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On null Lagrangians

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Abstract

We consider multiple-integral variational problems where the Lagrangian function, defined on a frame bundle, is homogeneous. We construct, on the corresponding sphere bundle, a canonical Lagrangian form with the property that it is closed exactly when the Lagrangian is null. We also provide a straightforward characterization of null Lagrangians as sums of determinants of total derivatives. We describe the correspondence between Lagrangians on frame bundles and those on jet bundles: under this correspondence, the canonical Lagrangian form becomes the fundamental Lepage equivalent. We also use this correspondence to show that, for a single-determinant null Lagrangian, the fundamental Lepage equivalent and the Carathéodory form are identical.

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1. Introduction

By a ‘null’ Lagrangian we mean one whose Euler–Lagrange equations vanish identically. Null Lagrangians are important in the context of the study of symmetries of Lagrangian systems [2], Carathéodory’s theory of fields of extremals, and integral invariants [4].

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The subject of null Lagrangians has an interestingly intermittent history. This may be because in the case of Lagrangians of the type that occur in classical dynamics, that is to say, ones which depend on a single independent variable x, a finite number of dependent variables u^α, $\alpha = 1, 2, \ldots, n$, and the (formal) first-order derivatives $\dot{u}^\alpha$ of the u^α with respect to x, the null Lagrangians are well known and easily described: a Lagrangian is null if and only if it is the total derivative of a function f of x and the u^α, so that

$$L(x, u^\alpha, \dot{u}^\alpha) = \frac{df}{dx} = \frac{\partial f}{\partial u^\alpha}\dot{u}^\alpha + \frac{\partial f}{\partial x}$$

(we use the summation convention for repeated indices throughout the paper).

On the other hand, the situation is neither quite so obvious nor so well known in the case of field-theoretic Lagrangians, even those of first order—that is to say, where there are several (but finitely many) independent variables x^i, $i = 1, 2, \ldots, m$, $m \geqslant 2$, and the Lagrangian is a function of these, the dependent variables u^α as before, and their formal first-order derivatives u^α_i. As a result, the field theoretical case has had to be rediscovered from time to time.

The 1983 paper of Hojman [6] is a case in point. This turned out to be seminal, because it led to Betounes's re-discovery of the so-called fundamental Lepage equivalent of a Lagrangian form—of which, more below. However, the whole question of first-order null Lagrangians had already been extensively analysed ten years or more before by both Krupka [7] and Rund [11]. After Hojman and Betounes, and apparently in ignorance of the preceding work, Olver and Sivaloganathan discussed the whole question afresh, and from a somewhat different point of view [10]. Other approaches to the problem, some including explicit formulae for higher-order null Lagrangians, may be found in [5] and the references therein.

Our excuse for revisiting the story is this. The theory of null first-order Lagrangians is remarkably transparent if the Lagrangians in question are assumed to be homogeneous, so that the variational integrals are parameter-independent. We can deduce the rather more complicated results in the jet bundle formalism by choosing special coordinates, called affine coordinates. By approaching the problem in this way, we have been led to a new interpretation of the fundamental Lepage equivalent, and thence to the discovery of a rather remarkable result, namely that for a single determinant null Lagrangian, the Carathéodory form and the fundamental Lepage equivalent are identical.

2. Properties of frame bundles

Suppose given a configuration manifold E with $\dim E = N = m + n$. By a (first-order) m-velocity at a point $u \in E$ we mean the 1-jet at the origin $0 \in \mathbf{R}^m$ of a smooth map ϕ of a neighbourhood of 0 into E with $\phi(0) = u$. The bundle of 1-jets at 0 of smooth maps $\mathbf{R}^m \to E$ is denoted by $T^1_{(m)}E$. By a first-order m-frame we mean the 1-jet of an immersion. The bundle of m-frames over E is denoted by $\mathcal{F}_{(m)}E$. We can also regard a point ξ of $\mathcal{F}_{(m)}E$ as an ordered linearly independent set $\{\xi_i\}$, $i = 1, 2, \ldots, m$, of elements of T_uE, $u \in E$. With this interpretation we see that $\mathcal{F}_{(m)}E$ is an open submanifold of the Whitney sum bundle $\bigoplus^m TE$ of arbitrary (not necessarily linearly independent) ordered m-tuples of tangent vectors.

We shall let $\tau^m : T^1_{(m)}E \to E$ denote the natural projection, and also its restriction to $\mathcal{F}_{(m)}E$. We write the coordinates on both $T^1_{(m)}E$ and $\mathcal{F}_{(m)}E$ as (u^A, u^A_i), where u^A, $A = 1, 2, \ldots, N$, are coordinates on E

and, for any ϕ defining the 1-jet,

$$u_i^A = \frac{\partial \phi^A}{\partial x^i}(0),$$

where the x^i are natural coordinates on $\mathbf{R}^m$. Then $\mathcal{F}_{(m)}E$ is defined by the condition that the matrix (u_i^A) has rank m. The frame corresponding to the point with coordinates (u^A, u_i^A) has for its ith element the vector $u_i^A \partial/\partial u^A$ at the point (u^A).

We shall also need to make use of $T^2_{(m)}E$, the bundle of 2-jets at 0 of maps $\mathbf{R}^m \to E$; we shall consider its restriction to a bundle over $\mathcal{F}_{(m)}E \subset T^1_{(m)}E$, with π the projection. We shall denote the extra fibre coordinates by u_{ij}^A, with the understanding that $u_{ji}^A = u_{ij}^A$ when $i \neq j$.

For each i, $i = 1, 2, \ldots, m$, we define an operator $S^i : T^*T^2_{(m)}E \to T^*T^2_{(m)}E$, linear over $C^\infty(T^2_{(m)}E)$, by

$$S^i(du^A) = 0, \quad S^i(du_j^A) = \delta^i_j\, du^A, \quad S^i(du_{jk}^A) = \delta^i_j\, du_k^A + \delta^i_k\, du_j^A.$$

Note that S^i restricts to a similar operator on $T^*T^1_{(m)}E$, which we denote by the same symbol. We can extend S^i to a derivation of degree 0 of $\bigwedge T^1_{(m)}E$.

We also define a derivation $d_i : T^*T^1_{(m)}E \to T^*T^2_{(m)}E$ by

$$d_i(du^A) = du_i^A, \quad d_i(du_j^A) = du_{ij}^A,$$

and for any $f \in C^\infty(T^1_{(m)}E)$,

$$d_i f = u_i^A \frac{\partial f}{\partial u^A} + u_{ij}^A \frac{\partial f}{\partial u_j^A}.$$

Finally, we define an operator $\varepsilon : T^*T^1_{(m)}E \to T^*T^2_{(m)}E$, the Euler–Lagrange operator, by

$$\varepsilon = \pi^* - d_i \circ S^i.$$

Then for any $\nu \in T^*T^1_{(m)}E$, say $\nu = \nu_A\, du^A + \nu_A^i\, du_i^A$, we have

$$\varepsilon(\nu) = \nu - d_i(\nu_A^i\, du^A) = \bigl(\nu_A - d_i(\nu_A^i)\bigr)\, du^A.$$

We note that $\varepsilon(\nu)$ is semi-basic over E, and that for any function L on $\mathcal{F}_{(m)}E$, $\varepsilon(dL) = 0$ is equivalent to the Euler–Lagrange equations for L.

3. Homogeneous Lagrangians

We consider $GL(m)^+$, the group of $m \times m$ matrices of positive determinant. This group acts on $\mathcal{F}_{(m)}E$ by $(a, \xi) \mapsto a \cdot \xi$ where, if $\xi = \{\xi_i\}$ and $a = (a_i^j)$, $a \cdot \xi = \{a_i^j \xi_j\}$. This action makes $\mathcal{F}_{(m)}E$ into a principal bundle; we denote the base by $\mathcal{S}_{(m)}E$, since it generalizes the sphere bundle of the case $m = 1$. A point of $\mathcal{S}_{(m)}E$ can be regarded as an oriented m-dimensional contact element at a point of E, or an oriented m-dimensional subspace of the tangent space at a point of E; in fact $\mathcal{S}_{(m)}E$ is a double cover of the Grassman m-plane bundle of E. We shall denote the natural projections by $\rho : \mathcal{F}_{(m)}E \to \mathcal{S}_{(m)}E$ and $\tau_+^m : \mathcal{S}_{(m)}E \to E$.

A function L on $\mathcal{F}_{(m)}E$ is said to be homogeneous if it satisfies

$$L(a\cdot\xi) = (\det a)L(\xi)$$

for all $a\in GL(m)^+$, or in coordinates

$$L(u^A, a_i^j u_j^A) = (\det a)L(u^A, u_i^A).$$

Given any function L and any immersion $\sigma:\mathbf{R}^m\to E$ we define the m-form $(\hat\sigma^* L)d^m x$ on $\mathbf{R}^m$, where $\hat\sigma:\mathbf{R}^m\to\mathcal{F}_{(m)}E$ is the 1-jet prolongation of σ. This m-form is to be thought of as the integrand of a variational problem; if L is homogeneous then the variational integral will not depend on the parametrization, provided the orientation is unchanged: that is, if we make an orientation-preserving parameter transformation $y^i = y^i(x)$ we will have

$$L\left(u^A\bigl(y(x)\bigr), \frac{\partial u^A}{\partial y^i}\bigl(y(x)\bigr)\right) d^m y = L\left(u^A(x), \frac{\partial u^A}{\partial x^i}(x)\right) d^m x.$$

Thus homogeneous Lagrangians are those that give rise to parameter-independent variational problems.

If we differentiate the determinantal homogeneity condition, in the coordinate form, partially with respect to a_i^j at the identity of $GL(m)^+$ we obtain

$$u_i^A \frac{\partial L}{\partial u_j^A} = \delta_i^j L.$$

This is in fact equivalent to the determinantal condition for $GL(m)^+$, because the vector fields $\Delta_i^j = u_i^A\partial/\partial u_j^A$ form a local basis (over $\mathbf{R}$) for the space of the fundamental vector fields corresponding to the $GL(m)^+$ action (see also [11]).

We can construct homogeneous Lagrangians on $\mathcal{F}_{(m)}E$ out of m-forms on $\mathcal{S}_{(m)}E$, as follows. A differential form on $\mathcal{S}_{(m)}E$ is semi-basic if it vanishes when contracted with any vector field vertical over τ_+^m. An m-form λ on $\mathcal{S}_{(m)}E$ which is semi-basic over E will be called a Lagrangian form. Now any Lagrangian form λ defines a Lagrangian function L on $\mathcal{F}_{(m)}E$ as follows. Let $\xi\in\mathcal{F}_{(m)}E$, with corresponding frame $\{\xi_i\}$. We may consider $\lambda_{\rho(\xi)}$ to be an element of $\bigwedge^m T^*E$ (rather than of $\bigwedge^m T^*\mathcal{S}_{(m)}E$) because λ is semi-basic. Now define $L(\xi)$ by

$$L(\xi) = \langle \xi_1\wedge\xi_2\wedge\cdots\wedge\xi_m, \lambda_{\rho(\xi)}\rangle = \langle \xi, \lambda_{\rho(\xi)}\rangle,$$

where the angle brackets denote the pairing of an m-vector and an m-form on E. As $\rho(a\cdot\xi) = \rho(\xi)$ for any $a\in GL(m)^+$, it follows immediately that

$$L(a\cdot\xi) = (\det a)L(\xi),$$

so the Lagrangian function defined by this construction is homogeneous.

Conversely, every homogeneous Lagrangian can be derived from a Lagrangian form. One way of doing this for nowhere-vanishing Lagrangians, which we introduced in [3], goes as follows. Given a non-vanishing homogeneous Lagrangian L, let Θ be the decomposable m-form defined by

$$\Theta = L^{-(m-1)}\bigwedge_{i=1}^m \frac{\partial L}{\partial u_i^A}\,du^A.$$

It can be shown that Θ is well-defined as an m-form on $\mathcal{F}_{(m)}E$; it is invariant under the $GL(m)^+$ action, and passes to the quotient to define a Lagrangian form $\widetilde{\Theta}$ on $\mathcal{S}_{(m)}E$; and the Lagrangian function associated with $\widetilde{\Theta}$ is L itself. Proofs of these assertions can be found in [3]. In the case $m = 1$, Θ is the Hilbert 1-form of Finsler geometry. On the other hand, for general m, Θ is closely related to the so-called Carathéodory form, as we shall explain below. We therefore call Θ the Hilbert–Carathéodory form associated with the homogeneous Lagrangian L.

There is another way of constructing a Lagrangian form from a homogeneous Lagrangian, which in the present context is more important; we shall call this the fundamental Lagrangian form, and discuss it in detail below. This second construction may be applied whether or not the Lagrangian vanishes anywhere.

The correspondence between Lagrangian forms and homogeneous Lagrangian functions is many-one. A form on $\mathcal{S}_{(m)}E$ which is pulled back to zero by the prolongation of every immersion defining the bundle is called a contact form. Two Lagrangian forms λ_1, λ_2 define the same Lagrangian function if and only if their difference $\lambda_1 - \lambda_2$ is a contact form.

For calculational purposes it is convenient to proceed as follows. A Lagrangian form can be regarded as a semi-basic m-form on $\mathcal{F}_{(m)}E$ invariant under the $GL(m)^+$ action. Note that, in particular, a basic m-form (that is, an m-form on E pulled back to $\mathcal{F}_{(m)}E$) is invariant. If μ is an invariant semi-basic m-form on $\mathcal{F}_{(m)}E$, the corresponding homogeneous Lagrangian L is given by

$$L = \mu(\Delta_1, \Delta_2, \dots, \Delta_m),$$

where the Δ_i are the locally defined vector fields given by

$$\Delta_i = u_i^A \frac{\partial}{\partial u^A}.$$

It does not matter that the Δ_i are defined only locally, since μ is semi-basic. Note that these local vector fields satisfy $[\Delta_i, \Delta_j] = 0$, a convenient property that we shall make use of later.

4. Null homogeneous Lagrangians

We shall devote the major part of this section to proving that a homogeneous first-order Lagrangian is null if and only if it can be derived from a closed basic Lagrangian form.

Suppose first that L is defined by a closed basic Lagrangian form. We express this by saying that there is a basic m-form μ on $\mathcal{F}_{(m)}E$ such that

$$L = \mu(\Delta_1, \Delta_2, \dots, \Delta_m);$$

we shall show that if μ is closed then L will be null.

From the assumed closure of μ, for any vector field X on $\mathcal{F}_{(m)}E$, we have

$$\begin{aligned}0 = d\mu(X, \Delta_1, \dots, \Delta_m) &= X\big(\mu(\Delta_1, \dots, \Delta_m)\big) + \sum_i (-1)^i \Delta_i\big(\mu(X, \Delta_1, \dots, \widehat{\Delta_i}, \dots, \Delta_m)\big)\\ &\quad + \sum_i (-1)^i \mu\big([X, \Delta_i], \Delta_1, \dots, \widehat{\Delta_i}, \dots, \Delta_m\big)\\ &= X(L) - \sum_i \big(\Delta_i\big(\mu^i(X)\big) - \mu^i\big([\Delta_i, X]\big)\big),\end{aligned}$$

where we have written

$$\mu^i = \mu(\Delta_1, \dots, \widehat{\Delta_i}, \dots, \Delta_m) = \mu^i_A\, du^A$$

say (μ^i is a semi-basic 1-form); we have here used the fact that $[\Delta_i, \Delta_j] = 0$. If we take $X = \partial/\partial u^A_i$ and $X = \partial/\partial u^A$ in turn, we find that

$$\frac{\partial L}{\partial u^i_A} = \mu^i_A; \qquad \frac{\partial L}{\partial u^A} = \Delta_i \mu^i_A.$$

From the first of these, $S^i\, dL = \mu^i$. Now

$$d_i \mu^i_A = \pi^* \Delta_i \mu^i_A + u^B_{ij} \frac{\partial \mu^i_A}{\partial u^B_j}.$$

But

$$\mu^i_A = \mu\left(\Delta_1, \dots, \Delta_{i-1}, \frac{\partial}{\partial u^A}, \Delta_{i+1}, \dots, \Delta_m\right),$$

so that for $i < j$,

$$\begin{aligned}\frac{\partial \mu^i_A}{\partial u^B_j} &= \mu\left(\Delta_1, \dots, \Delta_{i-1}, \frac{\partial}{\partial u^A}, \Delta_{i+1}, \dots, \Delta_{j-1}, \frac{\partial}{\partial u^B}, \Delta_{j+1}, \dots, \Delta_m\right)\\ &= -\mu\left(\Delta_1, \dots, \Delta_{i-1}, \frac{\partial}{\partial u^B}, \Delta_{i+1}, \dots, \Delta_{j-1}, \frac{\partial}{\partial u^A}, \Delta_{j+1}, \dots, \Delta_m\right) = -\frac{\partial \mu^j_A}{\partial u^B_i},\end{aligned}$$

and of course $\partial \mu^i_A / \partial u^B_i = 0$. Thus

$$u^B_{ij} \frac{\partial \mu^i_A}{\partial u^B_j} = 0,$$

whence

$$d_i \mu^i_A = \pi^* \Delta_i \mu^i_A = \pi^* \frac{\partial L}{\partial u^A},$$

and so

$$d_i \mu^i = d_i(\mu^i_A\, du^A) = (d_i \mu^i_A)\, du^A + \mu^i_A\, du^A_i = \pi^*\, dL.$$

It follows that

$$\varepsilon(dL) = \pi^*\, dL - d_i S^i\, dL = \pi^*\, dL - d_i \mu^i = 0,$$

and L is null as asserted.

We now show the converse: our argument is a variant of that given by Rund [11], but unlike him we emphasise the role of Lagrangian forms.

Let L be a null homogeneous first-order Lagrangian function. From the homogeneity condition

$$u^A_i \frac{\partial L}{\partial u^A_j} = \delta^j_i L$$

it follows, by induction and repeated differentiation, that for any $r = 1, 2, \dots$

$$u^{A_1}_{i_1} u^{A_2}_{i_2} \dots u^{A_r}_{i_r} \frac{\partial^r L}{\partial u^{A_1}_{j_1} \partial u^{A_2}_{j_2} \dots \partial u^{A_r}_{j_r}} = \delta^{j_1 j_2 \dots j_r}_{i_1 i_2 \dots i_r} L,$$

where δ is the generalized Kronecker delta (see for example [9]). Now by considering the coefficient of u^A_{ij} in the Euler–Lagrange equations, we see that if L is null,

$$\frac{\partial^2 L}{\partial u^B_i \partial u^A_j} = -\frac{\partial^2 L}{\partial u^A_i \partial u^B_j}.$$

It follows that if we set

$$\mu_{A_1 A_2 \dots A_m} = \frac{\partial^m L}{\partial u^{A_1}_1 \partial u^{A_2}_2 \dots \partial u^{A_m}_m}$$

then $\mu_{A_1 A_2 \dots A_m}$ is completely skew-symmetric in its indices; and on differentiating one more time we find that $\mu_{A_1 A_2 \dots A_m}$ is independent of the u^A_i. Under a coordinate transformation $u^A \mapsto v^A$ on E, the u^A_i transform like contravariant vectors:

$$v^A_i = \frac{\partial v^A}{\partial u^B} u^B_i.$$

It follows that for any $k_1, k_2, \dots, k_r$

$$\frac{\partial^r L}{\partial u^{A_1}_{k_1} \partial u^{A_2}_{k_2} \dots \partial u^{A_r}_{k_r}} = \frac{\partial v^{B_1}}{\partial u^{A_1}} \frac{\partial v^{B_2}}{\partial u^{A_2}} \cdots \frac{\partial v^{B_r}}{\partial u^{A_r}} \frac{\partial^r L}{\partial v^{B_1}_{k_1} \partial v^{B_2}_{k_2} \dots \partial v^{B_r}_{k_r}}.$$

Thus in particular

$$\frac{\partial^m L}{\partial v^{B_1}_1 \partial v^{B_2}_2 \dots \partial v^{B_m}_m} dv^{B_1} \wedge dv^{B_2} \wedge \cdots \wedge dv^{B_m} = \frac{\partial^m L}{\partial u^{A_1}_1 \partial u^{A_2}_2 \dots \partial u^{A_m}_m} du^{A_1} \wedge du^{A_2} \wedge \cdots \wedge du^{A_m},$$

which is to say that the right-hand side (say) is a well-defined semi-basic m-form on $\mathcal{F}_{(m)}E$. We set

$$\mu = \frac{1}{m!} \mu_{A_1 A_2 \dots A_m} du^{A_1} \wedge du^{A_2} \wedge \cdots \wedge du^{A_m}.$$

From the generalization of the homogeneity condition we obtain

$$\mu(\Delta_1, \Delta_2, \dots, \Delta_m) = u^{A_1}_1 u^{A_2}_2 \dots u^{A_m}_m \mu_{A_1 A_2 \dots A_m} = L.$$

If we now consider the remaining terms in the Euler–Lagrange equations we find that for any $A, A_1, A_2, \dots, A_m$

$$\sum_{i=1}^m \frac{\partial \mu_{A_1 \dots A_{i-1} A A_{i+1} \dots A_m}}{\partial u^{A_i}} = \frac{\partial \mu_{A_1 A_2 \dots A_m}}{\partial u^A}.$$

But

$$\begin{aligned} d\mu &= \frac{1}{m+1}\left(\frac{\partial \mu_{A_1A_2\dots A_m}}{\partial u^A}\,du^A\wedge du^{A_1}\wedge du^{A_2}\wedge\cdots\wedge du^{A_m}\right.\\ &\quad\left.+\sum_i \frac{\partial \mu_{A_1\dots A_{i-1}AA_{i+1}\dots A_m}}{\partial u^{A_i}}\,du^{A_i}\wedge du^{A_1}\wedge\cdots\wedge du^{A}\wedge\cdots\wedge du^{A_m}\right)\\ &= \frac{1}{m+1}\left(\frac{\partial \mu_{A_1A_2\dots A_m}}{\partial u^A}-\sum_i \frac{\partial \mu_{A_1\dots A_{i-1}AA_{i+1}\dots A_m}}{\partial u^{A_i}}\right)du^A\wedge du^{A_1}\wedge\cdots\wedge du^{A_m}=0, \end{aligned}$$

which is to say that μ is closed.

We have stated the result in terms of basic forms μ; but in fact it is only necessary to assume that μ is semi-basic, because a semi-basic form on $\mathcal{F}_{(m)}E$, which is closed, is necessarily basic. This is a consequence of a quite general result: if $\pi : B \to M$ is a bundle and ω is a closed semi-basic form on B then ω is the pull-back of a form on M. We have $V \lrcorner\, \omega = 0$ for any vertical vector field V since ω is semi-basic. Moreover, for any vertical V

$$\mathcal{L}_V\omega = V\lrcorner\, d\omega + d(V\lrcorner\,\omega) = 0;$$

thus ω 'passes to the quotient', i.e., defines a form on M by projection, of which it is the pull-back.

5. The fundamental Lagrangian form

The construction of a Lagrangian form described in the proof of the converse result above can be extended to any homogeneous Lagrangian, not just a null one.

Let L be a homogeneous first-order Lagrangian function. It remains true that for any $r = 1, 2, \dots$

$$u^{A_1}_{i_1}u^{A_2}_{i_2}\dots u^{A_r}_{i_r}\frac{\partial^r L}{\partial u^{A_1}_{j_1}\partial u^{A_2}_{j_2}\dots\partial u^{A_r}_{j_r}} = \delta^{j_1j_2\dots j_r}_{i_1i_2\dots i_r}L.$$

Also, by repeatedly using the commutator

$$\left[\Delta^j_i, \frac{\partial}{\partial u^A_k}\right] = -\delta^k_j\frac{\partial}{\partial u^A_i}$$

we obtain

$$\Delta^j_i\frac{\partial^r L}{\partial u^{A_1}_{k_1}\partial u^{A_2}_{k_2}\dots\partial u^{A_r}_{k_r}} = \delta^j_i\frac{\partial^r L}{\partial u^{A_1}_{k_1}\partial u^{A_2}_{k_2}\dots\partial u^{A_r}_{k_r}} - \sum_{s=1}^r\delta^{k_s}_i\frac{\partial^r L}{\partial u^{A_1}_{k_1}\dots\partial u^{A_s}_{j}\dots\partial u^{A_r}_{k_r}}.$$

As before,

$$\frac{1}{m!}\frac{\partial^m L}{\partial u^{A_1}_1\partial u^{A_2}_2\dots\partial u^{A_m}_m}\,du^{A_1}\wedge du^{A_2}\wedge\cdots\wedge du^{A_m}$$

is a well-defined semi-basic m-form on $\mathcal{F}_{(m)}E$; we denote it by λ. From the expression for Δ^j_i operating on the mth partial derivative we find that

$$\mathcal{L}_{\Delta^j_i}\lambda = 0,$$

so that λ is invariant (and defines a form on $\mathcal{S}_{(m)}E$). Finally, from the expression involving the generalized Kronecker delta, in the case $r = m$, we find as before that

$$\lambda(\Delta_1, \Delta_2, \ldots, \Delta_m) = L.$$

Thus λ is a Lagrangian form for L.

We call this Lagrangian form the fundamental Lagrangian form for the given Lagrangian.

We have shown that if L is null then its fundamental Lagrangian form is closed and basic. On the other hand, we know that if a homogeneous Lagrangian L admits a closed Lagrangian form μ then L must be null and μ must be basic. It is then easy to see, by repeatedly differentiating the equation $\mu(\Delta_1, \Delta_2, \ldots, \Delta_m) = L$ with respect to $u_i^{A_i}$, $i = 1, 2, \ldots, m$, that

$$\frac{\partial^m L}{\partial u_1^{A_1} \partial u_2^{A_2} \ldots \partial u_m^{A_m}} = \mu_{A_1 A_2 \ldots A_m},$$

so that $\mu = \lambda$. We therefore conclude that the necessary and sufficient condition for a homogeneous Lagrangian to be null is that its fundamental Lagrangian form is closed (and therefore basic).

We now give a useful representation of the fundamental Lagrangian form. First, it is easy to see that it can be written

$$\lambda = \frac{1}{m!} S^1 \, dS^2 \, d \ldots S^m \, dL.$$

It is useful to observe, in this context, that the operators $S^i \circ d$ and $S^j \circ d$ anti-commute.

Now consider a coordinate patch with coordinates (u^A, u_i^A). In this coordinate patch we shall (for the remainder of this section) use Latin indices $i, j, \ldots$ to represent indices $A, B, \ldots$ taking values in the range $\{1, \ldots, m\}$, and Greek indices $\alpha, \beta, \ldots$ to represent $A - m, B - m, \ldots$, where $A, B, \ldots$ lie in the range $\{m + 1, \ldots, N\}$: with this notation, the coordinates become $(u^j, u^\alpha, u_i^j, u_i^\alpha)$.

Using this notation, we restrict our attention to the open subset of the fibres in which the $m \times m$ matrix (u_i^j) is non-singular. Define functions $\bar{u}_i^j$ to be the entries in the inverse matrix, i.e., $\bar{u}_i^k u_k^j = \delta_i^j$. Then

$$\frac{\partial}{\partial u_i^j} = \bar{u}_j^k \left(\Delta_k^i - u_k^\alpha \frac{\partial}{\partial u_i^\alpha} \right),$$

and using this it turns out that S^i can be written as

$$S^i = \chi^j \otimes \Delta_j^i + \theta^\alpha \otimes \frac{\partial}{\partial u_i^\alpha},$$

where

$$\chi^j = \bar{u}_k^j \, du^k, \qquad \theta^\alpha = du^\alpha - u_j^\alpha \chi^j.$$

We see immediately that for homogeneous L

$$S^i \, dL = L\chi^i + \frac{\partial L}{\partial u_i^\alpha} \theta^\alpha.$$

Furthermore,

$$S^i \, d\chi^j = \chi^i \wedge \chi^j, \qquad S^i \, d\theta^\alpha = \chi^i \wedge \theta^\alpha,$$

and therefore

$$S^i d(f\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}\wedge\chi^{i_1}\wedge\cdots\wedge\chi^{i_s}) = \bigl(S^i df + (r+s)f\chi^i\bigr)\wedge\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_r}\wedge\chi^{i_1}\wedge\cdots\wedge\chi^{i_s}.$$

For any integers $i_1, i_2, \dots, i_p$, with $1\leqslant i_r\leqslant m$, $r=1,2,\dots,p$ (and $1\leqslant p\leqslant m$), set

$$\chi^{i_1 i_2\dots i_p} = \chi^{i_1}\wedge\chi^{i_2}\wedge\cdots\wedge\chi^{i_p};$$

and for any integers $j_1, j_2, \dots, j_q$, with $q\leqslant p$, $1\leqslant j_s\leqslant m$, $s=1,2,\dots,q$, set

$$\chi^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q} = \Delta_{j_q}\lrcorner\,\Delta_{j_{q-1}}\lrcorner\cdots\lrcorner\,\Delta_{j_1}\lrcorner\,\chi^{i_1 i_2\dots i_p}.$$

Note that the Δ_j satisfy $\langle\Delta_j,\chi^i\rangle = \delta^i_j$. Furthermore, $\chi^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q}$ is skew-symmetric in both sets of indices; it is zero unless the i_r are distinct, the j_s are distinct, and the latter comprise a subset of the former, in which case $\chi^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q}$ is a $(p-q)$-form which is, up to sign, the exterior product of 1-forms χ^k indexed by the complement of the j_s in the i_r.

Of course $\chi^i\wedge\chi^{i_1 i_2\dots i_p} = \chi^{i i_1\dots i_p}$, from which it follows that

$$(-1)^q\chi^i\wedge\chi^{i_1\dots i_p}_{j_1\dots j_q} = \chi^{i i_1\dots i_p}_{j_1\dots j_q} + \sum_{s=1}^{q}(-1)^s\delta^i_{j_s}\chi^{i_1\dots i_p}_{j_1\dots\widehat{j_s}\dots j_q},$$

an index to be omitted being indicated in the usual way. Slightly less obviously, we have

$$\chi^j\wedge\chi^{i_1\dots i_p}_{j j_1\dots j_q} = (-1)^q(p-q)\chi^{i_1\dots i_p}_{j_1\dots j_q}$$

(note that this time there is a sum over the repeated index j on the left). To see this, note first that both sides give zero unless the j_s are a subset of the i_r. In the latter case, without loss of generality we can write

$$(i_1, i_2, \dots, i_p) = (j_1, j_2, \dots, j_q, k_1, \dots, k_{p-q})$$

(as ordered sets), whence

$$\chi^{i_1\dots i_p}_{j_1\dots j_q} = \chi^{k_1}\wedge\cdots\wedge\chi^{k_{p-q}},$$

so that

$$\chi^{i_1\dots i_p}_{j j_1\dots j_q} = (-1)^q\Delta_j\lrcorner\,\chi^{k_1}\wedge\cdots\wedge\chi^{k_{p-q}},$$

from which the result follows.

We now show, inductively, that

$$\frac{1}{p!}S^{i_1}dS^{i_2}d\cdots S^{i_p}dL = \sum_{q=0}^{p}\frac{1}{(q!)^2}L^{j_1 j_2\dots j_q}_{\alpha_1\alpha_2\dots\alpha_q}\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q},$$

where for convenience we have written $L^{j_1 j_2\dots j_q}_{\alpha_1\alpha_2\dots\alpha_q}$ for

$$\frac{\partial^q L}{\partial u^{\alpha_1}_{j_1}\partial u^{\alpha_2}_{j_2}\dots\partial u^{\alpha_q}_{j_q}}.$$

We have already seen that this is correct when $p=1$. We now act with $p^{-1}S^{i_1}\circ d$ on

$$\sum_{q=0}^{p-1}\frac{1}{(q!)^2}L^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q},$$

and evaluate the result. We have, using an earlier remark,

$$S^{i_1}d\big(L^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}\big)=\big(S^{i_1}\,dL^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}+(p-1)L^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\chi^{i_1}\big)\wedge\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q},$$

and

$$S^{i_1}\,dL^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}=\Delta^{i_1}_j\big(L^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\big)\chi^j+L^{i_1j_1\dots j_q}_{\alpha\alpha_1\dots\alpha_q}\theta^\alpha=L^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\chi^{i_1}-\sum_{s=1}^{q}L^{j_1\dots i_1\dots j_q}_{\alpha_1\dots\alpha_s\dots\alpha_q}\chi^{j_s}+L^{i_1j_1\dots j_q}_{\alpha\alpha_1\dots\alpha_q}\theta^\alpha.$$

It follows that

$$\begin{aligned}S^{i_1}d\big(L^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}\big)&=(-1)^q pL^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_1}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}\\&\quad-(-1)^q\sum_{s=1}^{q}L^{j_1\dots i_1\dots j_q}_{\alpha_1\dots\alpha_s\dots\alpha_q}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{j_s}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}\\&\quad+L^{i_1j_1\dots j_q}_{\alpha\alpha_1\dots\alpha_q}\theta^\alpha\wedge\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}.\end{aligned}$$

We consider first of all the terms in which no χs occur. These are the terms like the last in the previous equation, for which $q=p-1$, and their contribution to the final sum can be written (remembering to insert the appropriate numerical factors)

$$\frac{1}{p!(p-1)!}L^{i_1j_1\dots j_{p-1}}_{\alpha_1\dots\alpha_p}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_p}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_{p-1}}.$$

The χ here is a 0-form, and in fact is just $\delta^{i_2\dots i_p}_{j_1\dots j_{p-1}}$. Taking into account the symmetries of $L^{i_1\dots i_p}_{\alpha_1\dots\alpha_p}$ we can write this sum as

$$\frac{1}{p!}L^{i_1\dots i_p}_{\alpha_1\dots\alpha_p}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_p},$$

which in turn is equal to

$$\frac{1}{(p!)^2}L^{j_1\dots j_p}_{\alpha_1\dots\alpha_p}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_p}\wedge\chi^{i_1\dots i_p}_{j_1\dots j_p},$$

as required.

For the terms which do involve χs, we have, when we collect together terms with the same number of factors θ,

$$\frac{1}{p}\sum_{q=0}^{p-1}\frac{1}{(q!)^2}\theta^{\alpha_1}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\left((-1)^q pL^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\chi^{i_1}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}-(-1)^q\sum_{s=1}^{q}L^{j_1\dots i_1\dots j_q}_{\alpha_1\dots\alpha_s\dots\alpha_q}\chi^{j_s}\wedge\chi^{i_2\dots i_p}_{j_1\dots j_q}+q^2L^{i_1j_1\dots j_{q-1}}_{\alpha_1\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots j_{q-1}}\right).$$

The first term inside the brackets, by the formula for $\chi^{i_1} \wedge \chi^{i_2\dots i_p}_{j_1\dots j_q}$, is

$$pL^{j_1\dots j_q}_{\alpha_1\dots\alpha_q}\chi^{i_1\dots i_p}_{j_1\dots j_q} + p\sum_{s=1}^{q}(-1)^s L^{j_1\dots i_1\dots j_q}_{\alpha_1\dots\alpha_s\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots\widehat{j_s}\dots j_q}.$$

Note that by using the symmetry of $L^{j_1\dots i_1\dots j_q}_{\alpha_1\dots\alpha_s\dots\alpha_q}$ we can move the index pair (i_1, α_s) to the front; then by relabelling the αs, and taking account of the skew symmetry in the αs coming from the summation over the θs, we can rewrite the sum above as

$$-p\sum_{s=1}^{q} L^{i_1 j_1\dots\widehat{j_s}\dots j_q}_{\alpha_1\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots\widehat{j_s}\dots j_q}.$$

Finally, we can relabel the js in the sum, to obtain

$$-p\sum_{s=1}^{q} L^{i_1 j_1\dots j_{q-1}}_{\alpha_1\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots j_{q-1}} = -pqL^{i_1 j_1\dots j_{q-1}}_{\alpha_1\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots j_{q-1}}.$$

The second term,

$$-(-1)^q\sum_{s=1}^{q} L^{j_1\dots i_1\dots j_q}_{\alpha_1\dots\alpha_s\dots\alpha_q}\chi^{j_s} \wedge \chi^{i_2\dots i_p}_{j_1\dots j_q},$$

can be rewritten, using the formula for the sum $\chi^j \wedge \chi^{i_2\dots i_p}_{jj_1\dots j_q}$, and similar rearrangements of indices, as

$$q(p-q)L^{i_1 j_1\dots j_{q-1}}_{\alpha_1\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots j_{q-1}}.$$

Taking account of the third term,

$$q^2 L^{i_1 j_1\dots j_{q-1}}_{\alpha_1\dots\alpha_q}\chi^{i_2\dots i_p}_{j_1\dots j_{q-1}},$$

we see that the terms involving $L^{i_1 j_1\dots j_{q-1}}_{\alpha_1\dots\alpha_q}$ cancel, and after division by p, and the reintroduction of the term with no χs, we are left with

$$\sum_{q=0}^{p}\frac{1}{(q!)^2}L^{j_1 j_2\dots j_q}_{\alpha_1\alpha_2\dots\alpha_q}\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\cdots\wedge\theta^{\alpha_q}\wedge\chi^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q}.$$

6. Some consequences

If λ is exact, say $\lambda = d\nu$ for some $(m-1)$-form on $\mathcal{F}_{(m)}E$, and we define functions ν^i on $\mathcal{F}_{(m)}E$ by

$$\nu^i = (-1)^{i-1}\nu(\Delta_1, \dots, \widehat{\Delta_i}, \dots, \Delta_m),$$

then

$$L = d\nu(\Delta_1, \dots, \Delta_m) = \Delta_i \nu^i.$$

It is easy to see, using an argument similar to one given previously, that

$$\frac{\partial \nu^j}{\partial u_i^A} = -\frac{\partial \nu^i}{\partial u_j^A},$$

whence

$$d_i \nu^i = \pi^* L,$$

which expresses L as a divergence (i.e., for any $\sigma : \mathbf{R}^m \to E$, $\hat{\sigma}^* L$ really is the divergence of the vector field whose components are $\hat{\sigma}^* \nu^i$).

One way of obtaining a closed basic m-form μ is to take functions $f^1, f^2, \dots, f^m$ on E and set

$$\mu = df^1 \wedge df^2 \wedge \cdots \wedge df^m;$$

then the corresponding null Lagrangian is

$$L = \det(\Delta_i f^j) = \det\left(u_i^A \frac{\partial f^j}{\partial u^A}\right).$$

In this case

$$\frac{\partial L}{\partial u_i^A} du^A = C_j^i \, df^j,$$

where C is the cofactor matrix of the matrix whose determinant is L; and therefore the Hilbert–Carathéodory form Θ is given by

$$\Theta = L^{-(m-1)} \bigwedge_{i=1}^{m} \frac{\partial L}{\partial u_i^A} du^A = \mu,$$

since $\det C = L^{m-1}$.

The Hilbert–Carathéodory form is always a Lagrangian form for its Lagrangian, but will not generally be equal to the fundamental Lagrangian form, even when the Lagrangian is null—it will differ from it by a contact form, and need not itself be closed. However, the immediately preceding argument shows that in the case where L is of the determinant form, $L = \det(\Delta_i f^j)$, the fundamental Lagrangian form and the Hilbert–Carathéodory form are the same.

If μ is a sum of terms of the form $df^1 \wedge df^2 \wedge \cdots \wedge df^m$ then the null Lagrangian L is the sum of the corresponding determinants. Now every closed form may be written (locally) as the sum of exterior products of exact differentials, and so every null homogeneous Lagrangian can be written as the sum of determinants. One way of writing a closed m-form as the sum of exterior products of exact differentials is to write it as the exterior derivative of an $(m-1)$-form: the $(m-1)$-form, when expressed in terms of N coordinate differentials, is the sum of ${}_N C_{m-1}$ terms, and when the exterior derivative is taken each of them gives the exterior product of m exact differentials. Thus every null homogeneous Lagrangian on $\mathcal{F}_{(m)} E$ can be written as the sum of at most ${}_N C_{m-1}$ determinants, where $N = \dim E$.

7. The jet bundle formalism

We can recover more conventional results about null Lagrangians from those obtained above for homogeneous ones by a special choice of coordinates, which we call affine coordinates: we choose coordinates

u^A on E such that $u^i = x^i$, $i = 1, 2, \dots, m$, and we set $u^{m+\alpha} = v^\alpha$, $\alpha = 1, 2, \dots, n$. With this choice we effectively restrict our attention to m-dimensional submanifolds of E which can be represented as graphs with respect to the first m coordinates, that is, in the form $v^\alpha = v^\alpha(x^i)$. The v_i^α can be regarded as coordinates on an open submanifold of each fibre of $\mathcal{S}_{(m)}E$, such that each suitable m-plane is coordinatized by its intersection with the affine n-plane $u_j^i = \delta_j^i$. Furthermore, we can regard E as fibred over an oriented m-dimensional base manifold B, whose coordinates are the x^i and which has a volume form $\omega = d^m x$; the v^α are the fibre coordinates, and the fibre dimension is n. The v_i^α are then the additional coordinates on the bundle $J^1\pi$ of 1-jets of sections of the fibration $\pi : E \to B$.

Let L be a homogeneous Lagrangian on $\mathcal{F}_{(m)}E$, and define $\check{L}$ by

$$\check{L}(x^i, v^\alpha, v_i^\alpha) = L(x^i, v^\alpha, \delta_i^j, v_i^\alpha);$$

then the extremals of $\check{L}$ are the extremals of L which are graphs in the sense described above. Given any function $\check{L}(x^i, v^\alpha, v_i^\alpha)$, one can reconstruct the homogeneous Lagrangian $L(u^A, u_i^A)$, at least locally. Geometric objects defined with respect to L in the homogeneous case, when they are expressed in terms of affine coordinates, take forms familiar from the usual jet bundle formulation of variational calculus for the field-theoretic Lagrangian $\check{L}$. For example, the 1-form χ^i becomes dx^i, the 1-form θ^α becomes the contact 1-form $dv^\alpha - v_i^a\, dx^i$, and the Hilbert–Carathéodory m-form becomes

$$\check{L}^{-(m-1)} \bigwedge_{i=1}^{m} \left(\check{L}\, dx^i + \frac{\partial \check{L}}{\partial v_i^\alpha} (dv^\alpha - v_j^\alpha\, dx^j) \right), \tag{1}$$

which is the Carathéodory form of $\check{L}$. Moreover, on restriction to affine coordinates $\chi^{12\dots m}_{j_1 j_2 \dots j_q}$ becomes $\omega_{j_1 j_2 \dots j_q} = d^{m-q} x_{j_1 j_2 \dots j_q}$.

Thus so far as local considerations and coordinate calculations are concerned, there is a complete equivalence between the jet bundle formalism and the homogeneous formalism.

In terms of affine coordinates we have

$$\Delta_i = \frac{\partial}{\partial x^i} + v_i^\alpha \frac{\partial}{\partial v^\alpha} = \frac{d}{dx^i}.$$

We conclude from our results in the homogeneous case that a Lagrangian L on $J^1\pi$ (we drop the notational distinction between a homogeneous Lagrangian and its jet bundle equivalent now) is null if and only if it can be written $L = \mu(\Delta_1, \Delta_2, \dots, \Delta_m)$, where μ is a closed m-form on E (and Δ_i is the local vector field given above). Thus L will be a polynomial in the jet coordinates v_i^α of order at most $\min(m, n)$. We can also express L as a divergence, $L = \Delta_i \nu^i$, where $\mu = d\nu$ and

$$\nu^i = (-1)^{i-1} \nu(\Delta_1, \dots, \widehat{\Delta_i}, \dots, \Delta_m).$$

For the basic type of null Lagrangian, in which μ is the exterior product of exact 1-forms, we have

$$L = \det\left(\frac{df^i}{dx^j} \right)$$

for functions $f^1, f^2, \dots, f^m$ on E. Any null Lagrangian can be written as a sum of ${}_N C_{m-1}$ terms of this type, where $N = m + n$. We hereby recover, in a more transparent way, the results of Olver and Sivaloganathan [10]. (These authors also use the term 'homogeneous', but with quite a different meaning from ours: for them, a homogeneous Lagrangian is one derived from an m-form μ on E which has constant coefficients when expressed as a linear combination of basis forms $dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dv^{\alpha_1} \wedge \cdots \wedge dv^{\alpha_{m-r}}$. This terminology strikes us as rather odd, since it is clearly coordinate dependent.)

Rund, in [11], also gives a determinantal formula for a null Lagrangian, depending on m arbitrary functions on E; the resulting Lagrangian is a homogeneous polynomial in the v_i^α of order M, where M is any preassigned integer with $M \leqslant \min(m, n)$. His construction works as follows, from our perspective. Let f^i, $i = 1, 2, \ldots, m$, be functions on E. For any set of M distinct integers $i_1, i_2, \ldots, i_M$ with $1 \leqslant i_1 < i_2 < \cdots < i_M \leqslant m$, we construct the m-form

$$dx^1 \wedge \cdots \wedge df^{i_1} \wedge \cdots \wedge df^{i_M} \wedge \cdots \wedge dx^m,$$

where the ith term is df^i if i belongs to the set $\{i_1, i_2, \ldots, i_M\}$, and dx^i otherwise. Now take the sum of all such terms, for all choices of the set $\{i_1, i_2, \ldots, i_M\}$ (for the chosen M). The corresponding null Lagrangian is the one given in Eq. (4.28) on p. 257 of [11].

8. Lepage equivalents

For any Lagrangian L on a jet bundle, an important construction is that of a Lepage equivalent of the m-form $L\,d^m x$. This is a form with the property that all its extremals are holonomic, and that these extremals are the same as those of the Lagrangian: if Φ is a Lepage equivalent then any section $\psi : B \to J^1\pi$ satisfying

$$\delta \int \psi^* \Phi = 0$$

must be a prolongation $\psi = j^1\phi$ for some section $\phi : B \to E$, and then also

$$\delta \int (j^1\phi)^* L\, d^m x = 0;$$

conversely if the latter condition holds then so does the former.

Any Lepage equivalent of $L\,d^m x$ is characterized by the conditions that $L\,d^m x - \Phi$ must be a contact form, and that for any vector field X defined on $J^1\pi$ and vertical over E, the contraction $X \lrcorner\, d\Phi$ must also be a contact form. In the present context, these conditions specify the 0-contact and 1-contact parts (respectively) of Φ, so that in coordinates we must have

$$\Phi = L\, d^m x + \frac{\partial L}{\partial v_i^\alpha} \theta^\alpha \wedge d^{m-1} x_i + \cdots,$$

where the $(m-1)$-form $d^{m-1}x_i$ is the contraction $\partial/\partial x^i \lrcorner\, d^m x$, the 1-form θ^α is the contact form $dv^\alpha - v_j^\alpha\, dx^j$, and the dots indicate terms that are 2-contact or more. These latter terms may be omitted completely, to give a well-defined Cartan form; there are, however, other possibilities, and two of them are particularly relevant to the present discussion. The first is the fundamental Lepage equivalent due to Krupka and Betounes [1,8], represented in coordinates as

$$\Phi = \sum_{q=0}^{\min\{m,n\}} \frac{1}{(q!)^2} \frac{\partial^q L}{\partial v_{i_1}^{\alpha_1} \ldots \partial v_{i_q}^{\alpha_q}} \theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_q} \wedge d^{m-q} x_{i_1 \ldots i_q};$$

this has the important property that it is closed precisely when the Lagrangian L is null [7].

It follows from the alternative representation of the fundamental Lagrangian form given above that the fundamental Lepage equivalent is just the fundamental Lagrangian form restricted to the jet bundle.

The other relevant Lepage equivalent is the Carathéodory form; this is a decomposable m-form defined for a non-vanishing Lagrangian and represented in coordinates as

$$\Theta = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \left(L\,dx^i + \frac{\partial L}{\partial v_i^\alpha} \theta^\alpha \right).$$

As we discussed above, in [3] we have described an invariant construction for a related m-form, the Hilbert–Carathéodory form, in the homogeneous situation, and shown how this projects to the Carathéodory form. Of course we should not in general expect that the Carathéodory form would have properties related specifically to null Lagrangians; it follows, however, from our results in the homogeneous case that when the null Lagrangian consists of a single determinant rather than a linear combination, the Carathéodory form and the fundamental Lepage equivalent are identical.

One further interesting new result follows from our analysis. The Carathéodory form is famous for being invariant under a general (rather than fibred) change of coordinates on E. But now we see that the fundamental Lepage equivalent must also be invariant in this way, because it comes from the frame bundle.

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