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Remarks on Some Series Expansions Associated with Certain Products of the Incomplete Gamma Functions

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Abstract—In several recent works, some interesting generalizations of the first-order Volterra-type integro-differential equation governing the unsaturated behavior of the free electron laser (FEL) were introduced and investigated by making use of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order). Among other things, it is observed here that an expansion formula for the confluent hypergeometric function in a series of the product of two entire (integral) incomplete Gamma functions does not hold true as asserted and applied in these earlier works. Some necessary corrections and possible remedies are also pointed out.

Keywords—Gamma and incomplete Gamma functions, FEL (free electron laser), Hypergeometric and confluent hypergeometric functions, Fractional calculus, Unilateral and bilateral expansions, Chu-Vandermonde theorem.

1. INTRODUCTION, DEFINITIONS, AND MOTIVATION

In terms of the familiar Gamma function $\Gamma(z)$ $(z \in \mathbb{C} \setminus \mathbb{Z}_0^-)$, the *incomplete Gamma function* $\gamma(z,\alpha)$ and its *complement* $\Gamma(z,\alpha)$ are given by (see, for details, [1, Chapter 9]; see also [2])

$$\gamma(z,\alpha) := \int_0^\alpha t^{z-1} e^{-t}\, dt = \Gamma(z) - \Gamma(z,\alpha)$$
$$(\Re(z) > 0;\ |\arg(\alpha)| \leqq \pi - \epsilon;\ 0 < \epsilon < \pi), \tag{1.1}$$

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$\mathbb{Z}_0^-$ being the set of *nonpositive* integers. For fixed α, $\Gamma(z,\alpha)$ is an *entire* (*integral*) function of z, while $\gamma(z,\alpha)$ is a *meromorphic* function of z with *simple* poles at the points

$$z = 0, -1, -2, \dots .$$

Thus, upon interchanging the rôles of z and α, we can make use of (1.1) in order to define the so-called *entire incomplete Gamma function* $\gamma^*(\alpha,z)$ by

$$z^\alpha \gamma^*(\alpha,z) := \frac{\gamma(\alpha,z)}{\Gamma(\alpha)} = 1 - \frac{\Gamma(\alpha,z)}{\Gamma(\alpha)} \qquad (|\arg(z)| \leqq \pi - \epsilon;\ 0 < \epsilon < \pi). \tag{1.2}$$

Definition (1.1) also yields the following representation:

$$\gamma(\alpha,z) = \alpha^{-1}\, z^\alpha\, {}_1F_1(\alpha;\alpha+1;-z) \tag{1.3}$$

in terms of the confluent hypergeometric ${}_1F_1$ function which corresponds to the special case

$$p = q = 1$$

of the generalized hypergeometric ${}_pF_q$ function (with p numerator *and* q denominator parameters) defined by

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p; & \\ & z \\ \beta_1,\dots,\beta_q; & \end{matrix}\right] &\equiv {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \end{aligned} \tag{1.4}$$

$$\left(p,q \in \mathbb{N}_0 := \{0,1,2,\dots\};\ p \leqq q+1;\ p \leqq q \text{ and } |z| < \infty; \right.$$

$$\left. p = q+1 \text{ and } |z| < 1;\ p = q+1,\ |z| = 1, \text{ and } \Re\left(\sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j\right) > 0 \right),$$

provided also that $\beta_j \notin \mathbb{Z}_0^-$ $(j = 1,\dots,q)$ (see, for details, [3, Chapter 4]); here, *and in what follows*, $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

given (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (\nu = n \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\};\ \lambda \in \mathbb{C}). \end{cases}$$

By means of Kummer's transformation (cf., e.g., [3, p. 253, equation 6.3 (7)]),

$$ {}_1F_1(\alpha;\beta;z) = e^z\, {}_1F_1(\beta-\alpha;\beta;-z), \tag{1.5}$$

the hypergeometric representation (1.3) can be rewritten in its *equivalent* form:

$$\gamma(\alpha, z) = \alpha^{-1} z^{\alpha} e^{-z} {}_1F_1(1; \alpha + 1; z). \tag{1.6}$$

Combining (1.6) with definition (1.2), we immediately obtain the following hypergeometric representation for $\gamma^*(\alpha, z)$:

$$\gamma^*(\alpha, z) = \frac{e^{-z}}{\Gamma(\alpha+1)} {}_1F_1(1; \alpha + 1; z) \tag{1.7}$$

or, equivalently,

$$\gamma^*(\alpha, z) = \frac{1}{\Gamma(\alpha+1)} {}_1F_1(\alpha; \alpha + 1; -z), \tag{1.8}$$

where we have made use of transformation (1.5) once again. Each of the confluent hypergeometric representations (1.7) and (1.8) reiterates the aforementioned fact that $\gamma^*(\alpha, z)$ is an *entire* (*integral*) function of z.

Recently, by employing the (Riemann-Liouville) operator D_z^{μ} of *fractional calculus*, defined by (cf., e.g., [4, p. 181 *et seq.*]; see also [5])

$$D_z^{\mu}\{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-\zeta)^{-\mu-1} f(\zeta)\, d\zeta & (\Re(\mu) < 0), \\[2ex] \dfrac{d^m}{dz^m} D_z^{\mu-m}\{f(z)\} & (m-1 \leqq \Re(\mu) < m;\ m \in \mathbb{N}), \end{cases} \tag{1.9}$$

provided that the integral exists, a number of workers (including, possibly among others, Boyadjiev *et al.* [6], Al-Shammery *et al.* [7,8], and Saxena and Kalla [9]) introduced *and* investigated several generalizations of the first-order Volterra-type integrodifferential equation governing the unsaturated behavior of the free electron laser (cf. [10,11]). In three of the aforecited recent works on *fractional* integrodifferential equations of Volterra type, use is also made of an expansion formula for the confluent hypergeometric ${}_1F_1$ function in a series of the product of two entire incomplete Gamma functions. We recall this *claimed* expansion formula in the following (*slightly modified*) form (cf. [7, p. 504, equation (15); 8, p. 86; 9, p. 93, equation (2.18)]):

$$\begin{aligned} {}_1F_1(a; c; z) = z^{1-a} e^z \Gamma(c) \sum_{n=0}^{\infty} \frac{(c-a)_n}{\Gamma(a-n)} \frac{(-z)^n}{n!} \\ \cdot \gamma^*(1-a+n, z)\, \gamma^*(c-a+n, z), \end{aligned} \tag{1.10}$$

which, in view of Kummer's transformation (1.5), can easily be rewritten in its *equivalent* form:

$$\begin{aligned} {}_1F_1(a; c; z) = (-z)^{1-c+a} \Gamma(c) \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(c-a-n)} \frac{z^n}{n!} \\ \cdot \gamma^*(a+n, -z)\, \gamma^*(1-c+a+n, -z), \end{aligned} \tag{1.11}$$

there being no constraints specified for the parameters a *and* c by the earlier workers.

The left-hand side of each of the expansion formulas (1.10) and (1.11) is indeed a power series in the argument z. However, for *unrestricted* parameters a and c, the right-hand sides of the expansion formulas (1.10) and (1.11) are obviously *not* power series in z, so these expansion formulas do not seem to be correct as asserted *and* applied in the earlier works [7–9]. In the present sequel to these earlier works, we aim at providing the corrected version as well as generalizations of the equivalent expansion formulas (1.10) and (1.11). We also point out how the aforementioned applications (in [6–9]) of such incorrect expansion formulas as (1.10) and (1.11) could possibly be remedied at least in some special situations.

2. DEMONSTRATION BASED UPON FRACTIONAL CALCULUS

First of all, for the Riemann-Liouville fractional differintegral operator D_z^μ, it is readily observed from definition (1.9) that

$$D_z^\mu\left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} z^{\lambda-\mu} \qquad (\lambda\in\mathbb{C}\setminus\mathbb{Z}^-;\ \mathbb{Z}^- := \mathbb{Z}_0^-\setminus\{0\};\ \mu\in\mathbb{C}), \tag{2.1}$$

which leads us to the following general fractional differintegral formula:

$$\begin{aligned} D_z^\mu&\left\{z^{\lambda-1}{}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}\ \omega z\right]\right\}\\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1}{}_{p+1}F_{q+1}\left[\begin{matrix}\lambda,\alpha_1,\dots,\alpha_p;\\ \lambda-\mu,\beta_1,\dots,\beta_q;\end{matrix}\ \omega z\right]\\ &\left(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \mu\in\mathbb{C};\ |\omega z|<\infty \text{ when } p\leqq q;\ |\omega z|<1, \text{ when } p=q+1\right),\end{aligned} \tag{2.2}$$

where (*and throughout this paper*) it is tacitly assumed that all multiple-valued functions take on their *principal* values (and *also* that *exceptional* parameter values, which would render either side of an assertion invalid or undefined, are excluded).

Now, in view of the confluent hypergeometric ${}_1F_1$ representation for $\gamma^*(\alpha, z)$ given by (1.7), we find from a special case of the operational formula (2.2) when

$$p=q=1, \qquad \alpha_1=1, \qquad \beta_1=\rho+1, \qquad \omega=1, \qquad \text{and} \qquad \lambda\longmapsto\lambda+1,$$

that

$$\begin{aligned} D_z^\mu\left\{z^\lambda e^z\gamma^*(\rho,z)\right\} &= \frac{1}{\Gamma(\rho+1)} D_z^\mu\left\{z^\lambda{}_1F_1(1;\rho+1;z)\right\}\\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\rho+1)\Gamma(\lambda-\mu+1)} z^{\lambda-\mu}\\ &\quad\cdot{}_2F_2\left[\begin{matrix}\lambda+1,1;\\ \lambda-\mu+1,\rho+1;\end{matrix}\ z\right]\\ &\left(\rho,\mu\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\mathbb{Z}^-\right),\end{aligned} \tag{2.3}$$

which, for $\rho=\lambda$, gives us the following interesting operational formula:

$$D_z^\mu\left\{z^\lambda e^z\gamma^*(\lambda,z)\right\} = z^{\lambda-\mu}e^z\gamma^*(\lambda-\mu,z) \qquad (\lambda,\mu\in\mathbb{C}). \tag{2.4}$$

In case we make use of the confluent hypergeometric ${}_1F_1$ representation given by (1.8), the operational formula (2.2) *with*

$$p=q=1, \qquad \alpha_1=\rho, \qquad \beta_1=\rho+1, \qquad \omega=-1, \qquad \text{and} \qquad \lambda\longmapsto\lambda+1$$

would yield

$$D_z^{\mu}\left\{z^{\lambda}\gamma^{*}(\rho,z)\right\} = \frac{1}{\Gamma(\rho+1)} D_z^{\mu}\left\{z^{\lambda}{}_1F_1(\rho;\rho+1;-z)\right\}$$
$$= \frac{\Gamma(\lambda+1)}{\Gamma(\rho+1)\Gamma(\lambda-\mu+1)} z^{\lambda-\mu} \cdot {}_2F_2\left[\begin{matrix}\lambda+1,\rho; \\ \lambda-\mu+1,\rho+1;\end{matrix} -z\right] \tag{2.5}$$
$$\left(\rho,\mu\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\mathbb{Z}^{-}\right).$$

Furthermore, since (cf. definition (1.4))

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} =: {}_0F_0(—;—;z), \tag{2.6}$$

a special case of the operational formula (2.2) when

$$p=q=0, \qquad \omega=1, \qquad \text{and} \qquad \lambda\longmapsto\lambda+1$$

immediately yields

$$D_z^{\mu}\left\{z^{\lambda}e^z\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} z^{\lambda-\mu}{}_1F_1(\lambda+1;\lambda-\mu+1;z) \tag{2.7}$$
$$\left(\mu\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\mathbb{Z}^{-}\right),$$

which, for $\lambda=0$, reduces to the elegant form:

$$D_z^{\mu}\{e^z\} = z^{-\mu}e^z\gamma^{*}(-\mu,z) \qquad (\mu\in\mathbb{C}), \tag{2.8}$$

by virtue of (1.7).

Next, we recall the following generalized Leibniz rule for fractional calculus (cf., e.g., [5, p. 317])

$$D_z^{\mu}\{f(z)g(z)\} = \sum_{n=-\infty}^{\infty}\kappa\binom{\mu}{\kappa n+\sigma} D_z^{\mu-\kappa n-\sigma}\{f(z)\}\,D_z^{\kappa n+\sigma}\{g(z)\}, \tag{2.9}$$
$$(\mu,\sigma\in\mathbb{C};\ 0<\kappa\leqq 1),$$

which, for $\sigma=0$ *and* $\kappa=1$, would reduce at once to a (relatively more familiar) *unilateral* form given by

$$D_z^{\mu}\{f(z)g(z)\} = \sum_{n=0}^{\infty}\binom{\mu}{n} D_z^{\mu-n}\{f(z)\}\,D_z^{n}\{g(z)\} \qquad (\mu\in\mathbb{C}), \tag{2.10}$$

D_z^n being the *ordinary* derivative operator of order $n\in\mathbb{N}_0$. By setting

$$f(z)=z^{\nu-\lambda}e^z \qquad \text{and} \qquad g(z)=z^{\lambda}\gamma^{*}(\rho,z)$$

in the general result (2.9), and making use of the operational formulas (2.3), (2.5), and (2.7), we obtain the following *bilateral* expansion:

$${}_2F_2\left[\begin{matrix}\nu+1,1; \\ \nu-\mu+1,\rho+1;\end{matrix} z\right] = \frac{\Gamma(\lambda+1)\Gamma(\nu-\lambda+1)\Gamma(\nu-\mu+1)}{\Gamma(\nu+1)}$$
$$\cdot\sum_{n=-\infty}^{\infty}\kappa\binom{\mu}{\kappa n+\sigma}\left\{\Gamma(\lambda-\kappa n-\sigma+1)\Gamma(\nu-\lambda-\mu+\kappa n+\sigma+1)\right\}^{-1}$$
$$\cdot{}_1F_1\left[\begin{matrix}\nu-\lambda+1; \\ \nu-\lambda-\mu+\kappa n+\sigma+1;\end{matrix} z\right]{}_2F_2\left[\begin{matrix}\lambda+1,\rho; \\ \lambda-\kappa n-\sigma+1,\rho+1;\end{matrix} -z\right] \tag{2.11}$$
$$\left(\mu,\nu,\sigma\in\mathbb{C};\ \nu-\lambda,\lambda,\rho\in\mathbb{C}\setminus\mathbb{Z}^{-};\ 0<\kappa\leqq 1\right),$$

which, in the special case when $\nu = \lambda$, yields

$$
\begin{gathered}
{}_2F_2\left[\begin{matrix}\lambda+1,1;\\ \lambda-\mu+1,\rho+1;\end{matrix} z\right] = e^z \sum_{n=-\infty}^{\infty} \kappa \binom{\mu}{\kappa n+\sigma} \frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda-\kappa n-\sigma+1)} \\
\cdot \gamma^*(\sigma+\kappa n-\mu, z)\, {}_2F_2\left[\begin{matrix}\lambda+1,\rho;\\ \lambda-\kappa n-\sigma+1,\rho+1;\end{matrix} -z\right] \\
\left(\mu,\sigma \in \mathbb{C};\ \lambda,\rho \in \mathbb{C}\setminus\mathbb{Z}^-;\ 0<\kappa \leqq 1\right).
\end{gathered}
\tag{2.12}
$$

Since [12, p. 326, equation 6.5 (13)]

$$
\begin{gathered}
\lim_{\delta\to -n}\left\{\frac{1}{\Gamma(\delta)}\, {}_pF_{q+1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \delta,\beta_1,\ldots,\beta_q;\end{matrix} z\right]\right\} \\
= \frac{\prod_{j=1}^{p} (\alpha_j)_{n+1}}{\prod_{j=1}^{q} (\beta_j)_{n+1}} \frac{z^{n+1}}{(n+1)!} \\
\cdot\, {}_pF_{q+1}\left[\begin{matrix}\alpha_1+n+1,\ldots,\alpha_p+n+1;\\ n+2,\beta_1+n+1,\ldots,\beta_q+n+1;\end{matrix} z\right] \qquad (n \in \mathbb{N}_0),
\end{gathered}
\tag{2.13}
$$

we can easily deduce the following *unilateral* expansions by letting

$$
\kappa = 1, \qquad \sigma = 0, \qquad \text{and} \qquad \lambda \longrightarrow 0
$$

in (2.11) and (2.12):

$$
\begin{gathered}
{}_2F_2\left[\begin{matrix}\nu+1,1;\\ \nu-\mu+1,\rho+1;\end{matrix} z\right] = \Gamma(\rho+1) \sum_{n=0}^{\infty} \binom{\mu}{n} \frac{(\rho)_n}{(\nu-\mu+1)_n} (-z)^n \\
{}_1F_1\left[\begin{matrix}\nu+1;\\ \nu-\mu+n+1;\end{matrix} z\right] \gamma^*(\rho+n, z) \\
\left(\mu,\nu \in \mathbb{C};\ \rho \in \mathbb{C}\setminus\mathbb{Z}^-\right)
\end{gathered}
\tag{2.14}
$$

and

$$
\begin{gathered}
{}_2F_2\left[\begin{matrix}1,1;\\ 1-\mu,\rho+1;\end{matrix} z\right] = \Gamma(\rho+1)\Gamma(1-\mu)\, e^z \sum_{n=0}^{\infty} \binom{\mu}{n} (\rho)_n (-z)^n \\
\cdot \gamma^*(n-\mu, z)\, \gamma^*(\rho+n, z) \\
\left(\mu \in \mathbb{C}\setminus\mathbb{N};\ \rho \in \mathbb{C}\setminus\mathbb{Z}^-\right).
\end{gathered}
\tag{2.15}
$$

This last expansion formula (2.15), which does involve a series of the product of two entire incomplete Gamma functions, would follow also from (2.14) upon setting $\nu = 0$.

3. BILATERAL EXPANSIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

Let

$$
F^{p:r;u}_{q:s;v} \qquad (p,q,r,s,u,v \in \mathbb{N}_0)
$$

denote a general (Kampé de Fériet's) double hypergeometric function defined by (cf., e.g., [12, p. 63, equation 1.7 (16)])

$$F^{p:r;u}_{q:s;v}\left[\begin{matrix}\alpha_1,\dots,\alpha_p:a_1,\dots,a_r;\ c_1,\dots,c_u;\\ \beta_1,\dots,\beta_q:b_1,\dots,b_s;\ d_1,\dots,d_v;\end{matrix}\ x,y\right]$$

$$:=\sum_{l,m=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_{l+m}\prod_{j=1}^{r}(a_j)_l\prod_{j=1}^{u}(c_j)_m}{\prod_{j=1}^{q}(\beta_j)_{l+m}\prod_{j=1}^{s}(b_j)_l\prod_{j=1}^{v}(d_j)_m}\frac{x^l}{l!}\frac{y^m}{m!},\tag{3.1}$$

where, for convergence of the double hypergeometric series,

$$p+r\leqq q+s+1\qquad\text{and}\qquad p+u\leqq q+v+1,$$

with equality only when

$$|x|^{1/(p-q)}+|y|^{1/(p-q)}<1\qquad(p>q)$$
$$\max\{|x|,|y|\}<1\qquad(p\leqq q).$$

From among many general families of bilateral expansions for multivariable functions with essentially arbitrary coefficients (see, for details, [13–16] *and* the references cited in each of these earlier works), we choose to recall here the following *special* case involving the Kampé de Fériet function defined by (3.1) [14, p. 198, equation (66)]:

$$F^{u+1:p;r}_{v+1:q;s}\left[\begin{matrix}\xi+\eta-1,\lambda_1,\dots,\lambda_u:\alpha_1,\dots,\alpha_p;\ a_1,\dots,a_r;\\ \xi+\eta-\mu-1,\mu_1,\dots,\mu_v:\ \beta_1,\dots,\beta_q;\ b_1,\dots,b_s;\end{matrix}\ xz,yz\right]$$
$$=\frac{\kappa\Gamma(\xi)\Gamma(\eta)\Gamma(\xi+\eta-\mu-1)\Gamma(\mu+1)}{\Gamma(\xi+\eta-1)}$$
$$\cdot\sum_{n=-\infty}^{\infty}\{\Gamma(\mu-\nu-\kappa n+1)\Gamma(\nu+\kappa n+1)\Gamma(\xi-\mu+\nu+\kappa n)\Gamma(\eta-\nu-\kappa n)\}^{-1}\tag{3.2}$$
$$\cdot F^{u:p+1;r+1}_{v:q+1;s+1}\left[\begin{matrix}\lambda_1,\dots,\lambda_u:\qquad\xi,\alpha_1,\dots,\alpha_p;\\ \mu_1,\dots,\mu_v:\xi-\mu+\nu+\kappa n,\beta_1,\dots,\beta_q;\end{matrix}\right.$$
$$\left.\begin{matrix}\eta,a_1,\dots,a_r;\\ \eta-\nu-\kappa n,b_1,\dots,b_s;\end{matrix}\ xz,yz\right]$$

$$(\Re(\xi)>0;\ \Re(\eta)>0;\ \Re(\xi+\eta)>1;\ \mu\notin\mathbb{C}\setminus\mathbb{Z}^-;\ 0<\kappa\leqq1),$$

provided that each side exists.

In its *further* special case when

$$u=v=0,\qquad\kappa=1,\qquad\text{and}\qquad\nu=0,$$

the bilateral expansion (3.2) would reduce immediately to the following unilateral expansion in series of the product of two generalized hypergeometric functions:

$$
F^{1:p;r}_{1:q;s}\left[\begin{matrix}\xi+\eta-1:\alpha_1,\dots,\alpha_p;a_1,\dots,a_r;\\ \xi+\eta-\mu-1:\beta_1,\dots,\beta_q;b_1,\dots,b_s;\end{matrix}\, xz, yz\right]
$$

$$
=\frac{\Gamma(\xi)\,\Gamma(\xi+\eta-\mu-1)}{\Gamma(\xi-\mu)\,\Gamma(\xi+\eta-1)}\sum_{n=0}^{\infty}\frac{(-\mu)_n\,(1-\eta)_n}{n!\,(\xi-\mu)_n}
$$

$$
\cdot\,{}_{p+1}F_{q+1}\left[\begin{matrix}\xi,\alpha_1,\dots,\alpha_p;\\ \xi-\mu+n,\beta_1,\dots,\beta_q;\end{matrix}\, xz\right] \tag{3.3}
$$

$$
\cdot\,{}_{r+1}F_{s+1}\left[\begin{matrix}\eta,a_1,\dots,a_r;\\ \eta-n,b_1,\dots,b_s;\end{matrix}\, yz\right]
$$

$$
(\Re(\xi)>0;\ \Re(\eta)>0;\ \Re(\xi+\eta)>1;\ \mu\in\mathbb{C}),
$$

it being assumed (as before) that each side of (3.3) exists.

Now, in view of the limit relationship (2.13), (3.3) when $\eta\longrightarrow 1$ yields

$$
F^{1:p;r}_{1:q;s}\left[\begin{matrix}\xi:\alpha_1,\dots,\alpha_p;a_1,\dots,a_r;\\ \xi-\mu:\beta_1,\dots,\beta_q;b_1,\dots,b_s;\end{matrix}\, xz, yz\right]
$$

$$
=\sum_{n=0}^{\infty}\frac{(-\mu)_n\prod_{j=1}^{r}(a_j)_n}{(\xi-\mu)_n\prod_{j=1}^{s}(b_j)_n}\frac{(-yz)^n}{n!}\,{}_{p+1}F_{q+1}\left[\begin{matrix}\xi,\alpha_1,\dots,\alpha_p;\\ \xi-\mu+n,\beta_1,\dots,\beta_q;\end{matrix}\, xz\right] \tag{3.4}
$$

$$
\cdot\,{}_{r}F_{s}\left[\begin{matrix}a_1+n,\dots,a_r+n;\\ b_1+n,\dots,b_s+n;\end{matrix}\, yz\right]\qquad(\Re(\xi)>0;\ \mu\in\mathbb{C}),
$$

Upon setting

$$
x=-y=1,\quad p=q=0,\quad\text{and}\quad r=s=1\qquad(a_1=\sigma;\ b_1=\rho)
$$

in (3.4), if we apply the Chu-Vandermonde theorem [12, p. 30, equation 1.2 (8)]:

$$
{}_2F_1(-N,b;c;1)=\frac{(c-b)_N}{(c)_N}\qquad\left(N\in\mathbb{N}_0;\ b\in C;\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right) \tag{3.5}
$$

in order to simplify the resulting double hypergeometric series on the left-hand side, we obtain

$$
\begin{aligned}
{}_2F_2\left[\begin{matrix}\xi,\rho-\sigma;\\ \xi-\mu,\rho;\end{matrix}\, z\right]&=\sum_{n=0}^{\infty}\frac{(-\mu)_n\,(\sigma)_n}{(\xi-\mu)_n\,(\rho)_n}\frac{z^n}{n!}\\
\cdot\,{}_1F_1\left[\begin{matrix}\xi;\\ \xi-\mu+n;\end{matrix}\, z\right]&{}_1F_1\left[\begin{matrix}\sigma+n;\\ \rho+n;\end{matrix}\, -z\right]\qquad(\Re(\xi)>0;\ \mu\in\mathbb{C}).
\end{aligned} \tag{3.6}
$$

In view of the confluent hypergeometric ${}_1F_1$ representations (1.7) and (1.8) for the entire incomplete Gamma function $\gamma^*(\alpha,z)$, a special case of (3.6) when

$$
\xi=1,\qquad \rho\longmapsto\rho+1,\qquad\text{and}\qquad\sigma=\rho
$$

would lead us once again to the expansion formula (2.15) in a series of the product of two entire incomplete Gamma functions.

For $\mu = N$ $(N \in \mathbb{N}_0)$, each of the infinite series occurring in (3.3), (3.4), and (3.6) would terminate, since

$$(-N)_n = 0 \qquad (n = N+1, N+2, N+3, \ldots). \tag{3.7}$$

Thus, if we let

$$\mu = N \qquad (N \in \mathbb{N}_0) \qquad \text{and} \qquad \xi \longrightarrow 1$$

in (for example) our expansion formula (3.6) and apply the limit relationship (2.13) once again, we get

$$\begin{aligned} {}_1F_1\left[\begin{matrix}\rho-\sigma+N; \\ \rho+N;\end{matrix} z\right] = \frac{(\rho)_N}{(\rho-\sigma)_N} e^z \sum_{n=0}^{N} (-1)^n \binom{N}{n} \frac{(\sigma)_n}{(\rho)_n} \\ \cdot {}_1F_1\left[\begin{matrix}\sigma+n; \\ \rho+n;\end{matrix} -z\right], \end{aligned} \tag{3.8}$$

which, for

$$\rho \longmapsto \rho+1 \qquad \text{and} \qquad \sigma = \rho,$$

immediately yields the following *special* case:

$${}_1F_1(N+1; \rho+N+1; z) = \frac{\Gamma(\rho+N+1)}{N!} e^z \sum_{n=0}^{N} (-1)^n \binom{N}{n} (\rho)_n \gamma^*(\rho+n, z). \tag{3.9}$$

Since

$$\begin{aligned} \lim_{\mu \to N} \{\gamma^*(n-\mu, z)\} &= \lim_{\mu \to N} \left\{ \frac{e^{-z}}{\Gamma(n-\mu+1)} {}_1F_1\left[\begin{matrix}1; \\ n-\mu+1;\end{matrix} z\right] \right\} \\ &= e^{-z} \lim_{\delta \to -(N-n)} \left\{ \frac{1}{\Gamma(1+\delta)} {}_1F_1\left[\begin{matrix}1; \\ 1+\delta;\end{matrix} z\right] \right\} \\ &= e^{-z} \cdot z^{N-n} e^z \\ &= z^{N-n} \qquad (0 \leqq n \leqq N;\ n, N \in \mathbb{N}_0), \end{aligned}$$

by means of the limit relationship (2.13), this last finite summation formula (3.9) would follow also from the expansion formula (2.15) upon letting $\mu \longrightarrow N$ $(N \in \mathbb{N}_0)$.

4. REMARKS AND OBSERVATIONS

For $\rho = 0$, both (2.14) and (2.15) hold true rather trivially, because the series in each case reduces (when $\rho = 0$) to its first term given by $n = 0$. Since these series reduce to their first term *also* when $\mu = 0$, the only *nontrivial* situations in which the hypergeometric ${}_2F_2$ function in (for example) the expansion formula (2.14) would become a confluent hypergeometric ${}_1F_1$ function occur when either

$$\nu = \rho \qquad \text{or} \qquad \nu = \mu.$$

We thus find from (2.14) that

$$\begin{aligned} {}_1F_1\left[\begin{matrix}1; \\ \rho-\mu+1;\end{matrix} z\right] = \Gamma(\rho+1) \sum_{n=0}^{\infty} \binom{\mu}{n} \frac{(\rho)_n}{(\rho-\mu+1)_n} (-z)^n \\ \cdot {}_1F_1\left[\begin{matrix}\rho+1; \\ \rho-\mu+n+1;\end{matrix} z\right] \gamma^*(\rho+n, z) \\ (\mu \in \mathbb{C};\ \rho \in \mathbb{C} \setminus \mathbb{Z}^-) \end{aligned} \tag{4.1}$$

or, equivalently,

$$\gamma^*(\rho-\mu,z)=e^{-z}\Gamma(\rho+1)\sum_{n=0}^{\infty}\binom{\mu}{n}\frac{(\rho)_n}{\Gamma(\rho-\mu+n+1)}(-z)^n \cdot {}_1F_1\left[\begin{matrix}\rho+1;\\ \rho-\mu+n+1;\end{matrix}\, z\right]\gamma^*(\rho+n,z) \qquad (\mu\in\mathbb{C};\ \rho\in\mathbb{C}\setminus\mathbb{Z}^-) \tag{4.2}$$

and

$${}_1F_1\left[\begin{matrix}\mu+1;\\ \rho+1;\end{matrix}\, z\right]=\Gamma(\rho+1)\sum_{n=0}^{\infty}\binom{\mu}{n}(\rho)_n\frac{(-z)^n}{n!} \cdot {}_1F_1\left[\begin{matrix}\mu+1;\\ n+1;\end{matrix}\, z\right]\gamma^*(\rho+n,z) \qquad (\mu\in\mathbb{C};\ \rho\in\mathbb{C}\setminus\mathbb{Z}^-). \tag{4.3}$$

Of the three expansion formulas (4.1), (4.2), and (4.3), only (4.3) expresses a general confluent hypergeometric ${}_1F_1$ function, but in a series which *also* involves an ${}_1F_1$ function. Consequently, even the expansion formula (4.3) would not serve the purpose in each of the earlier works [7–9], which was to express their ${}_1F_1$ solutions in series of the product of two relatively more familiar entire incomplete Gamma functions.

Our other expansion formula (2.15) does indeed involve a series of the product of two entire incomplete Gamma functions, but the expanded function in (2.15) is an ${}_2F_2$ function (*not* an ${}_1F_1$ function). And, as we pointed out in the preceding section in connection with the finite summation formula (3.9), the ${}_2F_2$ function in (2.15) would reduce to an ${}_1F_1$ function in the limit case when $\mu \longrightarrow N$ $(N\in\mathbb{N}_0)$. The resulting identity is the *special* finite summation formula (3.9) which ought to have been used (if at all applicable) in each of the works [7–9] in place of the obviously erroneous assertion (1.10) or (1.11).

Two *further* particular cases of the finite summation formula (3.9) (*with* $z=i\nu x$) when

$$N=2m-1 \quad\text{and}\quad \rho=m\alpha \qquad (m\in\mathbb{N})$$

and when

$$N=2m \quad\text{and}\quad \rho=(m+1)\alpha \qquad (m\in\mathbb{N}_0)$$

would provide the *corrected* versions of the results asserted *and* used by Boyadjiev *et al.* [6, p. 5, equations (14) and (15)].

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