A differential subordination and starlikeness of analytic functions

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Abstract

In the present work, we apply the general theory of differential subordination to obtain certain interesting criteria for starlikeness of an analytic function.

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1. Introduction

Let \( \mathcal{A} \) be the class of functions \( f \) which are analytic in the unit disc \( E = \{z : |z| < 1\} \) and are normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). Denote by \( \mathcal{A}' \) the class of functions \( f \) which are analytic in \( E \) and satisfy \( f'(0) = 1 \). Also, let \( K \), \( St \), \( St(\alpha) \) and \( C \) denote the usual classes of convex functions, starlike functions (with respect to the origin), starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), and close-to-convex functions in \( E \), respectively.

If \( f \) and \( g \) are analytic in \( E \), we say that \( f \) is subordinate to \( g \) in \( E \), written as \( f(z) \prec g(z) \) in \( E \), if \( g \) is univalent in \( E \), \( f(0) = g(0) \) and \( f(E) \subset g(E) \).

If \( \varphi : D \rightarrow \mathbb{C} (D \subset \mathbb{C}^2, \mathbb{C} \) is the complex plane) is an analytic function, \( p \) is a function analytic in \( E \) with \( (p(z),zp'(z)) \in D \) for \( z \in E \), and \( h \) is univalent in \( E \), then the function \( p \) is said to satisfy first order differential subordination provided that

\[
\varphi(p(z),zp'(z)) \prec h(z), z \in E, \varphi(p(0),0) = h(0).
\]

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A univalent function \( q \) is said to be the dominant of the differential subordination (1) if \( p(0) = q(0) \) and \( p < q \) for all \( p \) satisfying (1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q} < q \) for all dominants \( q \) of (1) is said to be the best dominant of (1).

For a real number \( \lambda, \lambda > 0 \), and a complex number \( \alpha \) with \( \Re \alpha > 0 \), we define
\[
\phi(\alpha, \lambda; p(z)) = (1 - \alpha)p(z) + \alpha(p(z))^2 + \alpha \lambda z p'(z),
\]
where \( p \) is an analytic function in \( E \) and satisfies the normalization condition \( p(0) = 1 \). In the present work, in Section 3, we study the first order differential subordination of the form
\[
\phi(\alpha, \lambda; p(z)) < \phi(\alpha, \lambda; q(z)), z \in E.
\] (2)

Our purpose is to find the conditions that the function \( q \) must satisfy so that it becomes the best dominant of the differential subordination (2).

Making appropriate choices for the functions \( p \) and \( q \) in (2), we then study the general differential subordination of the form
\[
\frac{zf'(z)}{f(z)} \left[ 1 - \alpha + \alpha(1 - \lambda) \frac{zf'(z)}{f(z)} + \alpha \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < \frac{zg'(z)}{g(z)}.
\]
\[
\left[ 1 - \alpha + \alpha(1 - \lambda) \frac{zg'(z)}{g(z)} + \alpha \lambda \left( 1 + \frac{zg''(z)}{g'(z)} \right) \right],
\] (3)

where \( f \) and \( g \) belong to \( \mathcal{A} \), \( \alpha \) is a complex number with \( \Re \alpha > 0 \) and \( \lambda, \lambda > 0 \), is real. As consequences, we obtain some interesting criteria for starlikeness by carefully selecting the function \( q \) in (2) (or the function \( g \) in (3)).

As a special case of differential subordination (3), we obtain the following differential subordination:
\[
\frac{zf'(z)}{f(z)} + \alpha z^2 \frac{f''(z)}{f(z)} < \frac{zg'(z)}{g(z)} + \alpha z^2 \frac{g''(z)}{g(z)},
\]

which generalizes the recent papers of Padmanabhan [1], Ramesha et al. [2] and Li and Owa [3].

2. Preliminaries

The following definitions and lemmas play a key role in the proofs of our results.

**Definition 2.1.** A function \( L(z, t), z \in E \) and \( t \geq 0 \) is said to be a subordination chain if \( L(\cdot, t) \) is analytic and univalent in \( E \) for all \( t \geq 0 \), \( L(z, \cdot) \) is continuously differentiable on \([0, \infty)\) for all \( z \in E \) and \( L(z, t_1) < L(z, t_2) \) for all \( 0 \leq t_1 \leq t_2 \).

**Lemma 2.1** ([4, pg. 159]). The function \( L(z, t) : E \times [0, \infty) \to \mathbb{C} \) of the form \( L(z, t) = a_1(t)z + \cdots \) with \( a_1(t) \neq 0 \) for all \( t \geq 0 \), and \( \lim_{t \to \infty} |a_1(t)| = \infty \), is a subordination chain if and only if \( \Re \left[ \frac{zL}{zL/zt} \right] > 0 \) for all \( z \in E \) and \( t \geq 0 \).

**Lemma 2.2** ([5]). Let \( F \) be analytic in \( E \) and let \( G \) be analytic and univalent in \( \overline{E} \) except for points \( \zeta \) such that \( \lim_{z \to \zeta} F(z) = \infty \), with \( F(0) = G(0) \). If \( F \) is not subordinate to \( G \) in \( E \), then there exist points \( z_0 \in E, \zeta_0 \in \partial E \) (boundary of \( E \)) and an \( m \geq 1 \) such that \( F(|z| < |z_0|) \subset G(E), F(z_0) = G(\zeta_0) \) and \( z_0 F'(z_0) = m\zeta_0 G'(\zeta_0) \).
3. Main results

First, we prove the following lemma.

**Lemma 3.1.** Let $\alpha$ be a complex number with $\text{Re}\, \alpha > 0$. Suppose that $q \in \mathcal{A}'$ is a convex univalent function which satisfies the following conditions:

(a) $\text{Re}\, q(z) > 0$, in $E$, when $\text{Re}\, \alpha \geq |\alpha|^2$;
(b) $\text{Re}\, q(z) > \frac{|\alpha|^2 - \text{Re}\, \alpha}{2|\alpha|^2}$, in $E$, when $\text{Re}\, \alpha < |\alpha|^2$.

For a real number $\lambda$, $\lambda > 0$, if a function $p \in \mathcal{A}'$ satisfies the differential subordination

$$\phi(\alpha, \lambda; p(z)) < \phi(\alpha, \lambda; q(z)),$$

in $E$, then $p(z) < q(z)$ in $E$ and $q$ is the best dominant.

**Proof.** Let

$$h(z) = \phi(\alpha, \lambda; q(z)) = (1 - \alpha)q(z) + \alpha(q(z))^2 + \alpha \lambda z q'(z).$$

Clearly $h$ is analytic in $E$ and $h(0) = 1$. First of all, we will prove that $h$ is univalent in $E$ so that the subordination (4) is well defined in $E$. From (5), we get

$$\frac{1}{\alpha} h'(z) = 2q(z) + \frac{\overline{\alpha} - |\alpha|^2}{|\alpha|^2} + \lambda \left(1 + \frac{z q''(z)}{q'(z)} \right).$$

In view of the conditions (a) and (b) above and the fact that $q$ is convex in $E$, we obtain

$$\text{Re} \frac{1}{\alpha} h'(z) > 0, \quad z \in E.$$

Since $\text{Re}\, \alpha > 0$, and $q$ is convex univalent in $E$, we conclude that $h$ is close-to-convex and, hence, univalent in $E$.

Without any loss of generality, we can assume $q$ to be analytic and univalent in $\overline{E}$ (closure of $E$). If not, then we can replace $p, q$ and $h$ by $p_r(z) = p(rz), q_r(z) = q(rz)$ and $h_r(z) = h(rz)$ respectively where $0 < r < 1$. These new functions satisfy the conditions of the theorem on $\overline{E}$. We can then prove that $p_r < q_r$, and by letting $r \to 1^-$, we obtain $p < q$.

We now show that $p < q$ in $E$. If possible, suppose that $p \neq q$ in $E$. Then, by Lemma 2.2, there exist points $z_0 \in E$ and $\xi_0 \in \partial E$ such that $p(|z| < |z_0|) \subset q(E)$, $p(z_0) = q(\xi_0)$ and $z_0 p'(z_0) = m \xi_0 q'(\xi_0), m \geq 1$. Thus,

$$(1 - \alpha) p(z_0) + \alpha(p(z_0))^2 + \alpha \lambda z_0 p'(z_0) = (1 - \alpha)q(\xi_0) + \alpha(q(\xi_0))^2 + m \alpha \lambda \xi_0 q'(\xi_0).$$

Consider a function

$$L(z, t) = (1 - \alpha)q(z) + \alpha(q(z))^2 + \alpha \lambda t z q'(z)$$

$$= h(z) - (1 - t)\alpha \lambda z q'(z)$$

$$= 1 + a_1(t)z + \cdots.$$

The function $L(z, t)$ is analytic in $E$ for all $t \geq 0$ and is continuously differentiable on $[0, \infty)$ for all $z \in E$. Now,

$$a_1(t) = \left. \frac{\partial L(z, t)}{\partial z} \right|_{(0, t)} = q'(0)(1 + \alpha + \alpha \lambda t).$$
As the function $q$ is univalent in $E$, we have $q'(0) \neq 0$. Also since $\Re \alpha > 0$, we get $|\arg(1 + \alpha + \alpha \lambda t)| < \pi/2$. Therefore, it follows that $a_1(t) \neq 0$ and $\lim_{t \to \infty} |a_1(t)| = \infty$. Further, a simple calculation yields

$$
\frac{\partial L}{\partial z} = \frac{1}{\lambda} \left( 2q(z) + \frac{\alpha - |\alpha|^2}{|\alpha|^2} \right) t \left( 1 + \frac{zf''(z)}{q'(z)} \right).
$$

In view of given conditions (a) and (b) and the facts that $\lambda > 0$ and $q$ is convex in $E$, we have

$$
\Re \left[ \frac{\partial L}{\partial z} \right] > 0,
$$

for all $z \in E$ and $t \geq 0$. Therefore, by Lemma 2.1, $L(z, t)$ is a subordination chain. So, $L(z, t_1) \prec L(z, t_2)$, for $0 \leq t_1 \leq t_2$. From (7), we have $L(z, 1) = h(z)$; thus we deduce that

$$
L(\xi_0, t) \notin h(E) \quad \text{for} \quad |\xi_0| = 1 \quad \text{and} \quad t \geq 1. \quad \text{(8)}
$$

From (6) and (7), we can write

$$
(1 - \alpha)p(z_0) + \alpha(p(z_0))^2 + \alpha \lambda z_0 p'(z_0) = L(\xi_0, m).
$$

In view of (8), we obtain $L(\xi_0, m) \notin h(E)$ for $z_0 \in E$, $|\xi_0| = 1$ and $m \geq 1$, which is a contradiction to (4). Hence, $p < q$ in $E$. Since $p = q$ satisfies (4), the function $q$ is the best dominant of the differential subordination (4).

We note that when $p(z) = \frac{zf'(z)}{f(z)}$,

$$
\phi \left( \alpha, \lambda; \frac{zf'(z)}{f(z)} \right) = \frac{zf'(z)}{f(z)} \left[ 1 - \alpha + \alpha (1 - \lambda) \frac{zf'(z)}{f(z)} + \alpha \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right],
$$

and

$$
\phi \left( \alpha, 1; \frac{zf'(z)}{f(z)} \right) = \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f'(z)}.
$$

Setting $p(z) = \frac{zf'(z)}{f(z)}$ in Lemma 3.1, we obtain the following general subordination result:

**Theorem 3.1.** Let $\alpha, \lambda$ and $q$ be as in Lemma 3.1. If a function $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies the differential subordination

$$
\phi \left( \alpha, \lambda; \frac{zf'(z)}{f(z)} \right) \prec \phi (\alpha, \lambda; q(z)), \quad z \in E,
$$

then

$$
\frac{zf'(z)}{f(z)} \prec q(z),
$$

for all $z$ in $E$. \(\square\)

Letting $\lambda = 1$ and $p(z) = \frac{zf'(z)}{f(z)}$ in Lemma 3.1, we obtain the following subordination result [6, Th. 3.2]:
Theorem 3.2. Let $\alpha$ and $q$ be as given in Lemma 3.1. If a function $f \in A$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies the differential subordination
\[\frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f(z)} < (1 - \alpha)q(z) + \alpha(q(z))^2 + \alpha z q'(z), \quad z \in E,\]
then
\[\frac{zf'(z)}{f(z)} < q(z),\]
for all $z$ in $E$. □

A function $f \in A$ is said to be $\alpha$-convex [7] if
\[\text{Re} \left[ (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0, \quad z \in E.\]

Our next result is a subordination theorem wherein we consider a combination of the analytic expressions for starlike and $\alpha$-convex functions and obtain an interesting criterion for starlikeness:

Theorem 3.3. Let $\lambda$ be a positive real number. Assume that $q \in A'$ is convex univalent in $E$ and $\text{Re} \ q(z) > 0$, $z \in E$. If a function $f \in A$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies the differential subordination
\[\frac{zf'(z)}{f(z)} \left[ (1 - \lambda)\frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < q(z) \left[ q(z) + \frac{\lambda z q'(z)}{q(z)} \right], \quad z \in E,\]
then
\[\frac{zf'(z)}{f(z)} < q(z),\]
for all $z$ in $E$.

Proof. The proof follows by letting $\alpha = 1$ in Theorem 3.1. □

4. Applications

In this section, we shall choose a distinguished function as the dominant $q$ which will satisfy the conditions of the general subordination theorem, Theorem 3.1. As a consequence, we shall obtain some new and interesting criteria for starlikeness of an analytic function.

Let us choose $q(z) = \frac{1 + az}{z}, -1 < a \leq 1$. Obviously, $q$ is convex univalent in $E$. It can be easily verified that the condition (a) of Theorem 3.1 (with $\alpha$ real) is obviously satisfied as $\text{Re} \ q(z) > \frac{1-a}{2a} > 0$, for all $z$ in $E$ and for all $\alpha$ and, therefore, for $0 \leq \alpha \leq 1$, in particular. However, when $\alpha > 1$, we get $\text{Re} \ q(z) > \frac{1-a}{2a} \geq \frac{a-1}{2a}$ provided $a \leq 1/\alpha$. Thus, in view of Theorem 3.1 (with $\alpha$ real), we obtain the following result:

Theorem 4.1. Let $\alpha$ and $\lambda$ be positive real numbers. Assume that $a, -1 < a \leq 1$, is a real number such that $a \leq 1/\alpha$ whenever $\alpha > 1$. Let $f \in A$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfy
\[\frac{zf'(z)}{f(z)} \left[ 1 - \alpha + \alpha(1 - \lambda)\frac{zf'(z)}{f(z)} + \alpha \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < h(z), \quad z \in E,\]
where
\[ h(z) = (1 - \alpha) \left( \frac{1 + az}{1 - z} \right) + \alpha \left( \frac{1 + az}{1 - z} \right)^2 + \alpha \lambda \left( \frac{(a + 1)z}{(1 - z)^2} \right). \]

Then,
\[ \frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z}, \]
for all \( z \) in \( E \).

We now investigate the image of the unit disc under the function \( h \). Rewriting \( h \) as
\[ h(z) = a[\alpha(a + 1) - 1] + (a + 1) \left[ \frac{1 - a\alpha + [2a\alpha + \alpha\lambda + \alpha - 1]z}{(1 - z)^2} \right], \]

we find that \( h(0) = 1 \) and \( h(-1) = \frac{a(a+1)(a-1-\lambda)+2(1-a)}{4} \). In view of Lemma 3.1, it is clear that the function \( h \) is close-to-convex in \( E \). The curve is symmetrical about the real axis and it intersects the real axis at one point only. The boundary of the curve is given by \( h(a^{\theta}) = u(\theta) + iv(\theta), \theta \in (-\pi, \pi) \), where
\[ u(\theta) = \frac{(1 - a) - \alpha(\lambda + 1)(1 + a) + \cos \theta[(a - 1) - a\alpha(a + 1)]}{2(1 - \cos \theta)} \]
and
\[ v(\theta) = \frac{(1 - a\alpha)(1 + a) \sin \theta}{2(1 - \cos \theta)}. \]

Eliminating \( \theta \), we get the equation of the boundary curve as
\[ v^2 = -\frac{(1 - a\alpha)^2(1 + a)}{\alpha(\lambda + 1 + a)} \left[ u - \frac{\alpha(a^2 - \lambda - a\lambda - 1 + 2(1 - a))}{4} \right] \]

which is a parabola opening towards the left, with its vertex at the point \( \left( \frac{\alpha(a^2 - \lambda - a\lambda - 1 + 2(1 - a))}{4}, 0 \right) \) and the negative real axis as the axis of parabola. Hence, \( h(E) \) is the exterior of this parabola and includes the right plane
\[ u \geq \frac{\alpha(a^2 - \lambda - a\lambda - 1 + 2(1 - a))}{4}. \]

**Remark 4.1.** When \( a\alpha = 1 \), the parabola given above degenerates into a straight line along the real axis \( v = 0 \) extending from the point \( (u, 0) \) to \( -\infty \), where
\[ u = -\frac{(\alpha - 1)^2 + \alpha\lambda(1 + a)}{4\alpha}. \]

Therefore, in this case, the region \( h(E) \) becomes the whole complex plane with slit along negative real axis from \( -\frac{(\alpha - 1)^2 + \alpha\lambda(1 + a)}{4\alpha} \) to \( -\infty \).

Setting \( a = 1 - 2\beta, 0 \leq \beta < 1 \) and \( \lambda = 1 \) in Theorem 4.1, we obtain the following result (also see [6, Cor. 4.1]):

**Corollary 4.1.** Let \( \alpha, \alpha > 0, \) be a real number. Assume that \( \beta, 0 \leq \beta < 1, \) is a real number such that \( \beta \geq \frac{1}{2} - \frac{1}{2\alpha} \) whenever \( \alpha > 1 \). For all \( z \in E, \) let \( f \in A, \frac{f(z)}{z} \neq 0 \) in \( E, \) satisfy the differential
Corollary 4.2. Let \( \alpha > 0 \) and \( \alpha \notin \text{Li and Owa } [3] \):

\[
\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} < h(z),
\]

where

\[
h(z) = (1 - \alpha) \left( \frac{1 + (1 - 2\beta)z}{1 - z} \right) + \alpha \left( \frac{1 + (1 - 2\beta)z}{1 - z} \right)^2 + \alpha \left( \frac{2(1 - \beta)z}{(1 - z)^2} \right).
\]

Then \( f \in \text{St}(\beta) \).

We note that if \( h(z) = u + iv \), then \( h(E) \) is the exterior of the parabola given by

\[
v^2 = -\frac{(1 - \alpha(1 - 2\beta))^2(2 - 2\beta)}{\alpha(3 - 2\beta)} \left[ u - \left( \alpha \beta \left( \beta - \frac{1}{2} \right) + \beta - \frac{\alpha}{2} \right) \right]
\]

with its vertex at \( \left( \alpha \beta \left( \beta - \frac{1}{2} \right) + \beta - \frac{\alpha}{2}, 0 \right) \).

Remark 4.2. (i) In the case when \( \beta = \frac{1}{2} \) or \( \beta = \frac{1}{2\alpha} \), the parabola given above degenerates into a straight line extending from the point \( -\left( \frac{2\alpha^2 - \alpha + 1}{4\alpha} \right) , 0 \) to \( -\infty \). Therefore, in this case, the region \( h(E) \) becomes the whole complex plane with a slit along the negative real axis from \( \left( \frac{2\alpha^2 - \alpha + 1}{4\alpha} \right) \) to \( -\infty \).

(ii) Corollary 4.1 necessarily improves the result of Ravichandran et al. [8].

(iii) For \( \alpha > 1 \), the condition that \( \beta \geq \frac{1}{2} \) or \( \beta \geq \frac{1}{2\alpha} \), in Corollary 4.1, restricts the range of \( \beta \) to \( (0, 1/2) \).

Taking \( a = 1 - \alpha \) and \( \lambda = 1 \) in Theorem 4.1, we obtain the following result which extends the result of Li and Owa [3]:

**Corollary 4.2.** Let \( \alpha, 0 < \alpha < 2 \), be a real number. If \( f \in A \), \( \frac{f(z)}{z} \neq 0 \) in \( E \), satisfies the differential subordination

\[
\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} < h(z), \quad z \in E,
\]

then \( f \in \text{St}(\alpha/2) \). Here

\[
h(z) = (1 - \alpha) \left( \frac{1 + (1 - \alpha)z}{1 - z} \right) + \alpha \left( \frac{1 + (1 - \alpha)z}{1 - z} \right)^2 + \alpha \left( \frac{(2 - \alpha)z}{(1 - z)^2} \right),
\]

and \( h(E) \) is the exterior of the parabola given by the equation

\[
v^2 = -\frac{(2 - \alpha)(1 - \alpha + \alpha^2)}{\alpha(3 - \alpha)} \left[ u - \left( \frac{\alpha^2(\alpha - 1)}{4} \right) \right]
\]

with its vertex at the point \( \left( -\frac{\alpha^2}{4}(1 - \alpha), 0 \right) \).

For \( a = \lambda = 1 \) in Theorem 4.1, we obtain the following result of Padmanabhan [1]:

**Corollary 4.3.** For \( 0 < \alpha \leq 1 \), if \( f \in A \), \( \frac{f(z)}{z} \neq 0 \) in \( E \), satisfies

\[
\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} < (1 - \alpha) \left( \frac{1 + z}{1 - z} \right) + \alpha \left( \frac{1 + z}{1 - z} \right)^2 + \alpha \frac{2z}{(1 - z)^2} = h(z), \quad \text{say},
\]
then $f \in St$. Here $h(E)$ is the exterior of the parabola given by the equation
\[ v^2 = -\frac{2(1-\alpha)^2}{3\alpha} \left[ u + \frac{\alpha}{2} \right] \]
with its vertex at $(-\alpha/2, 0)$.

Letting $\lambda = 1$ and $\alpha$ approach infinity in Theorem 4.1, we obtain the following result also given in [6]:

**Corollary 4.4.** For all $\alpha$, $-1 < \alpha \leq 0$, let an analytic function $f$ in $A$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfy
\[
\frac{z^2 f''(z)}{f(z)} < (a + 1) \frac{a z^2 + 2z}{1 - z^2} = h(z), \text{ say,}
\]
in $E$. Then $zf'(z) < \frac{1 + az}{1 - z}$ in $E$.

Working on similar lines to in Theorem 4.1, we obtain that $h(E)$ is the exterior of the parabola
\[ v^2 = -\frac{a^2(a + 1)}{a + 2} \left( u - \frac{a^2 - a - 2}{4} \right) \]
with its vertex at the point $\left( \frac{a^2 - a - 2}{4}, 0 \right)$.

If we take $\alpha = 1/2$ in Theorem 4.1, we get the following interesting result:

**Corollary 4.5.** Let $\lambda, \lambda > 0$ be a given real number. For $-1 < a \leq 1$, if $f \in A$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies the differential subordination
\[
\frac{zf'(z)}{f(z)} \left[ 1 + (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] < \frac{1 + az}{1 - z} + \left( \frac{1 + az}{1 - z} \right)^2 + \lambda \left( \frac{(a + 1)z}{(1 - z)^2} \right)
\]
in $E$, then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z},
\]
for all $z$ in $E$.

In this case, the image of the unit disc $E$ under the function $h(z) = u + iv$ is the exterior of the parabola
\[ v^2 = -\frac{(2 - a)^2(1 + a)}{(\lambda + 1 + a)} \left[ u - \frac{(a^2 - \lambda - a\lambda - 1) + 4(1 - a)}{4} \right] \]
which has its vertex at the point $\left( \frac{(a^2 - \lambda - a\lambda - 1) + 4(1 - a)}{4}, 0 \right)$.

Writing $\alpha = 1$ in Theorem 4.1, we obtain the following interesting criterion for starlikeness:

**Corollary 4.6.** Let $\lambda, \lambda > 0$ be a real number. For $-1 < a \leq 1$, if $f \in A$, $\frac{f(z)}{z} \neq 0$ in $E$, satisfies the differential subordination
\[
\frac{zf'(z)}{f(z)} \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] < \left( \frac{1 + az}{1 - z} \right)^2 + \lambda \left( \frac{(a + 1)z}{(1 - z)^2} \right) = h(z), \text{ say,}
\]
in $E$, then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z},
\]
for all $z$ in $E$.

It can be easily verified that the image of the open unit disc $E$ under the function $h$, i.e. $h(E)$, is the exterior of the parabola given by
\[
v^2 = -\frac{(1 - a)^2(1 + a)}{\lambda + 1 + a} \left[ u - \frac{(1 - a)^2 - \lambda(1 + a)}{4} \right]
\]
which has its vertex at the point \(\left(\frac{1 - a)^2 - \lambda(1 + a)}{4}, 0\right)\).

Example 4.1. Taking $a = 0$ and $\lambda = 1/2$ in Corollary 4.6, we obtain
\[
h(z) = \frac{2 + z}{2(1 - z)^2}.
\]
In this case, $h(E)$ is the exterior of the parabola given by
\[
v^2 = -\frac{2}{3} \left[ u - \frac{1}{8} \right]
\]
with its vertex at the point \(\left(\frac{1}{8}, 0\right)\).

Thus, an application of Corollary 4.6 gives:
If \( f \in A, \frac{f(z)}{z} \neq 0 \) in \( E \), satisfies the differential subordination

\[
zf'(z) \left[ \frac{zf'(z)}{f(z)} + \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < \frac{2 + z}{(1 - z)^2}, \quad z \in E,
\]

then \( \frac{zf'(z)}{f(z)} < \frac{1}{1-z} \) in \( E \), i.e. \( f \) is starlike of order 1/2.

Region \( h(E) \) has been shown shaded in Fig. 1.

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References