

Characterization of knot complements in the 4-sphere

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Abstract

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Knot complements in S^4 are characterized as follows: A connected open set $W \subset S^4$ is homeomorphic to the complement of some locally flat 2-sphere in S^4 if and only if $H_1(W)$ is infinite cyclic, W has one end, and the fundamental group of that end is infinite cyclic. Applications include a characterization of weakly flat 2-spheres in S^4 and a complement theorem for 2-spheres in S^4 .

Keywords: Knot, 2-sphere, 4-sphere, complement, weak flatness.

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Introduction

In this paper we characterize knot complements in S^4 . Our main theorem is the following.

Theorem 0.1. *Let W be a connected open subset of S^4 . Then $W \cong S^4 - K$ for some locally flat 2-sphere K in S^4 if and only if*

(0.1.1) $H_1(W) \cong \mathbb{Z}$, and

(0.1.2) W has one end ε with $\pi_1(\varepsilon)$ stable and $\pi_1(\varepsilon) \cong \mathbb{Z}$.

A locally flat 2-sphere in S^4 can, of course, be globally knotted, but homology does not detect knotting and so the homology of the complement of any locally flat

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2-sphere in S^4 will be the same as that of the standard S^2 in S^4 . In addition, a locally flat 2-sphere in S^4 always has a product neighborhood, and so the complement will have a stable fundamental group at infinity and the group will be isomorphic to \mathbb{Z} . Thus the two conditions in Theorem 0.1 are easily seen to be necessary.

A similar theorem for open subsets of S^n , $n \geq 6$, was proved by Daverman [3] with later improvements by Liem [13]. The theorems of Daverman and Liem are also valid in dimension five since it is possible to extend their techniques to dimension five using the machinery of [10]. The theorem in dimension four is more striking than the one in high dimensions (compare Theorem 0.1 with [3, Theorem 4]) because it turns out that some of the conditions which must be included as hypotheses in high dimensions are automatically satisfied in dimension four. For example, it is not necessary for us to assume that W has the homotopy type of a finite complex, but instead we prove that this follows from hypotheses (0.1.1) and (0.1.2). We can similarly dispense with higher homotopy conditions on ε and higher homology conditions on W .

The situation in dimension three is similar to that in dimension four. In fact, the paper of Daverman [2] implicitly contains a proof of the 3-dimensional case of Theorem 0.1.

Our theorem can be viewed as asserting that it is possible to add a boundary to W in the sense of Siebenmann's thesis [18]. Guilbault [11] has proved that this can be done in other special cases as well; in particular, his theorem covers the case in which W has the homotopy type of S^1 . At present it is not known whether the full generalization of Siebenmann's thesis holds in dimension four.

As an application of Theorem 0.1, we give a homotopy characterization of those 2-spheres $\Sigma \subset S^4$ having the property that $S^4 - \Sigma \cong S^4 - K$ for some locally flat 2-sphere $K \subset S^4$. As in the high dimensional case [3, 13], the characterization is stated in terms of the global 1-*alg* property.

Definition. A compact set X embedded in a manifold M is said to be *globally 1-*alg** if for every neighborhood U of X there exists a neighborhood V of X in U such that each loop in $V - X$ which is null-homologous in $V - X$ is null-homotopic in $U - X$. (The “*alg*” stands for “Abelian local groups”).

Corollary 0.2. *Let $\Sigma \subset S^4$ be a topologically embedded 2-sphere. Then $S^4 - \Sigma \cong S^4 - K$ for some locally flat 2-sphere $K \subset S^4$ if and only if Σ is globally 1-*alg*.*

Remark 0.3. It should be noted that a 2-sphere which satisfies the global 1-*alg* property need not be locally flat; examples are given in [11, 4]. A local version of the 1-*alg* property does imply local flatness for 2-spheres topologically embedded in a 4-manifold [7, Theorem 10].

As a second application of Theorem 0.1 we prove the following complement theorem for subsets of S^4 . A complement theorem is one which gives conditions

under which compacta with the same shape have homeomorphic complements and conversely. A comparable theorem for codimension two spheres in S^n , $n \geq 5$, is proved in [13]. The notation $\text{Sh}(X) = \text{Sh}(Y)$ means that X and Y have the same shape.

Theorem 0.4. *Let X be a compact subset of S^4 which is globally 1-alg. Then $S^4 - X \cong S^4 - K$ for some locally flat 2-sphere K in S^4 if and only if $\text{Sh}(X) = \text{Sh}(S^2)$.*

Suppose K is a locally flat 2-sphere in S^4 . It is a well-known folklore result that (Milnor duality [16] implies that) if $\pi_1(S^4 - K) \cong \mathbb{Z}$, then $S^4 - K$ has the homotopy type of S^1 . (Such a result certainly does not hold for codimension two spheres in S^n , $n \geq 5$.) In the course of proving Theorem 0.1, we prove the following stronger version of the folklore theorem which may be of independent interest.

Theorem 0.5. *Let W be an open subset of S^4 which has one end ε . If $\pi_1(W) \cong \mathbb{Z}$ and $\pi_1(\varepsilon) \cong \mathbb{Z}$, then W has the homotopy type of S^1 .*

Corollary 0.6. *If $\Sigma \subset S^4$ is a topologically embedded 2-sphere such that Σ is globally 1-alg and $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$, then $S^4 - \Sigma$ has the homotopy type of S^1 .*

Remark 0.7. The hypothesis that Σ be globally 1-alg is necessary; in [14] an example of a 2-sphere $\Sigma \subset S^4$ is constructed having the property that $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$ but $\pi_2(S^4 - \Sigma) \neq 0$.

As a consequence of Theorem 0.5 we are able to restate Guilbault's characterization [11] of weakly flat 2-spheres in S^4 .

Definition. A 2-sphere $\Sigma \subset S^4$ is *weakly flat* if $S^4 - \Sigma \cong S^4 - S^2 \cong S^1 \times \mathbb{R}^3$.

Theorem 0.8. *A 2-sphere $\Sigma \subset S^4$ is weakly flat if and only if Σ is globally 1-alg and $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$.*

The characterization of weakly flat 2-spheres in S^4 is, therefore, exactly the same as that of weakly flat 1-spheres in S^3 ; see [2, Theorem 3]. Instead of the π_1 condition, Guilbault uses the apparently stronger hypothesis that $S^4 - \Sigma$ has the homotopy type of S^1 . We should point out, however, that we do not give a new proof of Guilbault's result here; rather, we apply his theorem at a crucial spot in our proof. Hollingsworth and Rushing [12] have characterized weakly flat codimension two spheres in S^n , $n \geq 5$.

Remark 0.9. It is possible for $S^4 - \Sigma$ to have the homotopy type of S^1 even though Σ is not weakly flat; examples are given in [11, 4].

If we restate Theorem 0.8 in terms of complements, we get the following result which is more comparable to Theorem 0.1.

Theorem 0.10. *Let W be a connected open subset of S^4 . Then $W \cong S^4 - S^2 \cong S^1 \times \mathbb{R}^3$ if and only if*

(0.10.1) $\pi_1(W) \cong \mathbb{Z}$, and

(0.10.2) W has one end ε with $\pi_1(\varepsilon)$ stable and $\pi_1(\varepsilon) \cong \mathbb{Z}$.

1. Homotopy properties of W

In this section we explore some of the homotopy properties of W . We begin with some preliminary definitions and notation. We will use “ \cong ” to mean either “is homeomorphic to” or “is isomorphic to” (depending on the context), “ \simeq ” to mean “is homotopic to”, and “ \sim ” to mean “is homologous to”. The reader is referred to [18] for most definitions relating to ends of manifolds. In this paper we will be concerned only with manifolds W which have one end, ε . In this case, ε should be thought of as a sequence of connected open sets $V_1 \supset V_2 \supset V_3 \supset \dots$ in W such that $\bigcap_{i=1}^\infty V_i = \emptyset$ and each $W - V_i$ is compact. We say that $\pi_1(\varepsilon)$ is *stable* if the V_i can be chosen in such a way that the inclusion maps induce isomorphisms of all the images $\{\text{im}[\pi_1(V_{i+1}) \rightarrow \pi_1(V_i)]\}$. In case $\pi_1(\varepsilon)$ is stable and G is a group, we say that $\pi_1(\varepsilon) \cong G$ if each $\text{im}[\pi_1(V_{i+1}) \rightarrow \pi_1(V_i)]$ is isomorphic to G . Since every noncompact 4-manifold can be triangulated as a piecewise linear manifold [10, Theorem 8.2], we may assume that each of the 4-manifolds we work with is a PL manifold and we will use this PL structure whenever it is convenient to do so.

Lemma 1.1. *Suppose W is a connected open subset of a closed 4-manifold M^4 such that $H_1(M^4) = 0$. If W satisfies (0.1.1) and (0.1.2), then the natural map $\pi_1(\varepsilon) \rightarrow H_1(W)$ is an isomorphism.*

Proof. Let $U \subset W$ be a neighborhood of ε . Choose a connected PL manifold neighborhood V of ε so that $\text{im}[\pi_1(V) \rightarrow \pi_1(U)] \cong \mathbb{Z}$. Consider the commutative diagram

$$\begin{array}{ccccc} H_1(V) & \xrightarrow{\alpha} & H_1(W) & \longrightarrow & H_1(W, V) \\ \downarrow \iota_* & & \parallel & & \\ H_1(U) & \xrightarrow{\beta} & H_1(W) & & \end{array}$$

Using excision, we see that $H_1(W, V) \cong H_1(M^4, V \cup (M^4 - W))$. By considering the sequence of the pair $(M^4, V \cup (M^4 - W))$, we see that $H_1(M^4, V \cup (M^4 - W)) = 0$. Thus α is onto, which means that $\beta \iota_*$ is onto and so the inclusion induces an isomorphism of $\text{im}[H_1(V) \rightarrow H_1(U)]$ to $H_1(W)$. (Since $\text{im}[\pi_1(V) \rightarrow \pi_1(U)] \cong \mathbb{Z}$, $\text{im}[H_1(V) \rightarrow H_1(U)]$ is cyclic and the only onto homomorphisms from a cyclic group to \mathbb{Z} are isomorphisms.) Now the Hurewicz map is onto, so the composition of the Hurewicz map and the inclusion induced map gives an isomorphism $\text{im}[\pi_1(V) \rightarrow \pi_1(U)] \cong H_1(W)$. \square

Definition. A compact set X in a manifold satisfies property k -UV if for every neighborhood U of X there exists a neighborhood V of X such that any map of S^k into V is null-homotopic in U . We say that X is UV^k if X satisfies property i -UV for $0 \leq i \leq k$.

Notice that X is UV^0 if and only if X is connected.

Proposition 1.2. *If W is an open subset of a closed simply connected 4-manifold M^4 such that W satisfies (0.1.1) and (0.1.2) and $X = M^4 - W$, then X satisfies property 1-UV.*

Proof. Let $U \supset X$ be given. By Lemma 1.1 we can choose a compact PL manifold neighborhood V of X in U such that any loop in $V - X$ which is null-homologous in W is actually null-homotopic in $U - X$. Let l be a loop in V . We may assume that l is PL embedded. Since $l \approx 0$ in M^4 , l bounds a singular disk in M^4 . Approximate this disk by one which is PL and in general position. Any double points can be piped over the boundary to form a PL embedded disk Δ with $l = \partial\Delta$. Put Δ in general position with respect to ∂V . Then $\partial V \cap \Delta$ is a collection of circles. Consider such a circle C which is innermost on Δ . If C bounds in $M^4 - V$, then (by the choice of V) C is null-homotopic in $U - X$. Thus we can replace the subdisk of Δ which C bounds by one which is mapped into $U - X$.

Let $\mathcal{C} = \{C_i\}_{i=1}^n$ be the collection of all circles in $\Delta \cap \partial V$ such that the subdisk of Δ which each bounds is not contained in U . Consider a circle C_j which is innermost in \mathcal{C} and let Δ_j be the subdisk of Δ which C_j bounds. Then $\Delta_j \cap \partial V$ consists of a collection of circles. Let $\mathcal{C}^{(j)}$ be the subcollection of the circles in $\Delta_j \cap \partial V$ such that their union, together with C_j , bounds a surface in Δ_j whose image misses X . Join the circles in $\mathcal{C}^{(j)}$ with arcs on ∂V to form one loop l_j . Then $l_j \sim C_j$ in $M^4 - V \subset W$, so $l_j \approx C_j$ in $U - X$ (by the choice of V again). Hence there is a singular annulus in $U - X$ which joins l_j to C_j . Thus it is possible to replace Δ_j with a new disk which maps into U . This removes C_j from the collection \mathcal{C} . We can similarly remove each of the other loops from \mathcal{C} . \square

Corollary 1.3. *If W is an open subset of S^4 which satisfies (0.1.1) and (0.1.2), then $H_2(W) = 0$.*

Proof. By Alexander duality, $H_2(W) \cong \check{H}^1(S^4 - W)$. \square

Let $p: \tilde{W} \rightarrow W$ denote the universal cover of W .

Lemma 1.4. *Suppose W is an open subset of the closed, simply connected 4-manifold M^4 . If W has one end ε , $\pi_1(\varepsilon) \cong \mathbb{Z}$ and $\pi_1(W) \cong \mathbb{Z}$, then \tilde{W} has one end and that end is simply connected.*

Proof. Since $\pi_1(\varepsilon) \cong \mathbb{Z}$, we can choose a sequence $U_1 \supset U_2 \supset \dots$ of neighborhoods of ε such that the closure of each U_i is a PL manifold and $\text{im}[\pi_1(U_{i+1}) \rightarrow \pi_1(U_i)] \cong \mathbb{Z}$ for every i . Notice that Lemma 1.1 implies that each $p^{-1}(U_i)$ is connected. Choose $K \subset W$ such that $K - \text{Int}(p^{-1}(U_i))$ is a compact, connected PL manifold for each i and $p(K) = W$. Let t be a deck transformation of \tilde{W} which corresponds to a generator of $\pi_1(W)$. Define

$$V_n = \left(\bigcup_{i \leq -n} t^i(K) \right) \cup p^{-1}(U_n) \cup \left(\bigcup_{i \geq n} t^i(K) \right).$$

Now each V_n is connected, $\tilde{W} - V_n$ has compact closure, and $\bigcap_{n=1}^\infty V_n = \emptyset$. Therefore \tilde{W} has one end. In order to complete the proof, we must show the following: for every n there exists a k such that the inclusion induced homomorphism $\pi_1(V_{n+k+1}) \rightarrow \pi_1(V_n)$ is trivial. Fix n . Since $\pi_1(W, U_{n+1}) = 0$, there exists a homotopy h_{n+1} which pushes the 1-skeleton of W into U_{n+1} and is fixed on U_{n+2} . Let \tilde{h}_{n+1} be a homotopy of the 1-skeleton of \tilde{W} which covers h_{n+1} . Since $K - \text{Int}(p^{-1}(U_{n+2}))$ is compact, there exists a k such that

$$\tilde{h}_{n+1}(K^{(1)} \times I) \subset \bigcup_{-k \leq i \leq k} t^i(K).$$

Consider a loop l in V_{n+k+1} . Apply the homotopy \tilde{h}_{n+1} to l . The track of this homotopy will lie in V_{n+1} and the homotopy will push l to $l' \subset p^{-1}(U_{n+1})$. Then $p(l') \simeq 0$ in U_n (by Lemma 1.1) and so $l' \simeq 0$ in V_n . \square

2. Finiteness of W

This section contains the main technical lemmas of the paper. Our objective is to prove that an open set W satisfying the hypotheses of Theorem 0.1 has a tame end. In order to do so, we must show that W has the homotopy type of a finite complex.

Notation. For the rest of this section, W_1 will denote an open subset of a closed, simply connected 4-manifold M^4 such that W_1 has one end ε , $\pi_1(\varepsilon) \cong \mathbb{Z}$, and $\pi_1(W_1) \cong \mathbb{Z}$. By Lemma 1.1 there is a natural isomorphism between these groups. We will use J to denote $\pi_1(W_1)$ and Λ to denote the group ring $\mathbb{Z}[J]$. We use $p: \tilde{W}_1 \rightarrow W_1$ to denote the universal cover of W_1 (which, in this case, is the same as the infinite cyclic cover). We use $H_*(\tilde{W}_1)$ and $H_c^*(\tilde{W}_1)$ to denote the ordinary singular homology and cohomology with compact supports. Each of these groups has a natural Λ -module structure. Poincaré duality gives isomorphisms $H_k(\tilde{W}_1) \cong H_c^{4-k}(\tilde{W}_1)$. Fix a generator t of J . We identify t with the corresponding deck transformation on \tilde{W}_1 and also with the induced homomorphisms on $H_*(\tilde{W}_1)$. Notice that Λ can be thought of as the set of Laurent polynomials in t with integer coefficients.

Local coefficients. We find it convenient to use homology and cohomology with local coefficients (in the sense of Steenrod [20]). Let \mathcal{B} be a bundle over W_1 with fiber B , where B is a Λ -module. We use $H^*(W_1; \mathcal{B})$ to denote cohomology with coefficients in \mathcal{B} and $H_*^\infty(W_1; \mathcal{B})$ to denote the homology based on infinite chains with coefficients in \mathcal{B} . There are natural Poincaré duality isomorphisms

$$H^k(W_1; \mathcal{B}) \cong H_{4-k}^\infty(W_1; \mathcal{B})$$

(cf. [17, p. 388; 20, p. 620]).

Lemma 2.1. $H_3(\tilde{W}_1) = 0$.

Proof. By Lemma 1.4 we can choose a sequence $V_1 \supset V_2 \supset \dots$ of connected neighborhoods of the end of \tilde{W}_1 such that $\bigcap_{i=1}^\infty V_i = \emptyset$. Consider the sequence

$$\tilde{H}^0(V_n) \rightarrow H^1(\tilde{W}_1, V_n) \rightarrow H^1(\tilde{W}_1).$$

The first and last terms are zero and therefore $H^1(\tilde{W}_1, V_n) = 0$. Thus $H_c^1(\tilde{W}_1) \cong \lim_{\rightarrow n} H^1(\tilde{W}_1, V_n) = 0$. By Poincaré duality, $H_3(\tilde{W}_1) \cong H_c^1(\tilde{W}_1)$, so we have $H_3(\tilde{W}_1) = 0$. \square

Lemma 2.2. $H^3(W_1; \Lambda) = 0$.

Proof. By Poincaré duality, $H^3(W_1; \Lambda) \cong H_1^\infty(W_1; \Lambda)$, so we show that $H_1^\infty(W_1; \Lambda) = 0$. Let $c \in Z_1^\infty(W_1; \Lambda)$. Then c lifts to $\tilde{c} \in Z_1^\infty(\tilde{W}_1; \Lambda)$. Choose a sequence

$$W_1 = E_0 \supset E_1 \supset E_2 \supset \dots$$

of neighborhoods of ε such that $\bigcap_{i=0}^\infty E_i = \emptyset$ and each inclusion induced homomorphism $\pi_1(E_{i+1}) \rightarrow \pi_1(E_i)$ has image isomorphic to \mathbb{Z} . Let $\tilde{E}_i = p^{-1}(E_i)$ and notice that

$$\text{supp}(\tilde{c}) - \tilde{E}_j$$

is compact for every j . Furthermore, the inclusion induced homomorphism $\pi_1(\tilde{E}_{j+1}) \rightarrow \pi_1(\tilde{E}_j)$ is trivial for every j . Thus \tilde{c} is homologous to a 1-cycle \tilde{c}_1 in \tilde{E}_2 via a compact 2-chain $S_1 \subset \tilde{E}_0 = \tilde{W}_1$ and, inductively, \tilde{c}_{i-1} is homologous to a 1-cycle \tilde{c}_i in \tilde{E}_{i+1} via a compact 2-chain $S_i \subset \tilde{E}_{i-1}$. Then $S = S_1 \cup S_2 \cup S_3 \cup \dots$ is a (locally finite) infinite 2-chain in \tilde{W}_1 whose boundary is \tilde{c} . Moreover, since $S|_{\tilde{W}_1 - \tilde{E}_j} \subset S_1 \cup \dots \cup S_j$, we see that the image of S under the projection map p defines a (locally finite) infinite 2-chain in $C_2^\infty(W_1; \Lambda)$ whose boundary is c . \square

Lemma 2.3. $H^3(W_1; \mathcal{B}) = 0$ for every coefficient bundle \mathcal{B} whose fiber is a countably generated free Λ -module.

Proof. We must show that $H_1^\infty(W_1; \mathcal{B}) = 0$. Let B denote the fiber of \mathcal{B} ; then there is a decomposition

$$B = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \dots \tag{*}$$

in which each Λ_i is a copy of Λ . Enumerate the simplices of W_1 by $\{e_j^k \mid j = 1, 2, 3, \dots\}$, where $k = \dim e_j^k$, and let B_j^k denote the fiber of \mathcal{B} over the barycenter of e_j^k . Fix a basepoint in W_1 and choose a path from the basepoint to the barycenter of e_j^k for each j . These paths determine isomorphisms $h_j^k: B \rightarrow B_j^k$ and

$$B_j^k = h_j^k(\Lambda_1) \oplus h_j^k(\Lambda_2) \oplus h_j^k(\Lambda_3) \oplus \dots$$

Let $c \in Z_1^\infty(W_1; \mathcal{B})$ be an infinite 1-cycle representing an element of $H_1^\infty(W_1; \mathcal{B})$. Then c is determined by its values on the 1-simplices of W_1 . If e_j^1 is a 1-simplex, $c(e_j^1) \in B_j^1$, so we have $c(e_j^1) = \sum_{i=1}^\infty c_i(e_j^1)$ where $c_i(e_j^1) \in h_j^1(\Lambda_i)$. We can write $c = \sum_{i=1}^\infty c_i$; this sum makes sense because, for each j , at most finitely many of the terms $c_i(e_j^1)$ are nonzero. Since $\partial c = 0$ and the decomposition (*) is a Λ -module decomposition, we have that $\partial c_i = 0$ for every i . Notice that c_i can be thought of as an element of $Z_1^\infty(W_1; \Lambda_i)$ and thus we can proceed as in the proof of Lemma 2.2, working with a finite number of c_i at a time.

Let $E_0 \supset E_1 \supset E_2 \supset \dots$ be a sequence of neighborhoods of the end of W_1 as in the proof of Lemma 2.2. Since there are only a finite number of 1-simplices in $E_0 - E_2$, there exists a positive integer k_1 such that $c_i(e_j^1) = 0$ whenever $e_j^1 \subset E_0 - E_2$ and $i \geq k_1$. Just as in the proof of Lemma 2.2, there exists a finite 2-chain S_1 such that S_1 represents a homology from $\sum_{i \leq k_1} c_i$ to $\sum_{i \leq k_1} c_i^1$ where each $c_i^1 \in Z_1^\infty(W_1; \Lambda_i)$ and c_i^1 has its support in E_2 . Define $c_i^1 = c_i$ for $i > k_1$. Notice that S_1 defines a homology from $c = \sum_{i=1}^\infty c_i$ to $c^1 = \sum_{i=1}^\infty c_i^1$ and that c^1 has its support in E_2 . Similarly, there exists a finite 2-chain S_2 with support in E_1 such that S_2 represents a homology from c^1 to c^2 , where c^2 has support in E_3 . This process is continued inductively. Then $S = S_1 \cup S_2 \cup S_3 \cup \dots$ is a (locally finite) infinite 2-chain whose boundary is the original 1-cycle c . \square

Lemma 2.4. $H_2(\tilde{W}_1)$ is a projective Λ -module.

Proof. The proof uses Lemma 2.3 and is essentially the same as the proof of [21, Lemma 2.1], but we indicate how it goes for the sake of completeness. Since W_1 is connected we can cancel all the 4-handles and all but one of the 0-handles in a handle decomposition and so we see that W_1 has the homotopy type of a $\mathbb{C}W$ complex with one 0-cell and no k -cells for $k \geq 4$ [18, Lemma 5.3]. Thus, by [22, Lemma 1], we can define $H_*(\tilde{W}_1)$ using a chain complex of the form

$$0 \rightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

in which $C_0 \cong \Lambda$, \mathbb{Z} is the trivial Λ -module, and the augmentation ε is defined by

$$\varepsilon \left(\sum_i n_i t^i \right) = \sum_i n_i.$$

Let $\Lambda_0 = \ker \varepsilon$; then Λ_0 is the free submodule of $C_0 \cong \Lambda$ generated by $t - 1$. Therefore the sequence

$$0 \rightarrow \ker \partial_1 \hookrightarrow C_1 \xrightarrow{\partial_1} \Lambda_0 \rightarrow 0$$

splits and $C_1 \cong \ker \partial_1 \oplus \Lambda_0$. Thus $\ker \partial_1$ is projective and $\text{im } \partial_2 = \ker \partial_1$ (since $H_1(\tilde{W}_1) = 0$) so $\text{im } \partial_2$ is projective. Hence the sequence

$$0 \rightarrow \ker \partial_2 \hookrightarrow C_2 \rightarrow \text{im } \partial_2 \rightarrow 0$$

splits and so $C_2 \cong \ker \partial_2 \oplus \text{im } \partial_2$. Let $B = \partial_3(C_3) = \text{im } \partial_3 \subset \ker \partial_2 \subset C_2$. Then B determines a coefficient bundle \mathcal{B} over W_1 . Since $H_3(\tilde{W}_1) = 0$, ∂_3 is one-to-one, and so $B \cong C_3$, a countably generated free Λ -module. We can think of $\partial_3: C_3 \rightarrow B$ as an element of $\text{Hom}_\Lambda(C_3, B)$ and use the fact that $H^3(W_1; \mathcal{B}) = 0$ to conclude that there exists a Λ -homomorphism $\theta: C_2 \rightarrow B$ such that $\partial_3 = \delta_3(\theta) = \theta \circ \partial_3$. Thus $\theta|_{\ker \partial_2}$ splits the short exact sequence

$$0 \rightarrow B \rightarrow \ker \partial_2 \rightarrow H_2(\tilde{W}_1) \rightarrow 0$$

and we see that $C_2 \cong B \oplus H_2(\tilde{W}_1) \oplus \text{im } \partial_2$. \square

Construction of W_1 . Let W be an open subset of S^4 which satisfies (0.1.1) and (0.1.2). Since $\pi_1(\varepsilon) \cong \mathbb{Z}$, it follows that $\pi_1(W)$ is finitely generated. Therefore the kernel of the Hurewicz map $\pi_1(W) \rightarrow H_1(W)$ is normally generated by a finite set. Let W_1 be the manifold obtained from W by doing surgery on a finite set of curves which normally generate $\ker[\pi_1(W) \rightarrow H_1(W)]$. Specifically, choose a finite collection of PL embedded loops in W which normally generate $\ker[\pi_1(W) \rightarrow H_1(W)]$. These loops bound a collection of disjoint disks in S^4 and the disks determine framings for the loops. Note that a regular neighborhood of each loop is homeomorphic to $S^1 \times B^3$. Cut out each of these regular neighborhoods and replace each with a copy of $B^2 \times S^2$ to form W_1 . Notice three things about W_1 : First, the end of W_1 is exactly the same as that of W . Second, each 1-surgery corresponds to taking connected sum with a copy of $S^2 \times S^2$; i.e., $W_1 \subset M^4$, where M^4 is a connected sum of a finite number of copies of $S^2 \times S^2$, one for each curve in the finite set. Third, since we have killed the entire kernel of $\pi_1(W) \rightarrow H_1(W)$, it must be the case that $\pi_1(W_1) \cong \mathbb{Z}$.

Lemma 2.5. *Let W be an open subset of S^4 which satisfies (0.1.1) and (0.1.2). If W_1 is constructed from W by doing 1-surgeries as above, then $H_2(\tilde{W}_1)$ is finitely generated as a Λ -module.*

Proof. The first step of the proof consists of showing that $H_2(W_1)$ is a finitely generated free Abelian group. We do so by showing that the inclusion map induces an isomorphism $H_2(W_1) \cong H_2(M)$. Consider the exact sequence

$$H_3(M, W_1) \rightarrow H_2(W_1) \xrightarrow{\alpha} H_2(M) \rightarrow H_2(M, W_1) \xrightarrow{\beta} H_1(W_1) \rightarrow H_1(M).$$

The last term is obviously 0 and the first term is also 0 because Alexander duality and Proposition 1.2 show that $H_3(M, W_1) \cong \check{H}^1(M - W_1) = \check{H}^1(S^4 - W) = 0$. Similarly, we have that $H_2(M, W_1) \cong \check{H}^2(S^4 - W) \cong H_1(W) \cong \mathbb{Z}$. This means that β is an isomorphism and therefore α is as well.

We next consider $H_2(\tilde{W}_1) \otimes \mathbb{Q}$ and $H_2(W_1) \otimes \mathbb{Q}$. Since $H_2(W_1)$ is a finitely generated free \mathbb{Z} -module, $H_2(W_1) \otimes \mathbb{Q}$ will be a finitely generated vector space over \mathbb{Q} . Since $H_2(\tilde{W}_1)$ is projective over $\Lambda = \mathbb{Z}[J]$, $H_2(\tilde{W}_1) \otimes \mathbb{Q}$ will be projective over $\mathbb{Q}[J]$. But $\mathbb{Q}[J]$ is a principal ideal domain, so $H_2(\tilde{W}_1) \otimes \mathbb{Q}$ is a free $\mathbb{Q}[J]$ -module.

The third step of the proof consists of showing that $H_2(\tilde{W}_1) \otimes \mathbb{Q}$ is a finitely generated free $\mathbb{Q}[J]$ -module. Let $\psi: H_2(W_1) \rightarrow H_2(W_1) \otimes \mathbb{Q}$ be defined by $\psi(x) = x \otimes 1$ and define $\tilde{\psi}: H_2(\tilde{W}_1) \rightarrow H_2(\tilde{W}_1) \otimes \mathbb{Q}$ in a similar way. We then have the following commutative diagram:

$$\begin{array}{ccc} H_2(\tilde{W}_1) & \xrightarrow{\tilde{\psi}} & H_2(\tilde{W}_1) \otimes \mathbb{Q} \\ p_* \downarrow & & \downarrow p_* \otimes \text{id} \\ H_2(W_1) & \xrightarrow{\psi} & H_2(W_1) \otimes \mathbb{Q} \end{array}$$

The short exact sequence

$$0 \rightarrow C_*(\tilde{W}_1) \xrightarrow{t^{-1}} C_*(\tilde{W}_1) \xrightarrow{p_*} C_*(W_1) \rightarrow 0$$

of chain complexes gives rise to the exact sequence

$$0 = H_3(W_1) \rightarrow H_2(\tilde{W}_1) \xrightarrow{t^{-1}} H_2(\tilde{W}_1) \xrightarrow{p_*} H_2(W_1) \rightarrow H_1(\tilde{W}_1) = 0$$

of homology (cf. [16, p. 118]). Thus $p_*: H_2(\tilde{W}_1) \rightarrow H_2(W_1)$ is onto and

$$\ker p_* = (t - 1)(H_2(\tilde{W}_1)).$$

It follows that

$$\ker(p_* \otimes \text{id}) = (t - 1)(H_2(\tilde{W}_1) \otimes \mathbb{Q}).$$

This means that the image of $p_* \otimes \text{id}$ is a free \mathbb{Q} -module with one generator for each generator of $H_2(\tilde{W}_1) \otimes \mathbb{Q}$ as a free $\mathbb{Q}[J]$ -module. Thus $H_2(\tilde{W}_1) \otimes \mathbb{Q}$ is finitely generated as a $\mathbb{Q}[J]$ -module, having one generator corresponding to each element of a basis for $H_2(W_1) \otimes \mathbb{Q}$.

Finally, we show that $H_2(\tilde{W}_1)$ is finitely generated over Λ . Pick an arbitrary $x \in H_2(\tilde{W}_1)$. Let x_1, x_2, \dots, x_n be a collection of elements in $H_2(\tilde{W}_1)$ such that $\tilde{\psi}(x_1), \tilde{\psi}(x_2), \dots, \tilde{\psi}(x_n)$ is a basis for $H_2(\tilde{W}_1) \otimes \mathbb{Q}$. There exist elements q_1, q_2, \dots, q_n in $\mathbb{Q}[J]$ such that

$$\tilde{\psi}(x) = \sum_{i=1}^n q_i \cdot \tilde{\psi}(x_i).$$

Each q_i is a polynomial in t and t^{-1} with rational coefficients. If l is the lowest common denominator of the coefficients of the q_i , then

$$l \cdot \tilde{\psi}(x) = \sum_{i=1}^n \lambda_i \cdot \tilde{\psi}(x_i),$$

where $\lambda_i \in \Lambda$. Since $H_2(\tilde{W}_1)$ is projective, $\tilde{\psi}$ is one-to-one, and so we have that

$$l \cdot x = \sum_{i=1}^n \lambda_i \cdot x_i.$$

Now $H_2(\tilde{W}_1)$ is a direct summand of a free Λ -module $F = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \dots$, where each Λ_i is isomorphic to Λ . For each i there is an n_i such that $x_i \in \Lambda_1 \oplus \dots \oplus \Lambda_{n_i} \oplus 0 \oplus 0 \oplus \dots$. Thus there exists an integer k such that

$$H_2(\tilde{W}_1) \subset \Lambda_1 \oplus \dots \oplus \Lambda_k \oplus 0 \oplus 0 \oplus \dots.$$

The fact that $H_2(\tilde{W}_1)$ is a direct summand of F implies that there is a projection of F to $H_2(\tilde{W}_1)$ and the restriction of this projection takes $\Lambda_1 \oplus \dots \oplus \Lambda_k \oplus 0 \oplus 0 \oplus \dots$ onto $H_2(\tilde{W}_1)$. Therefore $H_2(\tilde{W}_1)$ is finitely generated. \square

Remark 2.6. The proof above could be simplified if we were to make use of the fact that every projective Λ -module is free [1, Theorem 1; 19, Theorem 2.3].

Proposition 2.7. *Suppose W is an open subset of S^4 which satisfies (0.1.1) and (0.1.2) and \tilde{W}_1 is constructed as above. Then W_1 has the homotopy type of a wedge of one 1-sphere and a finite number of 2-spheres.*

Proof. Notice that Lemma 2.5 asserts that W_1 satisfies condition NF2 of [21]. But Λ is Noetherian, so [21, Theorem B] implies that condition F2 is satisfied. Lemmas 2.3 and 2.4 show that W_1 satisfies condition D2 of [21], and so we can apply [21, Proposition 3.3] to conclude that W_1 has the homotopy type of a finite wedge of 1-spheres and 2-spheres. Since $\pi_1(W_1) \cong \mathbb{Z}$, there must be exactly one copy of S^1 in the wedge. \square

Corollary 2.8. *If W is an open subset of S^4 which satisfies (0.1.1) and (0.1.2), then the end of W is a tame end.*

Proof. By Proposition 2.7 and [18, Proposition 4.3] every 0-neighborhood of ε is finitely dominated. Therefore W_1 has a tame end in the sense of Siebenmann. Since \mathbb{Z} is finitely presented, W_1 also has a tame end in the sense of Freedman and Quinn (see [10, Section 11.9A]). But the end of W is the same as that of W_1 , so W has a tame end. \square

3. The $\pi_1 = \mathbb{Z}$ case

In this section we concentrate on the special case in which $\pi_1(W) \cong \mathbb{Z}$. We first prove that W must have the homotopy type of S^1 in this case.

Proof of Theorem 0.5. In case $\pi_1(W) \cong \mathbb{Z}$, we have that $W_1 = W$ and so we can apply Proposition 2.7 to conclude that W has the homotopy type of a wedge of a 1-sphere and a finite number of 2-spheres. But the number of 2-spheres must be 0 because $H_2(W) = 0$ (Corollary 1.3). \square

In our proof of Theorem 0.1 we will actually need a stronger version of Theorem 0.10 than the one stated in the Introduction, so we state the stronger version here before the proof. The stronger theorem merely states explicitly what is actually proved in [11].

Theorem 3.1. *Let W be a connected open subset of S^4 such that*

(3.1.1) $\pi_1(W) \cong \mathbb{Z}$, and

(3.1.2) W has one end ε with $\pi_1(\varepsilon)$ stable and $\pi_1(\varepsilon) \cong \mathbb{Z}$.

Then there is a PL submanifold N of S^4 such that $N \cong S^2 \times B^2$, $S^4 - \text{Int } N \cong S^1 \times B^3$, $S^4 - W \subset \text{Int } N$, and $W \cap N \cong \partial N \times [0, 1)$.

Proof. Theorem 0.5 implies that W has the homotopy type of S^1 and so Theorem 3.1 follows from the proof of [11, Theorem 4.3]. \square

Proof of Theorem 0.10. Let N be as in the conclusion of Theorem 3.1 and let $h: N \rightarrow S^2 \times B^2$ be a homeomorphism. If we take $K = h(S^2 \times \{0\})$, then K is a locally flat 2-sphere in S^4 such that $W \cong S^4 - K$. Furthermore, K is unknotted by [7, Theorem 6]. \square

4. The proof of Theorem 0.1

The proof of Theorem 0.1 uses techniques of Freedman developed in [5–9]. For convenience we refer only to [10] where all these techniques are presented together using consistent notation. According to [10], a *weak collar* of the end of W is a closed manifold neighborhood U of the end of W such that there is a proper map $U \times [0, 1) \rightarrow U$ which is the identity on $U \times \{0\}$.

Lemma 4.1. *If W is a connected open subset of S^4 which satisfies (0.1.1) and (0.1.2), then the end of W has a weak collar U .*

Proof. Let ε denote the end of W . By Corollary 2.8, ε is tame. Furthermore, \mathbb{Z} is a “good” group and $\tilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon)]) = 0$, so [10, Theorem 11.9B] implies that the end of W has a weak collar. \square

Lemma 4.2. *There exists a compact 4-manifold V such that $\partial V = \partial U$, $\pi_1(\partial V) \rightarrow \pi_1(V)$ is onto, and $\pi_1(V) \cong \mathbb{Z}$. Furthermore, V has the homotopy type of a 1-complex.*

Proof. By [10, Theorem 11.9C], $\pi_1(U) \cong \mathbb{Z}$ and $\ker[\pi_1(\partial U) \rightarrow \pi_1(U)]$ is a perfect group. Therefore there exists a Poincaré pair $(V, \partial U)$ with V a homotopy 1-complex

[10, Proposition 11.6C]. We may assume that V is a 4-manifold with $\partial V = \partial U$ by [10, Proposition 11.6A]. \square

Let $\hat{U} = U \cup (S^4 - W)$ and form the manifold $\mathcal{S} = V \cup_a \hat{U}$.

Lemma 4.3. $\mathcal{S} \cong S^4$.

Proof. By Proposition 0.1 and the definition of weak collar, \hat{U} is simply connected. Furthermore, $\pi_1(\partial U) \rightarrow \pi_1(V)$ is onto, so the Van Kampen theorem shows that \mathcal{S} is simply connected. Our conclusion will follow from the 4-dimensional Poincaré conjecture [5, Theorem 1.6] if we can just show that $H_2(\mathcal{S}) = 0$.

Consider the Mayer-Vietoris sequence

$$H_2(\partial U) \xrightarrow{\alpha} H_2(\hat{U}) \oplus H_2(V) \rightarrow H_2(\mathcal{S}) \rightarrow H_1(\partial U) \xrightarrow{\beta} H_1(\hat{U}) \oplus H_1(V).$$

Since $\ker[\pi_1(\partial U) \rightarrow \pi_1(V)]$ is perfect and $\pi_1(V) \cong \mathbb{Z}$, we have that $H_1(\partial U) \cong \mathbb{Z}$. Thus β is an isomorphism. Now $H_2(V) = 0$, so we will be finished if we can show that the inclusion induced map $\alpha' : H_2(\partial U) \rightarrow H_2(\hat{U})$ is onto. Next consider the sequence of the pair $(\hat{U}, \partial U)$.

$$H_2(\partial U) \xrightarrow{\alpha'} H_2(\hat{U}) \rightarrow H_2(\hat{U}, \partial U) \xrightarrow{\gamma} H_1(\partial U) \rightarrow H_1(\hat{U}).$$

By Poincaré and Alexander dualities, $H_2(\hat{U}, \partial U) \cong H^2(\hat{U}) \cong \check{H}^2(S^4 - W) \cong H_1(W) \cong \mathbb{Z}$. Furthermore, $H_1(\partial U) \cong \mathbb{Z}$ and $H_1(\hat{U}) = 0$, so γ is an isomorphism. Therefore α' is onto. \square

Proof of Theorem 0.1. Let $W' = V \cup U \subset \mathcal{S}$. Then $\pi_1(W') \cong \mathbb{Z}$ by the Van Kampen theorem, so we can apply Theorem 3.1 to $W' \subset \mathcal{S}$. There exists a compact manifold $N \subset \mathcal{S}$ such that $N \cong S^2 \times B^2$, $\mathcal{S} - W' \subset \text{Int } N$, and $N \cap W' \cong \partial N \times [0, 1)$. By pushing N along this product structure, we can arrange that $N \subset U$. Therefore $N \subset S^4$ and $N \cap W \cong \partial N \times [0, 1)$. If $h : S^2 \times B^2 \rightarrow N$ is a homeomorphism, then $K = h(S^2 \times \{0\})$ is the 2-sphere we are looking for. \square

We next prove a result which shows that conditions (0.1.1) and (0.1.2) determine the shape of the complement of W in S^4 . The reader is referred to [15] for definitions relating to shape theory.

Proposition 4.4. *Let W be an open subset of S^4 and let $X = S^4 - W$. Then W satisfies (0.1.1) and (0.1.2) if and only if $\text{Sh}(X) = \text{Sh}(S^2)$ and X is globally 1-*alg*.*

Proof. Suppose W satisfies (0.1.1) and (0.1.2). Let \mathcal{S} be as in the proof of Theorem 0.1. Then Theorem 3.1 shows that there exists a compact submanifold N of \mathcal{S} such that $X \subset \text{Int } N$, $N \cong S^2 \times B^2$ and $N - X \cong \partial N \times [0, 1)$. Since $N - X$ is a product, $\text{Sh}(X) = \text{Sh}(N) = \text{Sh}(S^2)$. Since $N - X \cong S^2 \times S^1 \times [0, 1)$, any loop in $N - X$ which is null-homologous in $N - X$ is also null-homotopic in $N - X$. Thus X is globally 1-*alg*.

Now suppose that $\text{Sh}(X) = \text{Sh}(S^2)$ and X is globally 1-*alg*. Then $H_1(W) \cong \check{H}^2(X)$ by Alexander duality, and so $H_1(W) \cong \mathbb{Z}$. Now W is connected since $\check{H}_0(W) \cong \check{H}^3(X) = 0$ and W has only one end since X is connected. Let ε denote the end of W . By [3, Lemma 1] (or by the proof of [11, Theorem 4.3]), $\pi_1(\varepsilon)$ is stable and the natural map $\pi_1(\varepsilon) \rightarrow H_1(W)$ is an isomorphism. Thus $\pi_1(\varepsilon) \cong \mathbb{Z}$. \square

Proof of Theorem 0.4. Theorem 0.4 follows immediately from Theorem 0.1 and Proposition 4.4. \square

Proof of Corollary 0.2. If Σ is globally 1-*alg*, then $S^4 - \Sigma \cong S^4 - K$ for some locally flat 2-sphere $K \subset S^4$ by Theorem 0.1 and Proposition 4.4. Conversely, if $S^4 - \Sigma \cong S^4 - K$ for some locally flat 2-sphere $K \subset S^4$, then the end of $S^4 - \Sigma$ is homeomorphic to the end of $S^4 - K$. Now K has a normal bundle in S^4 by [10, Theorem 9.3A] and this normal bundle is trivial since $H_2(S^4) = 0$. Therefore the end of $S^4 - K$ has a neighborhood homeomorphic to $S^1 \times S^2 \times \mathbb{R}^1$ and so the fundamental group at the end is \mathbb{Z} . Thus Σ is globally 1-*alg* by Proposition 4.4. \square

Proof of Corollary 0.6. Let $W = S^4 - \Sigma$. In order to apply Theorem 0.5, we need to check that π_1 of the end of W is isomorphic to \mathbb{Z} . This follows from Proposition 4.4. \square

Proof of Theorem 0.8. Theorem 0.8 follows from Corollary 0.6 and [11, Theorem 4.3]. \square

References

- [1] H. Bass, Projective modules over free groups are free, *J. Algebra* 1 (1964) 367–373.
- [2] R.J. Daverman, On weakly flat 1-spheres, *Proc. Amer. Math. Soc.* 38 (1973) 207–210.
- [3] R.J. Daverman, Homotopy classification of locally flat codimension two spheres, *Amer. J. Math.* 98 (1976) 367–374.
- [4] R.J. Daverman and T.B. Rushing, Weak flatness criteria for codimension 2 spheres in codimension 1 manifolds, *Gen. Topology Appl.* 6 (1976) 101–115.
- [5] M.H. Freedman, The topology of 4-manifolds, *J. Differential Geom.* 17 (1982) 357–453.
- [6] M.H. Freedman, A surgery sequence in dimension four: the relations with knot concordance, *Invent. math.* 68 (1982) 195–226.
- [7] M.H. Freedman, The disk theorem for four-dimensional manifolds, in: *Proceedings of the International Congress of Mathematicians, Warsaw (1983)* 647–663.
- [8] M.H. Freedman, A new technique for the link slice problem, *Invent. Math.* 80 (1985) 453–465.
- [9] M.H. Freedman, Whitehead₃ is a “slice” link, *Invent. Math.* 94 (1988) 175–182.
- [10] M.H. Freedman and F. Quinn, *Topology of 4-Manifolds* (Princeton Univ. Press, Princeton, NJ, 1990).
- [11] C. Guilbault, An open collar theorem for 4-manifolds, *Trans. Amer. Math. Soc.*, to appear.
- [12] J.G. Hollingsworth and T.B. Rushing, Homotopy characterization of weakly flat codimension two spheres, *Amer. J. Math.* 98 (1976) 385–394.
- [13] V.T. Liem, Homotopy characterization of weakly flat knots, *Fund. Math.* 102 (1979) 61–72.
- [14] V.T. Liem and G.A. Venema, On asphericity of knot complements, in preparation.
- [15] S. Mardešić and J. Segal, *Shape Theory* (North-Holland, Amsterdam, 1982).

- [16] J. Milnor, Infinite cyclic coverings, in: Conference on the Topology of Manifolds (Prindle, Weber & Schmidt, Boston, MA, 1968) 115-133.
- [17] J.R. Munkres, Elements of Algebraic Topology (Addison-Wesley, Reading, MA, 1984).
- [18] L.C. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, Ph.D. Dissertation, Princeton University, Princeton, NJ (1965).
- [19] P.F. Smith, A note on idempotent ideals in group rings, Arch. Math. 27 (1976) 22-27.
- [20] N.E. Steenrod, Homology with local coefficients, Ann. of Math. 44 (1943) 610-627.
- [21] C.T.C. Wall, Finiteness conditions for CW-complexes, Ann. of Math. 81 (1965) 56-69.
- [22] C.T.C. Wall, Finiteness conditions for CW complexes, II, Proc. Roy. Soc. Ser. A 295 (1966) 129-139.