt to define a fractional derivative, used a closed contour. Cauchy's integral formula is given by

(10) $$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(t)}{(t-z)^{n+1}}\,dt.$$

There is no problem in generalizing $n!$ to arbitrary values since $\nu! = \Gamma(\nu+1)$. However, when n is not an integer the integrand above no longer contains a pole, but a branch point. An appropriate contour would then require a branch cut which was not included in the work of Sonin and Letnikov.

It was not until H. Laurent [1884] published his paper that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. The theory of fractional calculus is intimately connected with the theory of operators. The operators D or d/dx and D^2 or d^2/dx^2 denote a rule of transformation of a function into other functions which are the first and second ordinary derivatives. The rule of transformation is familiar to all those who have studied calculus. At the present time, there are various notations in use which denote fractional operators. Although the authors cited in the text which follows employed operator notation of their own devising, the notation invented by Harold T. Davis will be used, namely

(11) $${}_cD_x^{-\nu} f(x), \qquad \mathrm{Re}(\nu)>0.$$

The operator ${}_cD_x^{-\nu}$ denotes integration of arbitrary order of the function f. The operator ${}_cD_x^{\nu}$ denotes differentiation of arbitrary order. The subscripts c and x denote the terminals of integration. The adjoining of these subscripts becomes a vital part of the operator symbol to avoid ambiguities in applications. The subject of fractional operator notation cannot be minimized. It has been said that the succinctness of this notation adds to the elegance of fractional calculus.

Before we turn our attention again to Laurent, it will be worthwhile to consider the problem of defining precisely integration and differentiation of arbitrary order. Most of the mathematicians mentioned so far were not merely formalizing, but were trying to solve a problem which they well understood but did not explicitly formulate. What is wanted is this: for every function f, of a sufficiently wide class, and any number ν -- fractional, irrational or complex--a function $D^{\nu} f(x) = g(x)$ should be assigned subject to the following criteria [Ross 1974], 5-6]:

1. If $f(z)$ is an analytic function of the complex variable z (or $z = x$ a real variable), the derivative $D^{\nu} f(z)$ is an analytic function of ν and z.

2. The operation $D^{\nu} f$ must produce the same result as ordinary differentiation when ν is a positive integer: $D^{\nu} f(x) = f^{(\nu)}(x)$. If $\nu = -n$, a negative integer, $D^{\nu} f(x)$ must produce the same result as ordinary n-fold integration, and $g(x) = D^{-n} f(x)$ must vanish together with all its n-1 derivatives at $x =$ the lower terminal of integration.

3. The fractional operators must be linear.

4. The operation of order zero leaves the function unchanged: $D^0 f = f$.

5. The law of exponents (indices) holds for integration of arbitrary order: $D^{-\mu}D^{-\nu} f = D^{-\mu-\nu} f$, $\mathrm{Re}(\mu)$ and $R(\nu)>0$.

Laurent's starting point was Cauchy's integral formula. His contour was an open circuit, in contrast to the closed circuit of Sonin and Letnikov. Make a cut along the real axis from x on the positive real axis to negative infinity. Laurent's contour, now called a Laurent loop, starts on the lower edge of the cut at a point $c < x$, goes to a point A, around the circle of radius ε whose center is x, in the positive sense to a point A' on the upper edge of the cut, and then along the upper edge of the cut to c'. Standard methods of contour integration yield the general result, now called the Riemann-Liouville integral:

$$(12) \qquad {}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-l)^{\nu-1} f(t)\, dt, \ \mathrm{Re}(\nu) > 0.$$

The method of contour integration, applied to Cauchy's integral formula by Laurent, produced the definition (12) for integration of arbitrary order. This definition satisfies the previous listed criteria. When $c = 0$, we have Riemann's definition, but without a complementary function. When $c = -\infty$, (12) can be shown to be equivalent to Liouville's first definition.

P. A. Nekrassov [1888] and A. Krug [1890] also obtained the fundamental definition (12) from Cauchy's integral formula, their methods differing in choice of a contour of integration. It remains a curious fact, however, that these generalized operators of integration and their connection with the Cauchy integral formula have succeeded in securing for themselves, to this day, only passing references in standard works in the theory of analytic functions.

One cannot replace $-\nu$ with ν formally in (12), expecting to obtain the derivative operator ${}_c\mathrm{D}_x^{\nu}$, because the integral $\int_c^x (x-t)^{-\nu-1} f(t)\,\mathrm{d}t$ would, in general, be divergent. It can be shown by analytic continuation that for differentiation of arbitrary order we have

$$
\begin{aligned}
{}_c\mathrm{D}_x^{\nu} f(x) &= {}_c\mathrm{D}_x^{m-p} f(x) = {}_c\mathrm{D}_x^{m}\, {}_c\mathrm{D}_x^{-p} f(x) \\
&= \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\frac{1}{\Gamma(p)}\int_c^x (x-t)^{p-1} f(t)\,\mathrm{d}t\right],
\end{aligned}
$$

where m is (for convenience) the least integer greater than ν, $\nu = m-p$, $0 < p \leqq 1$, and ${}_c\mathrm{D}_x^{m}$ is the ordinary differentiation operator $\mathrm{d}^m/\mathrm{d}x^m$. For $c = 0$ or $c = -\infty$, the integral above is a beta integral for a wide class of functions f and is readily evaluated.

For $f(x) = x^a$ and x^{-a}, $a > 0$, $\mathrm{Re}(\nu) > 0$, we have

$$
\left.
\begin{aligned}
&(14)\quad {}_0\mathrm{D}_x^{-\nu} x^a = \frac{\Gamma(a+1)}{\Gamma(a+\nu+1)}\, x^{a+\nu} \\
&(15)\quad {}_0\mathrm{D}_x^{\nu} x^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}\, x^{a-\nu}
\end{aligned}
\right\}\quad \text{Riemann}
$$

$$
\left.
\begin{aligned}
&(16)\quad {}_{-\infty}\mathrm{D}_x^{-\nu} x^{-a} = \frac{(-1)^{\nu}\Gamma(a-\nu)}{\Gamma(a)}\, x^{-a+\nu} \\
&(17)\quad {}_{-\infty}\mathrm{D}_x^{\nu} x^{-a} = \frac{(-1)^{\nu}\Gamma(a+\nu)}{\Gamma(a)}\, x^{-a-\nu}
\end{aligned}
\right\}\quad \text{Liouville}
$$

It is worthwhile noting that for $f(x) = x$ and $\nu = \frac{1}{2}$, formula (15) yields the same result as given by Lacroix in (2). We can also consider Center's observation concerning the derivative of arbitrary order of a constant. For $f(x) = 1$ and $\nu = \frac{1}{2}$, the definition (13) with $c = 0$ is

$$
\begin{aligned}
{}_0\mathrm{D}_x^{\frac{1}{2}}\, 1 &= {}_0\mathrm{D}_x^{1-\frac{1}{2}}\, 1 \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{-\frac{1}{2}}\, 1\,\mathrm{d}t\right]
\end{aligned}
$$

$$= \frac{d}{dx}\ \frac{1}{\Gamma(\frac{1}{2})}(2)x^{\frac{1}{2}}$$

(18) $$= \frac{1}{\Gamma(\frac{1}{2})}\ x^{-\frac{1}{2}}.$$

The same result could be obtained by use of formula (15) by taking a to be zero as did Center. But Center was incorrect when he said the Liouville definition yields zero for the arbitrary derivative of a constant. Definition (13) with $c = -\infty$ is

$$_{-\infty}D_x^{\frac{1}{2}}\ 1 = \frac{d}{dx}\left[\frac{1}{\Gamma(\frac{1}{2})}\int_{-\infty}^{x} (x-t)^{-\frac{1}{2}}\ 1\ dt\right]$$

This integral is divergent; hence the derivative of arbitrary order of a constant in the Liouville sense does not exist. Constants are in the Riemann class of functions.

THE LAST DECADE OF THE NINETEENTH CENTURY

Oliver Heaviside [1892] published a number of papers in which he showed how certain linear differential equations can be solved by the use of generalized operators. Heaviside was an untrained scientist, a fact which may explain his lack of rigor. His methods, which have proved to be useful to engineers in the theory of the transmission of electrical currents in cables, have been collected under the name *Heaviside operational calculus*.

The Heaviside operational calculus is concerned with linear functional operators. He denoted the operator d/dx by the letter p and treated it as if it were a constant in the solution of differential equations. D. F. Gregory [1841], said to be the founder of what was then called the *calculus of operations,* had put the solution of the heat equation into symbolic operator form:

$$z = Ae^{y\beta^{\frac{1}{2}}} + Be^{-y\beta^{\frac{1}{2}}},$$

where $\beta = a^{-1}(d/dx)$. But it was Heaviside's brilliant applications that accelerated the development of the theory of these generalized operators. He obtained correct results by expanding in powers of $p^{\frac{1}{2}}$, where $p^{\frac{1}{2}} = d^{\frac{1}{2}}/dx^{\frac{1}{2}} = D^{\frac{1}{2}}$. In the theory of electrical circuits, Heaviside found frequent use for the operator $p^{\frac{1}{2}}$. He interpreted $p^{\frac{1}{2}} \to 1$, that is, $D^{\frac{1}{2}}\ 1$ to mean $1/(\pi t)^{\frac{1}{2}}$ as in (18). Since $f(t) = 1$ is a function of the Riemann class, it is clear that Heaviside's operator must be interpreted in the context of the Riemann operator $_0D_x^{\nu}$.

His results were correct but he was unable to justify his procedures. Kelland, earlier, remarked on the ten-year interval between Fourier's publication and Liouville's applications. A similar situation followed Heaviside's publications, except that in this case, a much longer time elapsed before his procedures were justified by T. J. Bromwich [1919].

Harold T. Davis [1936, 16] said, "The period of the formal development of operational methods may be regarded as having ended by 1900. The theory of integral equations was just beginning to stir the imagination of mathematicians and to reveal the possibilities of operational methods." The author is preparing a paper on the sequel from 1900 to the present.

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