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Automata, groups, limit spaces, and tilings

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Abstract

We explore the connections between automata, groups, limit spaces of self-similar actions, and tilings. In particular, we show how a group acting “nicely” on a tree gives rise to a self-covering of a topological groupoid, and how the group can be reconstructed from the groupoid and its covering. The connection is via finite-state automata. These define decomposition rules, or self-similar tilings, on leaves of the solenoid associated with the covering.
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Keywords: Automata; Groups; Tilings; Self-coverings; Solenoids; Tree automorphisms

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1. Introduction

Groups generated by automata appeared in the early 60's, in particular through the work of Alešin [1]; important examples were studied later by Grigorchuk [6] and Gupta and Sidki [9]. A general theory of such groups has only recently started to emerge.

The situation changed considerably when the last author introduced the notion of "limit space" of a contracting group generated by automata. This provided a bridge with dynamical systems, by associating with such a group a space with covering map, and vice versa.

This paper explores the degree to which these two notions are equivalent. It turns out that the construction of a contracting group from a limit space, and of a limit space from a contracting group, are inverse to each other if the limit space is considered as a topological orbispace (Theorem 6.7).

The orbispaces considered in this paper are quite complicated: for example, they have uncountable fundamental group; the set of points with nontrivial isotropy groups is dense, etc. Other constructions would yield "nicer" limit orbispaces, e.g., such that the points with nontrivial isotropy groups form a closed nowhere dense set; but at the cost of making less transparent the similarity between self-coverings of orbispaces and self-similar groups.

This paper also attempts to present in a uniform way automata, groups, limit spaces, and the tilings that they carry. Many of the results are not new, although wherever possible they are stated in greater generality.

The main connections, starting from an automaton (Π), or a topological space with a branched covering (X), are as shown in Fig. 1. The connecting constructions are a limit space (L), a solenoid (S) and a discrete group (Γ). The commutativity of the diagram at † is proven in Theorem 6.7, and that at ‡ is proven in Theorem 6.1.

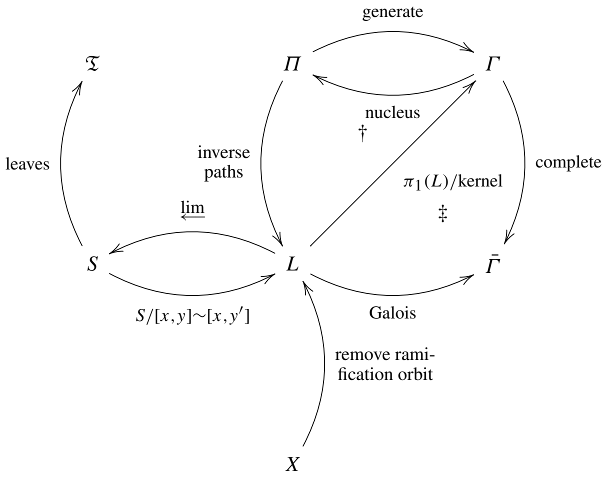

Fig. 1.

2. Definitions

We introduce in this section the main definitions used in the text. Most of them already appeared in [2] with examples and illustrations.

Our convention for $\mathbb{N}$ is that it does not contain 0.

2.1. Automata

An *automaton* Π is a pair A, Q of sets called respectively *alphabet* and *states*, with maps $\sigma : A \times Q \to A$ and $\tau : A \times Q \to Q$ called respectively the *output* and *transition* functions.

A *graph* is a pair V, E of sets called respectively *vertices* and *edges*, with maps $s, t : E \to V$ called respectively the *source* and *target* maps.[1]

The *graph* $\mathfrak{G}(\Pi)$ of an automaton Π is the graph with vertex set Q, containing for all $q \in Q$ and all $a \in A$ an edge from q to $\tau(a, q)$, labeled $a/\sigma(a, q)$.

The *dual* of Π is the automaton Π^* with alphabet Q, states A, output $\sigma^*(q, a) = \tau(a, q)$ and transition $\tau^*(q, a) = \sigma(a, q)$.

Let Π and Π' be two automata on the same alphabet A. The *product* of Π and Π' is the automaton $\Pi'' = \Pi * \Pi'$ with alphabet A, states $Q \times Q'$, output $\sigma''(a, (q, q')) = \sigma'(\sigma(a, q), q')$ and transition $\tau''(a, (q, q')) = (\tau(a, q), \tau'(\sigma(a, q), q'))$. The iterated product $\Pi * \cdots * \Pi$ is written Π^n.

We denote by $A^* = \bigcup_{n \geqslant 0} A^n$ the free monoid on A. It is conveniently represented as an $\#A$-regular rooted tree, with root the empty word, and with an edge from w to wa for all $w \in A^*, a \in A$.

[1] By "graph," we shall therefore always mean "oriented graph," with both loops and multiple edges allowed.

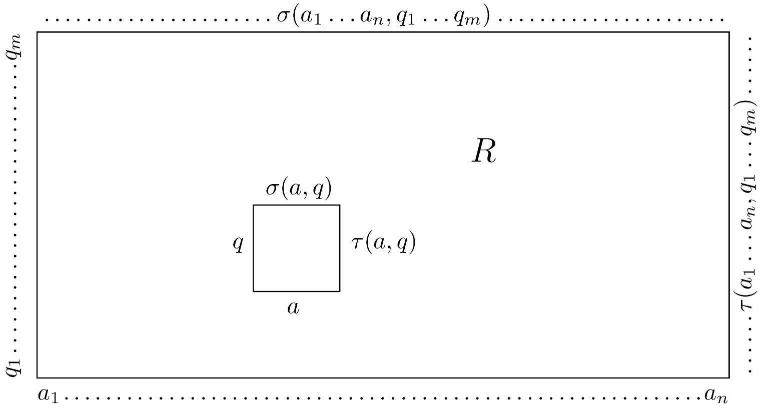

Fig. 2.

The output and transition functions are naturally extended to functions $\sigma : A^* \times Q^* \to A^*$ and $\tau : A^* \times Q^* \to Q^*$, by

$$\begin{aligned}
\sigma(a_1 a_2 \dots a_n, q) &= \sigma(a_1, q)\sigma\bigl(a_2 \dots a_n, \tau(a_1, q)\bigr),\\
\tau(a_1 a_2 \dots a_n, q) &= \tau\bigl(a_2 \dots a_n, \tau(a_1, q)\bigr),\\
\sigma(a_1 \dots a_n, q_1 q_2 \dots q_m) &= \sigma\bigl(\sigma(a_1 \dots a_n, q_1), q_2 \dots q_m\bigr),\\
\tau(a_1 \dots a_n, q_1 q_2 \dots q_m) &= \tau(a_1 \dots a_n, q_1)\tau\bigl(\sigma(a_1 \dots a_n, q_1), q_2 \dots q_m\bigr).
\end{aligned}$$

The extended input and output functions can be visualized in the following way: consider for all $a \in A, q \in Q$ a small square with lower side labeled a, left side q, top side $\sigma(a, q)$ and right side $\tau(a, q)$. Consider a large rectangle R with bottom label $a_1 \dots a_n$ and left label $q_1 \dots q_m$, and tile it by the above small squares. Then $\sigma(a_1 \dots a_n, q_1 q_2 \dots q_m)$ and $\tau(a_1 \dots a_n, q_1 q_2 \dots q_m)$ are the top and right labels of R (see Fig. 2).

Let Π and Π' be two automata on the same alphabet A. An *automaton homomorphism* $f : \Pi \to \Pi'$ is a map $f : Q \to Q'$ such that $\sigma'(a, f(q)) = \sigma(a, q)$ and $\tau'(a, f(q)) = f(\tau(a, q))$ for all $a \in A, q \in Q$. Note that there is a label-preserving graph homomorphism $\mathfrak{G}(\Pi) \to \mathfrak{G}(\Pi')$ if and only if there is an automaton homomorphism $\Pi \to \Pi'$. We call Π' a *subautomaton* of Π if f is injective and Π' is a *quotient* of Π if f is surjective.

2.2. Actions

An automaton Π, with a given state q, induces a transformation Π_q on the tree A^*, via σ; explicitly,

$$\Pi_q(a_1 a_2 \dots a_n) = \sigma(a_1, q)\Pi_{\tau(a_1, q)}(a_2 \dots a_n).$$

Note that $\Pi'_{q'}\Pi_q = (\Pi * \Pi')_{(q,q')}$.

Every orbit of this action lies in A^n for some $n \geqslant 0$.[2]

The transformations Π_q are invertible if $\sigma(-,q)$ is a permutation for all $q \in Q$. In that case Π is called *invertible*, and $\langle \Pi \rangle$, the *group of the automaton* Π, is defined as the group generated by the Π_q for all $q \in Q$.

Let Γ be a group acting on a set X, with generating set S. The *Schreier graph* of X is the graph with vertex set X, and for all $x \in X$ and $s \in S$ an edge from x to sx, labeled s.

Lemma 2.1. *The Schreier graph of $\langle \Pi \rangle$ on A^n is $\mathfrak{G}((\Pi^*)^n)$.*

Proof. Direct verification. Indeed the vertices of $\mathfrak{G}(\Pi^*)$ naturally identify with A, and its edges identify with $A \times Q$. Similarly, the vertices in $\mathfrak{G}((\Pi^*)^n)$ identify with A^n, and its edges identify with $A^n \times Q$. □

Definition 2.2. A group Γ acting on a rooted tree A^* is *spherically transitive* if for every n the action is transitive on A^n; in other words, Γ acts transitively on the spheres around the root vertex.

Definition 2.3. A group Γ is *recurrent* if for any vertex $w \in A^*$ the natural map $\mathsf{Stab}_\Gamma(w) \to \mathsf{Aut}(A^*)$, mapping $g \in \Gamma$ to its restriction on wA^* and identifying wA^* with A^*, has image Γ. Note that if Γ is generated by an automaton, then this image is by necessity a subgroup of Γ; in that case, the statement needs to be checked only for all $w \in A$.

An automaton Π is *recurrent* or *spherically transitive* if the associated group $\langle \Pi \rangle$ enjoys the respective property.

Lemma 2.4. *If Γ is recurrent and acts transitively on A, then it is spherically transitive.*

2.3. Contraction

Let Π be an automaton. It is *contracting* if there is a finite set $N \subset \langle \Pi \rangle$ such that for any $g \in \langle \Pi \rangle$ there is $n \in \mathbb{N}$ with $\tau(A^n, g) \subset N$. The minimal such set N is called the *nucleus* of $\langle \Pi \rangle$. It may be defined as

$$N = \bigcup_{g \in \langle \Pi \rangle} \bigcap_{n_0 \geqslant 0} \bigcup_{n \geqslant n_0} \tau\big(A^n, g\big).$$

Definition 2.5. The automaton is *nuclear* if it is contracting and its nucleus is equal to Q.

We remark that it is usually difficult to determine whether an automaton is contracting. On the other hand, it takes polynomial time to check whether it is nuclear.

If Π is a finite, contracting automaton, and is generated by its nucleus N, then we may replace Π's set of states with N, and obtain a nuclear automaton generating the same group.

[2] Equivalently, the action is *radial*.

Lemma 2.6. *If Π be a nuclear automaton, then $\tau : A \times Q \to Q$ is onto.*

Proof. Assume to the contrary that $q \notin \tau(A, Q)$. Then q could be removed from the nucleus, contradicting its minimality. □

Lemma 2.7. [12, Lemma 2.11.12] *If Π is nuclear, then there exists constants $\lambda < 1$, K and n such that $|\tau(x, g)| < \lambda|g| + K$ for all $g \in \langle \Pi \rangle$ and $x \in A^n$, where $|g|$ denotes the minimal length of $g \in \langle \Pi \rangle$ as a word over Q.*

We introduce also the following

Definition 2.8. Let Π be a contracting automaton. Its set of states $Q = N$ therefore contains a specific state, the identity, written ε. The automaton Π is *smooth* if τ is surjective and the following subgraph of $\mathfrak{G}(\Pi^*)$ is strongly connected: its vertices are letters $a \in A$; there is an edge from a to b for all $q \in Q$ with $\sigma(q, a) = b$ and $\tau(q, a) = \varepsilon$.

The following consequence of smoothness will not be used directly; its proof is implicit within the proof of Proposition 4.6: if an automaton is smooth, then it is recurrent and spherically transitive. More is true:

Lemma 2.9. *If Π is a smooth automaton, then for every pair of alphabet-words $a, b \in A^*$ of the same length, and any word $v \in Q^*$, there exists a word $w \in Q^*$ with $\sigma(a, w) = b$ and $\tau(a, w) = \varepsilon^{n_0} v_1 \varepsilon^{n_1} \dots v_k \varepsilon^{n_k}$ for some $n_0, \dots, n_k \geqslant 0$.*

2.4. Profinite groups

Let a group Γ act on a tree A^*. There is then a family of finite quotients Γ_n defined by restricting the action of Γ to A^n. These form a projective system $\Gamma_{n+1} \to \Gamma_n$, with inverse limit $\bar{\Gamma} = \varprojlim \Gamma_n$.

Lemma 2.10. *$\bar{\Gamma}$ is the closure of Γ in the topological group $\mathsf{Aut}(A^*)$, with its standard (compact-open) topology.*

Proof. In the standard topology of $W = \mathsf{Aut}(A^*)$, since A^* is discrete, a basis of neighbourhoods of the identity is given by subgroups fixing larger and larger finite sets of vertices. One may take as such subgroups the pointwise fixators $\mathsf{Stab}_W(n)$ of A^n; and then $\Gamma/(\Gamma \cap \mathsf{Stab}_W(n)) \cong \Gamma_n$.

$\bar{\Gamma}$ is then closed in W because it is an inverse limit of the closed subgroups $\Gamma_n \subset W/\mathsf{Stab}_W(n)$.

To show that Γ is dense in $\bar{\Gamma}$, pick $g = \varprojlim \gamma_n \in \bar{\Gamma}$, with $\gamma_n \in \Gamma_n$. Choose lifts $g_n \in \Gamma$ of γ_n. Since the action of g_n agrees with that of g on A^n, we have $g_n \to g$. □

3. Groupoids

In what follows we will use interchangeably the words orbispace and groupoid; more on that philosophy can be found in [10]. Recall that a groupoid is a graph (X, G, s, t) with a multiplication $\{(g, g') \in G \times G \mid s(g') = t(g)\} \to G$, an inverse $(\ )^{-1} : G \to G$ and an identity $1 : X \to G$, such that

$$s(gh) = s(g), \quad t(gh) = t(h), \quad gg^{-1} = 1_{s(g)}, \quad g^{-1}g = 1_{t(g)}, \quad 1_{s(g)}g = g = g1_{t(g)}.$$

More concisely, a groupoid is a small category in which all arrows are invertible. We will usually denote the groupoid simply by G.

The group $G_x = \{g \in G \mid s(g) = t(g) = x\}$ of self arrows of an object $x \in X$ is called the *isotropy* of x, and the subset $G^x = \{g \in G \mid s(g) = x\}$ is called the *fiber* of x.

By groupoid, we shall always mean topological groupoid, i.e., we shall assume the sets G, X will have natural topologies that make all structural maps—source, target, multiplication, inverse, unit—continuous.

As a trivial example of groupoid, we may take $G = X$ a topological space, with s, t the identity. An example that will be relevant to the sequel is the following: consider

$$X = \{0, 1\}^{\mathbb{N}}, \qquad I = \bigl\{(x, x)\bigr\}_{x \in X} \cup \bigl\{(w0\bar{1}, w1\bar{0})\bigr\}_{w \in X^*} \cup \bigl\{(w1\bar{0}, w0\bar{1})\bigr\}_{w \in X^*}, \tag{1}$$

where $\bar{0}$ and $\bar{1}$ denote the infinite words $00\ldots$ and $11\ldots$, respectively. The set X is given the Tychonoff (product) topology, and I is given the topology inherited from $X \times X$. Note that this makes I compact. Define $s(g)$ and $t(g)$ as the projections on the first and second coordinate, respectively; set $(x, y)^{-1} = (y, x)$, and $(x, y)(y, z) = (x, z)$. We shall see shortly (Lemma 3.6) that I should be understood as the interval $[0, 1]$.

A morphism of groupoids $G \to G'$ is a pair of continuous maps $\phi : G \to G'$, $\psi : X \to X'$, such that $\psi(s(g)) = s(\phi(g))$, $\psi(t(g)) = t(\phi(g))$, $\phi(g)^{-1} = \phi(g^{-1})$, and $\phi(gh) = \phi(g)\phi(h)$.

One can think of our groupoid (even though this is not equivalent) as a topological space $Z = X/\{s(g) = t(g)\}$, together with for every $z \in Z$ a small enough neighbourhood $U_z \ni z$ and a homeomorphism $U_z \cong \tilde{U}_z/G_z$, where G_z is a discrete group acting on a topological space $\tilde{U}_z$ and fixing a point $\tilde{z} \in \tilde{U}_z$ identified with z.

Let us give a vague outline of this construction in the special case when s is locally a homeomorphism. We denote by $\pi : X \to Z$ the projection. For a point $z = \pi(x) \in Z$, we set $G_z = G_x$, and choose $\tilde{U}_z$ as some neighbourhood of $\tilde{z} = x \in X$. We will define an action of G_z on this neighbourhood. Let $g \in G_z$ be an arrow. Since $s : G \to X$ is a local homeomorphism with $s(g) = x$, it admits a unique germ of section $\sigma_s : X \dashrightarrow G$, defined on a neighbourhood of x, and satisfying $\sigma_s(x) = g$. We define the action of g to be the composite $\phi_g = t\sigma_s : X \dashrightarrow X$. It is a germ of homeomorphism defined on a neighbourhood of x and fixing x. Clearly, if $\phi_g(x') = x''$ then $s(g) = x'$ and $t(g) = x''$ and hence $\pi(x') = \pi(x'')$, so π is locally a homeomorphism $\tilde{U}_z/G_z \to Z$ that[3] sends $\tilde{z} = x$ to $z = \pi(x)$.

[3] We are being sloppy here: first of all, it might not be possible to lift the action $G_z \to \{$germs of homeomorphisms$\}$ to an action defined on a neighbourhood of x. Secondly, Z might fail to be Hausdorff, in which case we could not conclude that $\tilde{U}_z/G_z \to Z$ is injective.

3.1. Geometric realizations

The following classical construction builds a topological space $|G|$ from a groupoid G, such that all considerations on G have equivalents on $|G|$. First, let

$$G_n = \big\{(x_0, g_1, x_1, \dots, g_n, x_n) \mid x_i \in X,\ g_i \in G,\ t(g_i) = x_i = s(g_{i+1}) \text{ for all } i\big\}$$

be the space of composable sequences of n arrows. There are face maps $d_i : G_n \to G_{n-1}$ and degeneracies $s_i : G_n \to G_{n+1}$, for $i \in \{0, \dots, n\}$, given by

$$\begin{aligned}
d_0(x_0, g_1, x_1, \dots, g_n, x_n) &= (x_1, \dots, g_n, x_n),\\
d_i(x_0, g_1, x_1, \dots, g_n, x_n) &= (x_0, \dots, x_{i-1}, g_i g_{i+1}, x_{i+1}, \dots, g_n, x_n),\\
d_n(x_0, g_1, x_1, \dots, g_n, x_n) &= (x_0, \dots, g_{n-1}, x_{n-1}),\\
s_i(x_0, g_1, x_1, \dots, g_n, x_n) &= (x_0, g_1, \dots, x_i, 1_{x_i}, x_i, \dots, g_n, x_n),
\end{aligned}$$

which turn the family $(G_n)_{n\geqslant 0}$ into a simplicial space. One then lets $|G|$ be the geometric realization of that simplicial space; namely, let Δ^n denote the standard n-simplex $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geqslant 0,\ \sum t_i = 1\}$, with its usual cofaces and codegeneracies

$$\begin{aligned}
\delta_i(t_0, \dots, t_{n-1}) &= (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}),\\
\sigma_i(t_0, \dots, t_{n+1}) &= (t_0, \dots, t_{i-1} + t_i, \dots, t_{n+1});
\end{aligned}$$

then we take

$$|G| = \coprod_{n\geqslant 0} G_n \times \Delta^n \Big/ \begin{matrix}\big(d_i(x), t\big) \sim \big(x, \delta_i(t)\big),\\ \big(s_i(x), t\big) \sim \big(x, \sigma_i(t)\big).\end{matrix}$$

Lemma 3.1. *If G is compact metrizable, then so is the k-skeleton $|G|_k$ of G:*

$$|G|_k = \coprod_{n=0}^{k} G_n \times \Delta^n/\sim.$$

Proof. Let us assume by induction that $|G|_{k-1}$ is compact metrizable. The space $|G|_k$ is obtained from $|G|_{k-1}$ by gluing $G_k \times \Delta^k$ via an attaching map $f : G_k \times \partial\Delta^k \to |G|_{k-1}$. Therefore, $|G|_k$ is the pushout of the following diagram of compact spaces:

$$\begin{tikzcd}
 & G_k \times \Delta^k \\
G_k \times \partial\Delta^k \arrow[ur, "\iota"] \arrow[dr, "f"'] & \\
 & {|G|_{k-1}}
\end{tikzcd}$$

The reader who is more concerned about mathematical rigor than geometric interpretation may disregard the above construction at no cost.

Since ι is a closed inclusion, the pushout $|G|_k$ is the quotient of $|G|_{k-1} \sqcup G_k \times \Delta^k$ by a closed equivalence relation, and is thus compact and metrizable. $\square$

We say a groupoid G is *connected* if it is impossible to disconnect $X = X' \sqcup X''$ and $G = G' \sqcup G''$ in disjoint open subsets such that (G', X') and $(G,'' X'')$ are groupoids by restriction.

Lemma 3.2. *G is connected if and only if $|G|$ is connected.*

Proof. If $|G|$ is not connected, write it as $|G| = A \sqcup B$, and set $X' = X \cap A$ and $X'' = X \cap B$, where X is viewed as a subspace of $|G|$. Set then $G' = \bigcup_{x \in X'} G^x$ and $G'' = \bigcup_{x \in X''} G^x$. The target map sends G' to X' and G'' to X'', because otherwise there would be an edge crossing from X' to X'', which is impossible by assumption.

If G is not connected, then $G = G' \sqcup G''$ and therefore $|G| = |G'| \sqcup |G''|$. $\square$

Definition 3.3. Let (ϕ, ψ) be a morphism of groupoids, $\phi : G \to G'$ and $\psi : X \to X'$. It is a *covering* if ψ is a covering, and

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "s"'] & G' \arrow[d, "s'"] \\ X \arrow[r, "\psi"'] & X' \end{tikzcd}$$

is a pull-back diagram.

This implies that ϕ and ψ have the same degree, which we call the degree of (ϕ, ψ). We shall actually abuse notation and denote both ϕ and ψ by the letter ϕ.

Lemma 3.4. *Let*

$$\begin{tikzcd} \tilde{R} \arrow[r, shift left, "\tilde{\alpha}"] \arrow[r, shift right, "\tilde{\beta}"'] \arrow[d, "p_R"'] & \tilde{X} \arrow[d, "p_X"] \\ R \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & X \end{tikzcd}$$

be a diagram of compact spaces, where $\alpha p_R = p_X \tilde{\alpha}$, $\beta p_R = p_X \tilde{\beta}$, and p_R, p_X are degree-d covering maps. Suppose that $Z = X/R$ is Hausdorff, and that the map of coequalizers $p_Z : \tilde{Z} = \tilde{X}/\tilde{R} \to Z$ has fibers of cardinality d. Then p_Z is a covering map.

Proof. Pick $z \in Z$ and let C be its preimage in X. The restriction of $\tilde{X}$ to C is trivialized by its map to $p_Z^{-1}(z)$. Extend that trivialization to an open neighbourhood $\mathcal{U} \supset C$. It induces trivializations of $\tilde{R}$ on $\alpha^{-1}(\mathcal{U})$ and $\beta^{-1}(\mathcal{U})$. Moreover, these trivializations agree on $\alpha^{-1}(C)=\beta^{-1}(C)$. The structure group being finite hence discrete, they must agree on some neighbourhood $\mathcal{V}\supset\alpha^{-1}(C)$ contained in $\alpha^{-1}(\mathcal{U})\cap\beta^{-1}(\mathcal{U})$. Pick a neighbourhood $\mathcal{U}''$ of C satisfying $\alpha^{-1}(\mathcal{U}'')\subset\mathcal{V}$ and $\beta^{-1}(\mathcal{U}'')\subset\mathcal{V}$.

Next, saturate $\mathcal{U}''$ under the equivalence relation generated by R, by setting $\mathcal{U}'=X\setminus p_1p_2^{-1}(X\setminus\mathcal{U}'')$, where p_1,p_2 are respectively the first- and second-coordinate projections $X\times_Z X\rightrightarrows X$; they are closed maps on the compact $X\times_Z X$, because Z was assumed Hausdorff.

Set $\mathcal{V}'=\alpha^{-1}(\mathcal{U}')=\beta^{-1}(\mathcal{U}')$. The trivializations induce isomorphisms

$$\left(\begin{array}{ccc}\tilde{R}|_{\mathcal{V}'} & \overset{\tilde\alpha}{\underset{\tilde\beta}{\rightrightarrows}} & \tilde{X}|_{\mathcal{U}'}\\ {\scriptstyle p_R}\downarrow & & \downarrow{\scriptstyle p_X}\\ \mathcal{V}' & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & \mathcal{U}'\end{array}\right)\cong\left(\begin{array}{ccc}\mathcal{V}'\times\{1,\dots,d\} & \overset{\alpha\times 1}{\underset{\beta\times 1}{\rightrightarrows}} & \mathcal{U}'\times\{1,\dots,d\}\\ \downarrow & & \downarrow\\ \mathcal{V}' & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & \mathcal{U}'\end{array}\right). \tag{2}$$

Therefore the coequalizers of (2) form a trivial covering. Now remember that the quotient $\mathcal{U}'/\mathcal{V}'$ is a neighbourhood of $z\in Z$. We have just exhibited a trivialization of $\tilde{Z}$ on $\mathcal{U}'/\mathcal{V}'$. It follows that the map $p_Z:\tilde{Z}\to Z$ is a covering. □

Proposition 3.5. *A homomorphism $\phi:G\to G'$ induces functorially a continuous map $|\phi|:|G|\to|G'|$. The map ϕ is a covering if and only if $|\phi|$ is a covering, and then both have the same degree.*

Proof. Let $(\phi:G\to G',\phi:X\to X')$ be our groupoid homomorphism. Define $|\phi|:|G|\to|G'|$ by

$$|\phi|(x_0,g_1,\dots,x_n;t_0,\dots,t_n)=\big(\phi(x_0),\phi(g_1),\dots,\phi(x_n);t_0,\dots,t_n\big).$$

Assume now that ϕ is a covering of degree some cardinal $\aleph$. First note that composable sequences of arrows $(x_0',g_1',\dots,x_n')\in G_n'$ satisfy the "unique lifting property." Namely, given $x_0\in X$ with $\phi(x_0)=x_0'$ there is a unique composable sequence of arrows $(x_0,g_1,\dots,x_n)\in G_n$ with $\phi(x_i)=x_i'$ and $\phi(g_i)=g_i'$. Indeed $g_1'\in(G')^{\phi(x_0)}$ has a unique lift $g_1\in G^{x_0}$; we set $x_1=t(g_1)$, and lift g_2', etc., until x_n'. Every sequence $(x_0',g_1',\dots,x_n')\in G_n'$ therefore has as many preimages in G_n as x_0' has preimages in X.

We have shown that $(G_n\to X)\to(G_n'\to X')$ is a pullback diagram, and hence that $G_n\to G_n'$ is a covering.[4] Therefore, by crossing with Δ^n, the map

$$\coprod_{n\geqslant 0}G_n\times\Delta^n\to\coprod_{n\geqslant 0}G_n'\times\Delta^n$$

is a covering of degree $\aleph$.

[4] Actually, we have only shown it at the level of sets, but an easy diagram chase shows that it is also a pullback of topological spaces.

Now recall that $|G|$ and $|G'|$ are obtained by quotienting, i.e., taking a coequalizer

$$\coprod_{n\geqslant 1}\coprod_{i=0}^{n} G_n \times \Delta^{n-1} \sqcup \coprod_{n\geqslant 0}\coprod_{i=0}^{n} G_n \times \Delta^{n+1} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \textstyle\coprod_{n\geqslant 0} G_n \times \Delta^n,$$

where

$$\alpha = \coprod_{n=1}^{k}\coprod_{i=0}^{n} d_i \times 1_{\Delta^{n-1}} \sqcup \coprod_{n=0}^{k-1}\coprod_{i=0}^{n} s_i \times 1_{\Delta^{n+1}}$$

and

$$\beta = \coprod_{n=1}^{k}\coprod_{i=0}^{n} 1_{G_n} \times \delta_i \sqcup \coprod_{n=0}^{k-1}\coprod_{i=0}^{n} 1_{G_n} \times \sigma_i.$$

In the following diagram both rows are coequalizers:

$$\begin{tikzcd}
\displaystyle\coprod_{n=1}^{k}\coprod_{i=0}^{n} G_n \times \Delta^{n-1} \sqcup \coprod_{n=0}^{k-1}\coprod_{i=0}^{n} G_n \times \Delta^{n+1} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\dot{\phi}"] & \displaystyle\coprod_{n\geqslant 0} G_n \times \Delta^n \arrow[r] \arrow[d, "\ddot{\phi}"] & {|G|_k} \arrow[d, "|\phi|"] \\
\displaystyle\coprod_{n=1}^{k}\coprod_{i=0}^{n} G'_n \times \Delta^{n-1} \sqcup \coprod_{n=0}^{k-1}\coprod_{i=0}^{n} G'_n \times \Delta^{n+1} \arrow[r, shift left, "\alpha'"] \arrow[r, shift right, "\beta'"'] & \displaystyle\coprod_{n\geqslant 0} G'_n \times \Delta^n \arrow[r] & {|G'|_k}
\end{tikzcd}$$

The space $|G|_k$ is the union of the $G_n \times (\Delta^n)^\circ$. On each such piece, $|\phi|$ is the product of $\phi : G_n \to G'_n$ and the identity on $(\Delta^n)^\circ$. In particular, $|\phi|$ is everywhere d-to-1. Applying Lemma 3.4, we see that $|G|_k \to |G'|_k$ is a covering, and by taking direct limits, so is $|G| \to |G'|$.

Conversely, if $|G| \to |G'|$ is a cover, then $X = |G|_0 \to X' = |G'|_0$ is a cover by restriction. The diagram $(G \to X) \to (G' \to X')$ yields a map $G \to X \times_{X'} G'$, for which we need to find an inverse. The mapping cylinder of s (respectively of s') $M_s = G \times [0,1] \cup_s X$ sits as a subspace of $|G|$ (respectively $|G'|$). It is the image of $G \times [0, \frac{1}{2}] \hookrightarrow G \times [0,1] \twoheadrightarrow |G|_1 \subset |G|$. The map $M_s \to M_{s'}$ is therefore also a cover. We have a map $(X \times_{X'} G') \times [0,1] \to M_{s'}$ defined by $(x, g', t) \mapsto (g', t)$. Using the unique homotopy lifting extension, we obtain a lift $(X \times_{X'} G') \times [0,1] \to M_s$ making the diagram

$$\begin{tikzcd}
(X \times_{X'} G') \times [0,1] \arrow[r, dotted] \arrow[rr, bend left] & M_s \arrow[r] & M_{s'} \\
(X \times_{X'} G') \times \{0\} \arrow[r] \arrow[u, hook] & X \arrow[r] \arrow[u, hook] & X' \arrow[u, hook]
\end{tikzcd}$$

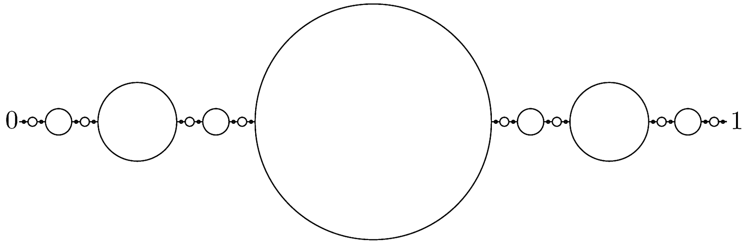

Fig. 3.

commute. Restriction to $(X \times_{X'} G') \to G \times \{1\}$ gives us our desired map. $\square$

Lemma 3.6. *The groupoid I defined in* (1) *admits an injective continuous map $\phi : [0,1] \to |I|$ with $\phi(0) = \bar{0}$ and $\phi(1) = \bar{1}$.*

Proof. We first define a map $\chi : X \to [0,1]$, by

$$\chi(w_1 w_2 \dots) = \sum_{i=1}^{\infty} 2w_i 3^{-i}.$$

Then we define $\psi : |I| \to [0,1]$ by

$$\psi(x_0, g_1, x_1, \dots, g_n, x_n; t_0, \dots, t_n) = \sum_{i=0}^{n} t_i \chi(x_i).$$

Consider the image $|I|^+$ of $(\{(x,x)\}_{x \in X} \cup \{(w0\bar{1}, w1\bar{0})\}_{w \in \{0,1\}^*}) \times \Delta^1$ in $|I|$; then ψ defines a continuous bijection between $|I|^+$ and $[0,1]$. Since $|I|^+$ is compact, ψ admits a continuous inverse ϕ. $\square$

Actually, $\phi([0,1])$ is a deformation retract of $|I|$: the n-skeleton $|I|_n$ is homeomorphic to the closure in $\mathbb{R}^{n+1}$ of the countable union of spheres (see also Fig. 3)

$$\bigcup_{w \in \{0,1\}^*} \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \left\| x - \left(\chi(w0\bar{1}) + \frac{3^{-|w|}}{6}, 0, \dots, 0 \right) \right\| = \frac{3^{-|w|}}{6} \right\}.$$

A nondegenerate n-simplex $|\Sigma|$ of $|I|_n$ comes from a sequence $\Sigma = (w0\bar{1}, w1\bar{0}, w0\bar{1}, \dots)$ or $(w1\bar{0}, w0\bar{1}, w1\bar{0}, \dots)$ in I_n. There are exactly two such simplices for every $(n-1)$-sphere in I_{n-1}, which glue on Σ as the two hemispheres attach to $|\Sigma|$ in $|I|_n$. One can retract $|I|$ to $|I|_1^+$ by successively sliding the n-spheres over the $(n+1)$-spheres, in shorter and shorter amounts of time as $n \to \infty$ so that the retraction be performed in finite total time.

We say that a groupoid G is *arcwise connected* if for every $x, y \in X$ there exists a homomorphism $\gamma : I \to G$ with $\gamma(\bar{0}) = x$ and $\gamma(\bar{1}) = y$.

Lemma 3.7. *G is arcwise connected if and only if $|G|$ is arcwise connected.*

Proof. Assume first that G is arcwise connected. Notice first that any point $(x_0, g_1, x_1, \dots, g_n, x_n; t_0, \dots, t_n) \in |G|$ is connected by an interval, within a simplex, to a point $(x_0, g_1, x_1, \dots, g_n, x_n; 1, 0, \dots, 0) \sim (x_0, 1)$ sitting in the 0-skeleton $G_0 \times \Delta^0 = X$ of G. It is therefore enough to show that any two points $x, y \in X \subseteq |G|$ are connected by an interval.

Now by assumption we have a homomorphism $\gamma : I \to G$ with $\gamma(\bar{0}) = x$ and $\gamma(\bar{1}) = y$, which induces a path $|\gamma| : |I| \to |G|$. Precomposing with the $\phi : [0,1] \to |I|$ from Lemma 3.6, we obtain a path $\delta = |\gamma|\phi : [0,1] \to |G|$ with $\delta(0) = x$ and $\delta(1) = y$.

Conversely, assume that $|G|$ is arcwise connected; let $x, y \in |G|_0$ be two points, and $\gamma : [0,1] \to |G|$ a path with $\gamma(0) = x$ and $\gamma(1) = y$. Set $J_n = \gamma^{-1}(|G|_n)$ for all $n \in \mathbb{N}$. Since the J_n are closed in $[0,1]$ there is by compactness of $[0,1]$ a minimal $n \in \mathbb{N}$ with $J_n = [0,1]$. If $n > 1$, we may perturb γ on $J_n \setminus J_{n-1}$ to make it avoid $G_n \times \{(\frac{1}{n+1}, \dots, \frac{1}{n+1})\} \setminus |G|_{n-1}$, and then homotope it within Δ^n so that its image is entirely contained in $|G|_{n-1}$. By repeating this argument, we may assume that $\gamma[0,1]$ lies in $|G|_1$ and is transverse to $G_1 \times \{(\frac{1}{2}, \frac{1}{2})\} \setminus |G|_0$.

Set now $C = \gamma^{-1}(G_1 \times \{(\frac{1}{2}, \frac{1}{2})\}) \setminus J_0$. It is discrete in $[0,1] \setminus J_0$, and therefore countable. Let Y be the topological space obtained by cutting $[0,1]$ at every point in C; more precisely, $Y = [0,1] \times \{0\} \cup C \times \{1\}$ with the topology inherited from the lexicographical ordering. Let $\pi : Y \to [0,1]$ be projection on the first coordinate; then $\gamma\pi : Y \to |G|_1$ admits an obvious retraction to $\delta : Y \to |G|_0$. We let K be the equivalence groupoid Y/π, namely $K = \{((t,\varepsilon),(t,\varepsilon')) \mid (t,\varepsilon),(t,\varepsilon') \in Y\}$; then the map δ extends to a morphism of groupoids $K \to G$ such that $\delta(0,0) = x$ and $\delta(1,0) = y$. It therefore suffices to exhibit a morphism ι of groupoids from I to K with $\iota(\bar{0}) = (0,0)$ and $\iota(\bar{1}) = (1,0)$.

Choose a surjective monotone map from the Cantor set $\{0,1\}^{\mathbb{N}}$ to $[0,1]$ such that all points in C are hit by a point of the form $w0\bar{1}$ (and hence also of the form $w1\bar{0}$). This map lifts to a map $\{0,1\}^{\mathbb{N}} \to K$ and induces the desired ι. □

3.2. Topological quotients

Another way of associating a space to a groupoid (X, G) is to quotient X by the equivalence relation $x \sim y$ if there exists $g \in G$ with $s(g) = x$ and $t(g) = y$. The *topological quotient* of G is $X/{\sim}$, denoted $G_\top$. It is less well behaved than the geometric realization $|G|$, for example, because if $\phi : G \to H$ is a covering of orbispaces then $G_\top \to H_\top$ is not necessarily a covering. It will typically be a *branched covering*, with branching locus the set of points $x \in G_\top$ where $G_x \to H_{\phi(x)}$ is not a group isomorphism. However, we still have the equivalences

Proposition 3.8.

$$G \text{ connected} \quad \Leftrightarrow \quad |G| \text{ connected} \quad \Leftrightarrow \quad G_\top \text{ connected},$$

$$G \text{ compact} \quad \Leftrightarrow \quad |G|_1 \text{ compact} \quad \Leftrightarrow \quad |G|_k \text{ compact} \quad \forall k \in \mathbb{N} \quad \Rightarrow \quad G_\top \text{ compact}.$$

Note that $\pi_1(|G|)$ and $\pi_1(G_\top)$ are very different; we are more interested in the former, and denote $\pi_1(G) = \pi_1(|G|)$.

3.3. Morita equivalence

In category theory, two categories can be equivalent even if their objects cannot be put in bijection.[5] The corresponding notion for topological groupoids is usually called *Morita equivalence*. Unlike abstract categories, two Morita equivalent groupoids do not always admit continuous functors between them:

Definition 3.9. Let (G, X) and (G', X') be two topological groupoids. They are *Morita equivalent* if there exists a topological space P, equipped with two maps $s_P : P \to X$ and $t_P : P \to X'$, such that

$$\begin{tikzcd} G'' = G \sqcup P \sqcup P \sqcup G' \arrow[d, shift right, "t_G \sqcup t_P \sqcup s_P \sqcup t_{G'}"'] \arrow[d, shift left, "s_G \sqcup s_P \sqcup t_P \sqcup s_{G'}"] \\ X'' = X \sqcup X' \end{tikzcd}$$

can be endowed with the structure of a groupoid. We further assume that the bijective maps $P/(p \sim ph) \to X$ and $P/(p \sim gp) \to X'$ are homeomorphisms.

Assume furthermore that G and G' come equipped with self-coverings f, f'. Then (G, f) and (G', f') are *Morita equivalent* if (G'', X'') admits a self-covering f'' whose restriction to G and G' yields f and f', respectively.

For instance, if $X = X' = \{\cdot\}$, then G and G' are Morita equivalent precisely when they are isomorphic groups. As another example, the groupoid I defined in (1) is Morita equivalent to the groupoid $G' = X' = [0, 1]$, by taking $P = \{0, 1\}^{\mathbb{N}}$ with $s_P(w) = w$ and $t_P(w) = \sum w_i 2^{-i}$.

Lemma 3.10. *If G and G' are two Morita equivalent groupoids, then*

(1) *their topological quotients $G_\top$ and $(G')_\top$ are homeomorphic*;
(2) *there is a functorial bijection $\mathcal{J} : \{$covers of $G\} \leftrightarrow \{$covers of $G'\}$*;
(3) *if furthermore (G, f) and (G', f') are Morita equivalent, then $\mathcal{J}$ associates the cover corresponding to f^n with the cover corresponding to $(f')^n$ for all $n \in \mathbb{N}$.*

Proof. For the first point, it suffices to write

$$G_\top = X/\big(s(g) \sim t(g)\big) = \big(P/(p \sim ph)\big)/(p \sim gp),$$
$$(G')_\top = X'/\big(s(h) \sim t(h)\big) = \big(P/(p \sim gp)\big)/(p \sim ph).$$

For the second point, let (H, Y) be a cover of (G, X) with maps $\pi : H \to G$ and $\pi : Y \to X$. Define

[5] For instance, the trivial category with one object and one isomorphism is equivalent to the category with two objects a, b and four isomorphisms, one between any two objects.

$$
\begin{aligned}
&Y' = \bigl\{(y, p) \in Y \times P \mid s_P(p) = \pi(y)\bigr\}, \quad \forall h \in H\colon \bigl(t(h), p\bigr) \sim \bigl(s(h), \pi(h)p\bigr),\\
&H' = \bigl\{(h, p, q) \in H \times P^2 \mid s_P(p) = s(\pi h),\ s_P(q) = t(\pi h)\bigr\},\\
&\quad \forall k, \ell \in H\colon (h, p, q) \sim \bigl(kh\ell^{-1}, \pi(k)p, \pi(\ell)q\bigr),
\end{aligned}
$$

which is a groupoid, with source $s(h, p, q) = (s(h), p)$, target $t(h, p, q) = (t(h), q)$, inverse $(h, p, q)^{-1} = (h^{-1}, q, p)$ and multiplication $(h, p, q) \cdot (k, q, r) = (hk, p, r)$. This construction associates to (H, Y) a groupoid (H', Y'), with a cover $\pi' : (H', Y') \to (G', X')$ given by $\pi'(y, p) = t_P(p) \in X'$ and $\pi'(h, p, q) = p^{-1}\pi(h)q \in G'$.

We now show that the map $\pi' : Y' \to X'$ is a cover. Indeed, it is the coequalizer of the diagram

$$
\begin{tikzcd}
H \times_X P \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Y \times_X P \arrow[r] \arrow[d] & Y' \arrow[d] \\
G \times_X P \arrow[r, shift left] \arrow[r, shift right] & P \arrow[r, "t_P"] & X'
\end{tikzcd}
$$

and the first two vertical arrows are pulled back from $Y \to X$, which was assumed to be a cover.

To apply Lemma 3.4, we need to show that $Y' \to X'$ is d-to-1. We first show that the fiber $(\pi')^{-1}(x')$ is isomorphic to the fiber $\pi^{-1}(s_P(p))$ via $F : y \mapsto [(y, p)]$, for some $p \in P$ satisfying $t_P(p) = x'$. To see that F is onto, notice that any $(\bar{y}, \bar{p}) \in (\pi')^{-1}(x')$ is equivalent to $(s(h), p)$ for some h lifting $p\bar{p}^{-1}$ and having target y. Also, the relation $(t(h), p) \sim (s(h), \pi(h)p)$ is transitive, and the only way to have $(y, p) \sim (y', p)$ is to let $h = 1$, so F is injective. It follows that $Y' \to X'$ is d-to-1 and by Lemma 3.4 it is a cover.

We next show that this construction is a bijection. For this, apply it again to (H', Y'), yielding a groupoid (H'', Y''). This new groupoid is canonically isomorphic to (H, Y). We show this on the level of Y'', and skip the details for H''.

By definition,

$$
\begin{aligned}
&Y'' = \bigl\{(y, p, p') \in Y \times P^2 \mid s_P(p) = \pi(y),\ t_P(p) = t_P(p')\bigr\},\\
&\quad \forall h \in H,\ g \in G'\colon \bigl(t(h), p, p'\bigr) \sim \bigl(s(h), \pi(h)pg, p'g\bigr).
\end{aligned} \tag{3}
$$

We defined two maps $Y \to Y''$, $Y'' \to Y$ and show that they are inverses to each other. First, given $y \in Y$, there exists $p \in P$ with $s_P(p) = \pi(y)$, and we define $Y \to Y''$ by $y \mapsto (y, p, p)$. If we had made another choice $p' \in P$, we would have obtained (y, p', p') which is equivalent to (y, p, p) by taking $h = 1_y, g = p^{-1}p'$ in (3); so the map is well defined.

Given $(y, p, p') \in Y''$, there exists a unique $k \in H$ with $s(k) = y$ and $\pi(k) = p(p')^{-1}$, because π is a covering. We define $Y'' \to Y$ by $(y, p, p') \mapsto t(k)$. If in (3) we consider two equivalent elements $(t(h), p, p')$ and $(s(h), \pi(h)pg, p'g)$, and find $k \in H$ with $s(k) = t(h)$ and $\pi(k) = p(p')^{-1}$, then $s(hk) = s(h)$ and $\pi(hk) = \pi(h)pg(p'g)^{-1}$; then $t(hk) = t(k)$ so the map is well defined.

Next, $Y \to Y'' \to Y$ is the identity, because $y \mapsto (y, p, p) \mapsto y$ using $k = 1_y$ in the paragraph above.

Finally, $Y'' \to Y \to Y''$ is the identity: we have $(y, p, p') \mapsto t(k) \mapsto (t(k), p', p')$ where $s(k) = y$ and $\pi(k) = p(p')^{-1}$. We take $h = k$, $g = 1_{t(p')}$ to show that $(t(k), p', p')$ and (y, p, p') are equivalent.

We now check the last assertion of the lemma. We again check it only on objects; the proof is similar for morphisms. Let f'' be the covering map of P given by the Morita equivalence. The cover of (G, X) associated to f^n is again (G, X), which yields

$$Y' = \{(x, p) \in X \times P \mid s_P(p) = f^n(x)\}, \quad \forall g \in G\colon \bigl(t(g), p\bigr) \sim \bigl(s(g), f^n(g)p\bigr),$$

with covering map $Y' \to X'$ given by $(x, p) \mapsto t_P(p)$. We show that this covering is equivalent to $X' \to X'$, $x' \mapsto (f')^n x'$.

Given $x' \in X'$, there exists $q \in P$ with $t_P(q) = x'$. Define $X' \to Y'$ by $x' \mapsto (s_P(q), (f'')^n(q))$. Conversely, given $(x, p) \in Y'$, there exists a unique $q \in P$ with $(f'')^n(q) = p$ and $s_P(q) = x$, because f'' is a cover. Define $Y' \to X'$ by $(x, p) \mapsto t_P(q)$. These two maps are mutual inverses. □

4. Orbispaces from automata

Let Π be a finite invertible automaton. The *action space* of Π is the groupoid $O(\Pi)$ with objects $A^{\mathbb{N}}$, with $\mathbb{N} = \{1, 2, \ldots\}$, with morphisms

$$O(\Pi) = \{(\alpha, \phi) \in A^{\mathbb{N}} \times \langle\Pi\rangle^{\mathbb{N}} \mid \phi(n+1) = \tau\bigl(\alpha(n), \phi(n)\bigr) \text{ for all } n\},$$

with source map $s(\alpha, \phi) = \alpha$ and target map $t(\alpha, \phi)(n) = \sigma(\alpha(n), \phi(n))$.

The *limit space* of Π is the groupoid $L(\Pi)$ with objects $A^{-\mathbb{N}}$, morphisms

$$L(\Pi) = \{(\alpha, \phi) \in A^{-\mathbb{N}} \times \langle\Pi\rangle^{-\mathbb{N}} \mid \phi(n+1) = \tau\bigl(\alpha(n), \phi(n)\bigr) \text{ for all } n, \text{ and } \phi(-\mathbb{N}) \text{ is finite}\},$$

with source map $s(\alpha, \phi) = \alpha$ and target map $t(\alpha, \phi)(n) = \sigma(\alpha(n), \phi(n))$. Note that by our convention $0 \notin \mathbb{N}$. We extend ϕ at 0 by $\phi(0) = \tau(\alpha(-1), \phi(-1))$.

The *solenoid space* of Π is the groupoid $S(\Pi)$ with objects $A^{\mathbb{Z}}$, morphisms

$$S(\Pi) = \{(\alpha, \phi) \in A^{\mathbb{Z}} \times \langle\Pi\rangle^{\mathbb{Z}} \mid \phi(n+1) = \tau\bigl(\alpha(n), \phi(n)\bigr) \text{ for all } n, \text{ and } \phi(\mathbb{Z}) \text{ is finite}\},$$

with source map $s(\alpha, \phi) = \alpha$ and target map $t(\alpha, \phi)(n) = \sigma(\alpha(n), \phi(n))$.

Note that the finiteness condition "$\phi(\mathbb{N})$ is finite" is automatically satisfied in $O(\Pi)$.

The sets A and $\langle \Pi \rangle$ are given the discrete topology. For all of these groupoids, the object set is given the Tychonoff (product) topology, in which a basis for the topology is given by cylinders

$$\mathcal{O}_{n_1,a_1,\dots,n_k,a_k} = \left\{\alpha\colon (\mathbb{N} \text{ or } -\mathbb{N} \text{ or } \mathbb{Z}) \to A \mid \alpha(n_1) = a_1, \dots, \alpha(n_k) = a_k\right\}.$$

Similarly, the morphism set is given the restriction of the product topology.

The action space is the usual groupoid considered in association with a group action: there is a morphism from $\alpha \in A^{\mathbb{N}}$ to $g(\alpha)$ for all $g \in \langle \Pi \rangle$. The action space has few useful topological properties, because in most cases the orbits of $\langle \Pi \rangle$ are dense in $A^{\mathbb{N}}$.

The limit space is endowed with a unilateral shift $f : L(\Pi) \to L(\Pi)$, given by $f(\alpha, \phi)(n) = (\alpha(n-1), \phi(n-1))$. This map defines a $\#A$-to-1 self-covering map of $L(\Pi)$.

The solenoid is endowed with a bilateral shift $f : S(\Pi) \to S(\Pi)$, given by $f(\alpha, \phi)(n) = (\alpha(n-1), \phi(n-1))$. This map defines a homeomorphism of $S(\Pi)$.

Lemma 4.1. [12, Proposition 3.2.8] *If Π is nuclear with nucleus N, then the groupoids $L(\Pi)$ and $S(\Pi)$ have finite fibers of cardinality at most $\#N$.*

Therefore the topological dimension of $L(\Pi)_{\top}$ and of $S(\Pi)_{\top}$ is at most $\#N - 1$.

Lemma 4.2. *If Π is nuclear, then*

$$L(\Pi) = \left\{(\alpha, \phi) \in A^{-\mathbb{N}} \times Q^{-\mathbb{N}} \mid \phi(n+1) = \tau\big(\alpha(n), \phi(n)\big) \textit{ for all } n\right\},$$
$$S(\Pi) = \left\{(\alpha, \phi) \in A^{\mathbb{Z}} \times Q^{\mathbb{Z}} \mid \phi(n+1) = \tau\big(\alpha(n), \phi(n)\big) \textit{ for all } n\right\}.$$

In other words, the morphisms of $L(\Pi)$ are given by negative-infinite paths in $\mathfrak{G}(\Pi)$, and the morphisms of $S(\Pi)$ are given by bi-infinite paths in $\mathfrak{G}(\Pi)$. In the other direction, the morphisms of $O(\Pi)$ are generated by positive-infinite paths in $\mathfrak{G}(\Pi)$, since these correspond to generators of $\langle \Pi \rangle$.

Corollary 4.3. *If Π is recurrent and nuclear, then $L(\Pi)$ is connected, metrizable, with topological dimension at most $\#Q - 1$.*

Theorem 4.4. [12, Proposition 5.7.8]

$$S(\Pi) = \varprojlim\big(L(\Pi) \xleftarrow{T} L(\Pi) \xleftarrow{T} \cdots\big).$$

Consider the Schreier graphs $\mathfrak{G}_n$ of $\Gamma = \langle \Pi \rangle$ on A^n. There is a graph covering map $\mathfrak{G}_{n+1} \to \mathfrak{G}_n$, given by the map

$$a_1 \dots a_{n+1} \mapsto a_1 \dots a_n$$

on their vertices. There is also a map between the geometric realizations[6] of $\mathfrak{G}_{n+1}$ and $\mathfrak{G}_n$, given as follows: first, map the vertices by

$$a_1a_2\dots a_{n+1} \mapsto a_2\dots a_{n+1}.$$

Then, map the edge labeled $q : a_1\dots a_{n+1} \to b_1\dots b_{n+1}$ to the edge labeled $\tau(a_1, q)$: $a_2\dots a_{n+1} \to b_2\dots b_{n+1}$.

Assume now that Π is smooth. Then an *expansion rule* for Π is the following: first, for every $q \in Q$, elements $e_q \in A$ and $v_q \in Q$ with $\tau(e_q, v_q) = q$. Second, for every $a, a' \in A$ a word $w_{a,a'} \in Q^*$ such that $\sigma(a, w_{a,a'}) = a'$ and $\tau(a, w_{a,a'}) = \varepsilon^{|w_{a,a'}|}$, a power of the identity state.

We note that the existence of an expansion rule is equivalent to the smoothness of the automaton.

Proposition 4.5. *If Π is nuclear, then $L(\Pi)$, as a topological graph,[7] is the inverse limit of the geometric realizations of the Schreier graphs $\mathfrak{G}_m$ of $\langle\Pi\rangle$ on A^m.*

Proof. Clearly $A^{-\mathbb{N}} = \varprojlim A^m$, where the map $A^{m+1} \to A^m$ is given by deletion of the first letter. Note that the edges of $\mathfrak{G}_m$ are in one-to-one correspondence with the set

$$\bigl\{(\alpha, \phi) \in A^m \times Q^m \mid \phi(n+1) = \tau\bigl(\alpha(n), \phi(n)\bigr) \text{ for all } n \in \{1, \dots, m-1\}\bigr\} \tag{4}$$

where to any edge q of $\mathfrak{G}_m$ from $a_1\dots a_m$ to $b_1\dots b_m$ one puts in correspondence the groupoid element (α, ϕ) given by $\alpha(n) = a_n$ and $\phi(n) = \tau(a_1, \dots, a_{n-1}, q)$. It is then clear by Lemma 4.2 that the inverse limit of (4) is $L(\Pi)$. $\square$

Proposition 4.6. *If Π is nuclear and spherically transitive, then $L(\Pi)$ is connected. If furthermore Π is smooth, then $L(\Pi)$ is arcwise connected.*

Proof. To prove that $L(\Pi)$ is connected it is sufficient, thanks to Proposition 4.5, to show that the graphs $\mathfrak{G}_n$ are connected; but this is precisely the condition that Π is spherically transitive.

Assume now that Π is smooth, and let $(\{e_q\}, \{v_q\}, \{w_{a,a'}\})$ be an expansion rule for Π. By Lemma 3.7, it is sufficient to construct, for any points $x, y \in A^{-\mathbb{N}} \subset L(\Pi)$, a path $\gamma : [0,1] \to |L(\Pi)|$ from x to y. We will define partial maps $\gamma_n : [0,1] \dashrightarrow |L(\Pi)|$ converging to γ.

Each γ_n will be defined on a finite union of closed subintervals of $[0,1]$, in such a way that if $[a,b]$ and $[c,d]$ are two consecutive intervals, then $\gamma_n(b)$ and $\gamma_n(c)$ are in the 0-skeleton of $|L(\Pi)|$ and have identical last n symbols.

[6] Here we mean the usual geometric realization of a graph, not to be confused with the geometric realization $|G|$ of a groupoid G.
[7] I.e., as a topological space X with edges $E \rightrightarrows X$.

We start by γ_0 defined only at 0 and 1, with $\gamma_0(0) = x$ and $\gamma_0(1) = y$. Assume now that γ_{n-1} has been defined; γ_n coincides with γ_{n-1} on the domain of γ_{n-1}. Consider two consecutive intervals $[a, b]$ and $[c, d]$ on which γ_{n-1} is defined, and write $\gamma_{n-1}(b) = \dots b_{-n} b_{-n+1} \dots b_{-1}$ and $\gamma_n(c) = \dots c_{-n} b_{-n+1} \dots b_{-1}$. Set $u = w_{b_{-n}, c_{-n}}$ and $\ell = |u|$, and cut the interval $[b, c]$ into $2\ell + 1$ parts $E_0, F_1, E_1, \dots, F_\ell, E_\ell$. Define γ_n on F_i as the linear map from F_i onto the geometric realization of the edge (α, ϕ) defined by

$$\alpha(m) = \begin{cases} b_m & \text{if } m > -n, \\ \sigma(b_{-n}, u_1 \dots u_{i-1}) & \text{if } m = -n, \\ e_{\phi(m+1)} & \text{if } m < -n, \end{cases} \qquad \phi(m) = \begin{cases} \varepsilon & \text{if } m > -n, \\ u_i & \text{if } m = -n, \\ v_{\phi(m+1)} & \text{if } m < -n. \end{cases}$$

There is clearly a partially defined map $\gamma'(t) = \lim_{n\to\infty} \gamma_n(t)$. On the intervals at which it is not defined, it can be extended by a constant path; we let γ be this extension $[0, 1] \to |L(\Pi)|$.

Because of smoothness, for every interval $[b, c]$ between two intervals of definition of γ_n there is a corresponding word $a \in A^n$ for which the following happens: Given a point $x \in [b, c]$, either x is mapped to an element $\alpha \in |L(\Pi)|_0 = A^{-\mathbb{N}}$ of the 0-skeleton whose last n entries agree with a, or x is mapped to some edge $(\alpha, \phi) \in |L(\Pi)|_1$, where the last n entries of α agree with a and the last n entries of ϕ are all the identity state ε.

It remains to show that γ is continuous; intuitively, this happens because, as $n \to \infty$ and the length of the intervals F_i on which the γ_n are defined tends to 0, the images of the γ_n are paths in $|L(\Pi)|$ which become closer and closer to identity morphisms, and degenerate to a single point in $|L(\Pi)|$.

Let us be more precise. The map $\gamma : [0, 1] \to |L(\Pi)|_1$ is clearly continuous on all F_i's interiors, so there remains to consider two cases: first, continuity at a point $x \in [0, 1] \setminus \bigcup F_i$. Its image $\gamma(x)$ lies in $A^* = |L(\Pi)|_0$. Viewing $|L(\Pi)|_1$ as a quotient of $L(\Pi)_1 \times \Delta^1$ by the relation $\sim$ that collapses all sets $\{1_w\} \times \Delta^1$ to a point, a neighbourhood of $\gamma(x) = 1_{\gamma(x)} \times \Delta^1/\sim$ can be chosen of the form $V \times \Delta^1/\sim$, where V is a neighbourhood of $1_{\gamma(x)}$ in $L(\Pi)_1$. We may assume furthermore V to be the cylinder of all elements in $L(\Pi)_1 \subset A^{-\mathbb{N}} \times Q^{-\mathbb{N}}$ with prescribed last n terms: the set of (α, ϕ) in which the last n letters of α agree with w, and the last n states of ϕ are all ε. Then $\gamma^{-1}(V \times \Delta^1/\sim)$ contains the interval $(b, c) \ni x$, where b and c are chosen such that $[a, b]$ and $[c, d]$ are two consecutive intervals on which γ_n is defined and $b < x < c$. We have shown that the preimage of a neighbourhood of $\gamma(x)$ is a neighbourhood of x, so γ is continuous at x.

Next, let us consider both left and right continuity at $x \in \partial F_i$. On the side of F_i it is clear from the definition of the γ_n, and on the other side the above argument applies. □

Note that we have actually shown slightly more in this proof; namely, since all the morphisms (α, ϕ) considered satisfy $\phi_0 = \varepsilon$, we have shown, anticipating Definition 5.1:

Proposition 4.7. *If Π is nuclear and smooth*[8] *then the standard tile $\mathcal{T}(\Pi)$ is arcwise connected.*

[8] Recall that smoothness implies spherical transitivity.

5. Tilings from automata

This section extends on Section 4, where a limit orbispace $L(\Pi)$ and a solenoid $S(\Pi)$ were constructed from an automaton Π. First, we define a few more geometric objects associated with an automaton Π.

Definition 5.1. The *standard tile* $\mathcal{T}(\Pi)$ is the groupoid with same objects $A^{-\mathbb{N}}$ as the limit space, but whose morphisms are inverse sequences starting at the trivial state*:*

$$\mathcal{T}(\Pi) = \{(\alpha, \phi) \in L(\Pi) \mid \phi(0) = \varepsilon\}.$$

The group Γ embeds in the group of homeomorphisms of $A^{\mathbb{N}}$. The *germ space* $\mathcal{G}(\Pi)$ has as objects $A^{\mathbb{N}}$, and has as set of morphisms from w to w' all germs of homeomorphisms mapping w to w' and coming from Γ. It is a quotient of the action groupoid: for instance, in the action groupoid, the automorphisms at w is the isotropy subgroup of w, while in the germ groupoid it is the quotient of the isotropy of w by the *stable isotropy* of w, i.e., those $g \in \Gamma$ that fix an open neighbourhood of w.

The geometric notions corresponding to the tiles are slightly more tricky to set up: the translates of the standard tile almost cover the solenoid, but for the discarded edges. We want to add "half of each edge" to the standard tile to obtain a covering of the solenoid with empty-interior intersections.

Call an edge (α, ϕ) of $L(\Pi)$ *critical* if $\phi_0 \neq \varepsilon$, and define the critical locus $C \subset |L(\Pi)|$ as follows. First let $C_1 \subset |L(\Pi)|_1$ be the set of middle points of critical edges. A point $x = (g, t) \in |L(\Pi)|_n$ with $g \in L(\Pi)_n$ and $t \in \Delta^n$ is in C if (one of) the closest point(s) to t in the 1-skeleton of Δ^n, call it t', satisfies $(g, t') \in C_1$. The critical locus is a hypersurface in $|L(\Pi)|$ that intersects all critical edges in their middle points.

Definition 5.2. The *geometric standard tile* $\bar{\mathcal{T}}$ is obtained from $|L(\Pi)|$ by cutting it along the critical locus C. We therefore have a surjective map $\bar{\mathcal{T}} \to |L(\Pi)|$ that is a homeomorphism away from C and that is generically 2-to-1 on C.

Lemma 5.3. *The natural inclusion $|\mathcal{T}(\Pi)| \subset \bar{\mathcal{T}}$ is a deformation retract.*

Proof. Consider a simplex $K = (x_0, g_1, \dots, x_n) \times \Delta^n \subset |L(\Pi)|$. We will define a retraction on each connected component K' of the preimage of K in $\bar{\mathcal{T}}$. We view K' as a subset of K. Let A and B be the set of vertices of K that belong, respectively do not belong, to K'. We define a retraction on K' as follows:

$$\rho_s\big((x_0, g_1, \dots, x_n), (t_0, \dots, t_n)\big) = \big((x_0, g_1, \dots, x_n), \big(u_0(s), \dots, u_n(s)\big)\big),$$

where

$$u_i(s) = \begin{cases} \big(1 + s\frac{\sum_{j\in B} t_j}{\sum_{j\in A} t_j}\big)t_i & \text{if the } i\text{th vertex of } K \text{ belongs to } A, \\ (1-s)t_i & \text{if the } i\text{th vertex of } K \text{ belongs to } B. \end{cases}$$

This retraction is compatible with the face maps and degeneracy maps.

We check that ρ_s is the identity on simplices in $|\mathcal{T}(\Pi)|$, because for these simplices we have $B = \emptyset$. $\square$

Within the limit space, we see a collection of copies of the standard tile, as follows: for $w \in A^*$, let $\mathcal{T}_w$ be the full subgroupoid on the objects $A^{-\mathbb{N}}w$. The same construction can be performed for the geometric standard tile: given $w \in |L|_0$, set

$$\bar{w} = \{ y \in \mathrm{Star}(w) \mid d(y, w) \leqslant d(z, w) \ \forall z \in \mathrm{Star}(w) \cap |L|_0 \}. \tag{5}$$

For any $T \subset |L|_0$, set $\bar{T} = \bigcup_{w \in T} \bar{w}$. Then for $w \in A^*$ we define the geometric tile $\overline{\mathcal{T}_w} \subset |L(\Pi)|$ as $\overline{A^{-\mathbb{N}}w}$.

Proposition 5.4. *Assume Π is nuclear; then for any $n \in \mathbb{N}$, the tiles $\{\overline{\mathcal{T}_w} \mid w \in A^n\}$ cover $|L(\Pi)|$, and two tiles $\overline{\mathcal{T}_w}, \overline{\mathcal{T}_{w'}}$ overlap if and only if w, w' are connected in the Schreier graph of $\langle \Pi \rangle$ on A^n.*

Furthermore each tile is the closure of its interior, and these tiles overlap with empty-interior intersection.

The tiling $\{\overline{\mathcal{T}_w} \mid w \in A^{n+1}\}$ is a refinement of the tiling $\{\overline{\mathcal{T}_w} \mid w \in A^n\}$, with the tile $\overline{\mathcal{T}_{xw}}$ being contained in the tile $\overline{\mathcal{T}_w}$.

Proof. If two tiles $\overline{\mathcal{T}_w}, \overline{\mathcal{T}_{w'}}$ overlap, then there is a morphism (α, ϕ) of $L(\Pi)$ in the nth preimage of the critical locus, i.e., satisfying $\phi(-n) \neq \varepsilon$. The Schreier graph contains an edge, labeled $\phi(-n)$, from w to w'.

Conversely, let $g : w \to w'$ be such an edge. By Lemma 2.6, there exists $(\alpha, \phi) \in L(\Pi)$ with $\phi(-n) = g$ and $\alpha([-n, -1]) = w$. The tiles $\overline{\mathcal{T}_w}$ and $\overline{\mathcal{T}_{w'}}$ overlap on the middle of that edge.

The second statement is obvious, because the interior of a tile $\overline{\mathcal{T}_w}$ is obtained by replacing $\leqslant$ by $<$ in the definition of $\bar{w}$. $\square$

Note that the tilings of the topological space $L(\Pi)_\top$ is much more complicated [2]. The automaton Π satisfies the *open set condition* if for every $q \in Q$ there exists $a \in A^n$ with $\tau(a, q) = \varepsilon$. This is equivalent to asking that every $g \in \langle \Pi \rangle$ have a trivial state.

Proposition 5.5. [12, Corollary 3.3.7] *If Π satisfies the open set condition, then every tile in $L(\Pi)_\top$ is the closure of its interior, and for every $n \geqslant 0$ the tiles in the nth subdivision of $L(\Pi)_\top$ have disjoint interior.*

On the other hand, if Π does not have the open set condition, then for every n large enough one can find a tile in the nth subdivision of $L(\Pi)_\top$ which is covered by the other tiles in the same subdivision.

Next, consider the solenoid $S(\Pi)$. This is naturally a foliated space: for $\mathcal{O}$ a Γ-orbit on $A^{\mathbb{N}}$, let $F_{\mathcal{O}}$ denote the full subgroupoid on the objects $A^{-\mathbb{N}}\mathcal{O}$. Note that there are no morphisms in $S(\Pi)$ between distinct leaves.

We endow the leaf $F_{\mathcal{O}}$ with the "left-Tychonoff, right-discrete" topology generated by open sets $\mathcal{O}_{n,w} = \{\alpha : \mathbb{Z} \to A \mid \alpha(n+i) = w_i \ \forall i \in \mathbb{N}\}$ for all $n \in \mathbb{Z}$ and $w \in A^{\mathbb{N}}$.

Lemma 5.6. *Assume Π is recurrent and spherically transitive. Then the shift map $f : S(\Pi) \to S(\Pi)$ preserves the foliation of $S(\Pi)$.*

Proof. It suffices to show that the partition of $A^{\mathbb{Z}}$ in $A^{-\mathbb{N}}\mathcal{O}$ is shift-invariant. For that purpose, we show that $\mathcal{O}$ is a Γ-orbit if and only if $A\mathcal{O}$ is a Γ-orbit.

Pick $g \in \Gamma$, $a \in A$ and $v \in \mathcal{O}$. Then $g \cdot av = \sigma(a,g)\tau(a,g) \cdot v \in A\mathcal{O}$. Conversely, pick $a, b \in A$ and $v, w \in \mathcal{O}$. Then there exists $g \in \Gamma$ with $g \cdot a = b$, whence $g \cdot av = bv'$ with $v' \in \mathcal{O}$, and $g' \in \Gamma$ with $g' \cdot v' = w$. Since Γ is recurrent, there exists $h \in \Gamma_b$ with $\tau(b,h) = g'$; therefore $hg \cdot av = bw$. □

The leaves of the solenoid are again tiled spaces: for $w \in \mathcal{O} \subset A^{\mathbb{N}}$, let $\mathcal{T}_w$ be the full subgroupoid on the objects $A^{-\mathbb{N}}w$. Define the geometric tiles $\overline{\mathcal{T}_w}$ as in (5).

Theorem 5.7. *Assume Π is nuclear, and let $\mathcal{O}$ be a Γ-orbit in $A^{\mathbb{N}}$. Then $|F_{\mathcal{O}}| \subset |S(\Pi)|$ is tiled by $\{\overline{\mathcal{T}_w} \mid w \in \mathcal{O}\}$.*

For any $w \in A^{\mathbb{N}}$, the adjacency graph of tiles on $|F_{\Gamma w}|$ is the Schreier graph of $\langle \Pi \rangle / \mathsf{Stab}(w)$.

Furthermore each tile is the closure of its interior, and these tiles overlap with empty-interior intersection.

The shift map $f : S(\Pi) \to S(\Pi)$ sends each leaf $F_{\mathcal{O}}$ to another one carrying a refinement of the tiling of $F_{\mathcal{O}}$. The tile $\overline{\mathcal{T}_{xw}} \subset F_{\mathcal{O}}$ is mapped into the tile $\overline{\mathcal{T}_w}$.

Proof. Analogous to that of Proposition 5.4. □

Every leaf $F_{\mathcal{O}}$ naturally covers $L(\Pi)$. The covering map $F_{\mathcal{O}} \to L(\Pi)$ is given by the restriction of the natural projection map $S(\Pi) \to L(\Pi)$ given by keeping only the negative part of objects and morphisms. This covering map folds every tile $\overline{\mathcal{T}_w}$ of $|F_{\mathcal{O}}|$ onto $|L(\Pi)|$.

We note that $S(\Pi) \to L(\Pi)$ has the structure of a foliated bundle $S(\Pi) = \tilde{L} \times_{\Gamma} A^{\mathbb{N}}$ in the sense of [11]. To see this, define the groupoid $\tilde{L}$ as follows: its objects are $A^{-\mathbb{N}} \times \Gamma$. Its set of morphisms is the set of $(\alpha, \phi, g) \in A^{-\mathbb{N}} \times \Gamma^{-\mathbb{N}} \times \Gamma$ such that $(\alpha, \phi) \in L(\Pi)$ and $\phi_0 = g$. There is a natural action of Γ on $A^{\mathbb{N}}$, and Γ acts on $\tilde{L}$ by acting on the second coordinate of $A^{-\mathbb{N}} \times \Gamma$.

In that context, we recall the notion of holonomy groupoid [11, p. 58], restricted to a transversal. Given a foliated space S over L, fix a point $w \in L$, and let F be its fiber. The *holonomy groupoid* has objects F, and has a morphism from $f \in F$ to $f' \in F$ for every path α in S from f to f'. Two paths α, α' are identified if they have the same holonomy, i.e., if both start at f, end at f', and $\alpha(\alpha')^{-1}$ induces by parallel transport the identity of F in the neighbourhood of f.

Proposition 5.8. *The holonomy groupoid of the foliation, restricted to a transversal, is the groupoid of germs $\mathcal{G}(\Pi)$ of Π.*

Proof. Fix a point $w \in L(\Pi)$; then its fiber in $S(\Pi)$ is $wA^{\mathbb{N}}$. The parallel transport along α in a leaf of S corresponds to the action of Γ on $A^{\mathbb{N}} \cong wA^{\mathbb{N}}$. Two paths $\alpha, \alpha' : wv \to wv'$ are equivalent if and only if $\alpha(\alpha')^{-1}$ is in the stable homotopy group of v. □

6. Automata from orbispaces

Let L be a connected orbispace with a d-to-1 covering map $f : L \to L$.

First, we may obtain a profinite group $\bar{\Gamma}$ as follows: pick a base point $* \in L$. For any $n \in \mathbb{N}$, set $A_n = f^{-n}(*)$. Consider the space $L_n = \{(g_0 \in L, g_1 \in f^{-1}(g_0), \dots, g_n \in f^{-1}(g_{n-1}))\}$. Then $L \cong L_n$, by projection on the last coordinate, and we may also view L_n as a degree-d^n covering space over L, with $f^n : L_n \to L$ realized as projection on the first coordinate.

Also, f^n can be viewed as a bundle with structure group the isometry group of an n-level $\#A$-regular rooted tree. Consider the associated principal bundle

$$Y_n = \left\{ \left(g \in L,\ p : \bigcup_{i=0}^{n} f^{-i}(g) \to \bigcup_{i=0}^{n} A_i \text{ a rooted tree isometry} \right) \right\},$$

with covering map $\tilde{f}_n$ the projection on the first coordinate; let Z_n be the connected component of $(*, 1, \dots, 1)$ in Y_n. Then Z_n is a Galois covering, with Galois group Γ_n. Therefore $\tilde{f} : Y_{n+1} \to Y_n$ given by $(g, p) \mapsto (g, p|_{\bigcup_{i=0}^{n} f^{-i}(g)})$ induces a projective system $\tilde{f} : \Gamma_{n+1} \to \Gamma_n$. Set $\bar{\Gamma}(L) = \varprojlim \Gamma_n$.

Theorem 6.1. *Let Π be nuclear and spherically transitive. Then $\bar{\Gamma}(L(\Pi))$ is the closure of $\langle \Pi \rangle$ in $\mathsf{Aut}(A^*)$.*

Proof. First, we remember by Lemma 2.10 that the closure of $\langle \Pi \rangle$ in $\mathsf{Aut}(A^*)$ is the inverse limit of its finite quotients acting on A^n. This finite quotient is nothing but the permutation group Γ_n in its action on A_n. □

We now seek a discrete group associated with L, and assume the geometric realization $|L|$ of L is arcwise connected. By Proposition 3.5, the map f induces a d-to-1 covering map $|f| : |L| \to |L|$.

We place ourselves in the following situation, that of an arcwise connected space X endowed with a d-to-1 covering map $f : X \to X$. Some of the interesting examples however, come from a slightly more general situation, where f needs only be a branched covering. This reduces to the previous situation by removing from X the branching locus, as well as all its iterated direct and inverse images.

We actually do not need to remove the inverse images of the branching locus; if we remove the forward images, we are led to consider a space X with a map f defined on a dense subset of X, and satisfying the unicity of path lifting property.

Here is the procedure for constructing an automaton $\Pi(X)$ out of our data [12, Proposition 5.2.2]. Pick a base point $x \in X$ and choose for all $y \in f^{-1}(x)$ a path ℓ_y from x to y.

Let $K \lhd \pi_1(X, x)$ be the subgroup consisting of all paths that induce the identity permutation on $f^{-n}(x)$, for all $n \in \mathbb{N}$. Our automaton is given as follows: its set of states Q is a subset of $\pi_1(X, x)$ that generates $\pi_1(X, x)/K$, and satisfies the condition (6) below. Its alphabet is $A = f^{-1}(x)$. Given $a \in A$ and $q \in Q$, consider the preimage γ of the path q with $\gamma(0) = a$. The output function of $\Pi(X)$ is $\sigma(a, q) = \gamma(1)$, and its transition function maps (a, q) to a path $\tau(a, q) \in Q$ congruent mod K to $\ell_a \gamma \ell_{\gamma(1)}^{-1}$. We therefore require for all $a \in A$ and $q \in Q$:

$$\ell_a \gamma \ell_{\gamma(1)}^{-1} K \cap Q \neq \emptyset. \tag{6}$$

Note that we may on one hand take $Q = \pi_1(X, x)$; however we wish usually the set of states to be finite, in which case (6) expresses a nontrivial restriction on Q.

Definition 6.2. [12, Chapter 5] We call $\Pi(X)$ the automaton constructed above, and, for L a groupoid, we set $\Pi(L) = \Pi(|L|)$. The automaton $\Pi(X)$ constructed depends of course on the choices made but we will show that the associated group $\langle \Pi(X) \rangle$ does not.

The group $\langle \Pi(X) \rangle$ is called the *iterated monodromy group* of f, written $\mathsf{IMG}(f)$.

The following proposition shows that the automaton associated with a *topologically expanding* map is contracting. A metric version was already proven in [12, Theorem 5.5.3].

Definition 6.3. Let X be a topological space and $f : X \to X$ a continuous map. It is *topologically expanding* if there exists an open subset $\mathcal{U} \subset X \times X$ such that $\bigcap_{n \geqslant 0} f^{-n}(\mathcal{U}) = \Delta$, where $\Delta = \{(x, x) \mid x \in X\}$ is the diagonal and f extends naturally to a function $X \times X \to X \times X$.

If $Y \subset X \times X$ contains the diagonal, let us write Y_0 for the connected component of Y containing the diagonal. We then say f is *smoothly topologically expanding* if there exists a $\mathcal{U} \subset X \times X$ as above, satisfying furthermore $f^{-1}(\mathcal{U})_0 \subset \mathcal{U}$ where $f^{-1}(\mathcal{U})_0$ denotes the connected component of $f^{-1}(\mathcal{U})$ containing Δ.

Proposition 6.4. *If X is compact, locally arcwise connected, and f is a smoothly topologically expanding degree-d cover, then the automaton $\Pi(X)$ is contracting.*

Proof. Set $A = \{1, \dots, d\}$. Let T be the standard d-regular rooted tree: it is the simplicial realization of the graph with vertices A^* and an edge between w and wa for all $w \in A^*$, $a \in A$. The standard compactification of T is $\bar{T} = T \cup \partial T$, with $\partial T = A^{\mathbb{N}}$.

We assume the automaton $\Pi(X)$ has been constructed, with choices of basepoint $*$ and connecting paths $\{\ell_a\}$. The paths ℓ_a extend to paths ℓ_w starting at $*$, defined as follows. For $w \in A^*, a \in A$, set $\ell_{wa} = \ell_w \tilde{\ell}_{w,a}$, where $\tilde{\ell}_{w,a}$ is the $|w|$th iterated f-preimage of ℓ_a starting at $\ell_w(1)$. This defines an embedding $T \to X$, mapping the vertex $w \in T$ to the extremity of ℓ_w. Furthermore, for each $w \in A^{\mathbb{N}}$ we have a path $\ell_w : \mathbb{R}_{\geqslant 0} \to X$ starting at $*$.

We show that these paths are actually finite, i.e., the map $T \to X$ extends to $\overline{T}$. For this purpose, let $Y \subset X \times \partial T$ be the space of accumulation points of rays ℓ_w:

$$Y = \bigl\{(x, w) \in X \times \partial T \mid x = \lim \ell_w(t_i) \text{ for a sequence } t_i \to \infty\bigr\}.$$

Then Y fibers over ∂T, with connected fibres since the ℓ_w are connected and X is compact. We set

$$Z = (Y \times_{\partial T} Y) \cap (\mathcal{U} \times \partial T).$$

By assumption, Z is stable under $f \times f \times$ 'shift,' so since f is expanding $Z \subset \Delta \times \partial T$. It follows that the fibres of $Y \to \partial T$ are discrete, so $Y \cong \partial T$. We have shown that each ray ℓ_w has a unique accumulation point $\lim(\ell_w)$. This proves that the map $T \to X$ extends to $\bar{T}$.

The relation $(x, y) \in \mathcal{U}$ means that "x and y are close." We need stronger notions of closeness, for which we introduce the notation $(x, y) \in \frac{1}{k}\mathcal{U}$, for $k \in \mathbb{N}$. By definition, if $\mathcal{V}$ is an open neighbourhood of Δ in $X \times X$, then $\frac{1}{k}\mathcal{V}$ denotes a choice of an open subset of $X \times X$ containing the diagonal and satisfying the condition:

$$\text{if } (x_i, x_{i+1}) \in \frac{1}{k}\mathcal{U} \text{ whenever } 0 \leqslant i < k, \quad \text{then } (x_0, x_k) \in \mathcal{V}.$$

Let us show that such sets always exist. For this, fix $\mathcal{V}$ and k, and let (Y_n) be a decreasing sequence (or net) of compact neighbourhoods of the diagonal Δ satisfying $\bigcap Y_n = \Delta$. Then $p_{12}^{-1}(Y_n) \cap \cdots \cap p_{k,k+1}^{-1}(Y_n) \searrow \Delta^{(k+1)}$, where $\Delta^{(k+1)}$ is the diagonal in X^{k+1}, and p_{ij} are the projections on two coordinates. Since images commute with decreasing intersections of compact sets, we get $Z_n = p_{1,k+1}(p_{12}^{-1}(Y_n) \cap \cdots \cap p_{k,k+1}^{-1}(Y_n)) \searrow \Delta$. For n big enough, we will then have $Z_n \subset \mathcal{V}$, and we set $\frac{1}{k}\mathcal{V} = (Y_n)^\circ$ for such a choice of n.

If $\gamma : [0, L] \to X$ is a path and $\mathcal{V}$ is an open neighbourhood of Δ, we will write $\gamma \Subset \mathcal{V}$ to mean $(\gamma(t), \gamma(t')) \in \mathcal{V}$ for all $t, t' \in [0, L]$; more generally, if $Y \subset X$, we write $Y \Subset \mathcal{V}$ to mean $(y, y') \in \mathcal{V}$ for all $y, y' \in Y$.

If $w \in T$, the *cone* $C(w)$ of w is the image in X of the subset spanned by $wA^* \cup wA^{\mathbb{N}}$ in $\bar{T}$.

Set $\mathcal{V} = \frac{1}{3}\frac{1}{2}\mathcal{U}$. We may choose $R \in \mathbb{N}$ such that $C(w) \Subset \mathcal{V}$ for all $w \in A^R$. To construct such an R, consider the function $h : \partial T \to \mathbb{N}$,

$$h(w) \triangleq \min\bigl\{M \in \mathbb{N} \mid C(w_1 \dots w_M) \Subset \mathcal{V}\bigr\}.$$

This function is continuous on a compact, so is bounded. Set $R = \max h(\partial T)$.

Define also $N, N', K' \subset \pi_1(X, *)$ by

$$\begin{aligned}
N &= \bigl\{\ell_v \rho \ell_w^{-1} \mid |v|, |w| \geqslant R \text{ and } \rho \Subset \mathcal{V}\bigr\}, \\
N' &= \left\{\ell_v \rho \ell_w^{-1} \mid |v| = |w| = R \text{ and } \rho \Subset \frac{1}{2}\mathcal{U}\right\}, \\
K' &= \bigl\{\ell_w \rho \ell_w^{-1} \mid \rho \Subset \mathcal{U}\bigr\}.
\end{aligned}$$

First, we claim that the elements of K' act trivially on the tree of preimages of $*$, and therefore are trivial in $\langle \Pi(X) \rangle$. Take $\gamma \in K'$. Then γ is a "balloon" $\rho \Subset \mathcal{U}$, attached by a "string" ℓ_w to $*$. Since $f^{-1}(\mathcal{U})_0 \subset \mathcal{U}$, an f-preimage of ρ is again of the form "balloon in

$\mathcal{U}$ attached by a string." They are all loops, so γ acts trivially by monodromy on $f^{-1}(*)$. Iterating this procedure, we see that γ acts trivially on $f^{-n}(*)$ for all $n \in \mathbb{N}$.

We next claim that $N \subset N'$. Take $\gamma \in N$, $\gamma = \ell_v \rho \ell_w^{-1}$. Set $v' = v_1 \dots v_R$ and $w' = w_1 \dots w_R$. Let $\ell_{v',v}$ be the image in X of the path from v' to v in T, and similarly for $\ell_{w',w}$. Then $\ell_{v',v}, \rho, \ell_{w',w}^{-1} \Subset \mathcal{V}$, so $\rho' \triangleq \ell_{v',v} \rho \ell_{w',w}^{-1} \Subset \frac{1}{2}\mathcal{U}$, and we can write $\gamma = \ell_{v'} \rho' \ell_{w'}^{-1} \in N'$.

We claim that N' is finite modulo K'. Indeed take two elements $\gamma, \gamma' \in N'$ with the same v, w: $\gamma = \ell_v \rho \ell_w^{-1}$, $\gamma' = \ell_v \rho' \ell_w^{-1}$. Their quotient is a balloon $\ell_v \rho (\rho')^{-1} \ell_v^{-1}$. It is in K' since $\rho(\rho')^{-1} \Subset \mathcal{U}$. We are left with finitely many choices for $v, w \in A^R$.

Finally, we claim that for any $\gamma \in \pi_1(X, *)$ there exists $n \geqslant R$ such that all f^{-n}-preimages of γ are contained in $\mathcal{V}$. Indeed partition γ in small segments $\gamma_1, \dots, \gamma_k$ such that $\gamma_i \Subset \mathcal{U}$ for all $i \in \{1, \dots, k\}$. Since $f^{-n}(\mathcal{U})$ converges to Δ, there exists $n \in \mathbb{N}$ such that $f^{-n}(\mathcal{U}) \Subset \frac{1}{k}\mathcal{V}$. The preimages of γ_i are in $\frac{1}{k}\mathcal{V}$, so all preimages of γ belong to $\mathcal{V}$, and therefore define an element of N.

We have shown that N contains the nucleus and that its image in $\langle \Pi(X) \rangle$ is finite. □

Note that this proposition applies in particular if $f \in \mathbb{C}(z)$ and the postcritical orbit of f does not intersect the Julia set of f; one then takes for X the Julia set of f [5].

Let A, A' be two sets of same cardinality. In the following definition, we relax the definition of automata in allowing the input and output alphabets to be respectively A and A'. More precisely, the transition function remains a function $\tau : A \times Q \to Q$, but the output function becomes a function $\sigma : A \times Q \to A'$. Such an automaton, with an initial state $q_0 \in Q$, defines a tree homomorphism $A^* \to (A')^*$, which is an isomorphism precisely when $\sigma(-, q)$ is a bijection for all $q \in Q$. The *states*[9] of $\phi : A^* \to (A')^*$ are the maps $\phi' : A^* \to (A')^*$ given by $\phi(vw) = v'\phi'(w)$ for some $v \in A^*$ and any $w \in A^*$. The set of states of the map $q_0 : A^* \to (A')^*$ is a subset of Q.

Definition 6.5. Two automata Π and Π' on alphabets A and A' are *equivalent* if $\#A = \#A'$, and there exists a tree isomorphism $\phi : A^* \to (A')^*$ given by a finite automaton in the sense described above, such that the states of ϕ are all of the form ϕg for some $g \in \langle \Pi \rangle$, and

$$\phi \circ \langle \Pi \rangle \circ \phi^{-1} = \langle \Pi' \rangle.$$

Proposition 6.6. *Let (X, f) be a space with a covering and let Π, Π' be two automata constructed as above from possibly different data x, ℓ_y, Q and x', ℓ'_y, Q'. Assume Π is contracting. Then Π and Π' are equivalent.*

Proof. Since equivalence of automata is transitive, we may assume $Q \subset \pi_1(X, x)$ and $Q' = \pi_1(X, x')$. Choose a path s from x to x' and use it to identify $\pi_1(X, x)$ with $\pi_1(X, x')$. We get an injection

$$s_Q : \langle Q \rangle \subset \pi_1(X, x) \xrightarrow{\sim} \pi_1(X, x') = \langle Q' \rangle.$$

[9] By abuse of notation, we refer to the states of ϕ to mean the states of an automaton defining the map ϕ.

The path s also induces a bijection between the preimages $A = f^{-1}(x)$ and $A' = f^{-1}(x')$, and similarly between higher-order preimages of x and x', by lifting appropriately s through powers of f. The paths ℓ_y can be used to identify the tree of preimages of x with A^* and similarly to identify the preimages of x' with $(A')^*$. Composing these three tree isomorphisms gives an isomorphism $s_A : A^* \to (A')^*$.

For all $a \in A$, let s_a be the f-preimage of s starting at a, and call its other extremity $a' \in A'$. Define $g_a \in \pi_1(X, x)$ by

$$g_a = \ell_a s_a \ell_{a'}^{-1} s^{-1}.$$

Define recursively $\phi : A^* \to (A')^*$ by $\phi(aw) = a'\phi(g_a w)$. Then $\phi = s_A$, and since Π is contracting, ϕ is automatic by [12, Corollary 2.11.7]. Furthermore, the states of ϕ are clearly of the form ϕg for some $g \in \langle \Pi \rangle$; for instance, on the first level, they are precisely the $\{\phi g_a\}$.

The best way to understand the definition of ϕ is as follows: forget for an instant that s is fixed, and denote the resulting ϕ by ϕ_s. Then

$$\phi_s \circ g_a = \phi_{g_a s} = \phi_{\ell_a s_a \ell_{a'}^{-1}},$$

in accordance the transition function defined above (6).

Clearly the actions of $\langle Q \rangle$ and $\langle Q' \rangle$ on A^* and $(A')^*$ are intertwined by s_Q and s_A. We are left to show that the induced map $s_\Pi : \langle \Pi \rangle \to \langle \Pi' \rangle$ is an isomorphism, where $\langle \Pi \rangle$ and $\langle \Pi' \rangle$ are the images of $\langle Q \rangle$ and $\langle Q' \rangle$ in $\mathsf{Aut}(A^*)$ and $\mathsf{Aut}((A')^*)$, respectively. The injectivity of s_Π follows from that of s_Q. To show the surjectivity of s_Π, let $g' \in \mathsf{Aut}((A')^*)$ be in $\langle \Pi' \rangle$, let $\gamma' \in \langle Q' \rangle = \pi_1(X, x')$ be a path representing it, and set $\gamma = s\gamma' s^{-1} \in \pi_1(X, x)$. We need to find a word in Q that acts on A^* the same way that γ does; such a word exists because Q was assumed to generate the image of $\pi_1(X, x)$ in $\mathsf{Aut}(A^*)$. □

Consider a manifold X with a branched covering map f, and let P be the smallest closed subset of X containing the critical values of f and $f(P)$. If P does not disconnect X we may consider the space $X' = X \setminus P$, with a partially defined covering map $\tilde{f} : X' \dashrightarrow X'$; we still have the unique lifting property for loops via $\tilde{f}$, so the construction of $\mathsf{IMG}(f)$ is unaffected by the fact that $\tilde{f}$ is not defined everywhere.

We may also consider the space $X'' = X \setminus \overline{\bigcup_{m,n \in \mathbb{N}} f^{-m} f^n(\text{critical points})}$, now with an everywhere defined covering map. An important source of examples is given by X the Riemann sphere, and f a rational map—see for instance [2].

Theorem 6.7. *If Π is nuclear and smooth, then Π is equivalent to $\Pi(L(\Pi))$; more precisely, the data x, ℓ_y, Q may be chosen in Definition* 6.2 *so that $\Pi = \Pi(L(\Pi))$.*

Recall the notion of critical edge from Definition 5.1.

Lemma 6.8. *Pick $x \in |L(\Pi)|_0$ and $\gamma \in \pi_1(|L(\Pi)|_1, x)$. Assume that γ is transverse to the midpoints of the critical edges. Let $u \in Q^*$ be the sequence of last states ϕ_0 of the critical edges (α, ϕ) crossed by γ. Then the action of γ on the tree of preimages of x only depends on u.*

Proof. Let $n > 0$ be an integer. We want to show that the action of γ on $f^{-n}(x)$ only depends on u. Let $y \in f^{-n}(x)$ be a preimage and $\tilde{\gamma}$ be the unique lift of γ starting at y. As before, $|L(\Pi)|$ has a tiling by tiles $\overline{\mathcal{T}_w}$, for all $w \in A^n$. The points where $\tilde{\gamma}$ crosses from a tile to another are exactly the preimages of the points where γ crosses a critical edge. So if $t \in [0,1]$ is one of these points and $\tilde{\gamma}(t-\epsilon) \in \overline{\mathcal{T}_w}$ for small $\epsilon > 0$, then $\tilde{\gamma}(t+\epsilon) \in \overline{\mathcal{T}_{\sigma(w,q)}}$, where $q = \phi_0$ is the last state of the critical edge (α, ϕ) on which $\gamma(t)$ lies. By induction, we get $\tilde{\gamma}(1) \in \overline{\mathcal{T}_{\sigma(w,u)}}$ where $w \in A^n$ is such that $y = \tilde{\gamma}(0)$ belongs to $\overline{\mathcal{T}_w}$. This shows that $\tilde{\gamma}(1)$ is the unique preimage of x lying in $\overline{\mathcal{T}_{\sigma(w,u)}}$ and is therefore determined by u and w only. □

Proof of Theorem 6.7. We show that a judicious choice of base point, connecting paths and generating set for $|L(\Pi)|$ gives $\Pi(|L(\Pi)|) \simeq \Pi$.

We start with any base point $x \in |L(\Pi)|_0$. For every $q \in Q \setminus \{1\}$, let $(\alpha_q, \phi_q): \alpha_q \to \beta_q$ be a critical edge with $(\phi_q)_0 = q$. Such an edge can be constructed inductively using the expansion rule: $(\alpha_q)_{n-1} \triangleq e_{(\phi_q)_n}$, $(\phi_q)_{n-1} \triangleq v_{(\phi_q)_n}$. Let $\gamma_q^1: x \to \alpha_q$ and $\gamma_q^2: \beta_q \to x$ be paths that cross no critical edge. These paths are images of paths in the standard tile, which is connected by Proposition 4.7. We let $\tilde{Q}$ be the set of paths $\gamma_q \triangleq \gamma_q^1(\alpha_q, \phi_q)\gamma_q^2 \subset \pi_1(|L(\Pi)|, x)$, parametrized by $q \in Q$. The ℓ_y's are taken to be paths from x to $y \in f^{-1}(x)$ that do not cross any critical edge. These paths give the natural identification of the tree of preimages of x with A^*, namely $y \in f^{-n}(x) \mapsto y_{-n}y_{-n+1}\dots y_{-1} \in A^*$.

We show that the γ_q generate the image of $\pi_1(|L(\Pi)|, x)$ in $\mathsf{Aut}(A^*)$. Let γ be any loop. By transversality, we may assume that it satisfies the hypothesis of Lemma 6.8. We get a word $w = w_1 \dots w_r \in Q^*$ and by Lemma 6.8, $\gamma_{w_1} \dots \gamma_{w_r}$ acts the same way as γ on A^*, which is what we wanted.

So we only need to show that $\Pi(|L(\Pi)|) \simeq (A, Q)$ has same output and transition functions as Π. We claim that if y is the preimage of x corresponding to a, then $\sigma(y, \gamma_q)$ is the preimage of x corresponding to $\sigma(a, q)$. Letting $\tilde{\gamma}$ be the lift of γ_q starting at y, the argument of Lemma 6.8 shows $\sigma(y, \gamma_q) = \tilde{\gamma}(1) \in \overline{\mathcal{T}_{\sigma(a,q)}}$, which proves our claim.

Last, we need to show that $\ell_y \tilde{\gamma} \ell_{\tilde{\gamma}(1)}^{-1}$ acts the same way on A^* as $\tau(a, q)$ does. By Lemma 6.8 it is enough to show the following: $\ell_y \tilde{\gamma} \ell_{\tilde{\gamma}(1)}^{-1}$ has at most one critical edge. If it has one, its last state is $\tau(a, q)$. And if it does not then $\tau(a, q) = 1$. Recall that the ℓ_y's were chosen without critical edges and that $\tilde{\gamma}$ is a preimage of γ, which has a unique critical edge (α_q, ϕ_q). So the only possible critical edge of $\ell_y \tilde{\gamma} \ell_{\tilde{\gamma}(1)}^{-1}$ is the preimage of (α_q, ϕ_q). Let us call $(\tilde{\alpha}, \tilde{\phi})$ that preimage. We need to show $\tilde{\phi}_0 = \tau(a, q)$; this is clear since $\tilde{\phi}_{-1} = (\phi_q)_0 = q$ and $\tilde{\alpha}$ is in the same tile $\overline{\mathcal{T}_a}$ as y, which implies $\tilde{\alpha}_{-1} = a$.

If $\Pi(|L(\Pi)|)$ had been built using different data, then by Proposition 6.6, we would still have Π equivalent to $\Pi(|L(\Pi)|)$. □

Theorem 6.9. *If Π and Π' are nuclear and equivalent, then $(L(\Pi), f)$ and $(L(\Pi'), f')$ are Morita equivalent.*

If furthermore Π and Π' are smooth, then Π and Π' are equivalent if and only if $(L(\Pi), f)$ and $(L(\Pi'), f')$ are Morita equivalent.

Proof. Let $\phi_0 : A^* \to (A')^*$ be an equivalence between Π and Π'. Denote $\langle \Pi \rangle$ by Γ, and set $\Phi = \phi_0 \Gamma$. The Morita equivalence is given by

$$P = \big\{(\alpha, \phi, \alpha') \in A^{-\mathbb{N}} \times \Phi^{-\mathbb{N}} \times (A')^{-\mathbb{N}} \mid \phi(n)\big(\alpha(n)w\big) = \alpha'(n)\phi(n+1)(w) \\ \text{for all } w \in A^{\mathbb{N}} \text{ and } n < -1, \text{ and } \phi(-\mathbb{N}) \text{ is finite}\big\},$$

with source and target maps $s_P(\alpha, \phi, \alpha') = \alpha$ and $t_P(\alpha, \phi, \alpha') = \alpha'$.

We first show that P is compact; this is because $W = \bigcup_{(\alpha,\phi,\alpha')\in P} \phi(-\mathbb{N})$ is finite; we actually show that W is contained in the Cartesian product of the set of states of ϕ and the nucleus of Π. Fix some $p = (\alpha, \phi, \alpha') \in P$, and consider the set of $g \in \Gamma$ that satisfy $\phi_0 g = \phi(n)$ for some $n < 0$. Since Π is nuclear, for every such g there exists a $k \in \mathbb{N}$ with $\tau(A^k, g) \subseteq Q$. Since there are finitely many such g's associated with p, there is a common such k for all g's. Fix now $n < 0$, and write $\phi(n) = \tau(w, \phi(n-k))$ with $w = \alpha(n-k)\dots\alpha(n-1)$. Write $\phi(n-k) = \phi_0 g'$ for a $g' \in \Gamma$. Then $\phi(n) = \tau(w, \phi_0 g') = \tau(\sigma(w, g'), \phi_0)\tau(w, g')$. We observe that $\tau(\sigma(w, g'), \phi_0)$ is a state of ϕ_0, and that $\tau(w, g') \in \tau(A^k, g') \subseteq Q$, which proves that W is finite; therefore P is closed in the compact $(A \times W \times A')^{-\mathbb{N}}$, so is compact.

We next show that s_P and t_P are onto. Choose any $\alpha \in A^{-\mathbb{N}}$. Then for every $m < 0$, there exists $(\alpha, \phi, \alpha') \in A^{-\mathbb{N}} \times \Phi^{-\mathbb{N}} \times (A')^{-\mathbb{N}}$ satisfying $\phi(n)(\alpha(n)w) = \alpha'(n)\phi(n+1)(w)$ for all $w \in A^{\mathbb{N}}$ and $n \geqslant m$; for instance, set $\phi(m) = \phi_0$. This determines $\phi(n)$ and $\alpha'(n)$ for all $n \geqslant m$. Choose the $\phi(n), \alpha'(n)$ arbitrarily for $n < m$. Since P is compact, there exists an accumulation point of the above choices as $m \to -\infty$, and s_P is onto. The same argument applies to t_P.

We then show that $L(\Pi) \sqcup P \sqcup P^{-1} \sqcup L(\Pi')$ is a groupoid. It suffices to check that given $x \in L(\Pi)$ and $y, z \in P$ with $t(x) = s(y)$, the product $xy \in P$ is well-defined; if $t(y) = t(z)$ then $yz^{-1} \in L(\Pi)$ is well-defined; and if $s(y) = s(z)$ then $y^{-1}z \in L(\Pi')$ is well-defined. Write $x = (\alpha, \phi)$ and $y = (\beta, \psi, \beta')$. Then $xy = (\alpha, \phi\psi, \beta')$ and $(\phi\psi)(-\mathbb{N})$ is finite because both $\phi(-\mathbb{N})$ and $\psi(-\mathbb{N})$ are finite, so $xy \in P$.

Similarly, write $z = (\alpha, \phi, \beta')$. Then $yz^{-1} = (\beta, \psi\phi^{-1}, \alpha)$; and $\psi(n)\phi(n)^{-1} = \phi_0 g_1 g_2^{-1} \phi_0^{-1} \in \Gamma$ assumes finitely many values, so $yz^{-1} \in L(\Pi)$.

Write also $z = (\beta, \phi, \alpha')$. Then $y^{-1}z = (\beta', \psi^{-1}\phi, \alpha')$; and $\psi(n)^{-1}\phi(n) = g_1^{-1} g_2 \in \Gamma$ assumes finitely many values, so $y^{-1}z \in L(\Pi)$.

Finally, the shift map clearly extends to P, by deleting the (-1)st entry of α, ϕ and α'.

Conversely, assume that Π and Π' are nuclear and smooth, and let P be a Morita equivalence between $G = L(\Pi)$ and $G' = L(\Pi')$. Set $G'' = G \sqcup P \sqcup P^{-1} \sqcup G'$. We obtain a diagram $|G| \hookrightarrow |G''| \hookleftarrow |G'|$, and the inclusions commute with the coverings f, f' and f''. By the second part of Theorem 6.7, we may pick data x, ℓ_y, Q such that $\Pi(|G|) = \Pi$, and similarly data $x', \ell_{y'}, Q'$ such that $\Pi(|G'|) = \Pi'$. We may then push these data forward into $|G''|$, obtaining two sets of data again giving Π and Π', respectively. It now follows from Proposition 6.6 that Π and Π' are equivalent automata. $\square$

7. Examples

This section describes some examples of automata and their associated limit spaces. We start by exhibiting various automata that fail to satisfy the various conditions: contraction, smoothness, etc. We then describe the favorable situation of the automaton associated to the covering $f(z) = z^2 - 1$ of the Riemann sphere.

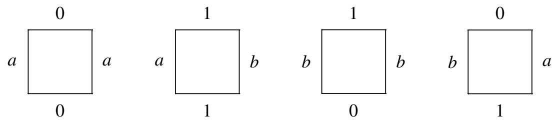

Fig. 4.

We will describe the group G of the automaton by giving its decomposition map $\psi : G \to G \wr \mathfrak{S}_A$ on generators of G. The associated automaton can be recovered by taking as states the generators of G, as alphabet A, and taking a transition from state q to state q' with input a and output a' precisely when $\psi(q) = (r, \pi)$ with $r(a) = q'$ and $\pi(a) = a'$.

The automaton corresponding to this description can be obtained by drawing square tiles with labels the input and output alphabet letters and states, by the procedure described in Section 2.1.

7.1. The Lamplighter group

Here, as in the next two examples, $A = \{0, 1\}$ and $\sigma = (0, 1)$ is the nontrivial element of $\mathfrak{S}_A$.

The Lamplighter group is the group $G = (\mathbb{Z}/2) \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} (\mathbb{Z}/2) \rtimes \mathbb{Z}$. It may be generated by an automaton as follows: $G = \langle a, b \rangle$ with $\psi(a) = (a, b)$ and $\psi(b) = (b, a)\sigma$. This automaton is not nuclear; indeed it is not even contracting, since a has infinite order and the projection of $\psi^n(a)$ on the first vertex is a. Its square description is shown in Fig. 4.

This automaton representation of G was used by Grigorchuk and Żuk in [8] to compute the ℓ^2 spectrum of the simple random walk on G.

The identification of G with the automaton can be understood as follows: identify the boundary of the tree $A^{\mathbb{N}}$ with $\mathbb{F}_2[[t]]$, under the map $(a_i) \mapsto \sum a_i t^i$. Then a and b identify respectively with the affine maps $f \mapsto (1+t)f$ and $f \mapsto (1+t)f + 1$ of $\mathbb{F}_2[[t]]$. Therefore G identifies with the maps of the form $f \mapsto (1+t)^n f + p$ for some $n \in \mathbb{Z}$ and $p \in \mathbb{F}_2[1+t, (1+t)^{-1}]$.

7.2. The Baumslag–Solitar group

Let m, n be two integers. The Baumslag–Solitar group $G_{m,n}$ is defined by its presentation

$$G_{m,n} = \langle a, t \mid t^{-1} a^m t = a^n \rangle.$$

It is residually finite precisely when $m = \pm n$ or $m = \pm 1$ or $n = \pm 1$. If $m = 1$, it may be represented by affine transformations as $a(X) = X + 1$ and $t(X) = nX$.

The group $G_{1,3}$ may be generated by an automaton as follows: $G = \langle a, b, c \rangle$ with $\psi(a) = (a, b)$ and $\psi(b) = (a, c)\sigma$ and $\psi(c) = (b, c)$. Again this automaton is not nuclear. Its square description is shown in Fig. 5.

The identification of G with the automaton can be understood as follows: identify the boundary of the tree $A^{\mathbb{N}}$ with $\mathbb{Z}_2$, under the map $(a_i) \mapsto \sum a_i 2^i$. Then a, b and c identify

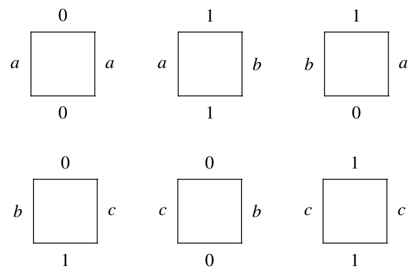

Fig. 5.

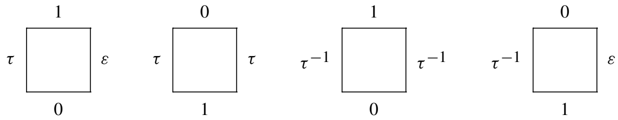

Fig. 6.

respectively with the affine maps $X \mapsto 3X$, $X \mapsto 3X+1$ and $X \mapsto 3X+2$. Therefore G identifies with the maps of the form $X \mapsto 3^n f + p$ for some $n \in \mathbb{Z}$ and $p \in \mathbb{Z}[1/3]$.

The similarity between the lamplighter and Baumslag–Solitar groups is not accidental; a common construction of $G_{1,m}$ and $(\mathbb{Z}/q) \wr \mathbb{Z}$ by automata is described in [4].

7.3. *The odometer*

Again identify the boundary $A^{\mathbb{N}}$ of the tree with $\mathbb{Z}_2$, and consider the subgroup $\mathbb{Z}$ of $\mathbb{Z}_2$. This cyclic group may be generated by an automaton as follows: $G = \langle \tau \rangle$ with $\psi(\tau) = (\varepsilon, \tau)\sigma$, where as before ε denotes the identity state. The associated automaton is contracting, with nucleus $\{\varepsilon, \tau, \tau^{-1}\}$. Its square description is shown in Fig. 6.

The limit space $L(\Pi)$ is $I/\{0^{\mathbb{N}} = 1^{\mathbb{N}}\}$ with I as in Lemma 3.6, and its topological quotient $L(\Pi)_{\top}$ is homeomorphic to the circle. The standard tile is I and its topological quotient is $[0,1]$. The topological quotient of the associated solenoid is the standard 2-adic solenoid: the inverse limit of

$$\cdots \longrightarrow S^1 \xrightarrow{(\ )^2} S^1 \xrightarrow{(\ )^2} S^1 .$$

We consider in the next three examples some contracting actions which exhibit various "pathologies."

7.4. *A nonrecurrent example*

Take now $A = \{0,1,2\}$, with $\sigma = (0,1,2)$ a three-cycle, and consider the action of $\mathbb{Z}$ defined as follows: $G = \langle \tau \rangle$, with $\psi(\tau) = (\varepsilon, \tau, \tau)\sigma$. The associated automaton is contract-

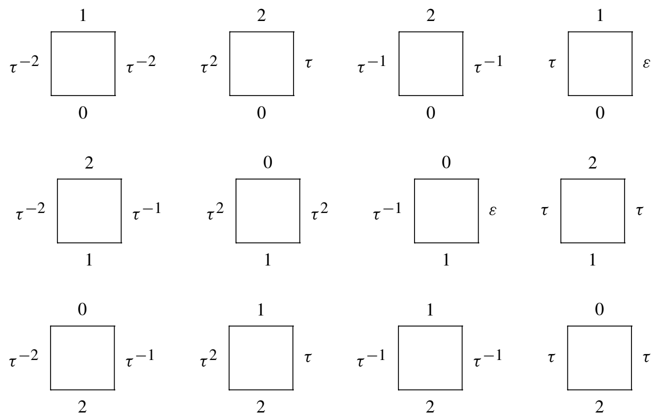

Fig. 7.

ing, with nucleus $\{\varepsilon, \tau^{\pm 1}, \tau^{\pm 2}\}$. The square description of its nucleus is shown in Fig. 7 (we omit the squares for the state ε, which have vertical labels ε and equal horizontal labels).

This group is not recurrent: indeed the stabilizer of a vertex (say 0) is $\langle \tau^3 \rangle$, and its projection on the subtree $0A^*$ is $\langle \tau^2 \rangle$.

The limit space $L(\Pi)_\top$ is the standard 2-adic solenoid. In particular, it is connected, but not arcwise connected. Its self-covering is the "triple-the-angle" map.

7.5. A nonsmooth example

Take again $A = \{0, 1, 2\}$ and $\sigma = (0, 1, 2)$, and consider the action of $\mathbb{Z}$ defined as follows: $G = \langle \tau \rangle$, with $\psi(\tau) = (\tau, \tau^{-1}, \tau)\sigma$. The associated automaton is contracting, with nucleus $\{\varepsilon, \tau^{\pm 1}, \tau^{\pm 2}\}$. The square description of its nucleus is shown in Fig. 8.

The stabilizer of a vertex (say 0) is $\langle \tau^3 \rangle$, which projects to G on the subtree $0A^*$; so G is recurrent.

The minimal automaton generating G, with set of states $Q = \{\varepsilon, \tau, \tau^{-1}\}$, is not smooth. However, since $\psi(\tau^2) = (\varepsilon, \varepsilon, \tau^2)\sigma^2$, the subgroup $\langle \tau^2 \rangle$ is smooth.

The limit space $L(\Pi)_\top$ is a circle, but $|L(\Pi)|$ looks more like a Möbius strip. The three tiles of $|L(\Pi)|$ project to the three overlapping intervals $[0, 2\pi/3]$, $[2\pi/3, \pi/3]$ and $[\pi/3, 0]$ of the circle (see Fig. 9).

7.6. A more complicated nonsmooth example

Take again $A = \{0, 1, 2\}$ and $\sigma = (0, 1, 2)$, and consider the action of $\mathbb{Z}$ defined as follows: $G = \langle \tau \rangle$, with $\psi(\tau) = (\tau^2, \varepsilon, \tau^{-1})\sigma$. The associated automaton is contracting, with nucleus $\{\varepsilon, \tau^{\pm 1}, \tau^{\pm 2}\}$. The square description of its nucleus is shown in Fig. 10.

The stabilizer of a vertex (say 0) is $\langle \tau^3 \rangle$, which projects to G on the subtree $0A^*$; so G is recurrent.

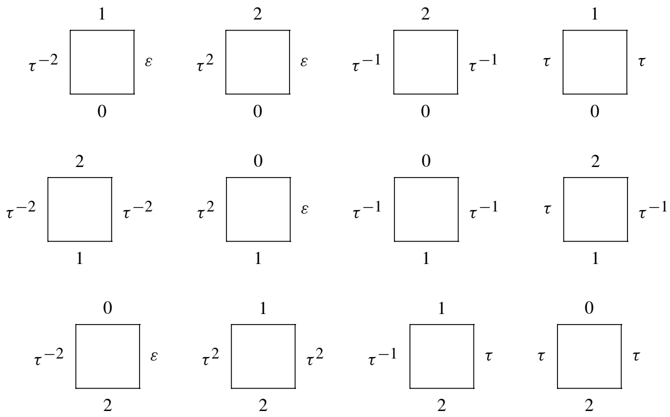

Fig. 8.

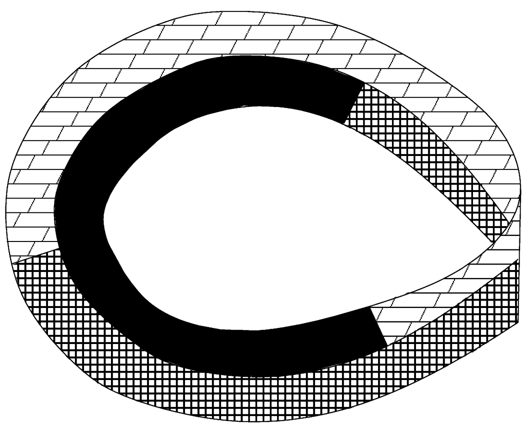

Fig. 9.

The automaton generating G is again nonsmooth, but this time in an essential way: there is no element of the nucleus sending 0 to 1 and projecting to a power of the trivial state. Indeed the only element with that property is τ^{-5} and, since it does not belong to the nucleus, there is no (bounded-length) path in $|L(\Pi)|$ between $w0$ and $w'1$ for any $w, w' \in A^{-\mathbb{N}}$.

The limit space $L(\Pi)$ is therefore connected, but not arcwise connected. Its topological quotient is as before a circle, since the group is recurrent. This example illustrates how the property of being arcwise connected is not invariant under Morita equivalence.

The topological quotient of the tile is the closure of the union of countably many closed intervals. Its boundary is a Cantor set.

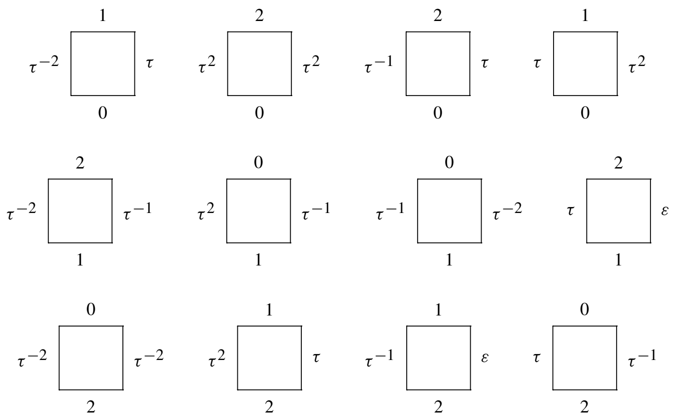

Fig. 10.

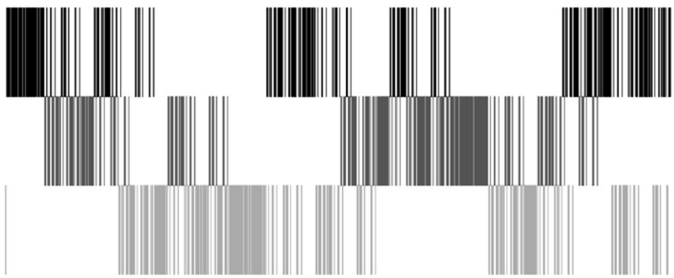

Fig. 11.

In Fig. 11 we represent the tiling of the circle (drawn as a horizontal interval) by three copies of the standard tile. We cross each copy of the tile by an interval in order to make the picture more legible, and draw each copy in a different tint.

7.7. *The "basilica group"*

This highly nontrivial example of group served as a motivation for the study of iterated monodromy groups and their general properties. It is defined, with $A = \{0, 1\}$ and $\sigma = (0, 1)$, as follows: $G = \langle a, b\rangle$, with $\psi(a) = (\varepsilon, b)\sigma$ and $\psi(b) = (\varepsilon, a)$. The associated automaton is contracting, with nucleus $\{\varepsilon, a^{\pm 1}, b^{\pm 1}, a^{-1}b, b^{-1}a\}$.

The group G is torsion-free, amenable, but cannot be obtained from groups of subexponential growth via direct limits, extensions, subgroups and quotients [3]. It was the first example with such a property. More details on G can be found in [7].

The topological quotient of the limit space is the Julia set of the complex map $f(z) = z^2 - 1$.

The odometer is conjugate, within $\mathsf{Aut}\, A^*$, to the subgroup $\langle a^{-1}b\rangle$ of G. This explains that the limit space of G is a quotient of the limit space of the odometer.

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