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Gaudin models with irregular singularities

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Abstract

We introduce a class of quantum integrable systems generalizing the Gaudin model. The corresponding algebras of quantum Hamiltonians are obtained as quotients of the center of the enveloping algebra of an affine Kac–Moody algebra at the critical level, extending the construction of higher Gaudin Hamiltonians from B. Feigin et al. (1994) [17] to the case of non-highest weight representations of affine algebras. We show that these algebras are isomorphic to algebras of functions on the spaces of opers on $\mathbb{P}^1$ with regular as well as irregular singularities at finitely many points. We construct eigenvectors of these Hamiltonians, using Wakimoto modules of critical level, and show that their spectra on finite-dimensional representations are given by opers with trivial monodromy. We also comment on the connection between the generalized Gaudin models and the geometric Langlands correspondence with ramification.

Keywords: Gaudin model; Irregular singularity; Kac–Moody algebra; Oper; Bethe Ansatz

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Contents

1. Introduction

Quantum integrable systems associated to simple Lie algebras are sometimes best understood in terms of the corresponding affine Kac–Moody algebras. A case in point is the Gaudin model [29]. Let us recall the setup (see, e.g., [17,22]): for each simple Lie algebra $\mathfrak{g}$ there is a collection of commuting quadratic Gaudin Hamiltonians Ξ_i, $i = 1, \dots, N$, in $U(\mathfrak{g})^{\otimes N}$, defined for any set of N distinct complex numbers $z_1, \dots, z_N$. A natural question is to find a maximal commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$ containing Ξ_i, $i = 1, \dots, N$. It turns out that such a subalgebra may be constructed with the help of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, which is the universal central extension of the formal loop algebra $\mathfrak{g}((t))$. The completed universal enveloping algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ of the affine Kac–Moody algebra at the *critical level* contains a large center [16,20]. It was shown in [17] (see also [22]) that the sought-after commutative algebra, called the *Gaudin algebra* in [22], may be obtained as a quotient of the center of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$, using the spaces of conformal blocks of $\widehat{\mathfrak{g}}$-modules. In particular, the quadratic Segal–Sugawara central elements of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ give rise to the quadratic Gaudin Hamiltonians, whereas higher order central elements of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ give rise to higher order generalized Gaudin Hamiltonians.

The center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ has been described in [16,20], where it was shown that $Z(\widehat{\mathfrak{g}})$ is isomorphic to the algebra of functions on the space of ${}^L G$-opers on the punctured disc (for an introduction to this subject, see [25]). Here ${}^L G$ is the group of inner automorphisms of the Langlands dual Lie algebra of ${}^L\mathfrak{g}$ (whose Cartan matrix is the transpose of the Cartan matrix of $\mathfrak{g}$). The notion of opers has been introduced by A. Beilinson and V. Drinfeld in [7] (following an earlier work [10]). Roughly speaking, a ${}^L G$-oper on X is a principal ${}^L G$-bundle on the punctured disc (or a smooth complex curve), equipped with a connection and a reduction to a Borel subgroup of ${}^L G$, satisfying a certain transversality condition with respect to the connection (we recall the precise definition in Section 4 below).

This description of the center was used in [22] to show that the Gaudin algebra is in fact isomorphic to the algebra of functions on the space of ${}^L G$-opers on $\mathbb{P}^1$ with regular singularities at the points $z_1, \dots, z_N$ and ∞. This implies that the joint eigenvalues of the Gaudin algebra on any module $M_1 \otimes \cdots \otimes M_N$ over $U(\mathfrak{g})^{\otimes N}$ are encoded by opers on $\mathbb{P}^1$ with regular singularities.

1.1. Non-highest weight representations and irregular singularities

In this paper we pursue further the connection between affine Kac–Moody algebras and quantum integrable systems. Recall from [17,22] that the action of the Gaudin algebra on the tensor product $M_1 \otimes \cdots \otimes M_N$ of $\mathfrak{g}$-modules comes about via its realization as the *space of coinvariants* of the tensor product of the induced modules $\mathbb{M}_1, \dots, \mathbb{M}_N$ over $\widehat{\mathfrak{g}}_{\kappa_c}$ of critical level, where we use the notation $\mathbb{M} = \operatorname{Ind}_{\mathfrak{g}[\![t]\!] \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c}} M$.

The notion of the space of coinvariants (which is the dual space to the so-called space of conformal blocks) comes from conformal field theory (see, e.g., [26] for a general definition). More precisely, we consider the curve $\mathbb{P}^1$ with the marked points $z_1, \dots, z_N$ and ∞. We attach the $\widehat{\mathfrak{g}}_{\kappa_c}$-modules $\mathbb{M}_1, \dots, \mathbb{M}_N$ to the points $z_1, \dots, z_N$ and another $\widehat{\mathfrak{g}}_{\kappa_c}$-module $\mathbb{M}_\infty = \operatorname{Ind}_{\mathfrak{g} \otimes t\mathbb{C}[\![t]\!] \oplus \mathbb{C}\mathbf{1}} \mathbb{C}$ to the point ∞. The corresponding space of coinvariants is isomorphic to $M_1 \otimes \cdots \otimes M_N$. It is shown in [17,22] that the center $Z(\widehat{\mathfrak{g}})$ acts on this space by functoriality, and its action factors through the Gaudin subalgebra of $U(\mathfrak{g})^{\otimes N}$.

In this construction the $\widehat{\mathfrak{g}}_{\kappa_c}$-modules $\mathbb{M}_i$ satisfy an important property: they are *highest weight modules* (provided that the M_i's are highest weight $\mathfrak{g}$-modules; in general, $\mathbb{M}_i$ is generated by vectors annihilated by the Lie subalgebra $\mathfrak{g} \otimes t\mathbb{C}[\![t]\!] \subset \widehat{\mathfrak{g}}_{\kappa_c}$). The representation theory of affine Kac–Moody algebras has up to now almost exclusively been concerned with highest weight representations, such as Verma modules, Weyl modules and their irreducible quotients.

But these are by far not the most general representations of $\widehat{\mathfrak{g}}_{\kappa_c}$. It is more meaningful to consider a larger category of all *smooth* representations. Those are generated by vectors annihilated by the Lie subalgebra $\mathfrak{g} \otimes t^N\mathbb{C}[\![t]\!]$ for some $N \in \mathbb{Z}_+$.

The main question that we address in this paper is the following:

What quantum integrable systems correspond to non-highest weight representations?

It is instructive to consider the following analogy: highest weight representations are like differential equations with the mildest possible singularities, namely, regular singularities, while non-highest weight representations are like equations with irregular singularities. Actually, this is much more than an analogy. The point is that the action of the center $Z(\widehat{\mathfrak{g}})$ on the spaces of coinvariants corresponding to non-highest weight representations of $\widehat{\mathfrak{g}}_{\kappa_c}$ factors through the algebra of functions on the space of *opers on* $\mathbb{P}^1$ *with irregular singularities at* $z_1, \dots, z_N$ *and* ∞. The order of the pole at each point is determined by the "depth" of the corresponding $\widehat{\mathfrak{g}}_{\kappa_c}$-module: if it is generated by vectors annihilated by the Lie subalgebra $\mathfrak{g} \otimes t^N\mathbb{C}[\![t]\!]$, then the order of the pole at this point is less than or equal to N. This may be summarized by the following diagram:

$$\boxed{\begin{array}{c}\text{highest weight}\\ \text{representations}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{opers with regular}\\ \text{singularities}\end{array}}$$

$$\boxed{\begin{array}{c}\text{non-highest weight}\\ \text{representations}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{opers with irregular}\\ \text{singularities}\end{array}}$$

Motivated by this picture, we call the corresponding quantum integrable systems the generalized *Gaudin models with irregular singularities*. Their classical limits may be identified with a class of Hitchin integrable systems on the moduli spaces of bundles on $\mathbb{P}^1$ with level structures at the points $z_1, \dots, z_N, \infty$ and with Higgs fields having singularities at those points of orders equal to the orders of the level structures. Classical integrable systems of this type have been considered in [2,6,9,12,37].

Thus, using non-highest weight representations, we obtain more general spaces of coinvariants (and conformal blocks) than those considered previously. At the critical level this yields interesting commutative subalgebras and quantum integrable systems, and away from the critical level, interesting systems of differential equations.

In particular, the Knizhnik–Zamolodchikov (KZ) equations may be obtained as the differential equations on the conformal blocks associated to highest weight representations of $\widehat{\mathfrak{g}}$ away from the critical level (see, e.g., [17]). On the other hand, there is a flat connection constructed by J. Millson and V. Toledano Laredo [36,52], and, independently, C. De Concini (unpublished; a closely related connection was also considered in [18]). Following [4], we will call it the DMT connection. We expect that this connection may also be obtained from a system of differential equations on the conformal blocks, but associated to non-highest weight representations of $\widehat{\mathfrak{g}}$, away from the critical level. An indication that this is the case comes from the work of P. Boalch [4], in which the quasi-classical limit of the DMT connection was related to certain isomonodromy equations, and the paper [3] where it was shown that isomonodromy equations of this type often arise as quasi-classical limits of equations on conformal blocks (for example, the so-called Schlesinger isomonodromy equations arise in the quasi-classical limit of the KZ equations, see [43]). We hope that our results on the generalized Gaudin models will help elucidate the relation between the DMT connection and conformal field theory.

1.2. The shift of argument subalgebra and the DMT Hamiltonians

What are the simplest Gaudin models with irregular singularities? They are obtained by allowing regular singularities at all but one point, where we allow a pole of order two. It is convenient to take $\infty \in \mathbb{P}^1$ as this special point. The corresponding Gaudin algebra is then a commutative subalgebra of $U(\mathfrak{g})^{\otimes N} \otimes S(\mathfrak{g})$, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. Here $S(\mathfrak{g})$ arises as the universal enveloping algebra of the commutative Lie algebra $\mathfrak{g} \otimes t\mathbb{C}[\![t]\!]/t^2\mathbb{C}[\![t]\!]$. This algebra naturally acts on the simplest non-highest weight $\widehat{\mathfrak{g}}$-module $\mathrm{Ind}_{\mathfrak{g}\otimes t^2\mathbb{C}[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{C}$ attached to the point ∞ (by endomorphisms commuting with the action of $\widehat{\mathfrak{g}}_{\kappa_c}$). We may then specialize this algebra at a point $\chi \in \mathfrak{g}^* = \operatorname{Spec} S(\mathfrak{g})$. As the result, we obtain a commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$ depending on $\chi \in \mathfrak{g}^*$.

Recently, this algebra has been constructed by L. Rybnikov [45], using the results of [16,20] on the center at the critical level and the method of [17]. Thus, the construction of [45] is essentially equivalent to our construction, applied in this special case.

Consider in particular the case when $N = 1$. Then the corresponding Gaudin algebra, which we denote by $\mathcal{A}_\chi$, is a subalgebra of $U(\mathfrak{g})$. It was shown in [45] that for regular semi-simple χ the algebra $\mathcal{A}_\chi$ is a quantization of the "shift of argument" subalgebra $\overline{\mathcal{A}}_\chi$ of the associated graded algebra $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ (we show below that this is true for all regular χ). The subalgebra $\overline{\mathcal{A}}_\chi$, introduced by A.S. Mishchenko and A.T. Fomenko in [38] (see also [35]), is the Poisson commutative subalgebra of[3] $S(\mathfrak{g}) = \operatorname{Fun} \mathfrak{g}^*$ generated by the derivatives of all orders in the direction of χ of all elements of $\operatorname{Inv} \mathfrak{g}^* = (\operatorname{Fun} \mathfrak{g}^*)^{\mathfrak{g}}$, the algebra of invariants in $\operatorname{Fun} \mathfrak{g}^*$. The algebra $\mathcal{A}_\chi$ is the quantization of $\overline{\mathcal{A}}_\chi$ in the sense that $\operatorname{gr} \mathcal{A}_\chi = \overline{\mathcal{A}}_\chi$.

We note that the problem of quantization of $\overline{\mathcal{A}}_\chi$ was posed by E.B. Vinberg [53]. Such a quantization has been previously constructed for $\mathfrak{g}$ of classical types in [42] (using twisted Yangians) and for $\mathfrak{g} = \mathfrak{sl}_n$ in [50] (using the symmetrization map) and [8] (using explicit formulas).[4]

Our general results on the structure of the Gaudin algebras identify $\mathcal{A}_\chi$ with the algebra of functions on the space $\mathrm{Op}_{^LG}(\mathbb{P}^1)_{\pi(-\chi)}$ of ^LG-opers on $\mathbb{P}^1$ with regular singularity at $0 \in \mathbb{P}^1$ and irregular singularity, of order 2, at $\infty \in \mathbb{P}^1$, with the most singular term $-\chi$ (where χ is any

[3] Here and below, for an affine algebraic variety Z we denote by $\operatorname{Fun} Z$ the algebra of regular functions on Z.

[4] In [46] it was shown that, if it exists, a quantization $\mathcal{A}_\chi$ is unique for generic χ.

regular element of $\mathfrak{g}^*$). This means that a joint generalized eigenvalue of the quantum shift of argument subalgebra $\mathcal{A}_\chi \subset U(\mathfrak{g})$, for regular semi-simple χ, on any $\mathfrak{g}$-module is encoded by a point in $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{\pi(-\chi)}$.

Note that $\overline{\mathcal{A}}_\chi$ is a graded subalgebra of $\mathrm{Fun}\,\mathfrak{g}^*$ with respect to the standard grading, so we have $\overline{\mathcal{A}}_\chi = \bigoplus_{i \geqslant 0} \overline{\mathcal{A}}_{\chi,i}$. It is easy to see that the degree 1 piece $\overline{\mathcal{A}}_{\chi,1}$ is the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (realized as degree one polynomial functions on $\mathfrak{g}^*$) which is the centralizer of χ. The degree two piece $\overline{\mathcal{A}}_{\chi,2}$ was determined in [53]: it is spanned by elements of the form

$$\overline{T}_\gamma(\chi) = \sum_{\alpha \in \Delta_+} \frac{(\alpha,\gamma)(\alpha,\alpha)}{(\alpha,\chi)} e_\alpha f_\alpha, \quad \gamma \in \mathfrak{h}^*,$$

where $\alpha \in \Delta_+$ is the set of positive roots of $\mathfrak{g}$ with respect to a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$, and $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ are generators of an $\mathfrak{sl}_2$ triple corresponding to α (we also use in this formula an inner product on $\mathfrak{h}^*$ corresponding to a non-degenerate invariant inner product on $\mathfrak{g}$). Since we could not find the proof of this result in the literature, we show that the elements $\overline{T}_\gamma(\chi)$ indeed belong to $\overline{\mathcal{A}}_\chi$ in Section 3.7.

Let now $T_\gamma(\chi)$ be the element of $U(\mathfrak{g})$ given by the formula

$$T_\gamma(\chi) = \frac{1}{2} \sum_{\alpha \in \Delta_+} \frac{(\alpha,\gamma)(\alpha,\alpha)}{(\alpha,\chi)} (e_\alpha f_\alpha + f_\alpha e_\alpha), \quad \gamma \in \mathfrak{h}^*, \tag{1.1}$$

so that the symbol of $T_\gamma(\chi)$ is equal to $\overline{T}_\gamma(\chi)$. These operators coincide with the connection operators of the DMT flat connection on $\mathfrak{h}_{\mathrm{reg}} \simeq \mathfrak{h}^*_{\mathrm{reg}}$ mentioned above. The flatness of this connection implies that these operators commute in $U(\mathfrak{g})$. Thus, we obtain that the quadratic part of the Poisson commutative algebra $\overline{\mathcal{A}}_\chi$ may be quantized by the DMT operators. It then follows from the results of [51] that $\mathcal{A}_\chi$ is a maximal commutative subalgebra of $U(\mathfrak{g})$ containing $\mathfrak{h}$ and the operators $T_\gamma(\chi)$.

To summarize, we find that the commutative algebra corresponding to the ordinary Gaudin model, and the quantum shift of argument subalgebra $\mathcal{A}_\chi$, arise as special cases of the general construction of Gaudin models with irregular singularities that we propose in this paper. In particular, the Gaudin operators and the DMT operators arise as the quadratic generators of the corresponding generalized Gaudin algebras.

Note that they both come from flat holomorphic connections, the KZ connection and the DMT connection, respectively. As shown in [52], the KZ and DMT connections are dual to each other in the case of $\mathfrak{gl}_N$. Therefore it is natural to expect that the corresponding Gaudin algebras are also dual in this case (some results in this direction are obtained in [39]). We hope to discuss this question elsewhere.

1.3. Diagonalizing quantum Hamiltonians

The next step is to consider the problem of diagonalization of the generalized Gaudin algebras on various representations.

For example, given a $\mathfrak{g}$-module M, we can try to find joint eigenvectors and eigenvalues of the algebra $\mathcal{A}_\chi$ acting on M, where χ is a regular semi-simple. We know from the above description of the spectrum that each joint eigenvalue of $\mathcal{A}_\chi$ on any $\mathfrak{g}$-module is encoded by a ${}^L G$-oper on $\mathbb{P}^1$ which belongs to $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{\pi(-\chi)}$. However, we would like to know which ${}^L G$-opers may be realized on a given $\mathfrak{g}$-module. For example, we show that if M is generated by a highest weight vector (for example, if M is a Verma module), then the ${}^LG$-opers arising from the joint eigenvalues on M have a fixed residue at the point $0 \in \mathbb{P}^1$ which is determined by the highest weight (this is similar to what happens in the Gaudin model, see [22]).

The most interesting case is when M is an irreducible finite-dimensional $\mathfrak{g}$-module. It was proved in [22] that in the case of the Gaudin model the joint eigenvalues of the Gaudin algebra on the tensor product of finite-dimensional irreducible $\mathfrak{g}$-modules are encoded by the ${}^LG$-opers on $\mathbb{P}^1$ with regular singularities at $z_1, \ldots, z_N$ and ∞, with prescribed residues at those points (determined by the highest weights of our modules), and with *trivial monodromy*. Moreover, it was conjectured in [22] that this sets up a bijection between the joint spectrum of the Gaudin algebra on this tensor product and this space of opers. This conjecture was motivated by the geometric Langlands correspondence (see below).

By applying the same argument in the irregular case, we show that the eigenvalues of $\mathcal{A}_\chi$ (for a regular semi-simple χ) on an irreducible finite-dimensional $\mathfrak{g}$-module V_λ are encoded by ${}^LG$-opers on $\mathbb{P}^1$ with irregular singularity of order 2 with the most singular term $-\chi$ at ∞, with regular singularity with residue λ at 0 and trivial monodromy.

Let $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}$ be the set of such opers. Then we obtain an injective map from the joint spectrum of $\mathcal{A}_\chi$ on V_λ to $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}$. We conjecture that this map is a bijection for any regular semi-simple χ (we also give a multi-point generalization of this conjecture). We expect that for generic χ the algebra $\mathcal{A}_\chi$ is diagonalizable on V_λ and has simple spectrum (this has been proved in [45] for $\mathfrak{g} = \mathfrak{sl}_n$). If this is so, then our conjecture would imply that there exists an eigenbasis of $\mathcal{A}_\chi$ in a $\mathfrak{g}$-module V_λ parameterized by the monodromy-free ${}^LG$-opers on $\mathbb{P}^1$ with prescribed singularities at two points.

In the case of the ordinary Gaudin model, there is a procedure for diagonalization of the Gaudin Hamiltonians called *Bethe Ansatz*. In [17,22] it was shown that this procedure can also be understood in the framework of coinvariants of $\widehat{\mathfrak{g}}$-modules of critical level. We need to use a particular class of $\widehat{\mathfrak{g}}$-modules, called the *Wakimoto modules*.

Let us recall that the Wakimoto modules of critical level are naturally parameterized by objects closely related to opers, which are called *Miura opers* [20]. They may also be described more explicitly as certain connections on a particular ${}^LH$-bundle $\Omega^{-\rho}$ on the punctured disc, where ${}^LH$ is the Cartan subgroup of ${}^LG$. The center acts on the Wakimoto module corresponding to a Cartan connection by the Miura transformation of this connection (see [20]). The idea of [17] was to use the spaces of conformal blocks of the tensor product of the Wakimoto modules to construct eigenvectors of the generalized Gaudin Hamiltonians. It was found in [17] that the eigenvalues of the Gaudin Hamiltonians on these vectors are encoded by the ${}^LG$-opers which are obtained by applying the Miura transformation to certain very simple Cartan connections on $\mathbb{P}^1$.

In this paper we apply the methods of [17,22] to define an analogue of Bethe Ansatz for the Gaudin models with irregular singularities. In particular, we use Wakimoto modules of critical level to construct eigenvectors of the generalized Gaudin algebras (such as the algebra $\mathcal{A}_\chi$) on Verma modules and finite-dimensional irreducible representations of $\mathfrak{g}$.

1.4. Connection to the geometric Langlands correspondence

One of the motivations for studying the generalized Gaudin systems and their connection to opers comes from the geometric Langlands correspondence. Here we give a very rough outline of this connection.

The ramified geometric Langlands correspondence proposed in [27] (see [24] and the last chapter of [25] for an exposition) assigns to holomorphic ${}^LG$-bundles with meromorphic connections on a Riemann surface X, certain categories of Hecke eigensheaves on the moduli stacks of G-bundles on X with level structures[5] at the positions of the poles of the connection of the orders equal to the orders of the poles of the connection (in the case of regular singularities, we may choose instead a parabolic structure, which is a reduction of the fiber of the bundle to a Borel subgroup of G). We note that recently the geometric Langlands correspondence with ramification has been related in [30] to the S-duality of four-dimensional supersymmetric Yang–Mills theory.

Now, the point is that the generalized Gaudin algebra $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ (and its versions with non-trivial characters (χ_i), χ_∞) introduced in this paper gives rise to a commutative algebra of global (twisted) differential operators on moduli stack $\mathrm{Bun}^{(m_i),m_\infty}_{G,(z_i),\infty}$ of G-bundles on $\mathbb{P}^1$ with level structures of orders m_i at the points z_i, $i=1,\dots,N$, and m_∞ at $\infty\in\mathbb{P}^1$. In the case when all $m_i=1$ this has been explained in detail in [19], following the seminal work [7] in the unramified case (which applies to curves of arbitrary genus), and in general the construction is similar.

In this paper we identify the Gaudin algebra $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ with the algebra of functions on the space $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$ of ${}^LG$-opers on $\mathbb{P}^1$ with singularities of orders m_i at the points z_i, $i=1,\dots,N$, and m_∞ at ∞. Thus, each $f\in\mathrm{Fun}\,\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$ gives rise to an element of the Gaudin algebra $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ and hence a differential operator on $\mathrm{Bun}^{(m_i),m_\infty}_{G,(z_i),\infty}$, which we denote by D_f.

For each point $\tau\in\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$ we may now write down the following system of differential equations on $\mathrm{Bun}^{(m_i),m_\infty}_{G,(z_i),\infty}$:

$$D_f\cdot\Psi=f(\tau)\Psi,\qquad f\in\mathrm{Fun}\,\mathrm{Op}_{{}^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty}. \tag{1.2}$$

This system defines a $\mathcal{D}$-module Δ_τ on $\mathrm{Bun}^{(m_i),m_\infty}_{G,(z_i),\infty}$, and by adapting an argument from [7] (which treated the unramified case), we obtain that Δ_τ is a Hecke eigensheaf, whose "eigenvalue" is the ${}^LG$-local system on $\mathbb{P}^1$ underlying the ${}^LG$-oper τ (see [19,23,24] for more details).

Thus, one can construct explicitly examples of Hecke eigensheaves on $\mathrm{Bun}^{(m_i),m_\infty}_{G,(z_i),\infty}$ corresponding to irregular connections by using the generalized Gaudin algebras introduced in this paper. These $\mathcal{D}$-modules provide us with a useful testing ground for the geometric Langlands correspondence.

The philosophy of the geometric Langlands correspondence also gives us insights into the structure of the spectra of the generalized Gaudin algebras on tensor products of finite-dimensional modules. Namely, the existence of such an eigenvector for a particular eigenvalue τ implies that the $\mathcal{D}$-module Δ_τ should be in some sense trivial. Therefore the ${}^LG$-local system underlying the ${}^LG$-oper τ should also be trivial (i.e., monodromy-free). For connections with regular singularities this is explained in [19,22], and in this paper we formulate conjectures to this effect in the case of connections with irregular singularities. This motivates, in particular, our conjecture, already discussed above in Section 1.3, that the spectrum of the Gaudin algebra $\mathcal{A}_\chi$

[5] Here by a level structure on a G-bundle on a curve X of order m at a point $x\in X$ we understand a trivialization of the bundle on the $(m-1)$st infinitesimal neighborhood of x.

on an irreducible finite-dimensional $\mathfrak{g}$-module is described by monodromy-free ${}^LG$-opers on $\mathbb{P}^1$ with singularities at 0 and ∞.

Another useful perspective is provided by the *separation of variables* method pioneered by E. Sklyanin [47]. The connection between the geometric Langlands correspondence and the separation of variables in the Gaudin model is discussed in detail in [19]. The point is that, roughly speaking, the Hecke property of the $\mathcal{D}$-module corresponding to the system (1.2) is reflected in the existence of separated variables for the system (1.2). These separated variables have been discovered by Sklyanin [47] in the case of $\mathfrak{g} = \mathfrak{sl}_2$ and regular singularities. One can approach more general Gaudin models with irregular singularities in a similar way. However, this subject is beyond the scope of the present paper.

1.5. Plan of the paper

The paper is organized as follows. In Section 2 we present a general construction of quantum Hamiltonians using spaces of coinvariants and the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of the vertex algebra $\mathbb{V}_0$ associated to the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level. We define the universal Gaudin algebra as a commutative subalgebra of $U(\mathfrak{g}[\![t]\!])^{\otimes N} \otimes U(t\mathfrak{g}[\![t]\!])$ and its various quotients. This generalizes the construction of the Gaudin algebra from [17,22] which appears as a special case. Another special case is a commutative subalgebra $\mathcal{A}_\chi$ of $U(\mathfrak{g})$. In Section 3 we describe the associated graded algebras of the Gaudin algebras constructed in Section 2 and relate them to the generalized Hitchin systems. In particular, we identify the associated graded algebra of $\mathcal{A}_\chi$ and the shift of argument subalgebra $\overline{\mathcal{A}}_\chi$ of $S(\mathfrak{g}) = \operatorname{Fun}\mathfrak{g}^*$ for regular χ (this has previously been done in [45] for regular semi-simple χ).

Next, we wish to identify the spectra of the Gaudin algebras with the appropriate spaces of ${}^LG$-opers on $\mathbb{P}^1$. We start by collecting in Section 4 various results on opers. Then we recall in Section 5 the isomorphism between $\mathfrak{z}(\widehat{\mathfrak{g}})$ and the algebra of functions on the space of ${}^LG$-opers on the disc from [16,20]. After that we identify in Section 5 the spectra of the universal Gaudin algebra and its quotients with various spaces of opers on $\mathbb{P}^1$ with singularities at finitely many points. In particular, we identify the spectrum of the algebra $\mathcal{A}_\chi$ with the space of ${}^LG$-opers on $\mathbb{P}^1$ with singularities of orders 1 and 2 at two marked points. We show, in the same way as in [22], that joint eigenvalues of the algebra $\mathcal{A}_\chi$ (and its generalizations) on finite-dimensional representations correspond to opers with trivial monodromy. Finally, in Section 6 we study the problem of diagonalization of the generalized Gaudin algebras. Applying the results of [17,22], we develop the Bethe Ansatz construction of eigenvectors of the Gaudin algebras, such as $\mathcal{A}_\chi$ and its multi-point generalizations, on tensor products of Verma modules and finite-dimensional representations of $\mathfrak{g}$.

2. Construction of generalized Gaudin algebras

In this section we introduce the universal Gaudin algebra and its various quotients by using the coinvariants construction for affine Kac–Moody algebras from [17,22].

2.1. Affine Kac–Moody algebra and its modules

Let $\mathfrak{g}$ be a simple Lie algebra. Recall that the space of invariant inner products on $\mathfrak{g}$ is one-dimensional. Choose a non-zero element κ in this space.

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$ is the (universal) extension of the Lie algebra $\mathfrak{g}((t)) = \mathfrak{g} \otimes \mathbb{C}((t))$ by the one-dimensional center $\mathbb{C}\mathbf{1}$:

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0. \tag{2.1}$$

The commutation relations in $\widehat{\mathfrak{g}}_\kappa$ read

$$\big[A \otimes f(t), B \otimes g(t)\big] = [A, B] \otimes fg - \kappa(A, B)\,\mathrm{Res}_{t=0}\, f dg \cdot \mathbf{1}. \tag{2.2}$$

The $\widehat{\mathfrak{g}}_\kappa$-modules on which $\mathbf{1}$ acts as the identity will be referred to as modules of *level* κ.

We introduce the following notation: for a Lie algebra $\mathfrak{l}$, a subalgebra $\mathfrak{m} \subset \mathfrak{l}$ and an $\mathfrak{m}$-module M, we denote by

$$\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{l}} M = U(\mathfrak{l}) \otimes_{U(\mathfrak{m})} M$$

the $\mathfrak{l}$-module induced from M.

Let $\widehat{\mathfrak{g}}_+$ be the Lie subalgebra $\mathfrak{g}[\![t]\!] \oplus \mathbb{C}\mathbf{1}$ of $\widehat{\mathfrak{g}}_\kappa$. Given a $\mathfrak{g}[\![t]\!]$-module M, we extend it to a $\widehat{\mathfrak{g}}_+$-module by making $\mathbf{1}$ act as the identity. Denote by $\mathbb{M}_\kappa$ the corresponding induced $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{M}_\kappa = \mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}_\kappa} M$$

of level κ.

For example, let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ its nilpotent radical and $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$. For any $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts by the character $\lambda : \mathfrak{h} \to \mathbb{C}$ and M_λ be the Verma module over $\mathfrak{g}$ of highest weight λ,

$$M_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda.$$

The corresponding induced $\widehat{\mathfrak{g}}_\kappa$-module $\mathbb{M}_{\lambda,\kappa}$ is the *Verma module* over $\widehat{\mathfrak{g}}_\kappa$ of level κ with highest weight λ.

For a dominant integral weight $\lambda \in \mathfrak{h}^*$, denote by V_λ the irreducible finite-dimensional $\mathfrak{g}$-module of highest weight λ. The corresponding induced module

$$\mathbb{V}_{\lambda,\kappa} = \mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}_\kappa} V_\lambda$$

is called the *Weyl module* of level κ with highest weight λ over $\widehat{\mathfrak{g}}_\kappa$.

A $\widehat{\mathfrak{g}}_\kappa$-module R is called a *highest weight module* if it is generated by a *highest weight vector*, that is a $v \in R$ such that $\widehat{\mathfrak{n}}_+ \cdot v = 0$, where

$$\widehat{\mathfrak{n}}_+ = (\mathfrak{n} \otimes 1) \oplus t\mathfrak{g}[\![t]\!],$$

and $\mathfrak{h} \otimes 1$ acts on v through a linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$. In this case we say that v (or M) has highest weight λ.

For example, both $\mathbb{M}_{\lambda,\kappa}$ and $\mathbb{V}_{\lambda,\kappa}$ (for a dominant integral weight λ) are highest weight modules, with highest weight λ.

Here is an example of a non-highest weight module, which we will use in this paper. Let $\chi : \mathfrak{g} \to \mathbb{C}$ be a linear functional. Observe that the composition

$$t\mathfrak{g}[\![t]\!] \to t\mathfrak{g}[\![t]\!]/t^2\mathfrak{g}[\![t]\!] \simeq \mathfrak{g} \xrightarrow{\chi} \mathbb{C}$$

defines a one-dimensional representation $\mathbb{C}_\chi$ of the Lie algebra $t\mathfrak{g}[\![t]\!]$. We extend it to the direct sum $t\mathfrak{g}[\![t]\!] \oplus \mathbb{C}\mathbf{1}$ by making $\mathbf{1}$ act as the identity. Now set

$$\mathbb{I}_{1,\chi,\kappa} = \operatorname{Ind}_{t\mathfrak{g}[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_\kappa} \mathbb{C}_\chi. \tag{2.3}$$

Note that $\mathbb{I}_{1,\chi,\kappa}$ may also be realized as

$$\operatorname{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}_\kappa} I_{1,\chi}, \quad \text{where} \quad I_{1,\chi} = \operatorname{Ind}_{t\mathfrak{g}[\![t]\!]}^{\mathfrak{g}[\![t]\!]} \mathbb{C}_\chi.$$

2.2. *Spaces of coinvariants*

We define spaces of coinvariants of the tensor products of $\widehat{\mathfrak{g}}_\kappa$-modules, associated to the projective line and a collection of marked points. These spaces and their duals, called spaces of conformal blocks, arise naturally in conformal field theory (for more on this, see, e.g., the book [26]).

Consider the projective line $\mathbb{P}^1$ with a global coordinate t and N distinct finite points $z_1, \ldots, z_N \in \mathbb{P}^1$. In the neighborhood of each point z_i we have the local coordinate $t - z_i$ and in the neighborhood of the point ∞ we have the local coordinate t^{-1}. Set

$$\widetilde{\mathfrak{g}}(z_i) = \mathfrak{g}((t - z_i)), \qquad \widetilde{\mathfrak{g}}(\infty) = \mathfrak{g}\big(\big(t^{-1}\big)\big)$$

and let $\widehat{\mathfrak{g}}_\kappa(z_i), \widehat{\mathfrak{g}}_\kappa(\infty)$ be the corresponding central extensions (2.1) respectively. Let $\widehat{\mathfrak{g}}_N$ be the extension of the Lie algebra $\bigoplus_{i=1}^N \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(\infty)$ by a one-dimensional center $\mathbb{C}\mathbf{1}$ whose restriction to each summand $\widetilde{\mathfrak{g}}(z_i)$ or $\widetilde{\mathfrak{g}}(\infty)$ coincides with $\widehat{\mathfrak{g}}_\kappa(z_i)$ or $\widehat{\mathfrak{g}}_\kappa(\infty)$. Thus, $\widehat{\mathfrak{g}}_N$ is the quotient of $\bigoplus_{i=1}^N \widehat{\mathfrak{g}}_\kappa(z_i) \oplus \widehat{\mathfrak{g}}_\kappa(\infty)$ by the subspace spanned by $\mathbf{1}_{z_i} - \mathbf{1}_{z_j}$, $1 \leqslant i < j \leqslant N$ and $\mathbf{1}_{z_i} - \mathbf{1}_\infty$.

In what follows, we will consider exclusively *smooth* $\mathfrak{g}[\![t]\!]$-modules. By definition, a $\mathfrak{g}[\![t]\!]$-module M is smooth if for any $v \in M$ we have $t^k\mathfrak{g}[\![t]\!]v = 0$ for sufficiently large $k \in \mathbb{Z}_+$. Equivalently, these are the $\mathfrak{g}[\![t]\!]$-modules such that for any $v \in M$ the map $\mathfrak{g} \to M, x \mapsto x \cdot v$ is continuous with respect to the t-adic topology on $\mathfrak{g}[\![t]\!]$ and the discrete topology on M. Note that if M is a smooth finitely generated $\mathfrak{g}[\![t]\!]$-module, the action of $\mathfrak{g}[\![t]\!]$ on M factors through the Lie algebra $\mathfrak{g}[\![t]\!]/t^k\mathfrak{g}[\![t]\!]$ for some $k \in \mathbb{Z}_+$.

Suppose we are given a collection $M_1, \ldots, M_N$ and M_∞ of $\mathfrak{g}[\![t]\!]$-modules. Then the Lie algebra $\widehat{\mathfrak{g}}_N$ naturally acts on the tensor product of the induced $\widehat{\mathfrak{g}}_\kappa$-modules $\bigotimes_{i=1}^N \mathbb{M}_i \otimes \mathbb{M}_\infty$ (in particular, $\mathbf{1}$ acts as the identity).

Let $\mathfrak{g}_{(z_i)} = \mathfrak{g}_{z_1,\ldots,z_N}$ be the Lie algebra of $\mathfrak{g}$-valued regular functions on $\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$ (i.e., rational functions on $\mathbb{P}^1$, which may have poles only at the points $z_1, \ldots, z_N$ and ∞). Clearly, such a function can be expanded into a Laurent power series in the corresponding local coordinates at each point z_i and at ∞. Thus, we obtain an embedding

$$\mathfrak{g}_{(z_i)} \hookrightarrow \bigoplus_{i=1}^N \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(\infty). \tag{2.4}$$

It follows from the residue theorem and formula (2.2) that the restriction of the central extension to the image of this embedding is trivial. Hence (2.4) lifts to an embedding $\mathfrak{g}_{(z_i)} \to \widehat{\mathfrak{g}}_N$.

Denote by $H(\mathbb{M}_{1,\kappa}, \ldots, \mathbb{M}_{N,\kappa}, \mathbb{M}_{\infty,\kappa})$ the space of *coinvariants* of $\bigotimes_{i=1}^N \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa}$ with respect to the action of the Lie algebra $\mathfrak{g}_{(z_i)}$:

$$H(\mathbb{M}_{1,\kappa}, \ldots, \mathbb{M}_{N,\kappa}, \mathbb{M}_{\infty,\kappa}) = \left(\bigotimes_{i=1}^N \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa} \right) \Big/ \mathfrak{g}_{(z_i)}.$$

By construction, we have a canonical embedding of a $\mathfrak{g}[\![t]\!]$-module M into the induced $\widehat{\mathfrak{g}}_\kappa$-module $\mathbb{M}_\kappa$:

$$x \in M \to 1 \otimes x \in \mathbb{M}_\kappa,$$

which commutes with the action of $\mathfrak{g}[\![t]\!]$ on both spaces. Thus, we have an embedding

$$\bigotimes_{i=1}^N M_i \otimes M_\infty \hookrightarrow \bigotimes_{i=1}^N \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa}.$$

The following result gives a description of $H(\mathbb{M}_1, \ldots, \mathbb{M}_N, \mathbb{M}_\infty)$ in terms of coinvariants for the finite-dimensional Lie algebra $\mathfrak{g} \subset \mathfrak{g}_{(z_i)}$. It will allow us to construct quantum Hamiltonians acting on tensor products of $\mathfrak{g}$-modules in Section 2.4 below.

Lemma 2.1. *(See [17, Lemma 1].) The composition of the above embedding and the projection*

$$\bigotimes_{i=1}^N \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa} \twoheadrightarrow H(\mathbb{M}_{1,\kappa}, \ldots, \mathbb{M}_{N,\kappa}, \mathbb{M}_{\infty,\kappa})$$

gives rise to an isomorphism

$$H(\mathbb{M}_{1,\kappa}, \ldots, \mathbb{M}_{N,\kappa}, \mathbb{M}_{\infty,\kappa}) \simeq \left(\bigotimes_{i=1}^N M_{i,\kappa} \otimes M_{\infty,\kappa} \right) \Big/ \mathfrak{g}.$$

Proof. Fix a point $u \in \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$ and let $\mathfrak{g}^0_{(z_i)} \subset \mathfrak{g}_{(z_i)}$ be the ideal of $\mathfrak{g}$-valued functions vanishing at u. Then, $\mathfrak{g}_{(z_i)}/\mathfrak{g}^0_{(z_i)} \cong \mathfrak{g}$, so that

$$\begin{aligned} H(\mathbb{M}_{1,\kappa}, \ldots, \mathbb{M}_{N,\kappa}, \mathbb{M}_{\infty,\kappa}) &= \left(\bigotimes_{i=1}^N \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa} \right) \Big/ \mathfrak{g}_{(z_i)} \\ &= \left(\left(\bigotimes_{i=1}^N \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa} \right) \Big/ \mathfrak{g}^0_{(z_i)} \right) \Big/ \mathfrak{g}. \end{aligned}$$

Since

$$\bigoplus_{i=1}^{N} \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(\infty) = \left(\bigoplus_{i=1}^{N} \mathfrak{g}[\![t - z_i]\!] \oplus \mathfrak{g}[\![t^{-1}]\!] \right) \oplus \mathfrak{g}^0_{(z_i)}$$

the $\widehat{\mathfrak{g}}_N$-module

$$\bigotimes_{i=1}^{N} \mathbb{M}_{i,\kappa} \otimes \mathbb{M}_{\infty,\kappa} = \operatorname{ind}^{\widehat{\mathfrak{g}}_N}_{\bigoplus_{i=1}^{N} \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(\infty) \oplus \mathbb{C}\mathbf{1}} \left(\bigotimes_{i=1}^{N} M_i \otimes M_\infty \right)$$

is freely generated over $U\mathfrak{g}^0_{(z_i)}$ by $\bigotimes_{i=1}^{N} M_i \otimes M_\infty$. The conclusion now follows. $\square$

2.3. The algebra of endomorphisms of $\mathbb{V}_{0,\kappa}$

Let

$$\mathbb{V}_{0,\kappa} = \operatorname{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\widehat{\mathfrak{g}}_+} \mathbb{C}$$

be the *vacuum module* of level κ, that is the Weyl module corresponding to the highest weight 0, and let v_0 be its generating vector.

We will be concerned with the algebra $\operatorname{End}_{\widehat{\mathfrak{g}}_\kappa}(\mathbb{V}_{0,\kappa})$ of endomorphisms of $\mathbb{V}_{0,\kappa}$. Note that we may identify this algebra with the space

$$\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) = \mathbb{V}_{0,\kappa}^{\mathfrak{g}[\![t]\!]}$$

of $\mathfrak{g}[\![t]\!]$-invariant vectors in $\mathbb{V}_{0,\kappa}$. Indeed, a $\mathfrak{g}[\![t]\!]$-invariant vector v gives rise to an endomorphism of $\mathbb{V}_{0,\kappa}$ commuting with the action of $\widehat{\mathfrak{g}}_\kappa$ which sends v_0 to v. Conversely, any $\widehat{\mathfrak{g}}_\kappa$-endomorphism of $\mathbb{V}_{0,\kappa}$ is uniquely determined by the image of v_0, which necessarily belongs to $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$. Thus, we obtain an isomorphism $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) \simeq \operatorname{End}_{\widehat{\mathfrak{g}}_\kappa}(\mathbb{V}_{0,\kappa})$ which gives $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ an algebra structure.

The space $\mathbb{V}_{0,\kappa}$ has the structure of a vertex algebra and it is easy to see that $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ is its center (see, e.g., [26]). This immediately implies the following:

Proposition 2.2. *The algebra $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ is commutative.*

Let $\widehat{\mathfrak{g}}_-$ be the Lie subalgebra $t^{-1}\mathfrak{g}[t^{-1}] \subset \widehat{\mathfrak{g}}_\kappa$. Let us identify $\mathbb{V}_{0,\kappa}$ with $U(\widehat{\mathfrak{g}}_-)$ as $\widehat{\mathfrak{g}}_-$-modules, with $\widehat{\mathfrak{g}}_-$ acting on $U(\widehat{\mathfrak{g}}_-)$ by left multiplication. This yields an embedding

$$\operatorname{End}_{\widehat{\mathfrak{g}}_\kappa}(\mathbb{V}_{0,\kappa}) \subset \operatorname{End}_{\widehat{\mathfrak{g}}_-}(\mathbb{V}_{0,\kappa}) \cong U(\widehat{\mathfrak{g}}_-)^{\mathrm{opp}}, \tag{2.5}$$

where the latter acts on $\mathbb{V}_{0,\kappa} \cong U(\widehat{\mathfrak{g}}_-)$ by right multiplication. Thus, $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ may be realized as a subalgebra of $U(\widehat{\mathfrak{g}}_-)^{\mathrm{opp}}$. This realization will be useful below.

Note that the Lie algebra $\operatorname{Der}(\mathbb{C}[\![t]\!]) = \mathbb{C}[\![t]\!]\partial_t$ of continuous derivations of $\mathbb{C}[\![t]\!]$ acts on $\mathfrak{g}(\!(t)\!)$ and leaves $\mathfrak{g}[\![t]\!]$ invariant. This action lifts to one on $\widehat{\mathfrak{g}}_\kappa$ since the cocycle appearing in (2.2) is invariant under coordinate changes, and induces one on $\mathbb{V}_{0,\kappa}$ which leaves $\mathbb{V}_{0,\kappa}^{\mathfrak{g}[\![t]\!]}$ invariant. Of particular relevance to us will be the translation operator $T = -\partial_t$ which acts on $\mathbb{V}_{0,\kappa}$ by

$$T v_0 = 0, \qquad [T, J_n] = -n J_{n-1}, \tag{2.6}$$

where, for $J \in \mathfrak{g}$ and $n \in \mathbb{Z}$ we denote $J \otimes t^n$ by J_n.

According to Theorem 5.1(1), $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) = \mathbb{C}$, unless κ takes a special value κ_c, called the *critical level*, and so a meaningful theory involving $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ exists only for the critical level. However, since the proofs of the results of the next section do not require it, we will postpone specializing κ to κ_c until Section 2.5.

2.4. Action of $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ on coinvariants

In order to simplify our notation, from now on we will omit the subscript κ in our formulas.

Fix a point $u \in \mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$. Recall that $\mathbb{V}_0$ denotes the vacuum module of $\widehat{\mathfrak{g}}_\kappa(u)$ and v_0 its generating vector. We define below a canonical action of the commutative algebra $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) \cong \mathrm{End}_{\widehat{\mathfrak{g}}_\kappa(u)}(\mathbb{V}_0)$ on spaces of coinvariants. This relies on the following well-known result.

Proposition 2.3. *For any modules $\{\mathbb{M}_i\}_{i=1}^N$ and $\mathbb{M}_\infty$ of $\widehat{\mathfrak{g}}_\kappa(z_i)$, $i = 1, \dots, N$ and $\widehat{\mathfrak{g}}_\kappa(\infty)$ respectively, the map*

$$\bigotimes_{i=1}^N \mathbb{M}_i \otimes \mathbb{M}_\infty \to \bigotimes_{i=1}^N \mathbb{M}_i \otimes \mathbb{M}_\infty \otimes \mathbb{V}_{0,\kappa}, \qquad \bigotimes_{i=1}^N v_i \otimes v_\infty \to \bigotimes_{i=1}^N v_i \otimes v_\infty \otimes v_0$$

induces an isomorphism

$$\jmath_u : H(\mathbb{M}_1, \dots, \mathbb{M}_N, \mathbb{M}_\infty) \to H(\mathbb{M}_1, \dots, \mathbb{M}_N, \mathbb{M}_\infty, \mathbb{V}_0).$$

Proof. Let $\mathfrak{g}_{(z_i,u)}$ be the Lie algebra of $\mathfrak{g}$-valued regular functions on $\mathbb{C} \setminus \{z_1, \dots, z_N, u\}$. Then, $\mathfrak{g}_{(z_i,u)} = \mathfrak{g}_{(z_i)} \oplus (t-u)^{-1}\mathfrak{g}[(t-u)^{-1}]$, as a vector space, so that

$$\begin{aligned} H(M_1, \dots, M_N, M_\infty, \mathbb{V}_0) &= \left(\bigotimes_{i=1}^N \mathbb{M}_i \otimes \mathbb{M}_\infty \otimes \mathbb{V}_0\right)\Big/\mathfrak{g}_{(z_i,u)} \\ &\simeq \left(\bigotimes_{i=1}^N \mathbb{M}_i\right)\Big/\mathfrak{g}_{(z_i)} \otimes \mathbb{V}_0/(t-u)^{-1}\mathfrak{g}\big[(t-u)^{-1}\big] \\ &\cong \left(\bigotimes_{i=1}^N \mathbb{M}_i \otimes \mathbb{M}_\infty\right)\Big/\mathfrak{g}_{(z_i)}, \end{aligned}$$

where the last isomorphism is due to the fact that $\mathbb{V}_0$ is a free module of rank 1 over $U((t-u)^{-1}\mathfrak{g}[(t-u)^{-1}])$. $\square$

Proposition 2.3 yields a canonical action of the algebra $\mathrm{End}_{\widehat{\mathfrak{g}}_\kappa(u)}(\mathbb{V}_0)$ on the space $H(\mathbb{M}_1, \dots, \mathbb{M}_N, \mathbb{M}_\infty)$ given by

$$X \cdot \left[\bigotimes_{i=1}^N v_i \otimes v_\infty\right] = \jmath_u^{-1}\left[\bigotimes_{i=1}^N v_i \otimes v_\infty \otimes X \cdot v_0\right], \tag{2.7}$$

where $[\bigotimes_{i=1}^N v_i \otimes v_\infty]$ denotes the image of

$$\bigotimes_{i=1}^N v_i \otimes v_\infty \in \bigotimes_{i=1}^N \mathbb{M}_i \otimes \mathbb{M}_\infty$$

in $H(\mathbb{M}_1, \dots, \mathbb{M}_\infty)$.

Let us work out this action explicitly. Set

$$\widehat{\mathfrak{g}}(u)_- = (t-u)^{-1}\mathfrak{g}\big[(t-u)^{-1}\big] \subset \widetilde{\mathfrak{g}}(u) \subset \widehat{\mathfrak{g}}_\kappa(u)$$

and identify $\mathbb{V}_0$ with $U(\widehat{\mathfrak{g}}(u)_-)$ as $\widehat{\mathfrak{g}}(u)_-$-modules, where $\widehat{\mathfrak{g}}(u)_-$ acts on $U(\widehat{\mathfrak{g}}(u)_-)$ by left multiplication. As explained in the previous section, this yields an embedding

$$\mathrm{End}_{\widehat{\mathfrak{g}}_\kappa(u)}(\mathbb{V}_0) \subset \mathrm{End}_{\widehat{\mathfrak{g}}(u)_-}(\mathbb{V}_0) \cong U\big(\widehat{\mathfrak{g}}(u)_-\big)^{\mathrm{opp}}, \tag{2.8}$$

where the latter acts on $\mathbb{V}_0 \cong U(\widehat{\mathfrak{g}}(u)_-)$ by right multiplication.

Notice next that the $\widetilde{\mathfrak{g}}(u)$-component of the embedding

$$\mathfrak{g}_{(z_i,u)} \hookrightarrow \bigoplus_{i=1}^N \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(u) \oplus \widetilde{\mathfrak{g}}(\infty)$$

has a section over $\widehat{\mathfrak{g}}(u)_-$ given by regarding an element of $\widehat{\mathfrak{g}}(u)_-$ as a regular function on $\mathbb{P}^1 \setminus \{u, \infty\}$. Composing with Laurent expansion at the points z_i and ∞ yields a Lie algebra homomorphism

$$\tau_{u,(z_i)} : \widehat{\mathfrak{g}}(u)_- \hookrightarrow \bigoplus_{i=1}^N \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(\infty). \tag{2.9}$$

Let $\Phi_{u,(z_i)}$ be the extension of $-\tau_{u,(z_i)}$ to an *anti*-homomorphism

$$\Phi_{u,(z_i)} : U\big(\widehat{\mathfrak{g}}(u)_-\big) \to U\,\widehat{\mathfrak{g}}_N.$$

Lemma 2.4. *Let $\mathbb{M}_i$, $\mathbb{M}_\infty$ and $\mathbb{M}_u$ be representations of $\widehat{\mathfrak{g}}_\kappa(z_i)$, $\widehat{\mathfrak{g}}_\kappa(\infty)$ and $\widehat{\mathfrak{g}}_\kappa(u)$ respectively. The following holds in $H(\mathbb{M}_1, \dots, \mathbb{M}_N, \mathbb{M}_\infty, \mathbb{M}_u)$ for any $v_i \in \mathbb{M}_i$, $v_\infty \in \mathbb{M}_\infty$, $v_u \in \mathbb{M}_u$ and $X \in U(\widehat{\mathfrak{g}}(u)_-)$ we have*

$$\left[\bigotimes_{i=1}^N v_i \otimes v_\infty \otimes X v_u\right] = \left[\Phi_{u,(z_i)}(X)\left(\bigotimes_{i=1}^N v_i \otimes v_\infty\right) \otimes v_u\right].$$

Proof. It suffices to prove this for $X \in \widehat{\mathfrak{g}}(u)_-$. In that case it follows because $X + \tau_{u,(z_i)}(X) \in \mathfrak{g}_{(z_i,u)}$, so that

$$\left[\tau_{u,(z_i)}(X) \cdot \left(\bigotimes_{i=1}^N v_i \otimes v_\infty\right) \otimes v_u\right] + \left[\bigotimes_{i=1}^N v_i \otimes v_\infty \otimes X \cdot v_u\right] = 0. \qquad \square$$

Proposition 2.5. *The action of*

$$X \in \mathrm{End}_{\widehat{\mathfrak{g}}_\kappa(u)}(\mathbb{V}_0) \subset U\big(\widehat{\mathfrak{g}}(u)_-\big)^{\mathrm{opp}}$$

on $H(\mathbb{M}_1, \dots, \mathbb{M}_N, \mathbb{M}_\infty)$ *defined by* (2.7) *is given by*

$$X\left[\bigotimes_{i=1}^{N} v_i \otimes v_\infty\right] = \left[\Phi_{u,(z_i)}(X)\left(\bigotimes_{i=1}^{N} v_i \otimes v_\infty\right)\right].$$

Proof. This follows at once from (2.7) and Lemma 2.4. □

For later use, we work out the homomorphism (2.9).

Proposition 2.6. *The following hold for any* $J \in \mathfrak{g}$, $m \geqslant 0$ *and* $n > 0$

$$\tau_{u,(z_i)}(J_m) = \sum_{i=1}^{N}\sum_{p=0}^{m}(-1)^{m-p}\binom{m}{p}(u - z_i)^{m-p} J_p^{(i)} + \sum_{p=0}^{m}(-1)^{m-p}\binom{m}{p} u^{m-p} J_{-p}^{(\infty)},$$

$$\tau_{u,(z_i)}(J_{-n}) = \frac{\partial_u^{n-1}}{(n-1)!}\left(-\sum_{i=1}^{N}\sum_{p\geqslant 0}\frac{J_p^{(i)}}{(u - z_i)^{p+1}} + \sum_{p\geqslant 0} u^p J_{p+1}^{(\infty)}\right),$$

where $J_p^{(i)} = J \otimes (t - z_i)^p \in \widetilde{\mathfrak{g}}(z_i)$ *and* $J_p^{(\infty)} = J \otimes t^{-p} \in \widetilde{\mathfrak{g}}(\infty)$.

Proof. The first identity follows from the fact that

$$(t - u)^m = \sum_{p=0}^{m}\binom{m}{p}(t - z_i)^p (z_i - u)^{m-p}$$

and

$$(t - u)^m = \sum_{p=0}^{m}(-1)^{m-p}\binom{m}{p} t^p u^{m-p}.$$

The second one follows from

$$\frac{1}{t-u} = -\frac{1}{u - z_i}\frac{1}{1 - \frac{t-z_i}{u-z_i}} = -\sum_{m\geqslant 0}\frac{(t - z_i)^m}{(u - z_i)^{m+1}}$$

and

$$\frac{1}{t-u} = \frac{1}{t}\sum_{m\geqslant 0}\left(\frac{u}{t}\right)^m. \qquad \square$$

2.5. The universal Gaudin algebra and its quotients

Now we identify $\widetilde{\mathfrak{g}}(u)$ and $\bigoplus_{i=1}^N \widetilde{\mathfrak{g}}(z_i) \oplus \widetilde{\mathfrak{g}}(\infty)$ with $\mathfrak{g}((t))$ and $\mathfrak{g}((t))^{\oplus(N+1)}$, respectively, by putting $\widetilde{\mathfrak{g}}(\infty)$ as the last component in the direct sum. We also identify

$$\widehat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}\big[t^{-1}\big] \simeq \widehat{\mathfrak{g}}(u)_- = (t-u)^{-1}\mathfrak{g}\big[(t-u)^{-1}\big].$$

Then we obtain an anti-homomorphism

$$\Phi_{u,(z_i)} : U(\widehat{\mathfrak{g}}_-) \to U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big).$$

By Proposition 2.6, $\Phi_{u,(z_i)}$ is given by the formula

$$\Phi_{u,(z_i)}\big(J^{a_1}_{-n_1} \dots J^{a_m}_{-n_m}\big) = \mathbf{J}^{a_m}_{-n_m}(u) \dots \mathbf{J}^{a_1}_{-n_1}(u), \tag{2.10}$$

where

$$\mathbf{J}_{-n}(u) = \frac{\partial_u^{n-1}}{(n-1)!}\left(\sum_{i=1}^N \sum_{p\geqslant 0} \frac{J_p^{(i)}}{(u-z_i)^{p+1}} - \sum_{p\geqslant 0} u^p J_{p+1}^{(\infty)}\right). \tag{2.11}$$

It is clear from the construction that $\Phi_{u,(z_i)}$ is independent of κ. What does depend on κ is the algebra $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ which we realize as a subalgebra of $U(\widehat{\mathfrak{g}}_-)^{\mathrm{opp}}$, and to which we then restrict the anti-homomorphism $\Phi_{u,(z_i)}$.

Note that $\mathfrak{g}$ acts on $\widehat{\mathfrak{g}}_-$ by adjoint action and that the induced action on $U(\widehat{\mathfrak{g}}_-)$ coincides with that on $\mathbb{V}_0$ under the identification $\mathbb{V}_0 \cong U(\widehat{\mathfrak{g}}_-)$. Since any $v \in \mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathbb{V}_0$ satisfies $\mathfrak{g} \cdot v = 0$, we find that $\Phi_{u,(z_i)}(v)$ is invariant under the diagonal action of $\mathfrak{g}$ on $U(\mathfrak{g}[\![t]\!])^{\otimes N} \otimes U(t\mathfrak{g}[\![t]\!])$. Thus, $\Phi_{u,(z_i)}$ restricts to an algebra homomorphism

$$\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) \to \big(U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big)\big)^{\mathfrak{g}}. \tag{2.12}$$

At this point we quote Theorem 5.1(1), according to which $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) = \mathbb{C}$, if $\kappa \neq \kappa_c$, where κ_c is the *critical* invariant inner product on $\mathfrak{g}$ defined by the formula[6]

$$\kappa_c(A, B) = -\frac{1}{2}\operatorname{Tr}_{\mathfrak{g}} \operatorname{ad} A \operatorname{ad} B.$$

On the other hand, the center $\mathfrak{z}_{\kappa_c}(\widehat{\mathfrak{g}})$ is non-trivial (see Theorem 5.1(2)).

Thus, the homomorphism (2.12) is non-trivial only for $\kappa = \kappa_c$. So from now on we will specialize κ to κ_c and omit the symbol κ from most of our formulas. In particular, we will write $\mathfrak{z}(\widehat{\mathfrak{g}})$ for $\mathfrak{z}_{\kappa_c}(\widehat{\mathfrak{g}})$, $\mathbb{M}$ for $\mathbb{M}_{\kappa_c}$, and so on.

Definition 2.7. The *universal Gaudin algebra*

$$\mathcal{Z}_{(z_i),\infty}(\mathfrak{g}) \subset \big(U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big)\big)^{\mathfrak{g}}$$

associated to $\mathfrak{g}$ and the set of points $z_1, \dots, z_N \in \mathbb{C}$ is the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under $\Phi_{u,(z_i)}$.

[6] Note that $\kappa_c = -h^\vee \kappa_0$, where κ_0 is the inner product normalized as in [32] (so that the square length of the maximal root is equal to 2) and $h^\vee$ is the dual Coxeter number.

Thus, $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is a commutative algebra, and the action of $\mathfrak{z}(\widehat{\mathfrak{g}})$ on coinvariants $H(\mathbb{M}_1, \dots, \mathbb{M}_N, \mathbb{M}_\infty)$ factors through $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ by Proposition 2.5.

We will show below that the algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is independent of the chosen point $u \in \mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$, which is the reason why we suppress its notational dependence on u. This is easier to establish in terms of the quotients $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ of $\mathcal{Z}_{(z_i),\infty}$ that we introduce presently.

Note first that the algebra $U(\mathfrak{g}[\![t]\!])^{\otimes N} \otimes U(t\mathfrak{g}[\![t]\!])$ is a complete topological algebra, whose topology is given as follows. For each positive integer m, the Lie subalgebra $t^m\mathfrak{g}[\![t]\!] \subset \mathfrak{g}[\![t]\!]$ is an ideal. The left ideal $\mathcal{I}_m$ it generates in $U(\mathfrak{g}[\![t]\!])$ is therefore a two-sided ideal and the quotient $U(\mathfrak{g}[\![t]\!])/\mathcal{I}_m$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g}_m)$, where $\mathfrak{g}_m = \mathfrak{g}[\![t]\!]/t^m\mathfrak{g}[\![t]\!]$. Similarly, the subalgebra $t^m\mathfrak{g}[\![t]\!] \subset t\mathfrak{g}[\![t]\!]$ generates a two-sided ideal $\bar{\mathcal{I}}_m$ of $U(t\mathfrak{g}[\![t]\!])$ and the quotient $U(t\mathfrak{g}[\![t]\!])/\bar{\mathcal{I}}_m$ is isomorphic to $U(\bar{\mathfrak{g}}_m)$, where $\bar{\mathfrak{g}}_m = t\mathfrak{g}[\![t]\!]/t^m\mathfrak{g}[\![t]\!]$.

The topology on $U(\mathfrak{g}[\![t]\!])^{\otimes N} \otimes U(t\mathfrak{g}[\![t]\!])$ is defined by declaring

$$\mathcal{I}_{(m_i)} = \bigoplus_{i=1}^{N} U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes(i-1)} \otimes \mathcal{I}_{m_i} \otimes U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes(N-i)} \otimes U\big(t\mathfrak{g}[\![t]\!]\big) \oplus U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes \bar{\mathcal{I}}_{m_\infty}$$

to be the base of open neighborhoods of 0. This algebra is complete in this topology, and it is the inverse limit

$$\begin{aligned} U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big) &= \varprojlim U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big)/\mathcal{I}_{(m_i)} \\ &= \varprojlim \bigotimes_{i=1}^{N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty}). \end{aligned}$$

For any collection of positive integers $m_1, \dots, m_N, m_\infty$, let

$$\Phi^{(m_i),m_\infty}_{u,(z_i)} : U(\widehat{\mathfrak{g}}_-) \to \bigotimes_{i=1}^{N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty}) \tag{2.13}$$

be the composition of $\Phi_{u,(z_i)}$ with the natural surjection

$$U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big) \twoheadrightarrow \bigotimes_{i=1}^{N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty}). \tag{2.14}$$

Thus, $\Phi^{(m_i),m_\infty}_{u,(z_i)}$ is obtained from the formulas (2.10) and (2.11) by setting $J^{(i)}_m = 0$ for $m > m_i$ and $J^{(\infty)}_m = 0$ for $m > m_\infty$.

Let

$$\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \subset \left(\bigotimes_{i=1}^{N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty}) \right)^{\mathfrak{g}}$$

be the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under $\Phi_{u,(z_i)}^{(m_i),m_\infty}$. The algebras $\mathcal{Z}_{(z_i),\infty}$ form an inverse system of commutative algebras, and

$$\mathcal{Z}_{(z_i),\infty}(\mathfrak{g}) = \varprojlim \mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g}). \tag{2.15}$$

Formulas (2.10)–(2.11) show that if u is considered as a variable, $\Phi_{u,(z_i)}^{(m_i),m_\infty}$ may be regarded as an anti-homomorphism

$$\Phi_{u,(z_i)}^{(m_i),m_\infty} : U(\widehat{\mathfrak{g}}_-) \to \left(\bigotimes_{i=1}^{N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty})\right) \otimes \mathbb{C}\big[(u-z_i)^{-1}\big]_{i=1,\dots,N} \otimes \mathbb{C}[u].$$

The following result gives an alternative description of $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$, which shows in particular that it is independent of u.

Proposition 2.8. *The algebra $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ is equal to the span of the coefficients of $\Phi_{u,(z_i)}^{(m_i),m_\infty}(A)$, $A \in \mathfrak{z}(\widehat{\mathfrak{g}})$, appearing in front of the monomials of the form $\prod_{i=1}^{N}(u-z_i)^{-n_i}u^{n_\infty}$.*

Proof. Since the translation operator acts as $-\partial_t = \partial_u$ on $\widehat{\mathfrak{g}}(t-u)_-$ and $\Phi_{u,(z_i)}^{(m_i),m_\infty}$ is given by Laurent expansion at the points $z_1, \dots, z_N$, it follows that, for any $A \in \mathfrak{z}(\widehat{\mathfrak{g}})$

$$\Phi_{u,(z_i)}^{(m_i),m_\infty}(TA) = \partial_u \Phi_{u,(z_i)}^{(m_i),m_\infty}(A),$$

where T is given by (2.6). For any $B \in \mathfrak{z}(\widehat{\mathfrak{g}})$, the expression $\Phi_{u,(z_i)}^{(m_i),m_\infty}(B)$, viewed as a rational function of u, is a finite linear combination of monomials of the form $\prod_{i=1}^{N}(u-z_i)^{-n_i}u^{n_\infty}$, where $n_i > 0, n_\infty \geqslant 0$. Such a linear combination is uniquely determined by its expansion in a power series at a fixed point (or equivalently, by the values of its derivatives at this point). Therefore the span of these expressions for a fixed u and all $B \in \mathfrak{z}(\widehat{\mathfrak{g}})$ of the form $T^n A, n \geqslant 0$, is the same as the span of the coefficients of $\Phi_{u,(z_i)}^{(m_i),m_\infty}(A)$, considered as a rational function in u, appearing in front of the monomials of the form $\prod_{i=1}^{N}(u-z_i)^{-n_i}u^{n_\infty}$. □

By (2.15) and Proposition 2.8, $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is also independent of u. Moreover, if u is regarded as a variable, and $\Phi_{u,(z_i)}$ as an anti-homomorphism

$$\Phi_{u,(z_i)} : U(\widehat{\mathfrak{g}}_-) \to \big(U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big)\big)\widehat{\otimes}\big(\mathbb{C}\big[(u-z_i)^{-1}\big]_{i=1,\dots,N} \otimes \mathbb{C}[u]\big),$$

where the tensor product is completed with respect to the natural topology on the Lie algebra $\mathfrak{g}[\![t]\!]^{\oplus(N+1)}$, then $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ may also be obtained as the completion of the span of the coefficients of the series $\Phi_{u,(z_i)}(A)$, $A \in \mathfrak{z}(\widehat{\mathfrak{g}})$, appearing in front of the monomials of the form $\prod_{i=1}^{N}(u-z_i)^{-n_i}u^{n_\infty}$.

It is instructive to think of the algebras $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ and their generalizations considered below as quotients of the universal Gaudin algebra. However, note that while $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is a complete topological algebra, the algebras $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ are its discrete quotients. For this reason in practice it is easier to work with the latter. In fact, for any collection of smooth finitely generated $\mathfrak{g}[\![t]\!]$-modules $M_1, \dots, M_N$, M_∞, the action of $\mathfrak{g}[\![t]\!]$ on M_i (resp., M_∞) factors through $\mathfrak{g}_{m_i}$

(resp., $\mathfrak{g}_{m_\infty}$). Therefore the action of $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ on the tensor product $\bigotimes_{i=1}^N M_i \otimes M_\infty$, or on the corresponding space of coinvariants

$$H(\mathbb{M}_1, \ldots, \mathbb{M}_N, \mathbb{M}_\infty) \cong \left(\bigotimes_{i=1}^N M_i \otimes M_\infty \right) \Big/ \mathfrak{g}$$

factors through that of $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$. Since the only $\mathfrak{g}[\![t]\!]$-modules that we consider are smooth finitely generated modules, we do not lose any generality by working with the quotients $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ rather than with the algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ itself.

2.6. Example: The Gaudin model

Consider the simplest case when all m_i and m_∞ are equal to 1. The corresponding algebra $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g})$ is a commutative subalgebra of $(U\mathfrak{g}^{\otimes N})^{\mathfrak{g}}$. The homomorphism $\Phi^{(1),1}_{u,(z_i)}(S)$ was defined in [17] (see also [22]).

Consider the Segal–Sugawara vector in $\mathbb{V}_0$:

$$S = \frac{1}{2} \sum_{a=1}^{d} J_{a,-1} J^a_{-1} v_0, \tag{2.16}$$

where $d = \dim \mathfrak{g}$ and $\{J_a\}, \{J^a\}$ are dual bases of $\mathfrak{g}$ with respect to a non-zero invariant inner product. It is easy to show (see, e.g., [26, Section 3.4.8]) that this vector belongs to $\mathfrak{z}(\widehat{\mathfrak{g}})$. Let us compute $\Phi^{(1),1}_{u,(z_i)}(S)$. Let

$$\Delta = \frac{1}{2} \sum_a J_a J^a \in U\mathfrak{g}$$

be the Casimir operator. Formulas (2.10)–(2.11) readily yield the following:

Lemma 2.9. *(See [17, Proposition 1].) We have*

$$\Phi^{(1),1}_{u,(z_i)}(S) = \sum_{i=1}^N \frac{\Delta^{(i)}}{(u - z_i)^2} + \sum_{i=1}^N \frac{\Xi_i}{u - z_i},$$

where the Ξ_i's are the Gaudin Hamiltonians

$$\Xi_i = \sum_{j \neq i} \sum_{a=1}^d \frac{J_a^{(i)} J^{a(j)}}{z_i - z_j}, \quad i = 1, \ldots, N. \tag{2.17}$$

By Proposition 2.8, $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g})$ contains each $\Delta^{(i)}$ and the Gaudin Hamiltonians Ξ_i, $i = 1,\dots,N$. Elements of $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g})$ are generalized Gaudin Hamiltonians which act on the tensor product $\bigotimes_{i=1}^N M_i$ of any N-tuple of $\mathfrak{g}$-modules or on the space of coinvariants

$$H(\mathbb{M}_1,\dots,\mathbb{M}_N,\mathbb{M}_\infty) = \left(\bigotimes_{i=1}^N M_i \otimes M_\infty\right)\Big/\mathfrak{g},$$

where each M_i is regarded as a $\mathfrak{g}[\![t]\!]$-module by letting $t\mathfrak{g}[\![t]\!]$ act by 0. If all M_i's and M_∞ are highest weight $\mathfrak{g}$-modules, then the corresponding induced modules $\mathbb{M}_i$ and $\mathbb{M}_\infty$ are highest weight $\widehat{\mathfrak{g}}_{\kappa_c}$-modules. Thus, the choice $m_i = 1, m_\infty = 1$ corresponds to highest weight modules. The spectrum of the corresponding algebra $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g})$ is the space of ${}^L G$-opers with regular singularities at the points $z_1,\dots,z_N$ and ∞ (see [22] and Section 5 below).

However, if we choose some of the m_i's (or m_∞) to be greater than 1, then for a general $\mathfrak{g}_{m_i}$-module M_i (or M_∞) the corresponding induced module $\mathbb{M}_i$ (or $\mathbb{M}_\infty$) will not be a highest weight module. The corresponding algebra $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ is isomorphic to the algebra of functions on the space ${}^L G$-opers with irregular singularities at the points z_i (and ∞) of orders m_i (resp., $\mathfrak{m}_\infty$), see Theorem 5.7. Therefore we refer to the corresponding integrable quantum models as *Gaudin models with irregular singularities*.

2.7. Another example: the algebra $\mathcal{A}_\chi$

Consider the case when there are two points: $z_1 \in \mathbb{C}$ and ∞, with $m_1 = 1$ and $m_\infty = 2$. This is the simplest model with an irregular singularity. The corresponding Gaudin algebra $\mathcal{Z}^{1,2}_{z_1,\infty}(\mathfrak{g})$ is a commutative subalgebra of $(U(\mathfrak{g}) \otimes U(\bar{\mathfrak{g}}_2))^{\mathfrak{g}}$. It is easy to see that it is independent of z_1, so we will set $z_1 = 0$.

The Lie algebra

$$\bar{\mathfrak{g}}_2 = t\mathfrak{g}[\![t]\!]/t^2\mathfrak{g}[\![t]\!]$$

is abelian and isomorphic to $\mathfrak{g}$ as a vector space. Any linear functional $\chi : \mathfrak{g} \to \mathbb{C}$ on $\mathfrak{g}$ therefore defines an algebra homomorphism $U(\bar{\mathfrak{g}}_2) \cong S\mathfrak{g} \to \mathbb{C}$. Let

$$\mathcal{A}_\chi = \mathrm{id} \otimes \chi\big(\mathcal{Z}^{1,2}_{0,\infty}(\mathfrak{g})\big) \subset U\mathfrak{g} \tag{2.18}$$

be the image of $\mathcal{Z}^{1,2}_{0,\infty}(\mathfrak{g})$ under the homomorphism $(U(\mathfrak{g}) \otimes U(\bar{\mathfrak{g}}_2))^{\mathfrak{g}} \to U(\mathfrak{g})$ given by applying χ to the second factor. It is clear that $\mathcal{A}_\chi$ is a commutative subalgebra of the centralizer $U(\mathfrak{g})^{\mathfrak{g}_\chi} \subset U(\mathfrak{g})$, where $\mathfrak{g}_\chi \subseteq \mathfrak{g}$ is the stabilizer of χ.

As a subalgebra of $U(\mathfrak{g})$, $\mathcal{A}_\chi$ acts on any $\mathfrak{g}$-module M. From the point of view of the coinvariants construction of Section 2.2, this action comes about as follows: we consider the $\widehat{\mathfrak{g}}_{\kappa_c}$-module $\mathbb{M}$ induced from M, attached to $0 \in \mathbb{P}^1$, and the non-highest weight $\widehat{\mathfrak{g}}_{\kappa_c}$-module $\mathbb{I}_{1,\chi}$ introduced in formula (2.3). Then the Lie algebra $\mathfrak{g}_{(z_i)}$ is $\mathfrak{g}[t,t^{-1}]$ and the space of coinvariants is

$$H(\mathbb{M},\mathbb{I}_{1,\chi}) = (\mathbb{M} \otimes \mathbb{I}_{1,\chi})/\mathfrak{g}\big[t,t^{-1}\big] \simeq (M \otimes I_\chi)/\mathfrak{g} \simeq M,$$

since $I_\chi = \mathrm{ind}^{\mathfrak{g}[\![t]\!]}_{t\mathfrak{g}[\![t]\!]} \mathbb{C}_\chi$ is isomorphic to $U\mathfrak{g}$ as a $\mathfrak{g}$-module.

The action of $\mathfrak{z}(\widehat{\mathfrak{g}})$ on $H(\mathbb{M},\mathbb{I}_{1,\chi}) \simeq M$ then factors through that of $\mathcal{A}_\chi$.

We note that it follows from the definition that $\mathcal{A}_\chi = \mathcal{A}_{c\chi}$ for any non-zero $c \in \mathbb{C}$ and that $\mathrm{Ad}_g(\mathcal{A}_\chi) = \mathcal{A}_{\mathrm{Ad}_g(\chi)}$ for any $g \in G$.

2.8. Multi-point generalization

The algebra $\mathcal{A}_\chi$ has a natural multi-point generalization. Namely, let m_i, $i = 1, \ldots, N$, and m_∞ be a collection of positive integers as in Section 2.5. Let us also fix characters $\chi_i : t^{m_i}\mathfrak{g}[\![t]\!] \to \mathbb{C}$ and $\chi_\infty : t^{m_\infty}\mathfrak{g}[\![t]\!] \to \mathbb{C}$. We will attach to this data a commutative algebra which simultaneously generalizes both $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}$ and $\mathcal{A}_\chi$. Note that a character

$$\chi : t^m\mathfrak{g}[\![t]\!] \to \mathbb{C}$$

has to vanish on the derived subalgebra of $t^m\mathfrak{g}[\![t]\!]$, that is on $t^{2m}\mathfrak{g}[\![t]\!]$. Hence, defining χ is equivalent to choosing an arbitrary linear functional on the abelian Lie algebra

$$t^m\mathfrak{g}[\![t]\!]/t^{2m}\mathfrak{g}[\![t]\!] \simeq \mathfrak{g} \otimes \mathbb{C}^m.$$

Let

$$I_{m,\chi} = \mathrm{Ind}^{\mathfrak{g}[\![t]\!]}_{t^m\mathfrak{g}[\![t]\!]}\mathbb{C}_\chi, \qquad \bar{I}_{m,\chi} = \mathrm{Ind}^{t\mathfrak{g}[\![t]\!]}_{t^m\mathfrak{g}[\![t]\!]}\mathbb{C}_\chi.$$

Note that the action of $\mathfrak{g}[\![t]\!]$ on $I_{m,\chi}$ factors through $\mathfrak{g}_{2m} = \mathfrak{g}[\![t]\!]/t^{2m}\mathfrak{g}[\![t]\!]$, and as a $\mathfrak{g}_{2m}$-module, $I_{m,\chi}$ is isomorphic to

$$I_{m,\chi} \simeq \mathrm{Ind}^{\mathfrak{g}_{2m}}_{t^m\mathfrak{g}_{2m}}\mathbb{C}_\chi$$

(and similarly for $\bar{I}_{m,\chi}$). Denote by $\mathcal{I}_{m,\chi}$ (resp., $\bar{\mathcal{I}}_{m,\chi}$) the annihilator of $I_{m,\chi}$ in $U(\mathfrak{g}_{2m})$ (resp., of $\bar{I}_{m,\chi}$ in $U(\bar{\mathfrak{g}}_{2m})$) and by $U_{m,\chi}$ (resp., $\overline{U}_{m,\chi}$) the quotient of $U(\mathfrak{g}_{2m})$ (resp., $U(\bar{\mathfrak{g}}_{2m})$) by $\mathcal{I}_{m,\chi}$ (resp., $\bar{\mathcal{I}}_{m,\chi}$). Thus, $U_{m,\chi}$ is the image of $U(\mathfrak{g}_{2m})$ and $U(\mathfrak{g}[\![t]\!])$ in $\mathrm{End}_{\mathbb{C}} I_{m,\chi}$, and $\overline{U}_{m,\chi}$ is the image of $U(\bar{\mathfrak{g}}_{2m})$ and $U(t\mathfrak{g}[\![t]\!])$ in $\mathrm{End}_{\mathbb{C}} I_{m,\chi}$.

Now, given the data of (m_i), m_∞, (χ_i), χ_∞, we obtain the algebra $\bigotimes_{i=1}^N U_{m_i,\chi_i} \otimes \overline{U}_{m_\infty,\chi_\infty}$, which is isomorphic to the quotient of $\bigotimes_{i=1}^N U(\mathfrak{g}_{2m_i}) \otimes U(\bar{\mathfrak{g}}_{2m_\infty})$ by a two-sided ideal. Let

$$\mathcal{A}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(\chi_i),\chi_\infty} \subset (U_{m_i,\chi_i} \otimes \overline{U}_{m_\infty,\chi_\infty})^{\mathfrak{g}}$$

be the image of

$$\mathcal{Z}^{(2m_i),2m_\infty}_{(z_i),\infty}(\mathfrak{g}) \subset (U_{2m_i} \otimes \overline{U}_{2m_\infty})^{\mathfrak{g}}.$$

Equivalently, $\mathcal{A}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(\chi_i),\chi_\infty}$ is the image of the universal Gaudin algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ under the homomorphism

$$U\big(\mathfrak{g}[\![t]\!]\big)^{\otimes N} \otimes U\big(t\mathfrak{g}[\![t]\!]\big) \to \bigotimes_{i=1}^N U_{m_i,\chi_i} \otimes \overline{U}_{m_\infty,\chi_\infty}. \tag{2.19}$$

Note in particular that

$$\mathcal{A}_\chi = \mathcal{A}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}.$$

The algebra $\mathcal{A}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(\chi_i),\chi_\infty}$ naturally acts on the tensor product $\bigotimes_{i=1}^N I_{m_i,\chi_i} \otimes I_{m_\infty,\chi_\infty}$, or on the corresponding space of coinvariants

$$H(\mathbb{I}_{m_1,\chi_1},\dots,\mathbb{I}_{m_N,\chi_N},\mathbb{I}_{m_\infty,\chi_\infty}) \cong \left(\bigotimes_{i=1}^N I_{m_i,\chi_i} \otimes I_{m_\infty,\chi_\infty}\right)\Big/\mathfrak{g} \simeq \left(\bigotimes_{i=1}^N I_{m_i,\chi_i} \otimes \bar{I}_{m_\infty,\chi_\infty}\right).$$

For $\chi = 0$, $U_{m,0} \simeq U(\mathfrak{g}_m)$ and $\overline{U}_{m,0} \simeq U(\bar{\mathfrak{g}}_m)$. But for a non-zero character χ the algebras $U_{m,\chi}$ and $\overline{U}_{m,\chi}$ are not in general isomorphic to universal enveloping algebras.

There is however one exception. If $\chi_\infty : t^m\mathfrak{g}[\![t]\!] \to \mathbb{C}$ factors as

$$t^m\mathfrak{g}[\![t]\!] \to \mathfrak{g}\otimes t^m \xrightarrow{\chi} \mathbb{C}$$

for some linear functional χ on $\mathfrak{g}$, then $\overline{U}_{m,\chi} \simeq \overline{U}(\mathfrak{g}_{m-1})$ for any χ. Therefore the algebra $\mathcal{A}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi_\infty}$ may be realized in $\bigotimes_{i=1}^N U_{m_i} \otimes \overline{U}_{m_\infty-1}$ in this case.

In particular, if $m=1$ then $\overline{U}_{1,\chi} \simeq \mathbb{C}$ regardless of χ. Therefore in the case when $\chi_i = 0$ for $i = 1,\dots,N$, and $m_\infty = 1$ the corresponding algebra $\mathcal{A}^{(m_i),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi_\infty}$ is a subalgebra of the universal enveloping algebra $\bigotimes_{i=1}^N U(\mathfrak{g}_{m_i})$.

In the case when $m_i = 1$, $i = 1,\dots,N$, we obtain a subalgebra of $U(\mathfrak{g})^{\otimes m}$ that has been previously constructed in [45]. This subalgebra $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ is obtained by applying the homomorphism

$$\mathrm{id}\otimes\chi : U(\mathfrak{g})^{\otimes m} \otimes S(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes m}$$

to the algebra $\mathcal{Z}^{(1),2}_{(z_i),\infty}(\mathfrak{g})$. Let us apply $\mathrm{id}\otimes\chi$ to $\Phi^{(1),2}_{u,(z_i)}(S)$, where S is the Segal–Sugawara element in $\mathbb{V}_0$. Then we find, in the same way as in Lemma 2.9, that

$$(\mathrm{id}\otimes\chi)\circ\Phi^{(1),2}_{u,(z_i)}(S) = \sum_{i=1}^N \frac{\Delta^{(i)}}{(u-z_i)^2} + \sum_{i=1}^N \frac{\Xi_{i,\chi}}{u-z_i} + (\chi,\chi),$$

where

$$\Xi_{i,\chi} = \sum_{j\neq i}\sum_{a=1}^d \frac{J_a^{(i)}J^{a(j)}}{z_i - z_j} + \chi^{(i)}, \quad i = 1,\dots,N, \tag{2.20}$$

where we identity $\mathfrak{h} \simeq \mathfrak{h}^*$ using the invariant inner product used in the definition of S. Thus, $\Xi_{i,\chi}$, $i = 1,\dots,N$, are elements of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$. These operators appeared in [18] in the study of generalized KZ equations. We note that in the case of $\mathfrak{g} = \mathfrak{sl}_2$ they were probably first considered in [47].

3. Associated graded algebras and Hitchin systems

In this section we consider generalized Hitchin systems on the moduli spaces of Higgs bundles, where the Higgs fields are allowed to have poles. Integrable systems of this type, generalizing the original Hitchin systems from [31] (which correspond to Higgs fields without poles), have been previously considered in [2,6,9,12,37]. Here we will focus on the case when the underlying curve is $\mathbb{P}^1$. We will construct algebras of Poisson commuting Hamiltonians in these systems in the standard way. We will then show that the generalized Gaudin algebras of commuting quantum Hamiltonians constructed in the previous section are quantizations of the Poisson commutative algebras of Hitchin Hamiltonians. The proof of this result relies on a local statement, due to [16,20], that the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathbb{V}_{0,\kappa_c}$ is a quantization of $S(\mathfrak{g}((t))/\mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}$. By definition, the Gaudin algebras are quotients of $\mathfrak{z}(\widehat{\mathfrak{g}})$, whereas the algebras of Hitchin Hamiltonians are quotients of $S(\mathfrak{g}((t))/\mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}$, hence the result.

The idea of using the center at the critical level for quantizing the Hitchin systems is due to Beilinson and Drinfeld [7], who used it to quantize the Hitchin systems defined on arbitrary smooth projective curves, without ramification. Here we develop this theory in a "transversal" direction: quantizing the Hitchin systems on $\mathbb{P}^1$, but with arbitrary ramification. The two scenarios may certainly be combined, giving rise to quantum integrable systems corresponding to arbitrary curves, with ramification. However, these systems are hard to analyze explicitly unless the underlying curve is $\mathbb{P}^1$ or an elliptic curve. In the general setting the focus shifts instead to the investigation of the salient features of the $\mathcal{D}$-modules on the moduli stacks of bundles with level (or parabolic) structures defined by the corresponding algebras of quantum Hamiltonians (see Section 1.4). It follows from the results of [7] that these $\mathcal{D}$-modules are Hecke eigensheaves, and hence they play an important role in the geometric Langlands correspondence.

On the other hand, in the case of $\mathbb{P}^1$ the generalized Gaudin algebras may be analyzed explicitly. In the case of regular singularities, corresponding to the ordinary Gaudin models, this has been done in [17,22] (see also [11,12] for a generalization to the case of elliptic curves). Here we look closely at another special case corresponding to irregular singularity of order 2 at one point of $\mathbb{P}^1$. The corresponding Poisson commutative algebra of Hitchin Hamiltonians may be identified with the shift of argument algebra $\overline{\mathcal{A}}_\chi$ introduced in [38]. We show that the quantum commutative algebra $\mathcal{A}_\chi$ introduced in Section 2.7 is the quantization of $\overline{\mathcal{A}}_\chi$ for all regular $\chi \in \mathfrak{g}^*$. This has been previously proved in [45] in the case when χ is regular semi-simple. In addition, we show that when χ is regular semi-simple the algebra $\mathcal{A}_\chi$ contains the DMT Hamiltonians (1.1).

3.1. Hitchin systems with singularities

Let us first recall the definition of the unramified Hitchin system. Denote by G the connected simply-connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{M}_G$ be the moduli space of stable G-bundles on a smooth projective curve X. The tangent space to $\mathcal{M}_G$ at $\mathcal{F} \in \mathcal{M}_G$ is isomorphic to $H^1(X, \mathfrak{g}_\mathcal{F})$, where $\mathfrak{g}_\mathcal{F} = \mathcal{F} \times_G \mathfrak{g}$. Hence, by the Serre duality, the cotangent space at $\mathcal{F}$ is isomorphic to $H^0(X, \mathfrak{g}^*_\mathcal{F} \otimes \Omega)$, where Ω is the canonical line bundle on X, by Serre duality. A vector $\eta \in H^0(X, \mathfrak{g}^*_\mathcal{F} \otimes \Omega)$ is referred to as a *Higgs field*. We construct the Hitchin map

$p : T^*\mathcal{M}_G \to \mathcal{H}_G$, where $\mathcal{H}_G$ is the *Hitchin space*

$$\mathcal{H}_G(X) = \bigoplus_{i=1}^{\ell} H^0\big(X, \Omega^{\otimes(d_i+1)}\big), \tag{3.1}$$

as follows. We use the following result of C. Chevalley.

Theorem 3.1. *The algebra* $\operatorname{Inv}\mathfrak{g}^*$ *of G-invariant polynomial functions on* $\mathfrak{g}^*$ *is isomorphic to the graded polynomial algebra* $\mathbb{C}[\overline{P}_1, \dots, \overline{P}_\ell]$, *where* $\deg \overline{P}_i = d_i + 1$, *and* $d_1, \dots, d_\ell$ *are the exponents of* $\mathfrak{g}$.

For $\eta \in H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega)$, $\overline{P}_i(\eta)$ is well-defined as an element of $H^0(X, \Omega^{\otimes(d_i+1)})$. By definition, the Hitchin map p takes $(\mathcal{F}, \eta) \in T^*\mathcal{M}_G$ to

$$\big(\overline{P}_1(\eta), \dots, \overline{P}_\ell(\eta)\big) \in \mathcal{H}_G.$$

Remark 3.2. This definition depends on the choice of generators of $\operatorname{Inv}\mathfrak{g}^*$, which is not canonical. To give a more canonical definition, let

$$\mathcal{P} = \operatorname{Spec}\operatorname{Inv}\mathfrak{g}^*.$$

Since $\operatorname{Inv}\mathfrak{g}^*$ is a graded algebra, we obtain a canonical $\mathbb{C}^\times$-action on $\mathcal{P}$. Now let $\Omega^\times$ be the $\mathbb{C}^\times$-bundle on X corresponding to the line bundle Ω. We then have a vector bundle

$$\mathcal{P}_\Omega = \Omega^\times \underset{\mathbb{C}^\times}{\times} \mathcal{P},$$

and we set

$$\mathcal{H}_G = H^0(X, \mathcal{P}_\Omega).$$

A choice of homogeneous generators $\overline{P}_i$, $i = 1, \dots, \ell$, of $\operatorname{Inv}\mathfrak{g}^*$ gives rise to a set of coordinates on $\mathcal{P}$ which enable us to identify $\mathcal{H}_G$ with (3.1). However, using this definition of $\mathcal{H}_G$, we obtain a definition of the Hitchin map that is independent of the choice of generators. Likewise, the generalized Hitchin map considered below may also be defined in a generator-independent way. But in order to simplify the exposition we will define them by using a particular choice of generators $\overline{P}_i$, $i = 1, \dots, \ell$, of $\operatorname{Inv}\mathfrak{g}^*$.

Now, given a linear functional $\phi : \mathcal{H}_G \to \mathbb{C}$, we obtain a function $\phi \circ p$ on $T^*\mathcal{M}_G$. Hitchin [31] has shown that for different ϕ's these functions Poisson commute with respect to the natural symplectic structure on $T^*\mathcal{M}_G$, and together they define an algebraically completely integrable system.

Let us generalize the Hitchin systems to the case of Higgs fields with singularities. Let x_i, $i = 1, \dots, n$ be a collection of distinct points on X. Let us choose a collection of positive integers m_i, $i = 1, \dots, n$. Denote by $\mathcal{M}_{G,(x_i)}^{(m_i)}$ the moduli space of semi-stable G-bundles on X with level structures of order m_i at x_i, $i = 1, \dots, n$. Recall that a level structure on a G-bundle $\mathcal{F}$ on a curve X of order m at a point $x \in X$ is a trivialization of $\mathcal{F}$ on the $(m-1)$st infinitesimal neighborhood of x. Then a cotangent vector to a point of $\mathcal{M}_{G,(x_i)}^{(m_i)}$ is a Higgs field with poles of orders at most m_i at the points x_i,

$$\eta \in H^0\big(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega(m_1x_1 + \cdots + m_nx_n)\big).$$

In the same way as above one constructs a generalized Hitchin map

$$p : T^*\mathcal{M}_{G,(x_i)}^{(m_i)} \to \mathcal{H}_{G,(x_i)}^{(m_i)} = \bigoplus_{j=1}^{\ell} H^0\big(X, \Omega(m_1x_1 + \cdots + m_nx_n)^{\otimes(d_j+1)}\big),$$

and shows that it defines an algebraically completely integrable system. These systems have been previously studied in [2,6,9,12,37].

We are interested in these integrable systems in the case when $X = \mathbb{P}^1$, with the marked points $z_1, \ldots, z_N, \infty$ (with respect to a global coordinate t, as before). In this case a G-bundle is semistable if and only if it is trivial. Thus, $\mathcal{M}_{G,(x_i)}^{(m_i)}$ may be identified in this case with the quotient

$$\prod_{i=1}^{N} G_{m_i} \times G_{m_\infty} \Big/ G_{\mathrm{diag}} \simeq \prod_{i=1,\ldots,N} G_{m_i} \times \overline{G}_{m_\infty}.$$

Here $G_m, \overline{G}_m$ are the Lie groups corresponding to the Lie algebras $\mathfrak{g}_m = \mathfrak{g}[\![t]\!]/t^m\mathfrak{g}[\![t]\!]$, $\bar{\mathfrak{g}}_m = t\mathfrak{g}[\![t]\!]/t^m\mathfrak{g}[\![t]\!]$. Hence $T^*\mathcal{M}_{G,(x_i)}^{(m_i)}$ is isomorphic to the Hamiltonian reduction of

$$T^*\left(\prod_{i=1}^{N} G_{m_i} \times G_{m_\infty}\right)$$

with respect to the diagonal action of G (and the 0 orbit in $\mathfrak{g}^*$). We can a will identify it with

$$\prod_{i=1,\ldots,N} G_{m_i} \times \overline{G}_{m_\infty} \times \prod_{i=1,\ldots,N} (\mathfrak{g}_{m_i})^* \times (\bar{\mathfrak{g}}_{m_\infty})^*. \tag{3.2}$$

A point of (3.2) is then a collection $(g_\alpha, A_\alpha(t)dt)_{\alpha=1,\ldots,N,\infty}$, where $g_i \in G_{m_i}$, $i = 1, \ldots, N$; $g_\infty \in \overline{G}_{m_\infty}$,

$$A_i(t)\,dt = \sum_{k=-m_i}^{-1} A_{i,k}t^k\,dt, \quad A_{i,k} \in \mathfrak{g}^*, \qquad A_\infty(t)\,dt = \sum_{k=-m_\infty}^{-2} A_{\infty,k}t^k\,dt, \quad A_{\infty,k} \in \mathfrak{g}^*.$$

We associate to it a $\mathfrak{g}^*$-valued one-form on $\mathbb{P}^1$ with singularities at the marked points (also known as an "L-operator"):

$$\eta = \sum_{i=1,\ldots,N} \mathrm{Ad}_{g_i}\big(A_i(t - z_i)\big)\,d(t - z_i) + \mathrm{Ad}_{g_\infty}\big(A_\infty\big(t^{-1}\big)\big)\,d\big(t^{-1}\big), \tag{3.3}$$

whose polar parts are given by the one-forms $A_i(t_\alpha)\,dt_\alpha$, conjugated by g_α's. The Hitchin map p takes this one-form to

$$\eta \mapsto \bigl(\overline{P}_1(\eta), \dots, \overline{P}_\ell(\eta)\bigr) \in \mathcal{H}_{G,(z_i),\infty}^{(m_i),m_\infty}$$
$$= \bigoplus_{j=1}^{\ell} H^0\bigl(\mathbb{P}^1, \Omega(m_1 z_1 + \dots + m_N z_N + m_\infty \infty)^{\otimes(d_j+1)}\bigr). \tag{3.4}$$

Now any linear functional $\phi : \mathcal{H}_{G,(z_i),\infty}^{(m_i),m_\infty} \to \mathbb{C}$ gives rise to a function $H_\phi = p \circ \phi$, and, according to the above general result of [2,6,9,12,31,37] these functions Poisson commute with each other.

By definition, the symplectic manifold (3.2) fibers over $(\prod_\alpha G_{m_\alpha})/G$. The fiber over the identify coset is the Poisson submanifold

$$\prod_{i=1,\dots,N} (\mathfrak{g}_{m_i})^* \times (\bar{\mathfrak{g}}_{m_\infty})^* \tag{3.5}$$

with its natural Kirillov–Kostant structure. Let $\widetilde{p}$ be the restriction of the Hitchin map to this subspace. Then we obtain a system of Poisson commuting Hamiltonians $\widetilde{p} \circ \phi$ on (3.5) for $\phi \in (\mathcal{H}_{G,(z_i),\infty}^{(m_i),m_\infty})^*$.

The Poisson algebra of polynomial functions on (3.5) is isomorphic to

$$\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty}). \tag{3.6}$$

Therefore $\widetilde{p}$ gives rise to a homomorphism of commutative algebras[7]

$$\overline{\Psi}_{(z_i),\infty}^{(m_i),m_\infty} : \operatorname{Fun} \mathcal{H}_{G,(z_i),\infty}^{(m_i),m_\infty} \to \bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty}) \tag{3.7}$$

(actually, it is easy to see that the image belongs to the subalgebra of G-invariants).

Lemma 3.3. *The homomorphism $\overline{\Psi}_{(z_i),\infty}^{(m_i),m_\infty}$ is injective.*

Proof. It is known that (in the general case) the Hitchin map is surjective, see [6,9,13,31,37]. This implies the statement of the lemma. □

Let $\overline{\mathcal{Z}}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ be the image of $\overline{\Psi}_{(z_i),\infty}^{(m_i),m_\infty}$. This is a Poisson commutative subalgebra of the Poisson algebra (3.6).

Now we can explain the connection between the ramified Hitchin systems and the generalized Gaudin algebras $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ introduced in Section 2.5. Note that the algebra $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ is defined as a quotient of a homomorphism of universal enveloping algebra and therefore inherits a filtration that is compatible with the PBW filtration on $\bigotimes_{i=1,\dots,N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty})$. The associated graded algebra to the latter is precisely $\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty})$.

[7] Recall that for an affine algebraic variety Z we denote by $\operatorname{Fun} Z$ the algebra of regular functions on Z.

Theorem 3.4. *The associated graded of the commutative subalgebra*

$$\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \subset \bigotimes_{i=1,\dots,N} U(\mathfrak{g}_{m_i}) \otimes U(\bar{\mathfrak{g}}_{m_\infty})$$

is the Poisson commutative subalgebra

$$\overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \subset \bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty}).$$

Thus, we obtain that $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ is a quantization of the Poisson commutative algebra $\overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ of Hitchin's Hamiltonians.

The proof of this result given below in Section 3.3 follows from the fact that the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$, whose quotient is $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$, is the quantization of the algebra of invariant functions on $\mathfrak{g}^*[\![t]\!]$. It is this local result that is responsible for the global quantization results such as Theorem 3.4 or Theorem 3.14 below (and the results of Beilinson and Drinfeld [7] for general curves in the unramified case). We will explain this local result in the next section.

3.2. *The associated graded algebra of $\mathfrak{z}(\widehat{\mathfrak{g}})$*

The PBW filtration on $U\widehat{\mathfrak{g}}_\kappa$ induces one on $\mathbb{V}_{0,\kappa}$, with associated graded

$$\mathrm{gr}(\mathbb{V}_{0,\kappa}) = S\big(\widehat{\mathfrak{g}}_\kappa((t))/\widehat{\mathfrak{g}}_+\big) = S\big(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!]\big).$$

The action of $\mathfrak{g}[\![t]\!]$ on $\mathbb{V}_{0,\kappa}$ is readily seen to preserve this filtration and therefore descends to one on $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])$. This latter action is independent of the level κ and is given by derivations induced by the adjoint action of $\mathfrak{g}[\![t]\!]$ on $\mathfrak{g}((t))/\mathfrak{g}[\![t]\!]$.

We therefore obtain an embedding

$$\mathrm{gr}\big(\mathbb{V}_{0,\kappa}^{\mathfrak{g}[\![t]\!]}\big) \subset S\big(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!]\big)^{\mathfrak{g}[\![t]\!]}$$

which for $\kappa = \kappa_c$ is in fact an equality, according to Theorem 3.8 below. We begin by describing the right-hand side.

Split $\mathfrak{g}((t))$ (considered as a vector space) as

$$\mathfrak{g}((t)) = \mathfrak{g}[\![t]\!] \oplus \widehat{\mathfrak{g}}_-, \tag{3.8}$$

where $\widehat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}]$, and identify $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])$ with $S(\widehat{\mathfrak{g}}_-)$. Under this identification, the adjoint action of $X \in \mathfrak{g}[\![t]\!]$ on $Y \in \widehat{\mathfrak{g}}_- \cong \mathfrak{g}((t))/\mathfrak{g}[\![t]\!]$ is given by

$$\mathrm{ad}(X)_- Y = [X, Y]_-,$$

where Z_- is the component of $Z \in \mathfrak{g}((t))$ along $\widehat{\mathfrak{g}}_-$.

Let $T = -\partial_t \in \mathrm{Der}\,\mathcal{O}$ be the translation operator acting on $\mathfrak{g}((t))$, so that $TX_n = -nX_{n-1}$ where $X_n = X \otimes t^n \in \widehat{\mathfrak{g}}_\kappa$, $X \in \mathfrak{g}$, $n \in \mathbb{Z}$. Then T preserves the decomposition (3.8) and extends to a derivation of $S(\widehat{\mathfrak{g}}_-)$.

Lemma 3.5.

$$\big[T, \mathrm{ad}(X)_-\big] = \mathrm{ad}(TX)_-.$$

Proof. Both sides are derivations and they coincide on $\widehat{\mathfrak{g}}_-$. □

For any $X \in \mathfrak{g}$, set

$$\overline{X}(z) = \sum_{n<0} \overline{X}_n z^{-n-1} \in (\widehat{\mathfrak{g}}_-)[\![z]\!] \subset S(\widehat{\mathfrak{g}}_-)[\![z]\!],$$

where $\overline{X}_n, n < 0$, denotes $X \otimes t^n$ considered as an element of $S(\widehat{\mathfrak{g}}_-)$. We extend the assignment $X \to \overline{X}(z)$ to an algebra homomorphism

$$S\mathfrak{g} \to S(\widehat{\mathfrak{g}}_-)[\![z]\!]$$

and denote the image of $P \in S\mathfrak{g}$ by $P(z) = \sum_{n<0} P_n z^{-n-1}$. The following summarizes the main properties of the map $P \mapsto P(z)$.

Lemma 3.6. *The following hold for any $P \in S\mathfrak{g}$, $X \in \mathfrak{g}$ and $k \in \mathbb{N}$,*

(i) $TP(z) = \frac{dP(z)}{dz}$,
(ii) $\mathrm{ad}(X_k)_- P(z) = z^k (\mathrm{ad}(X)P)(z)$.

Proof. Both sides of the identities are derivations which are readily seen to coincide on the elements $Y(z)$, $Y \in \mathfrak{g}$. □

Note that property (i) above is equivalent to the fact that, for any $P \in S\mathfrak{g}$ and $n > 0$,

$$P_{-n} = \frac{T^{(n-1)}}{(n-1)!} P_{-1} \tag{3.9}$$

while (ii) implies that the assignment $P \to P(z)$ maps $(S\mathfrak{g})^{\mathfrak{g}}$ to $S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}[\![z]\!]$.

Recall from Lemma 3.1 that $(S\mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra in $\ell = \mathrm{rk}(\mathfrak{g})$ generators and that one may choose a system of homogeneous generators $\overline{P}_1, \dots, \overline{P}_\ell$ such that $\deg \overline{P}_i = d_i + 1$, where the d_i's are the exponents of $\mathfrak{g}$.

The following theorem is due to A. Beilinson and V. Drinfeld [7] (see [20, Proposition 9.3], for an exposition).

Theorem 3.7. *The algebra $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])^{\mathfrak{g}[\![t]\!]}$ is freely generated by the elements $\overline{P}_{i,n}$, $i = 1, \dots, \ell$, $n < 0$.*

The following result, due to [16] (see [20, Theorem 9.6]), enables us to quantize the Hitchin Hamiltonians.

Theorem 3.8. *The inclusion*

$$\mathrm{gr}\,\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathrm{gr}\big(\mathbb{V}^{\mathfrak{g}}_{0,\kappa_c}[\![t]\!]\big) \hookrightarrow S\big(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!]\big)^{\mathfrak{g}[\![t]\!]}$$

is an isomorphism.

In other words, all $\mathfrak{g}[\![t]\!]$-invariants in $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])$ may be quantized.

In particular, each generator $\overline{P}_{i,-1}$ of $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])^{\mathfrak{g}[\![t]\!]}$ may be lifted to an element $S_i \in \mathfrak{z}(\widehat{\mathfrak{g}})$ whose symbol in $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])^{\mathfrak{g}[\![t]\!]}$ is equal to $\overline{P}_{i,-1}$. The element

$$\frac{T^{-n-1}}{(-n-1)!} S_i \in \mathfrak{z}(\widehat{\mathfrak{g}})$$

then gives us a lifting for $\overline{P}_{i,n}, n < -1$. Explicit formulas for these elements are unknown in general (however, recently some elegant formulas have been given in [8] in the case when $\mathfrak{g} = \mathfrak{sl}_n$). But for our purposes we do not need explicit formulas for the S_i's. All necessary information about their structure is in fact contained in the following lemma.

Let us observe that both $\mathbb{V}_{0,\kappa_c}$ and $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])$ are $\mathbb{Z}_+$-graded with the degrees assigned by the formula $\deg A_n = -n$, and this induces compatible $\mathbb{Z}_+$-gradings on $\mathfrak{z}(\widehat{\mathfrak{g}})$ and $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])^{\mathfrak{g}[\![t]\!]}$. This grading is induced by the vector field $L_0 = -t\partial_t$. It follows from the construction that $\deg \overline{P}_{i,-1} = d_i + 1$. Therefore we may, and will, choose $S_i \in \mathbb{V}_{0,\kappa_c}$ to be homogeneous of the same degree.

In $\mathbb{V}_0 = \mathbb{V}_{0,\kappa_c}$ we have a basis of lexicographically ordered monomials of the form

$$J^{a_1}_{-n_1} \dots J^{a_m}_{-n_m} v_0, \quad n_1 \geqslant \dots \geqslant n_m > 0,$$

where $\{J^a\}$ is a basis of $\mathfrak{g}$. The element $\overline{P}_{i,-1}$ of $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!])^{\mathfrak{g}[\![t]\!]}$ is a linear combination of monomials in $\overline{J}^a_{-1}$ of degree $d_i + 1$. Let P_i be the element of $\mathbb{V}_0$ obtained by replacing each of these monomials in the $\overline{J}^a_{-1}$'s by the corresponding lexicographically ordered monomial in the J^a_{-1}'s. Then P_i is the leading term of the sought-after element $S_i \in \mathfrak{z}(\widehat{\mathfrak{g}})$ (for $i = 1$ we actually have $S_1 = P_1$, but for $i > 1$ there are lower order terms). By taking into account the requirement that $\deg S_i = d_i + 1$, we obtain the following useful result.

Lemma 3.9. *The element $S_i \in \mathfrak{z}(\widehat{\mathfrak{g}})$ is equal to P_i plus the sum of lexicographically ordered monomials of orders less than $d_i + 1$. Each of these lower order terms contains at least one factor J^a_n with $n < -1$.*

3.3. Back to the Hitchin systems

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. Observe that the homomorphism $\Phi^{(m_i),m_\infty}_{u,(z_i)}$ given by formula (2.13) preserves filtrations. Therefore it gives rise to a homomorphism of associated graded algebras

$$\overline{\Phi}^{(m_i),m_\infty}_{u,(z_i)} : S(\widehat{\mathfrak{g}}_-) \to \bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty}).$$

Let us consider the image of $S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$ under this map. As in the quantum case, it may also be defined as the span of the coefficients of the series $\overline{\Phi}^{(m_i),m_\infty}_{u,(z_i)}(A)$, $A \in S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$, appearing in front of the monomials of the form $\prod_{i=1}^N (u - z_i)^{-n_i} u^{n_\infty}$.

It follows from the description of $S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$ given in Theorem 3.7 and the construction of the Hitchin map (3.4) that the image of $S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$ under $\overline{\Phi}^{(m_i),m_\infty}_{u,(z_i)}$ is precisely the subalgebra

$$\overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \subset \bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty}).$$

On the other hand, we know that $\overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ is the free polynomial algebra of regular functions on the graded vector space $\mathcal{H}^{(m_i),m_\infty}_{G,(z_i),\infty}$. From its description in formula (3.4) we obtain that we may choose as homogeneous generators of this polynomial algebra the images under $\overline{\Phi}^{(m_i),m_\infty}_{u,(z_i)}$ of the following generators of $S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$:

$$\overline{P}_{i,-n_i-1}, \quad i=1,\dots,\ell; \quad 0 \leqslant n_i \leqslant (d_i+1)\left(\sum_j m_j + m_\infty - 2\right)$$

(note that the number on the right is the degree of the line bundle $\Omega(m_1 z_1 + \cdots + m_N z_N + m_\infty \infty)^{\otimes (d_j+1)}$). Each of them lifts to a generator of $\mathfrak{z}(\widehat{\mathfrak{g}})$ (same notation, but without a bar). It is clear from the definition of $\Phi^{(m_i),m_\infty}_{u,(z_i)}$ that the images of these generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ in $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ under the homomorphism $\Phi^{(m_i),m_\infty}_{u,(z_i)}$ generate $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$.

Thus, we obtain two sets of generators of $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ and $\overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$, such that the latter are the symbols of the former. In addition, the latter are algebraically independent. Therefore we find that the symbol of any homogeneous polynomial in the generators of $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ is equal to the corresponding polynomial in their symbols. Thus, we obtain the desired isomorphism

$$\operatorname{gr} \mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \simeq \overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}).$$

Furthermore, it fits into a commutative diagram

$$\begin{tikzcd}
\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) \arrow[r, "\Phi^{(m_i),m_\infty}_{u,(z_i)}"] \arrow[d] & \operatorname{gr} \mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \arrow[d] \\
S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]} \arrow[r, "\overline{\Phi}^{(m_i),m_\infty}_{u,(z_i)}"] & \overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})
\end{tikzcd}$$

where the vertical maps are isomorphisms. □

Remark 3.10. The problem we had to deal with in the above proof is that *a priori* we do not have a well-defined map

$$\operatorname{gr} \mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \to \overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}). \tag{3.10}$$

However, such a map may be constructed following [7].

We have discussed above the Hitchin map on the moduli space of Higgs bundles on $\mathbb{P}^1$ corresponding to the *trivial* G-bundle (as this is the only G-bundle on $\mathbb{P}^1$ that is semi-stable). This moduli space is in fact an open substack in the moduli stack $\mathrm{Bun}_{G,(z_i),\infty}^{(m_i),m_\infty}$ of all Higgs bundles (corresponding to arbitrary G-bundles). The Hitchin map extends to a proper morphism from the entire stack of Higgs bundles to $\mathcal{H}_G$ (see, e.g., [7]). This implies that the algebra of global functions on the stack of Higgs bundles is equal the algebra of polynomial functions on the Hitchin space $\mathcal{H}_{G,(x_i)}^{(m_i)}$; that is, the algebra $\overline{\mathcal{Z}}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$. On the other hand, as in [7], the algebra $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ may be identified with a subalgebra (and, *a posteriori*, the entire algebra) of the algebra of (critically twisted) global differential operators on the stack $\mathrm{Bun}_{G,(z_i),\infty}^{(m_i),m_\infty}$ of G-bundles on $\mathbb{P}^1$ with the level structures. Now we obtain a well-defined injective map (3.10): it corresponds to taking the symbol of a differential operator. (Recall that the symbol of a global differential operator on a variety, or an algebraic stack, M is a function on the cotangent bundle to M. In our case, $M = \mathrm{Bun}_{G,(z_i),\infty}^{(m_i),m_\infty}$ and T^*M is isomorphic to the stack of Higgs bundles.) The surjectivity of (3.10) follows in the same way as above. Thus, we can obtain another proof of Theorem 3.4 this way.

3.4. Hitchin systems with non-trivial characters

In Section 2.8 we have generalized the definition of $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ to allow for non-trivial characters $\chi_i : t^{m_i}\mathfrak{g}[[t]] \to \mathbb{C}$ and $\chi_\infty : t^{m_\infty}\mathfrak{g}[[t]] \to \mathbb{C}$. A similar generalization is also possible classically. In order to define it, we apply a Hamiltonian reduction.

Let us recall that any character $\chi : t^m\mathfrak{g}[[t]] \to \mathbb{C}$ is necessarily trivial on the Lie subalgebra $t^{2m}\mathfrak{g}[[t]]$ and hence is determined by a character $t^m\mathfrak{g}[[t]]/t^{2m}\mathfrak{g}[[t]] \to \mathbb{C}$, which is just an arbitrary linear functional on $t^m\mathfrak{g}[[t]]/t^{2m}\mathfrak{g}[[t]]$ as this Lie algebra is abelian. We will write it in the form

$$\chi = \sum_{k=-m-1}^{-2m} \chi_k t^k\, dt, \qquad \chi_k \in \mathfrak{g}^*$$

(with respect to the residue pairing).

Now consider the cotangent bundle $T^*\mathcal{M}_{G,(z_i),\infty}^{(2m_i),2m_\infty}$ which is isomorphic to (3.2) with $m_i \mapsto 2m_i$, $m_\infty \mapsto 2m_\infty$. This is a Poisson manifold equipped with a Poisson action of the commutative Lie group $\prod_{\alpha=1,\dots,N,\infty} G_{m_\alpha,2m_\alpha}$, where $G_{m,2m}$ is the Lie group of $t^m\mathfrak{g}[[t]]/t^{2m}\mathfrak{g}[[t]]$. Now we apply the Hamiltonian reduction with respect to this action and the one-point orbit in the dual space to the Lie algebra of $\prod_{\alpha=1,\dots,N,\infty} G_{m_\alpha,2m_\alpha}$ corresponding to $((\chi_i), \chi_\infty)$.

Let us denote the resulting Poisson manifold by $\mathcal{M}_{G,(z_i),\infty;(\chi_i),\chi_\infty}^{(m_i),m_\infty}$. Its points may be identified with collections $((g_i), g_\infty; \eta)$, where $g_i \in G_{m_i}$, $g_\infty \in \overline{G}_{m_\infty}$ (where G_m is the Lie group of $\mathfrak{g}_m$ and $\overline{G}_m$ is the Lie group of $\bar{\mathfrak{g}}_m$) and η is a one-form (3.3), where $A_i(t-z_i)$ now has the form

$$A_i(t-z_i)d(t-z_i) = \left(\sum_{k=-m_i}^{-1} A_{i,k}(t-z_i)^k + \sum_{k=-m_i-1}^{-2m_i} \eta_k(t-z_i)^k\right) d(t-z_i), \qquad A_{i,k} \in \mathfrak{g}^*,$$

and similarly for $A_\infty(t^{-1})d(t^{-1})$. In other words, we allow the polar parts of η to have orders $2m_\alpha$, but fix the m_α most singular terms in the expansion to be given by the character $\mathrm{Ad}_{g_\alpha}(\chi_\alpha)$.

Consider now the Hitchin map defined by formula (3.4). Taking pull-backs of polynomial functions via the Hitchin map, we obtain a Poisson commutative subalgebra $\overline{\mathcal{A}}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})_{(\chi_i),\chi_\infty}$

in the algebra of functions on the Poisson manifold $\mathcal{M}^{(m_i),m_\infty}_{G,(z_i),\infty;(\chi_i),\chi_\infty}$ generalizing $\overline{\mathcal{Z}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$. This manifold fibers over $\prod_{i=1}^N G_{m_i} \times \overline{G}_{m_\infty}$. In the case when all $\chi_\alpha \equiv 0$ the fiber over the identify element in $\prod_{i=1}^N G_{m_i} \times \overline{G}_{m_\infty}$ is a Poisson submanifold isomorphic to $\prod_{i=1}^N \mathfrak{g}^*_{m_i} \times \bar{\mathfrak{g}}^*_{m_\infty}$ with its Kirillov–Kostant structure. This means that the restriction of the algebra of Hitchin Hamiltonians to this fiber gives rise to a Poisson commutative subalgebra of $\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty})$.

But for general characters (χ_i), χ_∞ the fiber over the identity element is no longer a Poisson submanifold (in other words, its defining ideal is not a Poisson ideal). This is due to the fact that $\mathfrak{g}_m$ acts non-trivially on non-zero characters $\chi : t^m\mathfrak{g}[\![t]\!] \to \mathbb{C}$. Therefore we cannot restrict the Hitchin system to this fiber. The best we can do is to consider the commutative Poisson subalgebra in the algebra of functions on the entire manifold $\mathcal{M}^{(m_i),m_\infty}_{G,(z_i),\infty;(\chi_i),\chi_\infty}$. This is analogous to the fact that we cannot realize the corresponding quantum algebras $\mathcal{A}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(\chi_i),\chi_\infty}$ as subalgebras of universal enveloping algebras, as we pointed out in Section 2.8.[8]

However, as in the quantum case, there one exception, namely, when the characters $\chi_i \equiv 0$ for all $i = 1,\dots,N$, and the character χ_∞ becomes zero when restricted to $t^{m+1}\mathfrak{g}[\![t]\!]$. It may then take non-zero values on $\mathfrak{g} \otimes t^m \simeq \mathfrak{g}$, determined by some $\chi \in \mathfrak{g}^*$. In this case the fiber of our Poisson manifold over the identity is Poisson and is isomorphic to $\prod_{i=1}^N \mathfrak{g}^*_{m_i} \times \bar{\mathfrak{g}}^*_{m_\infty-1}$ with its Kirillov–Kostant structure. This means that the restriction of the algebra of Hitchin Hamiltonians to this fiber gives rise to a Poisson commutative subalgebra of $\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty-1})$.

This Poisson algebra may be described in more concrete terms as follows. Note that the fiber of $\mathcal{M}^{(m_i),m_\infty}_{G,(z_i),\infty;(0),\chi}$ at the identity is the space of one-forms

$$\eta = \left(\sum_{i=1,\dots,N} \sum_{k=-m_i}^{-1} A_{i,k}(t-z_i)^k - \sum_{k=0}^{m_\infty-2} A_{\infty,-k-2} t^k - \chi t^{m_\infty-1} \right) dt, \tag{3.11}$$

where $A_{\alpha,k} \in \mathfrak{g}^*$. In other words, we fix the leading singular term of η at the point ∞ to be equal to $\chi \in \mathfrak{g}^*$.

This space is thus isomorphic to $\prod_{i=1}^N \mathfrak{g}^*_{m_i} \times \bar{\mathfrak{g}}^*_{m_\infty-1}$, and the algebra of polynomial functions on it is isomorphic, as a Poisson algebra, to $\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty-1})$, as we discussed above. Now we define $\overline{\mathcal{A}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ as the Poisson subalgebra of $\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty-1})$ generated by the coefficients of the invariant polynomials $\overline{P}_\ell(\eta)$ in front of the monomials in $(t-z_i)^{-n_i} t^{n_\infty}$. One shows in the same way as above that this is a Poisson commutative subalgebra of $\bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty-1})$.

Now, we claim that the algebra $\mathcal{A}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$, discussed at the very end of Section 2.8, is a quantization of the Poisson algebra $\overline{\mathcal{A}}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ for any $\chi \in \mathfrak{g}^*$. We will prove this in Section 3.6 in the special case when $N = 1$, $m_1 = 1$ and $m_2 = 1$ (we will also set $z_1 = 0$, but this is not essential). The proof in general is very similar and will be omitted.

We thus set out to prove that the algebra $\mathcal{A}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}$, denoted by $\mathcal{A}_\chi$ in Section 2.7, is a quantization of the algebra $\overline{\mathcal{A}}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}$, just defined. Let us denote the latter by $\overline{\mathcal{A}}_\chi$. By definition,

[8] This discussion suggests these algebras may instead be realized as commutative subalgebras of algebras of twisted differential operators on $\mathcal{M}^{(m_i),m_\infty}_{G,(z_i),\infty}$, with the twisting determined by the characters (χ_i), χ_∞.

$\mathcal{A}_\chi$ is a commutative subalgebra of $U(\mathfrak{g})$, and $\overline{\mathcal{A}}_\chi$ is a Poisson commutative subalgebra of $S(\mathfrak{g})$. Saying that $\mathcal{A}_\chi$ is a quantization of $\overline{\mathcal{A}}_\chi$ simply means that

$$\operatorname{gr}\mathcal{A}_\chi = \overline{\mathcal{A}}_\chi, \tag{3.12}$$

where $\operatorname{gr}\mathcal{A}_\chi$ is the associated graded algebra of $\mathcal{A}_\chi$ with respect to the PBW filtration on $U(\mathfrak{g})$.

First, we take a closer look at $\overline{\mathcal{A}}_\chi$ and show that it is nothing but the shift of argument subalgebra introduced in [38].

3.5. The shift of argument subalgebra $\overline{\mathcal{A}}_\chi$

Let us fix $\chi \in \mathfrak{g}^*$ and consider the space of one-forms (3.11) in the special case when $N = 1$, $z_1 = 0$, $m_1 = 1$ and $m_2 = 1$. This space consists of the one-forms

$$\eta = \frac{A}{t} - \chi, \quad A \in \mathfrak{g}^*,$$

and is therefore isomorphic to $\mathfrak{g}^*$. Following the above general definition, we define $\overline{\mathcal{A}}_\chi = \overline{\mathcal{A}}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}$ as the subalgebra of $S(\mathfrak{g}) = \operatorname{Fun}\mathfrak{g}^*$ generated by the coefficients of the polynomials in t,

$$\overline{P}_i\left(\frac{A}{t} - \chi\right), \quad i = 1, \dots, \ell.$$

Equivalently, $\overline{\mathcal{A}}_\chi$ is the subalgebra of $\operatorname{Fun}\mathfrak{g}^*$ generated by the iterated directional derivatives $D^i_\chi P$ of invariant polynomials $P \in (S\mathfrak{g})^{\mathfrak{g}} = \operatorname{Fun}\mathfrak{g}^*$ in the direction χ, where

$$D_\chi P(x) = \frac{d}{du}\bigg|_{u=0} P(x + u\chi). \tag{3.13}$$

This definition makes it clear that $\overline{\mathcal{A}}_\chi = \overline{\mathcal{A}}_{c\chi}$ for any non-zero $c \in \mathbb{C}$. We also have $\operatorname{Ad}_g(\overline{\mathcal{A}}_\chi) = \overline{\mathcal{A}}_{\operatorname{Ad}_g(\chi)}$ for any $g \in G$.

Note that for any $P \in S\mathfrak{g}$, $x \in \mathfrak{g}^*$ and $u \in \mathbb{C}$, we have

$$P(x + u\chi) = \sum_{m \geqslant 0} \frac{u^m}{m!} D^m_\chi P(x).$$

Therefore we find that $\overline{\mathcal{A}}_\chi$ may also be defined as the subalgebra of $S\mathfrak{g}$ generated by the shifted polynomials

$$P_{u\chi}(x) = P(x + u\chi), \tag{3.14}$$

where P varies in $(S\mathfrak{g})^{\mathfrak{g}}$ and $u \in \mathbb{C}$.

The algebra $\overline{\mathcal{A}}_\chi$ was introduced in [38], where it was shown to be Poisson commutative and of maximal possible transcendence degree $\dim\mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, provided χ is regular semi-simple element of $\mathfrak{g}^* \simeq \mathfrak{g}$. Here and below we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using a non-degenerate invariant inner product on $\mathfrak{g}$. These results were recently elucidated and extended by

B. Kostant to regular nilpotent elements χ [34]. We will show now that a slight generalization of the arguments used in [34,38] yields the same result for all regular elements. We recall that $a \in \mathfrak{g}$ is called *regular* if its centralizer in $\mathfrak{g}$ has the smallest possible dimension; namely, ℓ, the rank of $\mathfrak{g}$.

Theorem 3.11. *Let χ be any regular element of $\mathfrak{g}^* \simeq \mathfrak{g}$. Then $\overline{\mathcal{A}}_\chi$ is a free polynomial algebra in* $\dim \mathfrak{b}$ *generators*

$$D_\chi^{n_i} \overline{P}_i, \quad i = 1, \dots, \ell;\ n_i = 0, \dots, d_i, \tag{3.15}$$

where $\overline{P}_i$ is a generator of $S(\mathfrak{g})^{\mathfrak{g}}$ of degree $d_i + 1$ and D_χ is the derivative in the direction of χ given by formula (3.13).

Proof. It is clear from the definition that $\overline{\mathcal{A}}_\chi$ is generated by the iterated derivatives (3.15). Therefore we need to prove that the above polynomial functions on $\mathfrak{g}^*$ are algebraically independent for regular χ. For that it is sufficient to show that the differentials of these functions at a particular point $\eta \in \mathfrak{g}^*$ are linearly independent. Note that we have

$$\overline{P}_i(x + u\chi) = \sum_{m=0}^{d_i+1} \frac{u^m}{m!} D_\chi^m \overline{P}_i(x),$$

where the last coefficient, $D_\chi^{d_i+1} \overline{P}_i(x)$, is a constant. Thus, we need to show that the coefficients $C_{i,k} = C_{i,k}(\eta) \in \mathfrak{g}$ appearing in the u-expansion of the differential $d\overline{P}_i(x + u\chi)|_{x=\eta}$ of $\overline{P}_i(x + u\chi)$ (considered as a function of $x \in \mathfrak{g}^*$ with fixed u) at $\eta \in \mathfrak{g}^*$,

$$d\overline{P}_i(x + u\chi) = \sum_{k=0}^{d_i} C_{i,k} u^k,$$

are linearly independent.

Note that for any $\eta \in \mathfrak{g}^*$ and any $\mathfrak{g}$-invariant function $\overline{P}$ on $\mathfrak{g}^*$ we have $[\eta, d\overline{P}(\eta)] = 0$. Therefore we have

$$\big[\eta + u\chi, d\overline{P}_i(\eta + u\chi)\big] = 0.$$

Expanding the last equation in powers of u, we obtain the following system (see Lemma 6.1.1 of [38]):

$$\begin{gathered} [\eta, C_{i,0}] = 0, \\ [\eta, C_{i,k}] + [\chi, C_{i,k-1}] = 0, \quad 0 < k < d_i, \\ [\chi, C_{i,d_i}] = 0. \end{gathered} \tag{3.16}$$

Let us first prove that the elements $C_{i,k}$ are linearly independent when χ is a regular nilpotent element, following an idea from the proof of this theorem for regular semi-simple χ given in [38].

Let $\{e, 2\check{\rho}, f\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ such that $e = \chi$. We choose the element $\check{\rho}$ as our η – this is the point at which we evaluate the $C_{i,k}$'s. Under the adjoint action of $\check{\rho}$ the Lie algebra $\mathfrak{g}$ decomposes as follows:

$$\mathfrak{g} = \bigoplus_{i=-h}^{h} \mathfrak{g}_i,$$

where h is the Coxeter number. Here $\mathfrak{g}_0$ is the Cartan subalgebra containing $\check{\rho}$ and e is an element of $\mathfrak{g}_1$.

Let

$$W_j = \operatorname{span}\{C_{i,k} \mid i = 1, \dots, \ell;\ k = 0, \dots, j\},$$

where we set $C_{i,k} = 0$ for $k > d_i$. We will now prove that the following equality is true for all $j \geqslant 0$:

$$W_j = \bigoplus_{i=0}^{j} \mathfrak{g}_i. \tag{3.17}$$

Let us prove it for $j = 0$. The first equation in (3.16) implies that $C_{i,0}$, $i = 1, \dots, \ell$, belong to $\mathfrak{g}_0$. Furthermore, since $\check{\rho}$ is regular, they span $\mathfrak{g}_0$, by [33]. Therefore $W_0 = \mathfrak{g}_0$.

Now suppose that we have proved (3.17) for $j = 0, \dots, m$. Let us prove it for $j = m + 1$.

According to Eqs. (3.16), we have

$$\operatorname{ad} \check{\rho} \cdot W_{m+1} = \operatorname{ad} e \cdot W_m.$$

It follows from general results on representations of $\mathfrak{sl}_2$ that the map

$$\operatorname{ad} e : \mathfrak{g}_i \to \mathfrak{g}_{i+1}$$

is surjective for all $i \geqslant 0$. Therefore, using our inductive assumption, we obtain that $\operatorname{ad} e \cdot W_m = \bigoplus_{i=1}^{m+1} \mathfrak{g}_i$. Since $\operatorname{ad} \check{\rho}$ is invertible on this space and the kernel of $\operatorname{ad} \check{\rho}$ on the entire $\mathfrak{g}$ is equal to $\mathfrak{g}_0$, we find that W_{m+1} is necessarily contained in $\bigoplus_{i=0}^{m+1} \mathfrak{g}_i$ and its projection onto $\bigoplus_{i=1}^{m+1} \mathfrak{g}_i$ along $\mathfrak{g}_0$ is surjective. On the other hand, W_{m+1} contains $C_{i,0}$, $i = 1, \dots, \ell$, and therefore contains $\mathfrak{g}_0$. Hence we obtain the equality (3.17) for $j = m + 1$. This completes the inductive step and hence proves (3.17) for all $j \geqslant 0$.

Setting $j = h$, we obtain that the span of the elements C_{i,n_i}, $i = 1, \dots, \ell$; $n_i = 0, \dots, d_i$, is equal to $\bigoplus_{i=0}^{h} \mathfrak{g}_h$, which is a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Its dimension is equal to the number of these elements, which implies that they are linearly independent. This proves that the polynomials (3.15) are algebraically independent, and so $\overline{\mathcal{A}}_\chi$ is a free polynomial algebra in $\dim \mathfrak{b}$ generators, if χ is a regular nilpotent element.

Now we derive that the same property holds for $\overline{\mathcal{A}}_\chi$, where χ is an arbitrary regular element, following an argument suggested to us by L. Rybnikov (see also [34]). Consider the subset $S \subset \mathfrak{g}^* \simeq \mathfrak{g}$ of all elements satisfying the property that the polynomials (3.15) are algebraically independent. It is clear that this is a Zariski open subset of $\mathfrak{g}$, and it is non-empty since it contains regular nilpotent elements. Let χ be a regular element of $\mathfrak{g}^* \simeq \mathfrak{g}$. Recall that we have $\overline{\mathcal{A}}_\chi = \overline{\mathcal{A}}_{c\chi}$

for any non-zero $c \in \mathbb{C}$ and $\mathrm{Ad}_g(\overline{\mathcal{A}}_\chi) = \overline{\mathcal{A}}_{\mathrm{Ad}_g(\chi)}$ for any $g \in G$, where G is the adjoint group of $\mathfrak{g}$. This implies that S is a conic subset of $\mathfrak{g}$ that is invariant under the adjoint action of G. According to [33], the adjoint orbit of any regular element of $\mathfrak{g}$ contains an element of the form $e + A$, where e is a regular nilpotent element of an $\mathfrak{sl}_2$-triple $\{e, 2\check{\rho}, f\}$, and $A \in \bigoplus_{i \leqslant 0} \mathfrak{g}_i$. Hence we may restrict ourselves to the elements χ of this form.

Suppose that some χ of this form does not belong to the set S. Then neither do the elements

$$\chi_c = c \,\mathrm{Ad}\, \check{\rho}(c)^{-1}(\chi), \quad c \in \mathbb{C}^\times,$$

where $\check{\rho} : \mathbb{C}^\times \to G$ is the one-parameter subgroup of G corresponding to $\check{\rho} \in \mathfrak{g}$. But then the limit of χ_c as $c \to 0$ should not be in S. However, this limit is equal to e, which is a regular nilpotent element, and hence belongs to S. This is a contradiction, which implies that all regular elements of $\mathfrak{g}$ belong to S. This completes the proof. $\square$

The Poisson commutativity of $\overline{\mathcal{A}}_\chi$ follows from the above general results about the commutativity of the Hitchin Hamiltonians. It also follows from Theorem 3.14 below.

3.6. The quantization theorem

In [53] the problem of the existence of a quantization of $\overline{\mathcal{A}}_\chi$ was posed: does there exist a commutative subalgebra $\mathcal{A}_\chi$ of $U(\mathfrak{g})$ which satisfies (3.12)?

Such a quantization has been constructed for $\mathfrak{g}$ of classical types in [42], using twisted Yangians, and for $\mathfrak{g} = \mathfrak{sl}_n$ in [50], using the symmetrization map, and in [8], using explicit formulas. In [46] it was shown that, if it exists, a quantization of $\overline{\mathcal{A}}_\chi$ is unique for generic χ.

Recently, the quantization problem was solved in [45] for any simple Lie algebra $\mathfrak{g}$ and any regular semi-simple $\chi \in \mathfrak{g}^*$. More precisely, it was shown in [45] that the algebra $\mathcal{A}_\chi$, constructed in essentially the same way as in Section 2.7, is a quantization of $\overline{\mathcal{A}}_\chi$ for any regular semi-simple χ.

We will now prove that $\mathcal{A}_\chi$ is a quantization of $\overline{\mathcal{A}}_\chi$ for any regular $\chi \in \mathfrak{g}^*$. First, we prove the following statement, which is also implicit in [45].

Proposition 3.12. *The algebra $\overline{\mathcal{A}}_\chi$ is contained in* $\mathrm{gr}\,\mathcal{A}_\chi$ *for any $\chi \in \mathfrak{g}^*$.*

Proof. By definition (see Section 2.7) and Proposition 2.8, the algebra $\mathcal{A}_\chi$ is generated by the coefficients of the Laurent expansion of $\mathrm{id} \otimes \chi(\Phi^{1,2}_{u,(0)}(A))$ at $u = 0$, where A ranges over a system of generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$. Therefore we need to show that the symbol of each of these coefficients belongs to $\overline{\mathcal{A}}_\chi$.

Since $\Phi^{1,2}_{u,(0)}(TA) = \partial_u \Phi^{1,2}_{u,(0)}(A)$, where $T \in \mathrm{Der}\,\mathcal{O}$ is the translation operator given by (2.6), it suffices in fact to consider a system of generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ as an algebra endowed with the derivation T (equivalently, as a commutative vertex algebra).

By Theorem 3.8, $\mathrm{gr}(\mathfrak{z}(\widehat{\mathfrak{g}})) = S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$ and, by Theorem 3.7, $S(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[\![t]\!]}$ is generated, as an algebra endowed with the derivation T, by the polynomials P_{-1}, as P varies in $(S\mathfrak{g})^{\mathfrak{g}}$. It follows that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is generated, as an algebra with a derivation, by elements Π_{-1}, where Π_{-1} is such that its symbol is P_{-1}, and P varies in $(S\mathfrak{g})^{\mathfrak{g}}$.

The computation of the symbol of $\mathrm{id} \otimes \chi(\Phi^{1,2}_{u,(0)}(\Pi_{-1}))$ will be carried out in Lemma 3.13 below. To that end, it will be necessary to choose the lifts Π_{-1} in the following way. First, we will only consider homogeneous generators of $S\mathfrak{g}$. If $P \in S^d\mathfrak{g}$, then P_{-1} is of degree d with respect to the $\mathbb{Z}_+$-grading on $S(\widehat{\mathfrak{g}}_-)$ introduced in Section 3.2. We may, and will, assume that the lift Π_{-1} to $\mathfrak{z}(\widehat{\mathfrak{g}})$ is also homogeneous of degree d.

The proof of Proposition 3.12 is now completed by the following calculation. Let $\Pi \in U(\widehat{\mathfrak{g}}_-)$ be an element of order $d \in \mathbb{N}$ and degree d with respect to the $\mathbb{Z}_+$-grading on $U(\widehat{\mathfrak{g}}_-)$. Consider its symbol $\sigma(\Pi) \in S^d(\mathfrak{g} \otimes t^{-1})$ as an element $P \in S^d\mathfrak{g}$ via the linear isomorphism $\mathfrak{g} \otimes t^{-1} \cong \mathfrak{g}$. For each $n \in \mathbb{Z}$ denote by $((\mathrm{id} \otimes \chi) \circ \Phi^{1,2}_{u,(0)}(\Pi))_n$ the coefficient of u^n in $(\mathrm{id} \otimes \chi) \circ \Phi^{1,2}_{u,(0)}(\Pi)$. □

Lemma 3.13. *For any $k \geqslant 0$ the following holds: if $D^k_\chi P \neq 0$, then the symbol of $(\mathrm{id} \otimes \chi) \circ \Phi^{1,2}_{u,(0)}(\Pi)$ is equal to*

$$\sigma\big((\mathrm{id} \otimes \chi) \circ \Phi^{1,2}_{u,(0)}(\Pi)\big)_{-d+k} = \frac{(-1)^k}{k!} D^k_\chi P.$$

Proof. By assumption, Π is a linear combination of lexicographically ordered monomials of the form

$$\Pi_{(a_i,n_i,b_j)} = J^{a_1}_{-n_1} \cdots J^{a_p}_{-n_p} J^{b_1}_{-1} \cdots J^{b_q}_{-1},$$

where $n_1 \geqslant \cdots \geqslant n_p \geqslant 2$,

$$\mathrm{ord}(\Pi_{(a_i,n_i,b_j)}) = p + q \leqslant d \quad \text{and} \quad \deg(\Pi_{(a_i,n_i,b_j)}) = \sum_i n_i + q = d.$$

Note that if $p \neq 0$, then

$$\mathrm{ord}(\Pi_{(a_i,n_i,b_j)}) = p + q < \sum_i n_i + q = d,$$

since $n_i \geqslant 2$. Thus, only the monomials with $p = 0$ contribute to $\sigma(\Pi)$, so that

$$\sigma(\Pi) = \sum_{1 \leqslant b_1 \leqslant \cdots \leqslant b_d \leqslant \dim \mathfrak{g}} \alpha_{b_1,\dots,b_d} \bar{J}^{b_1}_{-1} \cdots \bar{J}^{b_d}_{-1}$$

for some constants $\alpha_{b_1,\dots,b_d} \in \mathbb{C}$, where the product is now that of $S(\widehat{\mathfrak{g}}_-)$ and

$$P = \sum_{1 \leqslant b_1 \leqslant \cdots \leqslant b_d \leqslant \dim \mathfrak{g}} \alpha_{b_1,\dots,b_d} J^{b_1} \cdots J^{b_d}.$$

For $P = \overline{P}_i$ this is in fact the statement of Lemma 3.9.

By (2.10),

$$\begin{aligned}(\mathrm{id} \otimes \chi) \circ \Phi^{1,2}_{u,(0)}(\Pi_{(a_i,n_i,b_j)}) &= \left(\frac{J^{b_q}}{u} - \chi\big(J^{b_q}\big)\right) \cdots \left(\frac{J^{b_1}}{u} - \chi\big(J^{b_1}\big)\right) \frac{J^{a_p}}{u^{n_p}} \cdots \frac{J^{a_1}}{u^{n_1}} \\ &= u^{-d}\big(J^{b_q} - u\chi\big(J^{b_q}\big)\big) \cdots \big(J^{b_1} - u\chi\big(J^{b_1}\big)\big) J^{a_p} \cdots J^{a_1}.\end{aligned}$$

The coefficient of u^{-d+k} in $(\mathrm{id}\otimes\chi)\circ\Phi^{1,2}_{u,(0)}(\Pi_{(a_i,n_i,b_j)})$ is therefore an element of $U\mathfrak{g}$ of order $\leqslant p+q-k$. If $p\neq 0$, this is strictly less than $\sum_i n_i+q-k=d-k$ since $n_i\geqslant 2$. If, on the other hand, $p=0$, then the coefficient of u^{-d+k} is proportional to a lexicographically ordered monomial in $U\mathfrak{g}$ of order d. It therefore follows that $\sigma((\mathrm{id}\otimes\chi)\circ\Phi^{1,2}_{u,(0)}(\Pi))_{-d+k}$ is the coefficient of u^k in

$$\sum_{1\leqslant b_1\leqslant\cdots\leqslant b_d\leqslant\dim\mathfrak{g}}\alpha_{b_1,\ldots,b_d}\big(J^{b_d}-u\chi\big(J^{b_d}\big)\big)\cdots\big(J^{b_1}-u\chi\big(J^{b_1}\big)\big)=u^{-d}P(\cdot-u\chi),$$

provided that this coefficient is non-zero (otherwise, we would obtain the symbol of the first non-zero lower order term). This implies the statement of the lemma. □

Theorem 3.14. *For regular $\chi\in\mathfrak{g}^*$ we have $\mathrm{gr}\,\mathcal{A}_\chi=\overline{\mathcal{A}}_\chi$, and so the commutative algebra $\mathcal{A}_\chi\subset U\mathfrak{g}$ is a quantization of the shift of argument subalgebra $\overline{\mathcal{A}}_\chi\subset S\mathfrak{g}$.*

Proof. We know that $\overline{\mathcal{A}}_\chi$ is generated by

$$D_\chi^{k_i}\overline{P}_i,\quad i=1,\ldots,\ell;\ 0\leqslant k_i\leqslant d_i.$$

On the other hand, it follows from the definition of $\mathcal{A}_\chi$ that it is generated by

$$\big((\mathrm{id}\otimes\chi)\circ\Phi^{1,2}_{u,(0)}(S_i)\big)_{-d_i-1+k_i},\quad i=1,\ldots,\ell;\ 0\leqslant k_i\leqslant d_i.$$

If χ is regular, then each $D_\chi^{k_i}\overline{P}_i\neq 0$, by Theorem 3.11, and therefore Lemma 3.13 implies that the generators of $\overline{\mathcal{A}}_\chi$ are equal to the symbols of the generators of $\mathcal{A}_\chi$, up to non-zero scalars. In addition, the generators $D_\chi^{k_i}\overline{P}_i$ of $\overline{\mathcal{A}}_\chi$ are algebraically independent for regular χ, by Theorem 3.11. Therefore, again applying Lemma 3.13, we obtain that the symbol of any non-zero element of $\mathcal{A}_\chi$ is a non-zero element of $\overline{\mathcal{A}}_\chi$. Therefore $\mathrm{gr}\,\mathcal{A}_\chi\subset\overline{\mathcal{A}}_\chi$. Combining this with Proposition 3.12, we obtain the assertion of the theorem. □

For non-regular χ, Proposition 3.12 implies that all elements of $\overline{\mathcal{A}}_\chi$ may be quantized, i.e., lifted to commuting elements of $\mathcal{A}_\chi$, but this still leaves open the possibility that the quantum algebra $\mathcal{A}_\chi$ is larger than its classical counterpart $\overline{\mathcal{A}}_\chi$ (we thank L. Rybnikov for pointing this out). However, we conjecture that this never happens:

Conjecture 1. *We have $\mathrm{gr}\,\mathcal{A}_\chi=\overline{\mathcal{A}}_\chi$ for all $\chi\in\mathfrak{g}^*$.*

By Theorem 3.14, this conjecture holds in the regular case, and it also holds in the most degenerate case when $\chi=0$. In that case $\mathcal{A}_\chi$ is the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ and $\overline{\mathcal{A}}_\chi=S(\mathfrak{g})^{\mathfrak{g}}$.

3.7. The DMT Hamiltonians

As we saw in Section 2.6, one can write down explicit formulas for quadratic generators of the algebra $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g})$; these are the original Gaudin Hamiltonians Ξ_i, $i=1,\ldots,N$.

In this section we determine quadratic generators of the algebra $\mathcal{A}_\chi$ for a regular semi-simple $\chi\in\mathfrak{g}$. They turn out to be none other than the DMT Hamiltonians discussed in the Introduction.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ containing χ and $\Delta \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$. For each $\alpha \in \Delta$, let $\mathfrak{sl}_2^\alpha = \langle e_\alpha, f_\alpha, h_\alpha \rangle \subset \mathfrak{g}$ be the corresponding three-dimensional subalgebra and denote by

$$C_\alpha = \frac{(\alpha, \alpha)}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha)$$

its truncated Casimir operator with respect to the restriction to $\mathfrak{sl}_2^\alpha$ of a fixed non-degenerate invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$. Note that C_α is independent of the choice of the root vectors e_α, f_α and satisfies $C_{-\alpha} = C_\alpha$. Let

$$\mathfrak{h}_{\mathrm{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta} \mathrm{Ker}(\alpha)$$

be the variety of regular elements in $\mathfrak{h}$, V a $\mathfrak{g}$-module and $\mathbf{V} = \mathfrak{h}_{\mathrm{reg}} \times V$ the holomorphically trivial vector bundle over $\mathfrak{h}_{\mathrm{reg}}$ with fiber V. Millson and Toledano Laredo [36,52], and, independently, De Concini (unpublished; a closely related connection was also considered in [18]) introduced the following holomorphic connection on $\mathbf{V}$:

$$\nabla_h = d - \frac{h}{2} \sum_{\alpha \in \Delta} \frac{d\alpha}{\alpha} \cdot C_\alpha = d - h \sum_{\alpha \in \Delta_+} \frac{d\alpha}{\alpha} \cdot C_\alpha,$$

where $\Delta = \Delta_+ \sqcup \Delta_-$ is the partition of Δ into positive and negative roots determined by a choice of simple roots $\alpha_1, \ldots, \alpha_n$ of $\mathfrak{g}$, and proved that ∇_h is flat for any value of the complex parameter h. Define, for every $\gamma \in \mathfrak{h}$, the function $T_\gamma : \mathfrak{h}_{\mathrm{reg}} \to \mathrm{End}(V)$ by

$$T_\gamma(\chi) = \sum_{\alpha \in \Delta_+} \frac{\alpha(\gamma)}{\alpha(\chi)} C_\alpha \tag{3.18}$$

and note that $T_{a\gamma + a'\gamma'} = aT_\gamma + a'T'_\gamma$. Then, the flatness of ∇_h is equivalent to the equations

$$[D_\gamma - hT_\gamma, D_{\gamma'} - hT_{\gamma'}] = 0$$

for any $\gamma, \gamma' \in \mathfrak{h}$, where $D_\gamma f(x) = \frac{d}{dt}\big|_{t=0} f(x + t\gamma)$. Dividing by h and letting h tend to infinity implies that, for a fixed $\chi \in \mathfrak{h}_{\mathrm{reg}}$ and any $\gamma, \gamma' \in \mathfrak{h}$,

$$\big[T_\gamma(\chi), T_{\gamma'}(\chi)\big] = 0.$$

In [53] it was shown that the algebra $\mathcal{A}_\chi$ contains the Hamiltonians $T_\gamma(\chi)$. We now give a proof of this result for completeness.

Let us identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using the invariant inner product $(\cdot,\cdot)$. For each homogeneous polynomial $p \in S\mathfrak{g}^*$ of degree d and $i = 1, \ldots, d$, let $p_\chi^{(i)} \in S^i\mathfrak{g}^*$ be defined by

$$p_\chi^{(i)} = \frac{D_\chi^{d-i} p}{(d-i)!},$$

where $D_\chi f(x) = \frac{d}{dt}\big|_{t=0} f(x+t\chi)$. Note that if p is invariant under $\mathfrak{g}$, then $p_\chi^{(i)}$ is invariant under the centralizer $\mathfrak{g}^\chi \subseteq \mathfrak{g}$ of χ. The following result shows that, when $\chi \in \mathfrak{h}_{\mathrm{reg}}$, the algebra $\overline{\mathcal{A}}_\chi$ contains the symbols $\overline{T}_\gamma(\chi)$ of the Hamiltonians $T_\gamma(\chi)$ defined by (3.18).

Proposition 3.15. *Assume that $\chi \in \mathfrak{h}_{\mathrm{reg}}$. Then,*

(i) *Let $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ be homogeneous and $q \in (S\mathfrak{h}^*)^W$ its restriction to $\mathfrak{h}$. Then*

$$p_\chi^{(2)} = q_\chi^{(2)} + \overline{T}_{\gamma_q}(\chi),$$

where $q_\chi^{(2)} = D_\chi^{d-2} q/(d-2)!$ and $\gamma_q = dq(\chi) \in T_\chi^*\mathfrak{h} \cong \mathfrak{h}$ is the differential of q at χ and the cotangent space $T_\chi^*\mathfrak{h}$ is identified with $\mathfrak{h}$ by means of the form $(\cdot,\cdot)$.

(ii) *As p varies over the homogeneous elements of $(S\mathfrak{g}^*)^{\mathfrak{g}}$, γ_q ranges over the whole of $\mathfrak{h}$.*

We will need the following

Lemma 3.16. *If $p \in S\mathfrak{g}^*$ is $\mathfrak{g}$-invariant and $\chi \in \mathfrak{h}$, then for any $\alpha \in \Delta$,*

$$\alpha(\chi) D_{e_\alpha} D_{f_\alpha} p_\chi^{(2)} = D_{h_\alpha} p(\chi).$$

Proof. Let $P \in (\mathfrak{g}^*)^{\otimes d}$ be the symmetric, multilinear function on $\mathfrak{g}$ defined by $p(x) = P(x, \dots, x)$. The $\mathfrak{g}$-invariance of p implies that

$$\begin{aligned} 0 &= \mathrm{ad}^*(e_\alpha) P(f_\alpha, \underbrace{\chi, \dots, \chi}_{d-1}) \\ &= -P(h_\alpha, \underbrace{\chi, \dots, \chi}_{d-1}) + (d-1)\alpha(\chi) P(f_\alpha, e_\alpha, \underbrace{\chi, \dots, \chi}_{d-2}) \\ &= -\frac{1}{d} D_{h_\alpha} p(\chi) + \alpha(\chi) \frac{d-1}{d!} D_{e_\alpha} D_{f_\alpha} D_\chi^{d-2} p, \end{aligned}$$

whence the claimed result. $\square$

Proof of Proposition 3.15. (i) Let $\nu : \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism of $\mathfrak{g}$-modules induced by $(\cdot,\cdot)$. Since $p_\chi^{(2)} \in S^2\mathfrak{g}^*$ is invariant under $\mathfrak{h} = \mathfrak{g}^\chi$, and $(e_\alpha, f_\alpha) = 2/(\alpha, \alpha)$, we have

$$\begin{aligned} p_\chi^{(2)} &= p_\chi^{(2)}\big|_{\mathfrak{h}} + \sum_{\alpha \in \Delta_+} \left(\frac{(\alpha,\alpha)}{2}\right)^2 \nu(e_\alpha)\nu(f_\alpha) D_{e_\alpha} D_{f_\alpha} p_\chi^{(2)} \\ &= p_\chi^{(2)}\big|_{\mathfrak{h}} + \sum_{\alpha \in \Delta_+} \left(\frac{(\alpha,\alpha)}{2}\right)^2 \nu(e_\alpha)\nu(f_\alpha) \frac{D_{h_\alpha} p(\chi)}{\alpha(\chi)} \\ &= p_\chi^{(2)}\big|_{\mathfrak{h}} + \sum_{\alpha \in \Delta_+} \frac{(\alpha,\alpha)}{2} \nu(e_\alpha)\nu(f_\alpha) \frac{dp(\chi)(\nu^{-1}(\alpha))}{\alpha(\chi)}, \end{aligned}$$

where the second equality follows from Lemma 3.16.

(ii) As p ranges over the homogeneous elements of $(S\mathfrak{g}^*)^{\mathfrak{g}}$, q ranges over those of $(S\mathfrak{h}^*)^W = \mathbb{C}[c_1, \ldots, c_n]$. The differential of the latter span $T^*_\chi\mathfrak{h}$ since the Jacobian of $c_1, \ldots, c_n$ at $\chi \in \mathfrak{h}$ is proportional to $\prod_{\alpha\in\Delta_+} \alpha(\chi)$ [5, V.5.4] and is therefore non-zero since χ is regular. $\square$

Now, according to Theorem 3.14, any element of $\overline{\mathcal{A}}_\chi$ may be lifted to $\mathcal{A}_\chi$. We already know that $\mathfrak{h} \subset \mathcal{A}_\chi$. An arbitrary lifting of $\overline{T}_\gamma(\chi) \in \overline{\mathcal{A}}_\chi$ to $U(\mathfrak{g})$ is equal to $T_\gamma(\chi) + J$, where $J \in \mathfrak{g}$. But the lifting to $\mathcal{A}_\chi$ has to commute with $\mathfrak{h}$. Therefore $J \in \mathfrak{h}$ and so $T_\gamma(\chi)$ itself belongs to $\mathcal{A}_\chi$ for all $\gamma \in \mathfrak{h}$.

4. Recollections on opers

In order to describe the universal Gaudin algebra and its various quotients introduced in Section 2 we need to recall the description of $\mathfrak{z}(\widehat{\mathfrak{g}})$. According to [16,20], $\mathfrak{z}(\widehat{\mathfrak{g}})$ is identified with the algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(D)$ of (regular) functions on the space $\operatorname{Op}_{^LG}(D)$ of ^LG-opers on the disc $D = \operatorname{Spec}\mathbb{C}[\![t]\!]$. Here ^LG is the Lie group of adjoint type corresponding to the Lie algebra $^L\mathfrak{g}$ whose Cartan matrix is the transpose of that of $\mathfrak{g}$. Note that ^LG is the Langlands dual group of the connected simply-connected Lie group G with Lie algebra $\mathfrak{g}$.

In this section we collect results on opers that we will need (for a more detailed exposition, see [25]). Then in the next section we describe $\mathfrak{z}(\widehat{\mathfrak{g}})$ and the Gaudin algebras in terms of opers.

4.1. Definition of opers

Let G be a simple algebraic group of adjoint type, B a Borel subgroup and $N = [B, B]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$. The quotient $H = B/N$ is a torus. Choose a splitting $H \to B$ of the homomorphism $B \to H$ and the corresponding splitting $\mathfrak{h} \to \mathfrak{b}$ at the level of Lie algebras. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. We choose generators $\{e_i\}$, $i = 1, \ldots, \ell$, of $\mathfrak{n}$ and generators $\{f_i\}$, $i = 1, \ldots, \ell$ of $\mathfrak{n}_-$ corresponding to simple roots, and denote by $\check{\rho} \in \mathfrak{h}$ the sum of the fundamental coweights of $\mathfrak{g}$. Then we have the following relations:

$$[\check{\rho}, e_i] = e_i, \qquad [\check{\rho}, f_i] = -f_i, \quad i = 1, \ldots, \ell.$$

A G-oper on a smooth curve X (or a disc $D \simeq \operatorname{Spec}\mathbb{C}[\![t]\!]$ or a punctured disc $D^\times = \operatorname{Spec}\mathbb{C}(\!(t)\!)$) is by definition a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal G-bundle $\mathcal{F}$ on X, ∇ is a connection on $\mathcal{F}$ and $\mathcal{F}_B$ is a B-reduction of $\mathcal{F}$ such that locally on X, in the analytic or étale topology, it has the following form. Choose a coordinate t and a trivialization of $\mathcal{F}_B$ on a sufficiently small open subset of X (in the analytic or étale topology, respectively) over which $\mathcal{F}_B$ may be trivialized. Then the connection operator ∇_{∂_t} corresponding to the vector field ∂_t has the form

$$\nabla_{\partial_t} = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t), \tag{4.1}$$

where each $\psi_i(t)$ is a nowhere vanishing function, and $\mathbf{v}(t)$ is a $\mathfrak{b}$-valued function. The space of G-opers on X is denoted by $\operatorname{Op}_G(X)$.

This definition is due to A. Beilinson and V. Drinfeld [7] (in the case when X is the punctured disc opers were first introduced in [10]).

In particular, suppose that $U = \operatorname{Spec} R$ and t is a coordinate on U. It is clear what this means if $U = D$ or $D^\times$, and if U is an affine curve with the ring of functions R, then t is an étale morphism $U \to \mathbb{A}^1$ (for example, if $U = \mathbb{C}^\times = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$, then t^n is a coordinate for any non-zero integer n). Then $\mathrm{Op}_G(U)$ has the following explicit realization: it is isomorphic to the quotient of the space of operators of the form[9]

$$\nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t), \quad e\mathbf{v}(t) \in \mathfrak{b}(R), \tag{4.2}$$

where each $\psi_i(t) \in R$ is a nowhere vanishing function, by the action of the group $B(R)$. Recall that the gauge transformation of an operator $\partial_t + A(t)$, where $A(t) \in \mathfrak{g}(R)$ by $g(t) \in G(R)$ is given by the formula

$$g \cdot \big(\partial_t + A(t)\big) = \partial_t + gA(t)g^{-1} - \partial_t g \cdot g^{-1}.$$

There is a unique element $g(t) \in H(R)$ such that the gauge transformation of the operator (4.2) by $H(R)$ has the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} f_i + \mathbf{v}(t), \quad \mathbf{v}(t) \in \mathfrak{b}(R). \tag{4.3}$$

This implies that $\mathrm{Op}_G(U)$ is isomorphic to the quotient of the space of operators of the form (4.3) by the action of the group $N(R)$.

4.2. Canonical representatives

Set

$$p_{-1} = \sum_{i=1}^{\ell} f_i.$$

The operator $\operatorname{ad} \check{\rho}$ defines the principal gradation on $\mathfrak{b}$, with respect to which we have a direct sum decomposition $\mathfrak{b} = \bigoplus_{i \geqslant 0} \mathfrak{b}_i$. Let p_1 be the unique element of degree 1 in $\mathfrak{n}$ such that $\{p_{-1}, 2\check{\rho}, p_1\}$ is an $\mathfrak{sl}_2$-triple. Let

$$V_{\mathrm{can}} = \bigoplus_{i \in E} V_{\mathrm{can},i}$$

be the space of $\operatorname{ad} p_1$-invariants in $\mathfrak{n}$, decomposed according to the principal gradation. Here

$$E = \{d_1, \dots, d_\ell\}$$

is the set of exponents of $\mathfrak{g}$ (see [33]). Then p_1 spans $V_{\mathrm{can},1}$. Choose a linear generator p_j of V_{can,d_j} (if the multiplicity of d_j is greater than one, which happens only in the case $\mathfrak{g} = D_{2n}$,

[9] In order to simplify notation, from now on we will write ∇ for ∇_{∂_t}.

$d_j = 2n$, then we choose linearly independent vectors in V_{can,d_j}). The following result is due to Drinfeld and Sokolov [10] (the proof is reproduced in the proof of Lemma 2.1 of [21]).

Lemma 4.1. *The gauge action of $N(R)$ on the space of operators of the form* (4.3) *is free, and each gauge equivalence class contains a unique operator of the form $\nabla = \partial_t + p_{-1} + \mathbf{v}(t)$, where $\mathbf{v}(t) \in V_{\mathrm{can}}(R)$, so that we can write*

$$\mathbf{v}(t) = \sum_{j=1}^{\ell} v_j(t) \cdot p_j. \tag{4.4}$$

Let x be a point of a smooth curve X and $D_x = \operatorname{Spec} \mathcal{O}_x$, $D_x^\times = \operatorname{Spec} \mathcal{K}_x$, where $\mathcal{O}_x$ is the completion of the local ring of x and $\mathcal{K}_x$ is the field of fractions of $\mathcal{O}_x$. Choose a formal coordinate t at x, so that $\mathcal{O}_x \simeq \mathbb{C}[[t]]$ and $\mathcal{K}_x = \mathbb{C}((t))$. Then the space $\mathrm{Op}_G(D_x)$ (resp., $\mathrm{Op}_G(D_x^\times)$) of G-opers on D_x (resp., $D_x^\times$) is the quotient of the space of operators of the form (4.1) where $\psi_i(t) \neq 0$ take values in $\mathcal{O}_x$ (resp., in $\mathcal{K}_x$) and $\mathbf{v}(t)$ takes values in $\mathfrak{b}(\mathcal{O}_x)$ (resp., $\mathfrak{b}(\mathcal{K}_x)$) by the action of $B(\mathcal{O}_x)$ (resp., $B(\mathcal{K}_x)$).

To make this definition coordinate-independent, we need to specify the action of the group of changes of coordinates on this space. Suppose that s is another coordinate on the disc D_x such that $t = \varphi(s)$. In terms of this new coordinate the operator (4.3) becomes

$$\nabla_{\partial_t} = \nabla_{\varphi'(s)^{-1}\partial_s} = \varphi'(s)^{-1}\partial_s + \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot \mathbf{v}\big(\varphi(s)\big).$$

Hence we find that

$$\nabla_{\partial_s} = \partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot \mathbf{v}\big(\varphi(s)\big).$$

4.3. Opers with singularities

A G-oper on D_x with singularity of order m at x is by definition (see [7, Section 3.8.8]) a $B(\mathcal{O}_x)$-conjugacy class of operators of the form

$$\nabla = \partial_t + \frac{1}{t^m}\left(\sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t)\right), \tag{4.5}$$

where $\psi_i(t) \in \mathcal{O}_x$, $\psi_i(0) \neq 0$, and $\mathbf{v}(t) \in \mathfrak{b}(\mathcal{O}_x)$. Equivalently, it is an $N(\mathcal{O}_x)$-equivalence class of operators

$$\nabla = \partial_t + \frac{1}{t^m}\big(p_{-1} + \mathbf{v}(t)\big), \quad \mathbf{v}(t) \in \mathfrak{b}(\mathcal{O}_x). \tag{4.6}$$

We denote the space of such opers by $\mathrm{Op}_G^{\leqslant m}(D_x)$.

Let $\mathfrak{m}_x = t\mathbb{C}[\![t]\!]$ be the maximal ideal of $\mathcal{O}_x = \mathbb{C}[\![t]\!]$. Then $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is naturally interpreted as the tangent space T_x to $x \in X$. Let $T_x^\times$ be the corresponding $\mathbb{C}^\times$-torsor. Now consider the affine space

$$\mathfrak{g}/G := \operatorname{Spec}(\operatorname{Fun}\mathfrak{g})^G = \operatorname{Spec}(\operatorname{Fun}\mathfrak{h})^W =: \mathfrak{h}/W.$$

It carries a natural $\mathbb{C}^\times$-action. We denote by $(\mathfrak{g}/G)_{x,n} = (\mathfrak{h}/W)_{x,n}$ its twist by the $\mathbb{C}^\times$-torsor $(T_x^\times)^{\otimes n}$:

$$(\mathfrak{g}/G)_{x,n} = \left(T_x^\times\right)^{\otimes n} \times_{\mathbb{C}^\times} \mathfrak{g}/G.$$

Define the *m-residue map*

$$\operatorname{res}_m : \operatorname{Op}_G^{\leqslant m}(D_x) \to (\mathfrak{g}/G)_{x,m-1} = (\mathfrak{h}/W)_{x,m-1} \tag{4.7}$$

sending ∇ of the form (4.6) to the image of $p_{-1} + \mathbf{v}(0)$ in $\mathfrak{h}/W$. It is clear that this map is independent of the choice of coordinate t (which was the reason for twisting by $(T_x^\times)^{\otimes(m-1)}$). This generalizes the definition of 1-residue given in [7, Section 3.8.11]. The 1-residue which takes values in $\mathfrak{g}/G = \mathfrak{h}/W$ and no twisting is needed. The definition of m-residue with $m > 1$ requires twisting by $(T_x^\times)^{\otimes(m-1)}$. Alternatively, we may view the m-residue map as a morphism from $\operatorname{Op}_G^{\leqslant m}(D_x)$ to the algebraic stack $\mathfrak{g}/(G \times \mathbb{C}^\times) \simeq \mathfrak{h}/(W \times \mathbb{C}^\times)$.

Introduce the following affine subspace $\mathfrak{g}_{\mathrm{can}}$ of $\mathfrak{g}$:

$$\mathfrak{g}_{\mathrm{can}} = \left\{ p_{-1} + \sum_{j=1}^{\ell} y_j p_j,\ y \in \mathbb{C} \right\}. \tag{4.8}$$

Recall from [33] that the adjoint orbit of any regular element in the Lie algebra $\mathfrak{g}$ contains a unique element which belongs to $\mathfrak{g}_{\mathrm{can}}$. Thus, the corresponding morphism $\mathfrak{g}_{\mathrm{can}} \to \mathfrak{g}/G = \mathfrak{h}/W$ is an isomorphism.

Proposition 4.2. *(See [10, Proposition 3.8.9].) The natural morphism* $\operatorname{Op}_G^{\leqslant m}(D_x) \to \operatorname{Op}_G(D_x^\times)$ *is injective. Its image consists of those G-opers on* $D_x^\times$ *whose canonical representatives have the form*

$$\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\ell} t^{-m(d_j+1)} u_j(t) p_j, \quad u_j(t) \in \mathbb{C}[\![t]\!]. \tag{4.9}$$

Moreover,

$$\operatorname{res}_m(\nabla) = p_{-1} + \left(u_1(0) + \frac{1}{4}\delta_{m,1}\right) p_1 + \sum_{j>1} u_j(0) p_j, \tag{4.10}$$

which is an element of $\mathfrak{g}_{\mathrm{can}} = \mathfrak{g}/G$ *(here we use the trivialization of* T_x *induced by the coordinate t in the definition of* res_m*).*

Proof. First, we bring an oper (4.5) to the form

$$\partial_t + \frac{1}{t^m}\left(p_{-1} + \sum_{j=1}^{\ell} c_j(t) p_j\right), \quad c_j(t) \in \mathbb{C}[[t]]$$

(in the same way as in the proof of Lemma 4.1). Next, we apply the gauge transformation by $\check{\rho}(t)^{-m}$ and obtain

$$\partial_t + p_{-1} + m\check{\rho}t^{-1} + \sum_{j=1}^{\ell} t^{-m(d_j+1)} c_j(t) p_j, \quad c_j(t) \in \mathbb{C}[[t]].$$

Finally, applying the gauge transformation by $\exp(-mp_1/2t)$, we obtain the operator

$$\partial_t + p_{-1} + \left(t^{-m-1}c_1(t) - \frac{m^2-2}{4}t^{-2}\right)p_1 + \sum_{j>1} t^{-m(d_j+1)} c_j(t) p_j,$$
$$c_j(t) \in \mathbb{C}[[t]]. \tag{4.11}$$

Thus, we obtain an isomorphism between the space $\mathrm{Op}_G^{\leqslant m}(D_x)$ and the space of opers on $D_x^\times$ of the form (4.9). Moreover, comparing formula (4.11) with formula (4.9) we find that $u_1(t) = c_1(t) - \frac{m^2-2}{4}t^{m-1}$ and $u_j(t) = c_j(t)$ for $j > 1$. Therefore the m-residue of the oper (4.9) is equal to (4.10). $\square$

If $m = 1$, then the corresponding opers are called opers with *regular singularity*. In $m > 1$, then they are called opers with *irregular singularity*. In what follows we will often refer to the 1-residue of an oper with regular singularity simply as residue.

Given $\nu \in \mathfrak{h}/W$, we denote by $\mathrm{Op}_G^{\leqslant 1}(D_x)_\nu$ the subvariety of $\mathrm{Op}_G^{\leqslant 1}(D_x)$ which consists of those opers that have residue $\nu \in \mathfrak{h}/W$.

In particular, the residue of a regular oper $\partial_t + p_{-1} + \mathbf{v}(t)$, where $\mathbf{v}(t) \in \mathfrak{b}(\mathcal{O}_x)$, is equal to $\varpi(-\check{\rho})$, where ϖ is the projection $\mathfrak{h} \to \mathfrak{h}/W$ (see [7]). Indeed, a regular oper may be brought to the form (4.6), using the gauge transformation with $\check{\rho}(t) \in B(\mathcal{K}_x)$, after which it takes the form

$$\partial_t + \frac{1}{t}\big(p_{-1} - \check{\rho} + t \cdot \check{\rho}(t)\big(\mathbf{v}(t)\big)\check{\rho}(t)^{-1}\big).$$

If $\mathbf{v}(t)$ is regular, then so is $\check{\rho}(t)(\mathbf{v}(t))\check{\rho}(t)^{-1}$. Therefore the residue of this oper in $\mathfrak{h}/W$ is equal to $\varpi(-\check{\rho})$, and so $\mathrm{Op}_G(D_x) = \mathrm{Op}_G^{\leqslant 1}(D_x)_{\varpi(-\check{\rho})}$.

Next, we consider opers with irregular singularities in more detail. Let $m > 1$. Denote by π the projection $\mathfrak{g} \to \mathfrak{g}/G \simeq \mathfrak{h}/W$. Any point in $\mathfrak{g}/G$ may be represented uniquely in the form $\pi(y)$, where y is a regular element of $\mathfrak{g}$ of the form

$$y = p_{-1} + \bar{y} = p_{-1} + \sum_{j=1}^{\ell} y_j p_j. \tag{4.12}$$

Let $N^{(1)}[\![t]\!]$ be the first congruence subgroup of $N[\![t]\!]$, which consists of all elements of $N[\![t]\!]$ congruent to the identity modulo t. The next lemma follows from the definition.

Lemma 4.3. *Let y be a regular element of $\mathfrak{g}$ of the form* (4.12)*. Then, for $m>1$, the space $\mathrm{Op}_G^{\leqslant m}(D_x)_{\pi(y)}$ of G-opers with singularity of order m and m-residue $\pi(y)$ is isomorphic to the quotient of the space of operators of the form*

$$\partial_t + \frac{1}{t^m}(p_{-1}+\bar{y}) + \mathbf{v}(t), \quad \mathbf{v}(t) \in t^{-1}\mathfrak{b}[\![t]\!],$$

by $N^{(1)}[\![t]\!]$. Equivalently, it is isomorphic to the space of operators of the form

$$\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\ell}\bigl(y_j t^{-m(d_j+1)} + t^{-m(d_j+1)+1} w_j(t)\bigr)p_j, \quad w_j(t) \in \mathbb{C}[\![t]\!].$$

4.4. Opers without monodromy

Suppose that $\check{\lambda}$ is a dominant integral coweight of $\mathfrak{g}$. Let $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ be the quotient of the space of operators of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell}\psi_i(t) f_i + \mathbf{v}(t), \tag{4.13}$$

where

$$\psi_i(t) = t^{\langle\alpha_i,\check{\lambda}\rangle}\bigl(\kappa_i + t(\ldots)\bigr) \in \mathcal{O}_x, \quad \kappa_i \neq 0,$$

and $\mathbf{v}(t) \in \mathfrak{b}(\mathcal{O}_x)$, by the gauge action of $B(\mathcal{O}_x)$. Equivalently, $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ is the quotient of the space of operators of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} t^{\langle\alpha_i,\check{\lambda}\rangle} f_i + \mathbf{v}(t), \tag{4.14}$$

where $\mathbf{v}(t) \in \mathfrak{b}(\mathcal{O}_x)$, by the gauge action of $N(\mathcal{O}_x)$. Considering the $N(\mathcal{K}_x)$-class of such an operator, we obtain an oper on $D_x^\times$. Thus, we have a map $\mathrm{Op}_G(D_x)_{\check{\lambda}} \to \mathrm{Op}_G(D_x^\times)$.

To understand better the image of $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ in $\mathrm{Op}_{^LG}(D_x^\times)$, we introduce, following [27, Section 2.9], a larger space $\mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}}$ as the quotient of the space of operators of the form (4.14), where now

$$\mathbf{v}(t) \in \mathfrak{h}(\mathcal{O}_x) \oplus t^{-1}\mathfrak{n}(\mathcal{O}_x),$$

by the gauge action of $N(\mathcal{O}_x)$.

Consider an operator of the form (4.14) with $\mathbf{v}(t) \in \mathfrak{h}(\mathcal{O}_x) \oplus t^{-1}\mathfrak{n}(\mathcal{O}_x)$. Denote by

$$\mathbf{v}_{-1} = \sum_{\alpha\in\Delta_+} \mathbf{v}_{\alpha,-1} e_\alpha \in \mathfrak{n},$$

the coefficient of $\mathbf{v}(t)$ in front of t^{-1}. Then, according to the definition, $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ is the subvariety of $\mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}}$ defined by the equations $\mathbf{v}_{\alpha,-1} = 0$, $\alpha \in \Delta_+$.

It is clear that the monodromy conjugacy class of an oper of the form (4.14) in $\mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}}$ is equal to $\exp(2\pi i \mathbf{v}_{-1})$. Therefore $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ is the locus of monodromy-free opers in $\mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}}$.

We have the following alternative descriptions of $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ and $\mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}}$.

Proposition 4.4. *(See [21,27].) For any dominant integral coweight $\check{\lambda}$ of G the map $\mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}} \to \mathrm{Op}_G(D_x^\times)$ is injective and its image is equal to $\mathrm{Op}_G^{\leqslant 1}(D_x)_{\varpi(-\check{\lambda}-\check{\rho})}$. Moreover, the points of $\mathrm{Op}_G(D_x)_{\check{\lambda}} \subset \mathrm{Op}_G(D_x)_{\check{\lambda}}^{\mathrm{nilp}}$ are precisely those G-opers with regular singularity and residue $\varpi(-\check{\lambda}-\check{\rho})$ which have no monodromy around x.*

In particular, we find that $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ is a subvariety in $\mathrm{Op}_G^{\leqslant 1}(D_x)_{\varpi(-\check{\lambda}-\check{\rho})}$ defined by $|\Delta_+|$ equations corresponding to $\mathbf{v}_{\alpha,-1} = 0$, $\alpha \in \Delta_+$.

One may rewrite the elements $\mathbf{v}_{\alpha,-1} = 0$, $\alpha \in \Delta_+$, which are the generators of the defining ideal of $\mathrm{Op}_G(D_x)_{\check{\lambda}}$ in $\mathrm{Fun}\,\mathrm{Op}_G^{\leqslant 1}(D_x)_{\varpi(-\check{\lambda}-\check{\rho})}$, as polynomials in the canonical coordinates $u_{j,n}$, $j = 1, \dots, \ell$; $n > 0$, on $\mathrm{Op}_G^{\leqslant 1}(D_x)_{\varpi(-\check{\lambda}-\check{\rho})}$ obtained via Proposition 4.2 (here $u_{j,n}$ is the t^n-coefficient of $u_j(t)$ in (4.9)). It is easy to see that these generators are homogeneous of degrees $\langle \alpha, \check{\lambda} + \check{\rho} \rangle$, $\alpha \in \Delta_+$, with respect to the grading on $\mathrm{Fun}\,\mathrm{Op}_{^LG}^{\leqslant 1}(D_x)_{\varpi(-\check{\lambda}-\check{\rho})}$ defined by the assignment $\deg u_{j,n} = n$. For instance, for $\check{\lambda} = 0$ these generators are u_{j,n_j}, $j = 1, \dots, \ell$; $n_j = 1, \dots, d_j$. Other examples are discussed in [19, Section 3.9].

5. Spectra of generalized Gaudin algebras and opers with irregular singularities

In this section we describe the algebra of endomorphisms of $\mathbb{V}_{0,\kappa_c}$, the center of the completed enveloping algebra $U_{\kappa_c}(\widehat{\mathfrak{g}})$ and the action of the center on various $\widehat{\mathfrak{g}}$-modules of critical level in terms of ^LG-opers. We then derive from this description and some general results on coinvariants [22,26] that the spectrum of the universal Gaudin algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is identified with the space of all ^LG-opers on $\mathbb{P}^1$ with singularities of arbitrary orders at $z_1, \dots, z_N$ and ∞. The spectrum of the quotient $\mathcal{Z}_{(z_i),\infty}^{(m_i),m_\infty}(\mathfrak{g})$ of $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is identified with the subspace of those ^LG-opers which have singularities of orders at most m_i at z_i and m_∞ at ∞. We also describe the spectrum of the algebra $\mathcal{A}_\chi$ for regular semi-simple and regular nilpotent χ, and the joint eigenvalues of $\mathcal{A}_\chi$, and its multi-point generalizations, on tensor products of finite-dimensional $\mathfrak{g}$-modules.

5.1. The algebra of endomorphisms of $\mathbb{V}_{0,\kappa}$

Let $\mathfrak{g}$ be a simple Lie algebra. Recall that, by definition, the Langlands dual Lie algebra $^L\mathfrak{g}$ is the Lie algebra whose Cartan matrix is the transpose to the Cartan matrix of $\mathfrak{g}$. In what follows we will choose Cartan decompositions of $\mathfrak{g}$ and $^L\mathfrak{g}$ and use the canonical identification

$$\mathfrak{h}^* = {}^L\mathfrak{h}$$

between $\mathfrak{h}^*$ and the Cartan subalgebra ${}^L\mathfrak{h}$ of the Langlands dual Lie algebra ${}^L\mathfrak{g}$. In particular, we will identify the weights and roots of $\mathfrak{g}$ with the coweights and coroots of ${}^L\mathfrak{g}$, respectively, and vice versa.

Recall that we denote by $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ the algebra of endomorphisms of the vacuum module $\mathbb{V}_{0,\kappa}$ of level κ (see Section 2.4). We will now describe $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ following [16,20] (for a detailed exposition, see [25]).

Let ${}^LG$ be the adjoint group of ${}^L\mathfrak{g}$ and $\mathrm{Op}_{{}^LG}(D)$ the space of ${}^LG$-opers on the disc $D = \mathrm{Spec}\,\mathbb{C}[\![t]\!]$ (see Section 4.1). Denote by $\mathrm{Fun}\,\mathrm{Op}_{{}^LG}(D)$ the algebra of regular functions on $\mathrm{Op}_{{}^LG}(D)$. In view of Lemma 4.1, it is isomorphic to the algebra of functions on the space of ℓ-tuples $(v_1(t), \ldots, v_\ell(t))$ of formal Taylor series, i.e., the space $\mathbb{C}[\![t]\!]^\ell$. If we write

$$v_i(t) = \sum_{n<0} v_{i,n} t^{-n-1},$$

then we obtain

$$\mathrm{Fun}\,\mathrm{Op}_{{}^LG}(D) \simeq \mathbb{C}[v_{i,n}]_{i=1,\ldots,\ell;\, n<0}. \tag{5.1}$$

Let $\mathrm{Der}\,\mathcal{O} = \mathbb{C}[\![t]\!]\partial_t$ be the Lie algebra of continuous derivations of the topological algebra $\mathcal{O} = \mathbb{C}[\![t]\!]$. The action of its Lie subalgebra $\mathrm{Der}_0\,\mathcal{O} = t\mathbb{C}[\![t]\!]\partial_t$ on $\mathcal{O}$ exponentiates to an action of the group $\mathrm{Aut}\,\mathcal{O}$ of formal changes of variables. Both $\mathrm{Der}\,\mathcal{O}$ and $\mathrm{Aut}\,\mathcal{O}$ naturally act on $\mathbb{V}_{0,\kappa}$ in a compatible way, and these actions preserve $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$. They also act on the space $\mathrm{Op}_{{}^LG}(D)$. One can check that the vector field $-t\partial_t$ defines a $\mathbb{Z}_+$-grading on $\mathrm{Fun}\,\mathrm{Op}_{{}^LG}(D)$ such that $\deg v_{i,n} = d_i - n$, and the vector field $-\partial_t$ acts as a derivation such that $-\partial_t \cdot v_{i,n} = -n v_{i,n-1}$.

Recall the critical invariant inner product κ_c introduced in Section 2.5.

Theorem 5.1. *(See [16,20].)*

(1) $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}}) = \mathbb{C}$ *if* $\kappa \neq \kappa_c$.
(2) *There is a canonical isomorphism*

$$\mathfrak{z}_{\kappa_c}(\widehat{\mathfrak{g}}) \simeq \mathrm{Fun}\,\mathrm{Op}_{{}^LG}(D)$$

of algebras which is compatible with the actions of $\mathrm{Der}\,\mathcal{O}$ *and* $\mathrm{Aut}\,\mathcal{O}$.

Since $\mathfrak{z}_\kappa(\widehat{\mathfrak{g}})$ is trivial for $\kappa \neq \kappa_c$, we will set $\kappa = \kappa_c$ and omit κ from our notation.

Let again x be a smooth point of a curve x and $\mathcal{O}_x \subset \mathcal{K}_x$ be as in Section 4.2. We have the Lie algebra $\mathfrak{g}(\mathcal{K}_x)$ and its central extension $\widehat{\mathfrak{g}}_{\kappa_c,x}$ defined by the commutation relations (2.2). Since the residue is coordinate-independent, the Lie algebra $\widehat{\mathfrak{g}}_{\kappa_c,x}$ is independent on the choice of an isomorphism $\mathcal{K}_x \simeq \mathbb{C}(\!(t)\!)$. We have a Lie subalgebra $\mathfrak{g}(\mathcal{O}_x) \subset \widehat{\mathfrak{g}}_{\kappa_c,x}$ and we define the corresponding vacuum module of level κ_c as

$$\mathbb{V}_{0,x} = \mathrm{Ind}_{\mathfrak{g}(\mathcal{O}_x)\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c,x}} \mathbb{C}.$$

Set

$$\mathfrak{z}(\widehat{\mathfrak{g}})_x = (\mathbb{V}_{0,x})^{\mathfrak{g}(\mathcal{O}_x)} = \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c,x}} \mathbb{V}_{0,x}.$$

Then the compatibility of the isomorphism of Theorem 5.1 with the action of $\operatorname{Aut}\mathcal{O}$ implies the existence of the following canonical (i.e., coordinate-independent) isomorphism:

$$\mathfrak{z}(\widehat{\mathfrak{g}})_x \simeq \operatorname{Fun}\operatorname{Op}_{{}^L G}(D_x). \tag{5.2}$$

Recall from Section 3.2 that the action of $L_0 = -t\partial_t \in \operatorname{Der}\mathcal{O}$ on the module $\mathbb{V}_0$ defines a $\mathbb{Z}$-grading on it such that $\deg v_0 = 0$, $\deg J_n^a = -n$. The action of $L_{-1} = -\partial_t \in \operatorname{Der}\mathcal{O}$ is given by the translation operator T defined by formula (2.6). Theorem 5.1 and the isomorphism (5.1) imply that there exist non-zero vectors $S_i \in \mathbb{V}_0^{\mathfrak{g}[\![t]\!]}$ of degrees $d_i + 1$, $i = 1, \dots, \ell$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\big[S_i^{(n)}\big]_{i=1,\dots,\ell;\, n\geqslant 0} v_0,$$

where $S_i^{(n)} = T^n S_i$, and under the isomorphism of Theorem 5.1 we have

$$S_i^{(n)} \mapsto n! v_{i,-n-1}, \quad n \geqslant 0. \tag{5.3}$$

The $\mathbb{Z}$-gradings on both algebras get identified and the action of T on $\mathfrak{z}(\widehat{\mathfrak{g}})$ becomes the action of $-\partial_t$ on $\operatorname{Fun}\operatorname{Op}_{{}^L G}(D)$.

Now recall from Section 3.2 that the PBW filtration on $U(\widehat{\mathfrak{g}}_{\kappa_c})$ induces a filtration on $\mathbb{V}_0$ such that the associated graded is identified with

$$S\big(\mathfrak{g}(\!(t)\!)/\mathfrak{g}[\![t]\!]\big) = \operatorname{Fun}\big(\mathfrak{g}(\!(t)\!)/\mathfrak{g}[\![t]\!]\big)^* = \operatorname{Fun}\big(\mathfrak{g}^*[\![t]\!]\,dt\big) \simeq \operatorname{Fun}\mathfrak{g}^*[\![t]\!],$$

where we use the canonical non-degenerate pairing

$$\phi(t)\,dt \in \mathfrak{g}^*[\![t]\!]\,dt, \quad A(t) \in \mathfrak{g}(\!(t)\!)/\mathfrak{g}[\![t]\!] \mapsto \operatorname{Res}_{t=0}\big\langle\phi(t), A(t)\big\rangle\,dt,$$

and a coordinate t on D. Let

$$\operatorname{Inv}\mathfrak{g}^*[\![t]\!] = \big(\operatorname{Fun}\mathfrak{g}^*[\![t]\!]\big)^{\mathfrak{g}[\![t]\!]}$$

be the algebra of $\mathfrak{g}[\![t]\!]$-invariant functions on $\mathfrak{g}^*[\![t]\!]$. According to Theorem 3.8, the map

$$\operatorname{gr}\mathfrak{z}(\widehat{\mathfrak{g}}) \to \operatorname{gr}\mathbb{V}_0 = \operatorname{Fun}\mathfrak{g}^*[\![t]\!]$$

gives rise to an isomorphism

$$\operatorname{gr}\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Inv}\mathfrak{g}^*[\![t]\!].$$

In particular, the symbols of the generators $S_i^{(n)}$ of $\mathfrak{z}(\widehat{\mathfrak{g}})$ have the following simple description.

Let $\operatorname{Inv}\mathfrak{g}^*$ be the algebra of $\mathfrak{g}$-invariant functions on $\mathfrak{g}^*$. By Theorem 3.1,

$$\operatorname{Inv}\mathfrak{g}^* = \mathbb{C}[\overline{P}_i]_{i=1,\dots\ell},$$

where the generators $\overline{P}_i$ may be chosen in such a way that they are homogeneous and $\deg\overline{P}_i = d_i + 1$.

As in Section 3.2, we use the elements $\overline{P}_i$ to construct generators of the algebra $\operatorname{Inv}\mathfrak{g}^*[\![t]\!]$. We will use the generators $\overline{J}^a_n$, $n<0$, of $S(\mathfrak{g}((t))/\mathfrak{g}[\![t]\!]) = \operatorname{Fun}\mathfrak{g}^*[\![t]\!]$, which are the symbols of $J^a_n|0\rangle \in \mathbb{V}_0$. These are linear functions on $\mathfrak{g}^*[\![t]\!]$ defined by the formula

$$\overline{J}^a_n\big(\phi(t)\big) = \operatorname{Res}_{t=0}\big\langle\phi(t), J^a\big\rangle t^n\,dt. \tag{5.4}$$

We will also write

$$\overline{J}^a(z) = \sum_{n<0} \overline{J}^a_n z^{-n-1}.$$

Let us write $\overline{P}_i$ as a polynomial in the $\overline{J}^a$'s, $\overline{P}_i = \overline{P}_i(\overline{J}^a)$. Define a set of elements $\overline{P}_{i,n} \in \operatorname{Fun}\mathfrak{g}^*[\![t]\!]$ by the formula

$$\overline{P}_i\big(\overline{J}^a(z)\big) = \sum_{n<0} \overline{P}_{i,n} z^{-n-1}. \tag{5.5}$$

Note that each of the elements $\overline{P}_{i,n}$ is a finite polynomial in the $\overline{J}^a_n$'s. Now Theorems 3.7 and 3.8 imply the following:

Lemma 5.2. *The generators $\overline{P}_i$ of* $\operatorname{Inv}\mathfrak{g}^*$ *may be chosen in such a way that the symbol of $S^{(n)}_i \in \mathfrak{z}(\widehat{\mathfrak{g}})$ is equal to $n!\overline{P}_{i,-n-1}$.*

For example, we may choose the degree 2 vector S_1 to be the Segal–Sugawara vector (2.16) (it is unique up to a non-zero scalar). Its symbol is equal to

$$\overline{P}_{1,-1} = \frac{1}{2}\sum_a \overline{J}_{a,-1}\overline{J}^a_{-1},$$

where $\overline{P}_1 = \frac{1}{2}\sum_a J_a J^a$ is the quadratic Casimir generator of $\operatorname{Inv}\mathfrak{g}^*$.

The algebra $\operatorname{Fun}\operatorname{Op}_{{}^L G}(D)$ has a canonical filtration such that the associated graded algebra is isomorphic to

$$\operatorname{Inv}{}^L\mathfrak{g}[\![t]\!] = \big(\operatorname{Fun}{}^L\mathfrak{g}[\![t]\!]\big)^{{}^L G[\![t]\!]}$$

(see [20, Section 11.3]). The spectrum of this algebra is the jet scheme of

$${}^L\mathfrak{g}/{}^L G = \operatorname{Spec}\big(\operatorname{Fun}{}^L\mathfrak{g}\big)^{{}^L G}.$$

Using the canonical isomorphisms

$${}^L\mathfrak{g}/{}^L G = {}^L\mathfrak{h}/W = \mathfrak{h}^*/W = \mathfrak{g}^*/G,$$

we identify the jet scheme of ${}^L\mathfrak{g}/{}^L G$ with the jet scheme of $\mathfrak{g}^*/G$. Therefore we have a canonical isomorphism

$$\operatorname{Inv}{}^L\mathfrak{g}[\![t]\!] = \operatorname{Inv}\mathfrak{g}^*[\![t]\!]. \tag{5.6}$$

On the other hand, we know that

$$\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) = \operatorname{Inv} \mathfrak{g}^* [\![t]\!].$$

The following result is proved in [20, Theorem 11.4].

Proposition 5.3. *The isomorphism $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} \operatorname{Op}_{^L G}(D)$ of Theorem* 5.1(2) *preserves the filtrations on both algebras, and the corresponding isomorphism of the associated graded algebras is the isomorphism* (5.6) *multiplied by $(-1)^n$ on the subspaces of degree n.*

5.2. The center of the completed enveloping algebra

Recall from [26] that $\mathbb{V}_0 = \mathbb{V}_{0,\kappa_c}$ is a vertex algebra, and $\mathfrak{z}(\widehat{\mathfrak{g}})$ is its commutative vertex subalgebra; in fact, it is the center of $\mathbb{V}_0$. We will also need the center of the completed universal enveloping algebra of $\widehat{\mathfrak{g}}$ of critical level. This algebra is defined as follows.

Let $U_{\kappa_c}(\widehat{\mathfrak{g}})$ be the quotient of the universal enveloping algebra $U(\widehat{\mathfrak{g}}_{\kappa_c})$ of $\widehat{\mathfrak{g}}_{\kappa_c}$ by the ideal generated by $(\mathbf{1} - 1)$. Define its completion $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ as follows:

$$\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}}) = \varprojlim U_{\kappa_c}(\widehat{\mathfrak{g}}) / U_{\kappa_c}(\widehat{\mathfrak{g}}) \cdot \left(\mathfrak{g} \otimes t^N \mathbb{C}[\![t]\!]\right).$$

It is clear that $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ is a topological algebra which acts on all smooth $\widehat{\mathfrak{g}}_{\kappa_c}$-module, i.e. such that any vector is annihilated by $\mathfrak{g} \otimes t^N \mathbb{C}[\![t]\!]$ for sufficiently large N, and the central $\mathbf{1}$ acts as the identity. Let $Z(\widehat{\mathfrak{g}})$ be the center of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$.

Denote by $\operatorname{Fun} \operatorname{Op}_{^L G}(D^\times)$ the algebra of regular functions on the space $\operatorname{Op}_{^L G}(D^\times)$ of $^L G$-opers on the punctured disc $D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)$. In view of Lemma 4.1, it is isomorphic to the algebra of functions on the space of ℓ-tuples $(v_1(t), \dots, v_\ell(t))$ of formal Laurent series, i.e., the ind-affine space $\mathbb{C}(\!(t)\!)^\ell$. If we write $v_i(t) = \sum_{n \in \mathbb{Z}} v_{i,n} t^{-n-1}$, then we obtain that $\operatorname{Fun} \operatorname{Op}_{^L G}(D^\times)$ is isomorphic to the completion of the polynomial algebra $\mathbb{C}[v_{i,n}]_{i=1,\dots,\ell;\, n \in \mathbb{Z}}$ with respect to the topology in which the basis of open neighborhoods of zero is formed by the ideals generated by $v_{i,n}$, $i = 1, \dots, \ell$; $n \geqslant N$, for $N \geqslant 0$.

Theorem 5.4. *(See [16,20].) There is a canonical isomorphism*

$$Z(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} \operatorname{Op}_{^L G}\left(D^\times\right)$$

of complete topological algebras which is compatible with the actions of $\operatorname{Der} \mathcal{O}$ *and* $\operatorname{Aut} \mathcal{O}$.

For a smooth point $x \in X$ as above, we have the Lie algebra $\widehat{\mathfrak{g}}_{\kappa_c,x}$. We define the completed enveloping algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}}_x)$ of $\widehat{\mathfrak{g}}_{\kappa_c,x}$ in the same way as above. Let $\mathbb{Z}(\widehat{\mathfrak{g}})_x$ be its center. Then Theorem 5.4 implies the following isomorphism:

$$Z(\widehat{\mathfrak{g}})_x \simeq \operatorname{Fun} \operatorname{Op}_{^L G}\left(D_x^\times\right).$$

Each element $A \in \mathbb{V}_0$ gives rise to a "vertex operator" which is a formal power series

$$Y[A, z] = \sum_{n \in \mathbb{Z}} A_{[n]} z^{-n-1}, \quad A_{[n]} \in \widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$$

(see [26, Section 4.2]). In particular, we have the elements $S_{i,[n]}$ attached to the generators $S_i \in \mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathbb{V}_0$. Under the isomorphism of Theorem 5.4 we have

$$S_{i,[n]} \to v_{i,n}. \tag{5.7}$$

The algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ has a PBW filtration, and its associated graded algebra is the completed symmetric algebra $\widetilde{S}(\mathfrak{g}((t)))$ of $\mathfrak{g}((t))$, which we identify with the topological algebra $\operatorname{Fun}\mathfrak{g}^*((t))$. Let $\overline{P}_{i,n}$ be the symbol of the element $S_{i,[n]}$ in $\operatorname{Fun}\mathfrak{g}^*((t))$. These elements are given by the formula

$$\overline{P}_i\big(\overline{J}^a(z)\big) = \sum_{n\in\mathbb{Z}} \overline{P}_{i,n} z^{-n-1}, \tag{5.8}$$

where

$$\overline{J}^a(z) = \sum_{n\in\mathbb{Z}} \overline{J}^a_n z^{-n-1},$$

and the $\overline{J}^a_n$'s are the generators of $\operatorname{Fun}\mathfrak{g}^*((t))$, defined by the formula

$$\overline{J}^a_n\big(\phi(t)\big) = \operatorname{Res}_{t=0}\langle\phi(t), J^a\rangle t^n\,dt, \quad \phi(t) \in \mathfrak{g}^*((t)),\ n \in \mathbb{Z}$$

(compare with formula (5.4)).

It is easy to see that the elements $\overline{P}_{i,n}$ are $G((t))$-invariant elements of $\operatorname{Fun}\mathfrak{g}^*((t))$. Moreover, they are topological generators of the algebra $(\operatorname{Fun}\mathfrak{g}^*((t)))^{G((t))}$. More precisely, $(\operatorname{Fun}\mathfrak{g}^*((t)))^{G((t))}$ is isomorphic to a completion of the free polynomial algebra in $\overline{P}_{i,n}$, $i = 1,\dots,\ell$; $n \in \mathbb{Z}$, see [7, Theorem 3.7.5].

On the other hand, the algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(D^\times)$ also has a canonical filtration such that the associated graded algebra is canonically isomorphic to $(\operatorname{Fun}{}^L\mathfrak{g}((t)))^{^LG((t))}$. In the same way as at the end of Section 5 we obtain a canonical isomorphism

$$\big(\operatorname{Fun}{}^L\mathfrak{g}((t))\big)^{^LG((t))} \simeq \big(\operatorname{Fun}\mathfrak{g}^*((t))\big)^{G((t))}.$$

We can now take the symbol of $v_{i,n}, n \in \mathbb{Z}$, in $(\operatorname{Fun}\mathfrak{g}^*((t)))^{G((t))}$ and compare it with $\overline{P}_{i,n} = \sigma(S_{i,[n]})$, where $v_{i,n}$ and $S_{i,[n]}$ are related by formula (5.7). Using the commutative vertex algebra structures on $\mathfrak{z}(\widehat{\mathfrak{g}})$ and $\operatorname{Fun}\operatorname{Op}_{^LG}(D)$ and Proposition 5.3, we obtain the following:

Lemma 5.5. *The symbol of $v_{i,n}$ is equal to $(-1)^{d_i+1}\overline{P}_{i,n}$.*

If M is a smooth $\widehat{\mathfrak{g}}_{\kappa_c}$-module, then the action of $Z(\widehat{\mathfrak{g}})$ on M gives rise to a homomorphism

$$Z(\widehat{\mathfrak{g}}) \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} M.$$

For example, if $M = \mathbb{V}_0$, then using Theorems 5.1 and 5.4 we identify this homomorphism with the surjection

$$\operatorname{Fun}\operatorname{Op}_{^LG}\big(D^\times\big) \twoheadrightarrow \operatorname{Fun}\operatorname{Op}_{^LG}(D)$$

induced by the natural embedding

$$\mathrm{Op}_{^L G}(D) \hookrightarrow \mathrm{Op}_{^L G}(D^\times).$$

Recall that the Harish–Chandra homomorphism identifies the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ with the algebra $(\mathrm{Fun}\,\mathfrak{h}^*)^W$ of polynomials on $\mathfrak{h}^*$ which are invariant with respect to the action of the Weyl group W. Therefore a character $Z(\mathfrak{g}) \to \mathbb{C}$ is the same as a point in $\mathrm{Spec}(\mathrm{Fun}\,\mathfrak{h}^*)^W$ which is the quotient $\mathfrak{h}^*/W$. For $\lambda \in \mathfrak{h}^*$ we denote by $\varpi(\lambda)$ its projection onto $\mathfrak{h}^*/W$. In particular, $Z(\mathfrak{g})$ acts on M_λ and V_λ via its character $\varpi(\lambda+\rho)$. We also denote by I_λ the maximal ideal of $Z(\mathfrak{g})$ equal to the kernel of the homomorphism $Z(\mathfrak{g}) \to \mathbb{C}$ corresponding to the character $\varpi(\lambda+\rho)$.

Recall that we have an isomorphism

$$\mathfrak{g}^*/G \simeq \mathfrak{h}^*/W = {}^L\mathfrak{h}/W.$$

Let us denote by π the corresponding map $\mathfrak{g}^* \to {}^L\mathfrak{h}/W$.

Theorem 5.6.

(1) *Let*

$$\mathbb{U}_m = \mathrm{Ind}_{t^m\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{C}.$$

The homomorphism $Z(\widehat{\mathfrak{g}}) \to \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{U}_m$ factors as

$$Z(\widehat{\mathfrak{g}}) \simeq \mathrm{Fun}\,\mathrm{Op}_{^L G}(D^\times) \twoheadrightarrow \mathrm{Fun}\,\mathrm{Op}_{^L G}^{\leqslant m}(D) \hookrightarrow \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{U}_m.$$

(2) *Let*

$$\mathbb{I}_{m,\chi} = \mathrm{Ind}_{t^m\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{C}_\chi,$$

where $\chi \in \mathfrak{g}^*$ and $t^m\mathfrak{g}[[t]]$ acts on $\mathbb{C}_\chi$ via

$$t^m\mathfrak{g}[[t]] \to \mathfrak{g}\otimes t^m \xrightarrow{\chi} \mathbb{C}.$$

The homomorphism $Z(\widehat{\mathfrak{g}}) \to \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{I}_{m,\chi}$ factors as

$$Z(\widehat{\mathfrak{g}}) \simeq \mathrm{Fun}\,\mathrm{Op}_{^L G}(D^\times) \twoheadrightarrow \mathrm{Fun}\,\mathrm{Op}_{^L G}^{\leqslant(m+1)}(D)_{\pi(-\chi)} \to \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{I}_{m,\chi}, \tag{5.9}$$

where $\mathrm{Op}_{^L G}^{\leqslant(m+1)}(D)_{\pi(-\chi)}$ is the space of opers with singularity of order $m+1$ and the $(m+1)$-residue $\pi(-\chi)$.

In addition, if χ is a regular element of $\mathfrak{g}^*$, then the last map in (5.9) *is injective.*

(3) *Let M be a $\mathfrak{g}$-module on which the center $Z(\mathfrak{g})$ acts via its character $\varpi(\lambda+\rho)$, and let $\mathbb{M}$ be the induced $\widehat{\mathfrak{g}}_{\kappa_c}$-module. Then the homomorphism $Z(\widehat{\mathfrak{g}}) \to \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{M}$ factors as*

$$Z(\widehat{\mathfrak{g}}) \simeq \mathrm{Fun}\,\mathrm{Op}_{^L G}(D^\times) \twoheadrightarrow \mathrm{Fun}\,\mathrm{Op}_{^L G}^{\leqslant 1}(D)_{\varpi(-\lambda-\rho)} \to \mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{M}.$$

(4) *For an integral dominant weight $\lambda \in \mathfrak{h}^*$ the homomorphism*

$$\operatorname{Fun}\operatorname{Op}_{{}^L G}(D^\times) \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_\lambda$$

factors as

$$\operatorname{Fun}\operatorname{Op}_{{}^L G}(D^\times) \to \operatorname{Fun}\operatorname{Op}_{{}^L G}(D)_\lambda \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_\lambda.$$

Proof. According to Proposition 4.2, the ideal of $\operatorname{Op}^{\leqslant m}_{{}^L G}(D)$ in $\operatorname{Fun}\operatorname{Op}_{{}^L G}(D^\times)$ is generated by v_{i,n_i}, $i = 1, \dots, \ell$; $n_i \geqslant m(d_i + 1)$. By formula (5.7), these correspond to $S_{i,[n_i]}$, $i = 1, \dots, \ell$; $n_i \geqslant m(d_i + 1)$. It follows from the definition of vertex operators that for any $A \in \mathbb{V}_0$ of degree N, the operators $A_{[n]}$, $n \geqslant mN$, act by zero on any vector that is annihilated by $\mathfrak{g} \otimes t^m \mathbb{C}[[t]]$. Since $\deg S_i = d_i + 1$, we obtain that the homomorphism $Z(\widehat{\mathfrak{g}}) \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{U}_m$ factors as

$$Z(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun}\operatorname{Op}_{{}^L G}(D^\times) \twoheadrightarrow \operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D) \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{U}_m.$$

To complete the proof of part (1), we need to show that the last homomorphism is injective. It suffices to prove that the map

$$\operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D) \to \mathbb{U}_m,$$

applied by acting on the generating vector of $\mathbb{U}_m$, is injective. This map preserves natural filtrations on both spaces, and it is sufficient to show that the corresponding map of the associated graded is injective.

The PBW filtration on $U(\widehat{\mathfrak{g}}_{\kappa_c})$ induces a filtration on $\mathbb{U}_m$, and the associated graded space is identified with the symmetric algebra $S(\mathfrak{g}((t))/t^m \mathfrak{g}[[t]])$. On the other hand, we have, by Proposition 4.2,

$$\operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D) \simeq \mathbb{C}[v_{i,n_i}]_{i=1,\dots,\ell;\, n<m(d_i+1)}.$$

Now Lemma 5.5 implies that

$$\operatorname{gr}\operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D) \simeq \mathbb{C}[\overline{P}_{i,n_i}]_{i=1,\dots,\ell;\, n<m(d_i+1)}$$

(see formula (5.8)). Thus, we need to show that the map

$$\mathbb{C}[\overline{P}_{i,n_i}]_{i=1,\dots,\ell;\, n<m(d_i+1)} \to S\big(\mathfrak{g}((t))/t^m \mathfrak{g}[[t]]\big),$$

is injective. Let us apply the automorphism $\overline{J}^a_n \mapsto \overline{J}_{n-m}$ of $\mathfrak{g}((t))$ (considered as a vector space) to both sides. Then we have $\overline{P}_{i,n} \mapsto \overline{P}_{i,n-m(d_i+1)}$, and so the above map becomes

$$\mathbb{C}[\overline{P}_{i,n_i}]_{i=1,\dots,\ell; n<0} \to S\big(\mathfrak{g}((t))/\mathfrak{g}[[t]]\big),$$

which is injective by Theorem 3.7. This completes the proof of part (1).

Next, we prove part (2). We will work with $\mathbb{I}_{m-1,\chi}$ instead of $\mathbb{I}_{m,\chi}$. According to the result of part (1), we have an injective homomorphism from the algebra

$$\operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D^\times) \simeq \mathbb{C}[v_{i,n_i}]_{i=1,\dots,\ell;\, n_i<m(d_i+1)},$$

to $\operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{U}_m$, sending

$$v_{i,n_i} \mapsto S_{i,[n_i]}.$$

In addition, the commutative algebra

$$S(\mathfrak{g}) = S\big(\mathfrak{g} \otimes t^{m-1}\big) = \mathbb{C}\big[\bar{J}^a_{m-1}\big]_{a=1\dots,\dim\mathfrak{g}}$$

also acts on $\mathbb{U}_m$ and commutes with the action $\operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D^\times)$. Let us denote the generating vector of $\mathbb{U}_m$ by v_m. Since the elements J^a_n, $n \geqslant m$, of $\widehat{\mathfrak{g}}_{\kappa_c}$ annihilate v_m, we obtain from Lemma 3.9 that the generator $S_{i,[m(d_i+1)-1]}$ acts on the generating vector of $\mathbb{U}_m$ by multiplication by $\overline{P}_i(\bar{J}^a_{m-1})$.

It follows from the definition that $\mathbb{I}_{m-1,\chi}$ is the quotient of $\mathbb{U}_m$ by the maximal ideal in $S(\mathfrak{g}) = \operatorname{Fun}\mathfrak{g}^*$ corresponding to $\chi \in \mathfrak{g}^*$. Therefore we find that $S_{i,[m(d_i+1)-1]} \in Z(\widehat{\mathfrak{g}})$ acts on the generating vector of $\mathbb{I}_{m-1,\chi}$, and hence on the entire module $\mathbb{I}_{m-1,\chi}$, by multiplication by the value of $\overline{P}_i \in \operatorname{Fun}\mathfrak{g}^*$ at $\chi \in \mathfrak{g}^*$. By Lemma 5.5 and Proposition 4.2, this means that the action of $\operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D^\times)$ on $\mathbb{I}_{m-1,\chi}$ factors through the algebra of functions on opers with singularity of order m and the m-residue $\pi(-\chi)$.

Thus, we obtain that the homomorphism $Z(\widehat{\mathfrak{g}}) \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{I}_{m-1,\chi}$ factors as

$$Z(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun}\operatorname{Op}_{{}^L G}\big(D^\times\big) \twoheadrightarrow \operatorname{Fun}\operatorname{Op}^{\leqslant m}_{{}^L G}(D)_{\pi(-\chi)} \to \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{I}_{m-1,\chi}.$$

To complete the proof of part (2), we need to show that the last homomorphism is injective if χ is regular. It suffices to show that the natural map

$$\mathbb{C}[S_{i,[n_i]}]_{i=1,\dots,\ell;\, n_i<m(d_i+1)-1} \to \mathbb{I}_{m-1,\chi} \tag{5.10}$$

obtained by acting on the generating vector of $\mathbb{I}_{m-1,\chi}$, is injective. As in the proof of part (1), we will derive this from the injectivity of the corresponding maps of associated graded spaces.

The associated graded space of $\mathbb{I}_{m-1,\chi}$ with respect to the PBW filtration on the universal enveloping algebra of $\widehat{\mathfrak{g}}_{\kappa_c}$ is naturally identified with

$$S\big(\mathfrak{g}((t))/t^{m-1}\mathfrak{g}[[t]]\big) \simeq \operatorname{Fun}\big(t^{-m+1}\mathfrak{g}^*[[t]]\,dt\big) \simeq \operatorname{Fun}\big(t^{-m+1}\mathfrak{g}^*[[t]]\big).$$

Now, using Lemma 3.9, we obtain that the symbol of the image of a polynomial R in $S_{i,[n_i]}$ under the map (5.10), considered as a function on $t^{-m+1}\mathfrak{g}^*[[t]]$, is equal to the same polynomial R in which we make a replacement

$$S_{i,[n]} \mapsto \overline{P}_{i,n}\big(\cdot + \chi t^{-m}\big)$$

followed by the shift of argument by χt^{-m}. Thus, injectivity of the associated graded map of (5.10) is equivalent to the algebraic independence of the restrictions of

$$\overline{P}_{i,n_i}, \quad i = 1,\dots,\ell;\ n_i < m(d_i+1) - 1,$$

which are functions on $\mathfrak{g}^*((t))$, to $\chi t^{-m} + t^{-m+1}\mathfrak{g}^*[[t]]$. Let us prove this algebraic independence.

To simplify the argument, let us apply the automorphism of $\mathfrak{g}((t))$ sending $\overline{J}^a_n$ to $\overline{J}^a_{n-m}$. Then $\chi t^{-m} + t^{-m+1}\mathfrak{g}^*[[t]]$ becomes $\chi + t\mathfrak{g}^*[[t]]$ and $\overline{P}_{i,n} \mapsto \overline{P}_{i,n-m(d_i+1)}$. We therefore need to prove that the restrictions of the polynomials $\overline{P}_{i,n}$, $i = 1, \dots, \ell$; $n < -1$, to $\chi + t\mathfrak{g}^*[[t]] \subset \mathfrak{g}^*[[t]]$ are algebraically independent. This would follow if we show that their differentials at the point χ are linearly independent.

Let us identify the cotangent space to $\chi \in \mathfrak{g}^*$ with $(\mathfrak{g}^*)^* = \mathfrak{g}$ and the cotangent space to $\chi \in \chi + t\mathfrak{g}^*[[t]]$ with $(t\mathfrak{g}^*[[t]])^* = \mathfrak{g}((t))/t^{-1}\mathfrak{g}[[t]]$. Since χ is regular, we obtain from [33] that the values $d\overline{P}_i|_\chi$ of the differentials $d\overline{P}_i$ of the generators $\overline{P}_i$ of the algebra $\operatorname{Inv}\mathfrak{g}^*$ at $\chi \in \mathfrak{g}^*$ are linearly independent vectors in the centralizer $\mathfrak{g}_\chi$ of χ, considered as a subspace of $\mathfrak{g}$. Using the explicit formula (5.5) for $\overline{P}_{i,n}$, we find that the value of $d\overline{P}_{i,n}$, $n < -1$, at $\chi \in \mathfrak{g}^*[[t]]$ is equal to

$$(d\overline{P}_i|_\chi) \otimes t^n \in \mathfrak{g}((t))/t^{-1}\mathfrak{g}[[t]].$$

These vectors are linearly independent for $i = 1, \dots, \ell$ and $n < -1$. Therefore the restrictions of the polynomials $\overline{P}_{i,n}$, $i = 1, \dots, \ell$; $n < -1$, to $\chi + t\mathfrak{g}^*[[t]] \subset \mathfrak{g}^*[[t]]$ are algebraically independent, and so the map (5.10) is injective. This completes the proof of part (2).

Part (3) of the Theorem is obtained by combining Theorem 12.4, Lemma 9.4 and Proposition 12.8 of [20].

Part (4) is established in [28, Lemma 1.7]. □

We note that parts (1) and (2) of the theorem may be interpreted as saying that the supports of the $Z(\widehat{\mathfrak{g}})$-modules $\mathbb{U}_m$ and $\mathbb{I}_{m,\chi}$, considered as subvarieties in $\operatorname{Op}_{{}^LG}(D^\times)$, are equal to $\operatorname{Op}^{\leqslant(m+1)}_{{}^LG}(D)$ and $\operatorname{Op}^{\leqslant(m+1)}_{{}^LG}(D)_{\pi(-\chi)}$, respectively. If χ were not regular, then the support of $\mathbb{I}_{m,\chi}$ would still be contained in $\operatorname{Op}^{\leqslant(m+1)}_{{}^LG}(D)_{\pi(-\chi)}$, but it would not be equal to it (equivalently, the last map in (5.9) would not be injective). For example, if $\chi = 0$, we have $\mathbb{I}_{m,0} = \mathbb{U}_m$, and so the support of $\mathbb{I}_0$ is equal to $\operatorname{Op}^{\leqslant m}_{{}^LG}(D)$. This explains the special role played by regular characters χ.

5.3. Description of the Gaudin algebras

Now we are ready to show that the Gaudin algebras $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$, which were introduced in Section 2.5, are isomorphic to algebras of functions on opers on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$ with appropriate singularities at the points $z_1, \dots, z_N$ and ∞. Here we follow the analysis of [22, Section 2.5], where the Gaudin algebras were described in the case regular singularities (when all m_i and m_∞ are equal to 1).

Let us denote by

$$\operatorname{Op}_{{}^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty}$$

the space of ${}^LG$-opers on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$ whose restriction to $D^\times_{z_i}$ belongs to

$$\operatorname{Op}^{\leqslant m_i}_{{}^LG}\big(D^\times_{z_i}\big) \subset \operatorname{Op}_{{}^LG}\big(D^\times_{z_i}\big), \quad i = 1, \dots, N,$$

and whose restriction to $D^\times_\infty$ belongs to

$$\operatorname{Op}^{\leqslant m_\infty}_{{}^LG}\big(D^\times_\infty\big) \subset \operatorname{Op}_{{}^LG}\big(D^\times_\infty\big).$$

Thus, points of $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$ are ${}^LG$-opers on $\mathbb{P}^1$ with singularities at z_i, $i=1,\dots,N$, and ∞ of orders m_i, $i=1,\dots,N$, and m_∞, respectively (and regular elsewhere).

Theorem 5.7.

(1) *There is an isomorphism of algebras*

$$\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}) \simeq \operatorname{Fun}\mathrm{Op}_{^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty}.$$

Let us fix a $\mathfrak{g}_{m_i}$-module M_i for each $i=1,\dots,N$, and a $\bar{\mathfrak{g}}_{m_\infty}$-module M_∞. Then the following holds:

(2) *Suppose that we have $m_j=1$ and let M_j be a $\mathfrak{g}$-module on which the center $Z(\mathfrak{g})$ acts via its character $\varpi(\lambda+\rho)$. Then the action of $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ on $\bigotimes_{i=1}^N M_i \otimes M_\infty$ factors through the algebra of functions on a subset of $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$, which consists of the opers with regular singularity and* 1*-residue $\varpi(-\lambda_j-\rho)$ at z_j.*
(3) *Under the assumptions of part* (2)*, suppose in addition that λ_j is an integral dominant weight and $M_j=V_{\lambda_j}$ is the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ_j. Then the action of $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ on $\bigotimes_{i=1}^N M_i \otimes M_\infty$ factors through the algebra of functions on a subset of $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$, which consists of the opers with regular singularity at z_j,* 1*-residue $\varpi(-\lambda_j-\rho)$, and trivial monodromy around z_j.*
(4) *Now let $\chi\in\mathfrak{g}^*$ be a regular element. Set $M_j=\mathbb{I}_{m_j,\chi}$. Then the action of $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$ on $\bigotimes_{i=1}^N M_i \otimes M_\infty$ factors through the algebra of functions on a subset of $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(m_i),m_\infty}_{(z_i),\infty}$, which consists of the opers with singularity of order m_j+1 at z_j and (m_j+1)-residue $\pi(-\chi)$ (here we trivialize the tangent space to z_j using the global coordinate t on $\mathbb{P}^1$).*

An analogous result also holds with z_j replaced by ∞.

Proof. Proof is a word-for-word repetition of the argument used in the proof of Theorem 2.7 of [22] (which corresponds to the special case of parts (1)–(3) of the above theorem when all the m_i's and m_∞ are equal to 1). Using the general results of [26] about the action of commutative vertex algebras on coinvariants, we show the following. Let $\mathbb{M}_1,\dots,\mathbb{M}_N$ and $\mathbb{M}_\infty$ be $\widehat{\mathfrak{g}}_{\kappa_c}$-modules. Then, as explained in Section 2.5, the universal Gaudin algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ acts on the corresponding space $H(\mathbb{M}_1,\dots,\mathbb{M}_N,\mathbb{M}_\infty)$ of coinvariants. Suppose that the action of $Z(\widehat{\mathfrak{g}})\simeq \operatorname{Fun}\mathrm{Op}_{^LG}(D^\times)$ on $\mathbb{M}_i$ factors through $\operatorname{Fun}\mathrm{Op}^{M_i}_{^LG}(D^\times)$, where $\mathrm{Op}^{M_i}_{^LG}(D^\times)\subset \mathrm{Op}_{^LG}(D^\times)$. Then, in the same way as in the proof of Theorem 2.7 of [22] we obtain that the action of $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ factors through the space of ${}^LG$-opers on $\mathbb{P}^1\setminus\{z_1,\dots,z_N,\infty\}$ whose restriction to the punctured disc $D^\times_{z_i}$, $i=1,\dots,N$ (resp., $D^\times_\infty$) belongs to $\mathrm{Op}^{M_i}_{^LG}(D^\times_{z_i})$ (resp., $\mathrm{Op}^{M_i}_{^LG}(D^\times_\infty)$).

Now we combine this result with the local statements of Theorem 5.6 describing the action of the center on $\widehat{\mathfrak{g}}_{\kappa_c}$-modules. This gives us the sought-after assertions about the factorization of the action of the universal Gaudin algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ on particular modules.

To complete the proof of part (1), we need to show that the homomorphism

$$\operatorname{Fun}\mathrm{Op}_{^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty} \to \bigotimes_{i=1}^N U(\mathfrak{g}_{m_i})\otimes U(\bar{\mathfrak{g}}_{m_\infty}) \tag{5.11}$$

obtained this way is injective. But its image is, by definition, the algebra $\mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$, and so the injectivity of this homomorphism implies that we have an isomorphism

$$\operatorname{Fun}\operatorname{Op}_{{}^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty} \simeq \mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g}), \tag{5.12}$$

as stated in part (1) of the theorem.

In order to prove the injectivity we pass to the associated graded algebras. According to Theorem 3.4, at the level of associated graded algebras the homomorphism (5.11) becomes the homomorphism

$$\overline{\Psi}^{(m_i),m_\infty}_{(z_i),\infty} : \operatorname{Fun}\mathcal{H}^{(m_i),m_\infty}_{G,(z_i),\infty} \to \bigotimes_{i=1,\dots,N} S(\mathfrak{g}_{m_i}) \otimes S(\bar{\mathfrak{g}}_{m_\infty})$$

defined in formula (3.7). By Lemma 3.3, $\overline{\Psi}^{(m_i),m_\infty}_{(z_i),\infty}$ is injective. This implies the injectivity of (5.11). Therefore we obtain an isomorphism (5.12). $\square$

Note that the embedding

$$\operatorname{Op}_{{}^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty} \hookrightarrow \operatorname{Op}_{{}^LG}(D_u)$$

obtained by restricting an oper to the disc D_u around a point $u \in \mathbb{P}^1\{z_1, \dots, z_N, \infty\}$ gives rise to a surjective homomorphism

$$\operatorname{Fun}\operatorname{Op}_{{}^LG}(D_u) \twoheadrightarrow \operatorname{Fun}\operatorname{Op}_{{}^LG}\big(\mathbb{P}^1\big)^{(m_i),m_\infty}_{(z_i),\infty}.$$

The corresponding homomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}})_u \twoheadrightarrow \mathcal{Z}^{(m_i),m_\infty}_{(z_i),\infty}(\mathfrak{g})$$

is nothing but the homomorphism $\Phi^{(m_i),m_\infty}_{(z_i),\infty}$ from Section 2.5.

Observe also that part (1) of the theorem implies that the universal Gaudin algebra $\mathcal{Z}_{(z_i),\infty}(\mathfrak{g})$ is isomorphic to the (topological) algebra functions on the ind-affine space of all (meromorphic) ${}^LG$-opers on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$.

5.4. *The case of singularity of order* 2

Consider a special case of the theorem when we have two points $z_1 = 0$ and ∞, set $m_1 = 1$ and $m_\infty = 1$ and choose a regular $\chi \in \mathfrak{g}^*$ corresponding to the point ∞. The corresponding Gaudin algebra $\mathcal{A}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}$ is the algebra $\mathcal{A}_\chi$ introduced in Section 2.7.

Let $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)}$ be the space of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularity at the point 0 and with singularity of order 2 at ∞, with 2-residue $\pi(-\chi)$. This space has the following concrete realization for regular χ.

Let us pick an element of the form

$$-p_{-1} - \overline{\chi} = -p_{-1} - \sum_{j=1}^{\ell} \overline{\chi}_j p_j \in {}^L\mathfrak{g}_{\mathrm{can}}$$

(see formula (4.8) for the definition of ${}^L\mathfrak{g}_{\mathrm{can}}$) in the conjugacy class $\pi(-\chi) \in \mathfrak{g}^*/G = {}^L\mathfrak{g}/{}^LG$. Then it follows from Lemma 4.3 that on the punctured disc $D_\infty^\times$ at ∞ (with coordinate $s = t^{-1}$) each element of $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)}$ may be uniquely represented by a connection operator of the form

$$\partial_s - p_{-1} - \sum_{j=1}^{\ell} \bigl(\overline{\chi}_j s^{-2d_j-2} + s^{-2d_j-1} u_j(s)\bigr) p_j, \qquad u_j(s) = \sum_{n \geqslant 0} u_{j,n} s^n \in \mathbb{C}[\![s]\!]$$

(the sign in front of p_{-1} may be eliminated by a gauge transformation with $\check{\rho}(-1)$, which would result in multiplication of $u_j(s)$ by $(-1)^j$, but we prefer not to do this). This oper extends to $\mathbb{P}^1 \setminus 0$ if and only if each $u_j(s)$ belongs to $\mathbb{C}[s]$. To understand its behavior at 0, we apply the change of variables $s = t^{-1}$. After applying the gauge transformation with $\check{\rho}(t^{-2})$, we find that the restriction of this oper to the punctured disc $D_0^\times$ at $0 \in \mathbb{P}^1$ is equal to

$$\partial_t + p_{-1} + \frac{2\check{\rho}}{t} + \sum_{j=1}^{\ell} \bigl(\overline{\chi}_j + \widetilde{u}_j(t)\bigr) p_j, \quad \widetilde{u}_j(t) = t^{-1} u_j\bigl(t^{-1}\bigr).$$

Next, we apply the gauge transformation with $\exp(-p_1/t)$ and obtain

$$\partial_t + p_{-1} + \sum_{j=1}^{\ell} \bigl(\overline{\chi}_j + \widetilde{u}_j(t)\bigr) p_j, \quad \widetilde{u}_j(t) = t^{-1} u_j\bigl(t^{-1}\bigr).$$

This oper has regular singularity at 0 if and only if

$$u_j(s) = \sum_{n=0}^{j} u_{j,n} s^n.$$

Thus, we find that each element of $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)}$ may be represented uniquely by an operator of the form

$$\partial_t + p_{-1} + \sum_{j=1}^{\ell} \left(\overline{\chi}_j + \sum_{n=0}^{d_j} u_{j,n} t^{-n-1} \right) p_j. \tag{5.13}$$

Note that according to Proposition 4.2, its 1-residue at 0 is equal to

$$p_{-1} + \sum_{j=1}^{\ell} \left(u_{j,d_j} + \frac{1}{4}\delta_{j,1} \right) p_j \in {}^L\mathfrak{g}_{\mathrm{can}} \simeq {}^L\mathfrak{g}/{}^LG = \mathfrak{g}/G. \tag{5.14}$$

In particular, we obtain that

$$\mathrm{Fun}\,\mathrm{Op}_{{}^LG}\bigl(\mathbb{P}^1\bigr)_{\pi(-\chi)} \simeq \mathbb{C}[u_{j,n_j}]_{j=1,\dots,\ell;\, n_j=0,\dots,d_j}. \tag{5.15}$$

Theorem 5.8. *If $\chi \in \mathfrak{g}^*$ is regular, then the algebra $\mathcal{A}_\chi$ is isomorphic to the algebra of functions on* $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)}$.

Proof. According to Theorem 5.7(4), we have a surjective homomorphism

$$\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{P}^1)_{\pi(-\chi)} \to \mathcal{A}_\chi. \tag{5.16}$$

To show that it is an isomorphism, it is sufficient to prove that the corresponding homomorphism of the associated graded algebras is an isomorphism. According to [45] and Theorem 3.14, $\operatorname{gr}\mathcal{A}_\chi = \overline{\mathcal{A}}_\chi$. But for regular χ the algebra $\overline{\mathcal{A}}_\chi$ is a free polynomial algebra with generators $D_\chi^{n_i}\overline{P}_i$, $i = 1, \dots, \ell$; $n_i = 0, \dots, d_i + 1$, by [34,38] and Theorem 3.11.

On the other hand, it follows from formula (5.15), Lemma 5.5 and the discussion in the proof of Theorem 3.14 that $\operatorname{gr}\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{P}^1)_{\pi(-\chi)}$ is isomorphic to the same free polynomial algebra. Hence the map (5.16) is an isomorphism. $\square$

5.5. Joint eigenvalues on finite-dimensional modules

Let us first recall the results of [21,22] (see also [17,19]) on the joint eigenvalues of the ordinary Gaudin algebra $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ on the tensor products $\bigotimes_{i=1}^N V_{\lambda_i}$ of irreducible finite-dimensional $\mathfrak{g}$-modules V_{λ_i}. Let $\operatorname{Op}_{^LG}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$ be the set of ^LG-opers on $\mathbb{P}^1$ with regular singularities at z_i, $i = 1, \dots, N$, and ∞, with residues $\varpi(-\lambda_i - \rho)$ and $\varpi(-\lambda_\infty - \rho)$, and with trivial monodromy representation. Then according to [21, Corollary 4.8] (see also [22, Theorem 2.7(3)] and Theorem 5.7(3) above), we have the following description of the joint eigenvalues of $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ on $\bigotimes_{i=1}^N V_{\lambda_i}$.

Theorem 5.9. *The set of joint eigenvalues of* $\mathcal{Z}^{(1),1}_{(z_i),\infty}(\mathfrak{g})$ *on* $\bigotimes_{i=1}^N V_{\lambda_i}$ (*without multiplicities*) *is a subset of* $\operatorname{Op}_{^LG}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$.

Further, Conjecture 1 of [22] states that this inclusion is actually a bijection. Now we discuss analogous results and conjectures for the generalized Gaudin algebras corresponding to irregular singularity of order 2 at ∞.

We start with the simplest such Gaudin algebra, namely, the algebra $\mathcal{A}_\chi \subset U(\mathfrak{g})$. Consider its action on the irreducible finite-dimensional $\mathfrak{g}$-module V_λ, where λ is a dominant integral weight. Note that from the point of view of the general construction of Section 2.8 this action comes about through the action of $\mathcal{A}_\chi = \mathcal{A}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}$ on the space of coinvariants

$$H(\mathbb{V}_\lambda \otimes \mathbb{I}_{1,\chi}) \simeq (V_\lambda \otimes I_\chi)/\mathfrak{g} \simeq V_\lambda.$$

Suppose that χ is regular semi-simple. Then $\mathcal{A}_\chi$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is the centralizer of χ in $\mathfrak{g}$. Therefore the action of $\mathcal{A}_\chi$ preserves the weight decomposition of V_λ with respect to the $\mathfrak{h}$-action. It is natural to ask what are the joint generalized eigenvalues of $\mathcal{A}_\chi$ on these components.

Let

$$\operatorname{Op}_{^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)} \subset \operatorname{Op}_{^LG}(\mathbb{P}^1)_{\pi(-\chi)}$$

be the set of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularity at the point 0 with the 1-residue $\varpi(-\lambda-\rho)$, singularity of order 2 at the point ∞ with the 2-residue $\pi(-\chi)$ and trivial monodromy. Then, according to Theorem 5.7(3), the action of $\mathcal{A}_\chi$ on $U(\mathfrak{g})$ factors through the homomorphism

$$\mathcal{A}_\chi \simeq \operatorname{Fun}\operatorname{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)} \twoheadrightarrow \operatorname{Fun}\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}.$$

In other words, we obtain the following description of the joint generalized eigenvalues of the commutative algebra $\mathcal{A}_\chi$ (which are the points of $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)}$, according to Theorem 5.8) on its generalized eigenvectors in V_λ.

Theorem 5.10. *For regular semi-simple $\chi \in \mathfrak{g}^*$ the set of joint generalized eigenvalues of $\mathcal{A}_\chi$ on V_λ (without multiplicities) is a subset of* $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}$.

Concretely, elements of $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}$, whose 1-residue at 0 is equal to $\varpi(-\lambda-\rho)$ are represented by the connections of the form (5.13) with the expression (5.14) equal to $\varpi(-\lambda-\rho)$. As we explained after Proposition 4.4, the condition that this connection has trivial monodromy around 0 imposes a set of $\dim {}^L\mathfrak{n}$ algebraic equations on the oper, and $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}$ is just the set of solutions of these equations plus $\dim {}^L\mathfrak{h}$ equations corresponding to the 1-residue condition at 0. Note that the dimension of $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)_{\pi(-\chi)}$ is equal to $\dim {}^L\mathfrak{b}$, and so it is reasonable to expect that $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\chi)}$ is a finite set.

Conjecture 2. *The injective map of Theorem* 5.10 *is a bijection for any regular semi-simple element $\chi \in \mathfrak{g}^*$ and dominant integral weight λ.*

If the action of the algebra $\mathcal{A}_\chi$ on V_λ is diagonalizable (which we expect to happen for generic regular semi-simple χ), then Proposition 5.10 would give us a labeling of an eigenbasis of $\mathcal{A}_\chi$ in V_λ by monodromy-free ${}^LG$-opers on $\mathbb{P}^1$ with prescribed singularities at 0 and ∞.

Proposition 5.10 has the following multi-point generalization. Consider the algebra

$$\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi} \subset U(\mathfrak{g})^{\otimes N}$$

(see Section 2.8 and [45]). It may be obtained as the quotient of

$$\mathcal{Z}^{(1),2}_{(z_i),\infty}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N} \otimes U(\bar{\mathfrak{g}}_2)$$

obtained by applying the character $\bar{\mathfrak{g}}_2 \to \mathfrak{g} \xrightarrow{\chi} \mathbb{C}$ along the last factor.

Let

$$\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^{(1),2}_{(z_i);\pi(-\chi)} \subset \operatorname{Op}_{{}^LG}(\mathbb{P}^1)^{(1),2}_{(z_i),\infty}$$

be the space of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularities at the points z_i, $i=1,\dots,N$, and with singularity of order 2 at the point ∞ with the 2-residue $\pi(-\chi)$.

Conjecture 3. *For any regular $\chi \in \mathfrak{g}^*$*

$$\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi} \simeq \operatorname{Fun}\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^{(1),2}_{(z_i);\pi(-\chi)}.$$

Now let $\lambda_1, \dots, \lambda_N$ be a collection of dominant integral weights of $\mathfrak{g}$. Consider the action of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on the tensor product $\bigotimes_{i=1}^N V_{\lambda_i}$. We will now assume that χ is regular semisimple.

Let

$$\mathrm{Op}_{^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\chi)} \subset \mathrm{Op}_{^L G}(\mathbb{P}^1)^{(1),2}_{(z_i);\pi(-\chi)}$$

be the set of $^L G$-opers on $\mathbb{P}^1$ with regular singularities at the points z_i, $i = 1, \dots, N$, with the 1-residues $\varpi(-\lambda_i - \rho)$ and trivial monodromy around these points, and with singularity of order 2 at the point ∞ with the 2-residue $\pi(-\chi)$. Then Theorem 5.7 implies the following result.

Theorem 5.11. *There is an injective map from the set of joint generalized eigenvalues of the commutative algebra* $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ *on* $\bigotimes_{i=1}^N V_{\lambda_i}$ *(without multiplicities) to* $\mathrm{Op}_{^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\chi)}$.

We propose the following analogue of Conjecture 2:

Conjecture 4. *The injective map of Theorem* 5.11 *is a bijection.*

This should be viewed as an analogue of Conjecture 1 of [22]. The motivation for both conjectures comes from the geometric Langlands correspondence (see the discussion in [22] after Conjecture 1).

6. Bethe Ansatz in Gaudin models with irregular singularities

In this section we develop an analogue of the Bethe Ansatz method for constructing eigenvectors of the Gaudin algebra in the case of irregular singularities. For definiteness, we will consider here the case of the Gaudin algebras $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$, but our methods may be generalized to yield eigenvectors in other generalized Gaudin models.

The construction of the Bethe Ansatz for Gaudin models with regular singularities is explained in detail in [22, Sections 4–5], following [17] (for another approach, see [44]). In this section we will follow the same approach, using notation and results of [22].

6.1. Wakimoto modules

The construction of eigenvectors of the Hamiltonians of the ordinary Gaudin model developed in [17,22] utilizes a class of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules called *Wakimoto modules*. These modules were defined in [54] for $\mathfrak{g} = \mathfrak{sl}_2$ and in [14,15,20] for a general simple Lie algebra $\mathfrak{g}$ (for a detailed exposition, see [25]). Here we will follow the notation of [20,22], where we refer the reader for more details.

Wakimoto modules over $\widehat{\mathfrak{g}}_{\kappa_c}$ are parameterized by connections on an $^L H$-bundle $\Omega^{-\rho}$ on the punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((t))$. Here $\Omega^{-\rho}$ is defined as the push-forward of the $\mathbb{C}^\times$-bundle corresponding to the canonical line bundle Ω on $D^\times$ under the homomorphism $\mathbb{C}^\times \to {^L H}$ corresponding to the integral coweight $-\rho$ of $^L H$ (we recall that $^L H$ is a Cartan subgroup of the group $^L G$ of adjoint type). A choice of coordinate t on the disc D gives rise to a trivialization of Ω, and hence of $\Omega^{-\rho}$. A connection on $\Omega^{-\rho}$ may then be written as an operator

$$\overline{\nabla} = \partial_t + \nu(t), \quad \nu(t) \in {^L\mathfrak{h}}((t)) = \mathfrak{h}^*((t))$$

(see [20, Section 5.5]). If s is another coordinate such that $t = \varphi(s)$, then this connection will be represented by the operator

$$\partial_s + \varphi'(s)\nu\big(\varphi(s)\big) + \rho \cdot \frac{\varphi''(s)}{\varphi'(s)}. \tag{6.1}$$

Let $\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ be the space of all connections on the ${}^L H$-bundle $\Omega^{-\rho}$ on $D^\times$. Denote by $b_{i,n}$, $n \in \mathbb{Z}$, the function on $\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ defined by the formula

$$\partial_t + \nu(t) \mapsto \mathrm{Res}_{t=0}\big\langle \check{\alpha}_i, \nu(t)\big\rangle t^n \, dt.$$

The algebra $\mathrm{Fun}\,\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ of functions on $\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ is a complete topological algebra

$$\mathrm{Fun}\,\mathrm{Conn}\big(\Omega^{-\rho}\big)_{D^\times} \simeq \varprojlim \mathbb{C}[b_{i,n}]_{i=1,\ldots,\ell;\, n\in\mathbb{Z}}/I_N,$$

where I_N is the ideal generated by $b_{i,n}$, $i = 1, \ldots, \ell$; $n \geqslant N$. A module over this algebra is called smooth if every vector is annihilated by an ideal I_N for large enough N. In particular, each $\overline{\nabla} \in \mathrm{Conn}(\Omega^{-\rho})_{D^\times}$ gives rise to a one-dimensional smooth module over $\mathrm{Fun}\,\mathrm{Conn}(\Omega^{-\rho})_{D^\times}$, which we denote by $\mathbb{C}_{\overline{\nabla}}$. Equivalently, $\mathbb{C}_{\overline{\nabla}}$ may be viewed as a module over the commutative vertex algebra $\pi_0 = \mathrm{Fun}\,\mathrm{Conn}(\Omega^{-\rho})_D$ (see [20, Section 4.2]).

Next, we define the Weyl algebra $\mathcal{A}^{\mathfrak{g}}$ with generators $a_{\alpha,n}$, $a^*_{\alpha,n}$, $\alpha \in \Delta_+$, $n \in \mathbb{Z}$, and relations

$$\big[a_{\alpha,n}, a^*_{\beta,m}\big] = \delta_{\alpha,\beta}\delta_{n,-m}, \qquad [a_{\alpha,n}, a_{\beta,m}] = \big[a^*_{\alpha,n}, a^*_{\beta,m}\big] = 0. \tag{6.2}$$

Let $M_{\mathfrak{g}}$ be the Fock representation of $\mathcal{A}^{\mathfrak{g}}$ generated by a vector $|0\rangle$ such that

$$a_{\alpha,n}|0\rangle = 0, \quad n \geqslant 0; \qquad a^*_{\alpha,n}|0\rangle = 0, \quad n > 0.$$

It carries a vertex algebra structure (see [20]). It follows from the general theory of [26, Chapter 5], that a module over the vertex algebra $M_{\mathfrak{g}}$ is the same as a smooth module over the Weyl algebra $\mathcal{A}^{\mathfrak{g}}$, i.e., one such that every vector is annihilated by $a_{\alpha,n}, a^*_{\alpha,n}$ for large enough n.

According to [20, Theorem 4.7], we have a homomorphism of vertex algebras

$$w_{\kappa_c} : \mathbb{V}_0 \to M_{\mathfrak{g}} \otimes \pi_0,$$

which sends the center $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathbb{V}_0$ to π_0. Moreover, the corresponding homomorphism of commutative algebras

$$\mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D) \to \mathrm{Fun}\,\mathrm{Conn}\big(\Omega^{-\rho}\big)_D$$

is induced by the *Miura transformation*

$$\mathrm{Conn}\big(\Omega^{-\rho}\big)_D \to \mathrm{Op}_{{}^L G}(D)$$

defined in [20, Section 10.3], following [10]. We recall that this map sends a Cartan connection $\overline{\nabla} = \partial_t + \nu(t)$ to the ${}^L G$-oper which is the $N[\![t]\!]$-gauge equivalence class of $\nabla = \partial_t + p_{-1} + \nu(t)$. There is a similar map over the punctured disc $D^\times$ or a smooth curve.

This result implies that for any smooth $\mathcal{A}^{\mathfrak{g}}$-module L and any smooth module R over $\operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{D^\times}$ the tensor product $L \otimes R$ is a $\widehat{\mathfrak{g}}_{\kappa_c}$-module. In particular, taking $R = \mathbb{C}_{\overline{\nabla}}$, we obtain a $\widehat{\mathfrak{g}}_{\kappa_c}$-module $L \otimes \mathbb{C}_{\overline{\nabla}}$. These are the Wakimoto modules. The following is proved in [20, Theorem 12.6].

Theorem 6.1. *The center $Z(\widehat{\mathfrak{g}}) = \operatorname{Fun}\operatorname{Op}_{{}^L G}(D^\times)$ acts on $L \otimes \mathbb{C}_{\overline{\nabla}}$ via the character corresponding to the ${}^L G$-oper $\bar{\mathbf{b}}^*(\overline{\nabla})$, where*

$$\bar{\mathbf{b}}^*(\overline{\nabla}) : \operatorname{Conn}\bigl(\Omega^{-\rho}\bigr)_{D^\times} \to \operatorname{Op}_{{}^L G}\bigl(D^\times\bigr)$$

is the Miura transformation on $D^\times$. In particular, the action of $Z(\widehat{\mathfrak{g}})$ is independent of the choice of the module L.

6.2. Coinvariants of Wakimoto modules

The idea of [17,22] is to construct eigenvectors of Gaudin algebras using spaces of coinvariants of the tensor products of Wakimoto modules. These coinvariants are defined (following the general definition of [26, Chapter 10]) with respect to the vertex algebra $M_{\mathfrak{g}} \otimes \pi_0$. By functoriality of coinvariants, the above homomorphism w_{κ_c} of vertex algebras gives rise to linear maps from the spaces of coinvariants with respect to $\mathbb{V}_0 = \mathbb{V}_{0,\kappa_c}$, which are just the spaces of $\mathfrak{g}_{(z_i)}$-coinvariants introduced in Section 2, to the spaces of coinvariants with respect to $M_{\mathfrak{g}} \otimes \pi_0$. Because $M_{\mathfrak{g}} \otimes \pi_0$ is a much simpler vertex algebra, its coinvariants are easy to compute, and in the cases we consider below they turn out to be one-dimensional. Therefore we obtain linear functionals on the spaces of $\mathfrak{g}_{(z_i)}$-coinvariants that are of interest to us. It then follows from the construction that these linear functionals are eigenvectors of the corresponding Gaudin algebra.

Let us choose a set of distinct points $x_1, \ldots, x_p$ on $\mathbb{C} \subset \mathbb{P}^1$. We attach to each of these points, and to the point ∞, an $M_{\mathfrak{g}}$-module and a π_0-module. As the $M_{\mathfrak{g}}$-modules attached to x_i, $i = 1, \ldots, p$, we choose $M_{\mathfrak{g}}$ itself, and to ∞ we attach another module $M'_{\mathfrak{g}}$ generated by a vector $|0\rangle'$ such that

$$a_{\alpha,n}|0\rangle' = 0, \quad n > 0; \qquad a^*_{\alpha,n}|0\rangle' = 0, \quad n \geqslant 0.$$

As the π_0-modules attached to x_i, $i = 1, \ldots, p$, we take the one-dimensional modules $\mathbb{C}_{\overline{\nabla}_i} = \mathbb{C}_{\nu_i(z)}$, where

$$\overline{\nabla}_i = \partial_z + \nu_i(z), \quad \nu_i(z) \in \mathfrak{h}^*((z)) = {}^L\mathfrak{h}((z)),$$

and as the π_0-module attached to the point ∞ we take $\mathbb{C}_{\overline{\nabla}_\infty} = \mathbb{C}_{\nu_\infty(z)}$, where $\overline{\nabla}_\infty = \partial_z + \nu_\infty(z)$.

The corresponding space of coinvariants for $M_{\mathfrak{g}} \otimes \pi_0$ is the tensor product of the spaces

$$H_{M_{\mathfrak{g}}}\bigl(\mathbb{P}^1; (x_i), \infty; (M_{\mathfrak{g}}), M'_{\mathfrak{g}}\bigr) \quad \text{and} \quad H_{\pi_0}\bigl(\mathbb{P}^1, (x_i), \infty; (\mathbb{C}_{\nu_i(z)}), \mathbb{C}_{\nu_\infty}(z)\bigr)$$

of coinvariants for the $M_{\mathfrak{g}}$-modules and the π_0-modules, respectively (see [22] for their definition). The following result is proved in [22, Proposition 4.9] (see also [17, Proposition 4]).

Proposition 6.2.

(1) *The space* $H_{M_{\mathfrak{g}}}(\mathbb{P}^1; (x_i), \infty; (M_{\mathfrak{g}}), M'_{\mathfrak{g}})$ *is one-dimensional and the projection of the vector* $|0\rangle^{\otimes N} \otimes |0\rangle'$ *on it is non-zero.*
(2) *The space* $H_{\pi_0}(\mathbb{P}^1, (x_i), \infty; (\mathbb{C}_{\nu_i(z)}), \mathbb{C}_{\nu_\infty}(z))$ *is one-dimensional if and only if there exists a connection* $\overline{\nabla}$ *on the* ${}^L H$*-bundle* $\Omega^{-\rho}$ *on* $\mathbb{P}^1 \setminus \{x_1, \ldots, x_p, \infty\}$ *whose restriction to the punctured disc at each* x_i *is equal to* $\partial_t + \nu_i(t - x_i)$*, and whose restriction to the punctured disc at* ∞ *is equal to* $\partial_{t^{-1}} + \nu_\infty(t^{-1})$.

Otherwise, $H_{\pi_0}(\mathbb{P}^1, (x_i), \infty; (\mathbb{C}_{\nu_i(z)}), \mathbb{C}_{\nu_\infty}(z)) = 0$.

Formula (6.1) shows that if we have a connection on $\Omega^{-\rho}$ over $\mathbb{P}^1$ whose restriction to $\mathbb{P}^1 \setminus \infty$ is represented by the operator $\partial_t + \nu(t)$, then its restriction to the punctured disc $D_\infty^\times$ at ∞ reads, with respect to the coordinate $s = t^{-1}$

$$\partial_s - s^{-2}\nu\big(s^{-1}\big) - 2\rho s^{-1}.$$

In [17,22] we chose $\nu_i(z)$ to be of the form

$$\nu_i(z) = \frac{\nu_i}{z} + \sum_{n \geqslant 0} \nu_{i,n} z^n,$$

and $\nu_\infty(z)$ to be of the form

$$\nu_\infty(z) = \frac{\nu_\infty}{z} + \sum_{n \geqslant 0} \nu_{\infty,n} z^n.$$

In other words, we consider connections on $\Omega^{-\rho}$ with regular singularities. The condition of the proposition is then equivalent to saying that the restriction of $\overline{\nabla}$ to $\mathbb{P}^1 \setminus \infty$ is represented by the operator $\partial_t + \nu(t)$, where

$$\nu(t) = \sum_{i=1}^{p} \frac{\nu_i}{t - x_i},$$

and $\partial_t + \nu_i(t - x_i)$ is the expansion of $\nu(t)$ at x_i, $i = 1, \ldots, p$, while $\nu_\infty(t^{-1})$ is the expansion of $-t^2\nu(t) - 2\rho t$ in powers of t^{-1}.

Now we will choose $\nu_i(z)$ to be the same as above, but we will choose $\nu_\infty(z)$ to be of the form

$$\nu_\infty(z) = \frac{\chi}{z^2} + \sum_{n \geqslant -1} \nu_{\infty,n} z^n,$$

where $\chi \in \mathfrak{h}^*$. Then the condition of the proposition means that the restriction of $\overline{\nabla}$ to $\mathbb{P}^1 \setminus \infty$ is represented by the operator

$$\partial_t - \chi + \sum_{i=1}^{p} \frac{\nu_i}{t - x_i}.$$

This connection has irregular singularity at ∞. Note also that we have $\nu_{\infty,-1} = -2\rho - \sum_{i=1}^{p} \nu_i$.

Using the homomorphism w_{κ_c}, we obtain $\widehat{\mathfrak{g}}_{\kappa_c}$-module structures on $M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_i(z)}$ and on $M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_\infty(z)}$. As explained in [22, Section 4.2], functoriality of coinvariants implies that there is a natural map from the space of $\mathfrak{g}_{(x_i)}$-coinvariants

$$H\big(M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_1(z)}, \dots, M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_p(z)}, M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_\infty(z)}\big),$$

defined in Section 2.2, to the corresponding space of coinvariants with respect to $M_{\mathfrak{g}} \otimes \pi_0$, which is

$$H_{M_{\mathfrak{g}}}\big(\mathbb{P}^1; (x_i), \infty; (M_{\mathfrak{g}}), M'_{\mathfrak{g}}\big) \otimes H_{\pi_0}\big(\mathbb{P}^1, (x_i), \infty; (\mathbb{C}_{\nu_i(z)}), \mathbb{C}_{\nu_\infty}(z)\big).$$

Now, if $\nu_i(z)$ and $\nu_\infty(z)$ are as above, the latter space is one-dimensional, by Proposition 6.2. Hence we obtain a non-zero linear functional

$$\tau_{(x_i)} : H\big(M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_1(z)}, \dots, M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_p(z)}, M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_\infty}(z)\big) \to \mathbb{C}, \tag{6.3}$$

which we normalize so that its value on $|0\rangle^{\otimes N} \otimes |0\rangle'$ is equal to 1.

6.3. *Construction of the Bethe vectors*

We are now ready to construct eigenvectors of the Gaudin algebra $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$. The idea is to use the space of coinvariants of the tensor product of specially selected Wakimoto modules. We attach them to the points z_i, $i = 1, \dots, \ell$, and ∞, and also to additional points $w_1, \dots, w_m$. The modules attached to the points z_i, $i = 1, \dots, \ell$ will be of the form $M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_i(z)}$, where the connection $\partial_z + \lambda_i(z)$ has regular singularity. For such modules we have a homomorphism $\mathbb{M}^*_{\lambda_i} \to M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_i(z)}$, where λ_i is the most singular coefficient of $\lambda_i(z)$. The module attached to ∞ will be of the form $M'_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_\infty(z)}$, where the connection $\partial_z + \lambda_\infty(z)$ has singularity of order 2. We then have a homomorphism $\mathbb{I}_{1,\chi} \to M'_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_\infty(z)}$, where χ is the most singular coefficient of $\lambda_\infty(z)$. Finally, the module attached to w_j will be of the form $M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)}$, where $\mu_j(z) = -\alpha_{i_j}/z + \cdots$. Considered as a $\widehat{\mathfrak{g}}_{\kappa_c}$-module, this module contains a vector annihilated by $\mathfrak{g}[\![t]\!]$, provided that a certain system of equations, called *Bethe Ansatz equations*, is satisfied. If it is satisfied, then we can use these $\mathfrak{g}[\![t]\!]$-invariant vectors to construct eigenvectors of the Gaudin algebra $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ in $\bigotimes_{i=1}^N M_{\lambda_i}$.

Now we explain this in more detail. Let us look more closely at the Wakimoto modules $M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_i(z)}$ and $M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu_\infty(z)}$. Let M^*_λ be the $\mathfrak{g}$-module contragredient to the Verma module M_λ, and $\mathbb{M}^*_\lambda$ the corresponding induced $\widehat{\mathfrak{g}}_{\kappa_c}$-module. As shown in [22, Section 4.4], we have a non-trivial homomorphism of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules

$$\mathbb{M}^*_\lambda \to M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu(z)}, \quad \text{if } \nu(z) = \frac{\nu}{z} + \sum_{n\geqslant 0} \nu_n z^n.$$

Now suppose that $\nu = -\alpha_i$, $i = 1, \dots, \ell$. Consider the vector

$$e^R_{i,-1}|0\rangle \in M_{\mathfrak{g}} \otimes \mathbb{C}_{\nu(z)}, \qquad \nu(z) = -\frac{\alpha_i}{z} + \sum_{\mathfrak{n}\geqslant 0} \nu_n z^n \tag{6.4}$$

(see [22, formula (4.7)] for the definition of $e^R_{i,-1}$).

According to [17, Lemma 2] (see also [22, Lemma 4.5]), we have

Lemma 6.3. *The vector* (6.4) *is annihilated by* $\mathfrak{g}[[t]]$ *if and only if we have*

$$\langle \check{\alpha}_i, \nu_0 \rangle = 0. \tag{6.5}$$

Finally, consider the $\widehat{\mathfrak{g}}_{\kappa_c}$-module $M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu(z)}$, where

$$\nu(z) = \frac{\chi}{z^2} + \sum_{n \geqslant -1} \nu_n z^n, \qquad \chi \in \mathfrak{h}^*. \tag{6.6}$$

Using the explicit formulas for the homomorphism w_{κ_c} (see [22, Theorem 4.1]), we obtain that the vector $|0\rangle' \in M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu(z)}$ satisfies

$$t^2\mathfrak{g}[[t]] \cdot |0\rangle' = 0, \qquad (A \otimes t) \cdot |0\rangle' = \chi(A)|0\rangle', \quad A \in \mathfrak{g},$$

where we extend $\chi \in \mathfrak{h}^*$ to a linear functional on $\mathfrak{g}^*$ via the projection $\mathfrak{g} \to \mathfrak{h}$ obtained using the Cartan decomposition of $\mathfrak{g}$ (abusing notation, we will denote this extension by the same symbol χ).

This implies the following:

Lemma 6.4. *For any* $\chi \in \mathfrak{h}^*$ *there is a homomorphism of* $\widehat{\mathfrak{g}}_{\kappa_c}$*-modules*

$$\mathbb{I}_{1,\chi} \to M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu(z)},$$

where $\nu(z)$ *is given by formula* (6.6)*, sending the generating vector of* $\mathbb{I}_{1,\chi}$ *to* $|0\rangle' \in M'_{\mathfrak{g}} \otimes \mathbb{C}_{\nu(z)}$.

Now let us fix an N-tuple of distinct points $z_1, \ldots, z_N$ of $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$, an element $\chi \in \mathfrak{h}^*$, a set of weights $\lambda_i \in \mathfrak{h}^*$, $i = 1, \ldots, N$, and a set of simple roots α_{i_j}, $j = 1, \ldots, m$, of $\mathfrak{g}$. Consider a connection $\overline{\nabla}$ on $\Omega^{-\rho}$ on $\mathbb{P}^1$ whose restriction to $\mathbb{P}^1 \setminus \infty$ is equal to $\partial_t + \lambda(t)$, where

$$\lambda(t) = -\chi + \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} - \sum_{j=1}^{m} \frac{\alpha_{i_j}}{t - w_j}, \tag{6.7}$$

where $w_1, \ldots, w_m$ is an m-tuple of distinct points of $\mathbb{P}^1 \setminus \{\infty\}$ such that $w_j \neq z_i$. Denote by $\lambda_i(t - z_i)$ the expansions of $\lambda(t)$ at the points z_i, $i = 1, \ldots, N$, and by $\mu_j(t - w_j)$ the expansions of $\lambda(t)$ at the points w_j, $j = 1, \ldots, m$. We have:

$$\lambda_i(z) = \frac{\lambda_i}{z} + \cdots, \qquad \mu_j(z) = -\frac{\alpha_{i_j}}{z} + \mu_{j,0} + \cdots,$$

where

$$\mu_{j,0} = -\chi + \sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{\alpha_{i_s}}{w_j - w_s}. \tag{6.8}$$

Finally, the expansion of this connection near ∞ reads $\partial_s + \lambda_\infty(s)$, where $s = t^{-1}$. Then $\lambda_\infty(s) = s^{-2}\lambda(s^{-1}) - 2\rho s^{-1}$. Therefore we have

$$\lambda_\infty(z) = \frac{\chi}{z^2} - \frac{\sum_{i=1}^N \lambda_i - \sum_{j=1}^m \alpha_{i_j} + 2\rho}{z} + \cdots. \tag{6.9}$$

In the previous subsection we constructed a non-zero linear functional $\tau_{(z_i),(w_j)}$ on the corresponding space of $\mathfrak{g}_{(z_i),(w_j)}$-coinvariants

$$\tau_{(z_i),(w_j)} : H\big((M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_i(z)}), (M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)}), M'_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_\infty(z)}\big) \to \mathbb{C}. \tag{6.10}$$

(In particular, this implies that this space of coinvariants is itself non-zero.)

Next, we use Lemma 6.3, according to which the vectors $e^R_{i_j,-1}|0\rangle \in M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)}$ are $\mathfrak{g}[\![t]\!]$-invariant if and only if the equations $\langle \check{\alpha}_{i_j}, \mu_{j,0}\rangle = 0$ are satisfied, where $\mu_{j,0}$ is the constant coefficient in the expansion of $\lambda(t)$ at w_j given by formula (6.8). This yields the following system of equations:

$$\sum_{i=1}^N \frac{\langle \check{\alpha}_{i_j}, \lambda_i\rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_j}, \alpha_{i_s}\rangle}{w_j - w_s} = \langle \check{\alpha}_{i_j}, \chi\rangle, \quad j = 1, \ldots, m. \tag{6.11}$$

These are the *Bethe Ansatz equations* of our Gaudin model.

This is a system of equations on the complex numbers w_j, $j = 1, \ldots, m$, to each of which we attach a simple root α_{i_j}. We have an obvious action of a product of symmetric groups permuting the points w_j corresponding to simple roots of the same kind. In what follows, by a *solution* of the Bethe Ansatz equations we will understand a solution defined up to these permutations. We will adjoin to the set of all solutions associated to all possible collections $\{\alpha_{i_j}\}$ of simple roots of $\mathfrak{g}$, the unique "empty" solution, corresponding to the empty set of simple roots (when this system of equations is empty).

Suppose that Eqs. (6.11) are satisfied. Then we obtain a homomorphism of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules

$$\bigotimes_{i=1}^N \mathbb{M}^*_{\lambda_i} \otimes \mathbb{V}_0^{\otimes m} \otimes \mathbb{I}_{1,\chi} \to \bigotimes_{i=1}^N M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_i(z)} \otimes \bigotimes_{j=1}^m M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)} \otimes \big(M'_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_\infty(z)}\big),$$

which sends the vacuum vector v_0 in the jth copy of $\mathbb{V}_0$ to $e^R_{i_j,-1}|0\rangle \in M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)}$. Hence we obtain the corresponding map of the spaces of $\mathfrak{g}_{(z_i),(w_j)}$-coinvariants

$$H\big((\mathbb{M}^*_{\lambda_i}), (\mathbb{V}_0), \mathbb{I}_{1,\chi}\big) \to H\big((M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_i(z)}), (M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)}), M'_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_\infty(z)}\big).$$

But the insertion of $\mathbb{V}_0$ does not change the space of coinvariants, according to Proposition 2.3. Hence the first space of coinvariants is isomorphic to the space of $\mathfrak{g}_{(z_i)}$-coinvariants

$$H\big((\mathbb{M}^*_{\lambda_i}), \mathbb{I}_{1,\chi}\big) \simeq \left(\bigotimes_{i=1}^N M^*_{\lambda_i} \otimes I_\chi\right)\Big/\mathfrak{g} \simeq \bigotimes_{i=1}^N M^*_{\lambda_i}.$$

Composing the corresponding map

$$\bigotimes_{i=1}^{N} M_{\lambda_i}^* \to H\big((M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_i(z)}), (M_{\mathfrak{g}} \otimes \mathbb{C}_{\mu_j(z)}), M'_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda_\infty(z)}\big)$$

with the linear functional $\tau_{(z_i),(w_j)}$ (see (6.10)), we obtain a linear functional

$$\psi\big(w_1^{i_1}, \dots, w_m^{i_m}\big) : \bigotimes_{i=1}^{N} M_{\lambda_i}^* \to \mathbb{C}. \tag{6.12}$$

The functional $\psi(w_1^{i_1}, \dots, w_m^{i_m})$ coincides with the functional denoted in the same way in [22, Section 4.4]. Indeed, it is obtained by computing the same coinvariants as in the setting of [22] and here. What is different here is that the numbers w_j's are solutions of the Bethe Ansatz equations (6.11), whereas in [22] they are the solutions of another set of equations, namely Eqs. (4.18) of [22], which correspond to the special case $\chi = 0$ (note that in this case $\mathbb{I}_{1,\chi} = \mathbb{U}_1$, which is the module that was attached to the point ∞ in [22]).

According to formula (4.22) of [22] (based on the computations performed in [1] and in [17, Lemma 3]), $\psi(w_1^{i_1}, \dots, w_m^{i_m})$ corresponds to the vector

$$\phi\big(w_1^{i_1}, \dots, w_m^{i_m}\big) \in \bigotimes_{k=1}^{N} M_{\lambda_k}$$

given by the formula

$$\phi\big(w_1^{i_1}, \dots, w_m^{i_m}\big) = \sum_{p=(I^1,\dots,I^N)} \bigotimes_{k=1}^{N} \prod_{s \in I^k} \frac{f_{i_s}}{(w_s^{i_s} - w_{s+1}^{i_{s+1}})} \cdot v_{\lambda_k} \tag{6.13}$$

(up to a scalar). Here f_i denotes a generator of the Lie algebra $\mathfrak{n}_- \subset \mathfrak{g}$ corresponding to the ith simple root, and v_{λ_k} is a highest weight vector in M_{λ_k}. The summation is taken over *all ordered partitions* $I^1 \cup I^2 \cup \dots \cup I^N$ of the set $\{1, \dots, m\}$, where $I^k = \{j_1^k, j_2^k, \dots, j_{a_j}^k\}$, and the product is taken from left to right, with the convention that the w with the lower index $j_{a_j}^k + 1$ is z_k. Note that we differentiate between partitions obtained by permuting elements within each subset I^k.

Thus, $\psi(w_1^{i_1}, \dots, w_m^{i_m})$ is the linear functional on $\bigotimes_{i=1}^N M_{\lambda_i}^*$ equal to the pairing with $\phi(w_1^{i_1}, \dots, w_m^{i_m})$.

In particular, we find that $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ has weight

$$\sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{m} \alpha_{i_j}. \tag{6.14}$$

We call $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ the *Bethe vector* corresponding to a solution of the Bethe Ansatz equations (6.11).

Theorem 6.5. *If the Bethe Ansatz equations* (6.11) *are satisfied, then the vector* $\phi(w_1^{i_1},\dots,w_m^{i_m})$ *given by formula* (6.13) *is either equal to zero or is an eigenvector of the Gaudin algebra* $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ *in* $\bigotimes_{i=1}^N M_{\lambda_i}$.

Proof. Proof is identical to the proof of Theorem 4.11 of [22] (which is based on [17, Theorem 3]). Let us look at the linear functional $\psi(w_1^{i_1},\dots,w_m^{i_m})$ as a map of coinvariants

$$\bigotimes_{i=1}^N M^*_{\lambda_i} \simeq H\big((\mathbb{M}^*_{\lambda_i}),\mathbb{I}_{1,\chi}\big) \to H\big((M_{\mathfrak{g}}\otimes\mathbb{C}_{\lambda_i(z)}),(M_{\mathfrak{g}}\otimes\mathbb{C}_{\mu_j(z)}),M'_{\mathfrak{g}}\otimes\mathbb{C}_{\lambda_\infty(z)}\big)\simeq\mathbb{C}.$$

By functoriality of coinvariants, this map intertwines the natural actions of

$$\mathfrak{z}(\widehat{\mathfrak{g}})_u = \operatorname{Fun}\operatorname{Op}_{{}^LG}(D_u)$$

on the left- and right-hand sides. By Theorem 5.7, on the left hand side this action corresponds to the action of the Gaudin algebra $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on $\bigotimes_{i=1}^N M^*_{\lambda_i}$. On the other hand, by Theorem 6.1, the action of $\operatorname{Fun}\operatorname{Op}_{{}^LG}(D_u)$ on the right-hand side factors through the Miura transformation

$$\operatorname{Fun}\operatorname{Op}_{{}^LG}(D_u)\to\operatorname{Fun}\operatorname{Conn}\big(\Omega^{-\rho}\big)_{D_u}.$$

This means that $\psi(w_1^{i_1},\dots,w_m^{i_m})$ (and hence $\phi(w_1^{i_1},\dots,w_m^{i_m})$) is an eigenvector of $\mathfrak{z}(\widehat{\mathfrak{g}})_u = \operatorname{Fun}\operatorname{Op}_{{}^LG}(D_u)$ whose eigenvalue is the ${}^LG$-oper on $\mathbb{P}^1$ whose restriction to D_u is the Miura transformation of the connection $\partial_t+\lambda(t)$ restricted to D_u. □

In particular, $\phi(w_1^{i_1},\dots,w_m^{i_m})$ is an eigenvector of the operators $\Xi_{i,\chi}$, $i=1,\dots,N$, given by formula (2.20) and of the DMT Hamiltonians $T_\gamma(\chi)$ (acting via the diagonal action of $\mathfrak{g}$ on $\bigotimes_{i=1}^N M_{\lambda_i}$). The Bethe Ansatz procedure for these quadratic Hamiltonians was considered, from a different perspective, in [39], for $\mathfrak{g}=\mathfrak{sl}_n$. The results of [39] are in agreement with Theorem 6.5. We also note that in the case when $\mathfrak{g}=\mathfrak{sl}_2$ the diagonalization problem for the operators $\Xi_{i,\chi}$, $i=1,\dots,N$, was studied in [47].

6.4. *Eigenvalues on Bethe vectors*

Next, we compute the joint eigenvalues of the algebra $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on the Bethe vector $\phi(w_1^{i_1},\dots,w_m^{i_m})$.

It follows from Theorem 5.7 that the action of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on the space of coinvariants $H((\mathbb{M}^*_{\lambda_i}),\mathbb{I}_{1,\chi})$ factors through the algebra

$$\operatorname{Fun}\operatorname{Op}^{\mathrm{RS}}_{{}^LG}\big(\mathbb{P}^1\big)^{(\lambda_i)}_{(z_i);\pi(-\chi)},$$

where $\operatorname{Op}^{\mathrm{RS}}_{{}^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\chi)}$ is the space of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularities at the points z_i, $i=1,\dots,N$, with the 1-residues $\varpi(-\lambda_i-\rho)$, and with singularity of order 2 at the point ∞ with the 2-residue $\pi(-\chi)$. Therefore the joint eigenvalues of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on $\phi(w_1^{i_1},\dots,w_m^{i_m})$ are recorded by a point in $\mathrm{Op}^{\mathrm{RS}}_{^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\chi)}$.

To describe this point, we let

$$\mathrm{Conn}\big(\Omega^{-\rho}\big)_{\mathbb{P}^1\setminus\{(z_i),(w_j),\infty\}}$$

denote the space of connections on $\Omega^{-\rho}$ over $\mathbb{P}^1\setminus\{(z_i),(w_j),\infty\}$. Then we have a Miura transformation

$$\mu_{(z_i),(w_j),\infty}:\mathrm{Conn}\big(\Omega^{-\rho}\big)_{\mathbb{P}^1\setminus\{(z_i),(w_j),\infty\}}\to \mathrm{Op}_{^LG}\big(\mathbb{P}^1\backslash\big\{(z_i),(w_j),\infty\big\}\big).$$

Lemma 6.6. *Let $\overline{\nabla}=\partial_t+\lambda(t)$, where $\lambda(t)$ is given by formula* (6.7)*, and the numbers w_j satisfy the Bethe Ansatz equations* (6.11)*. Then*

$$\mu_{(z_i),(w_j),\infty}(\overline{\nabla})\in \mathrm{Op}^{\mathrm{RS}}_{^LG}\big(\mathbb{P}^1\big)^{(\lambda_i)}_{(z_i);\pi(-\chi)}\subset \mathrm{Op}_{^LG}\big(\mathbb{P}^1\backslash\big\{(z_i),(w_j),\infty\big\}\big).$$

Proof. We need to show that the restriction of $\mu_{(z_i),(w_j),\infty}(\overline{\nabla})$ to $D^\times_{w_j}$, $j=1,\dots,m$, belongs to the subspace $\mathrm{Op}_{^LG}(D_{w_j})$ of regular opers, and the restriction of $\mu_{(z_i),(w_j),\infty}(\overline{\nabla})$ to $D^\times_\infty$ belongs to $\mathrm{Op}^{\leqslant 2}(D_\infty)_{\pi(-\chi)}$.

To see the former, we recall that the restriction of the connection $\overline{\nabla}$ to $D^\times_{w_j}$ has the form

$$\partial_t-\frac{\alpha_{i_j}}{t-w_j}+\mu_{j,0}+\cdots,$$

where $\mu_{j,0}$ is given by formula (6.8). The Bethe Ansatz equations (6.11) are equivalent to the equations $\langle\check{\alpha}_{i_j},\mu_{j,0}\rangle=0$. According to [22, Lemma 3.5], this implies that the Miura transformation of this connection is regular at w_j.

To prove the latter, we recall that the restriction of $\overline{\nabla}$ to $D^\times_\infty$ has the form (6.9):

$$\partial_s+\frac{\chi}{s^2}+\cdots,\qquad s=t^{-1}.$$

The Miura transformation of this connection is the ^LG-oper which is the $N((s))$-gauge equivalence class of the operator

$$\nabla=\partial_s+p_{-1}+\frac{\chi}{s^2}+\cdots.$$

Applying the gauge transformation with $\check{\rho}(s)^2$, we obtain the connection

$$\partial_s+\frac{1}{s^2}(p_{-1}+\chi)-\frac{2\check{\rho}}{s}+\cdots.$$

This oper has singularity of order 2, and its 2-residue is equal to $\pi(-p_{-1}-\chi)=\pi(-\chi)$, since $\chi\in\mathfrak{h}^*\subset\mathfrak{g}^*$. $\square$

Now we are ready to describe the joint eigenvalues of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on the Bethe vectors.

Theorem 6.7. *The joint eigenvalues of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on*

$$\phi\bigl(w_1^{i_1},\dots,w_m^{i_m}\bigr)\in\bigotimes_{i=1}^N M_{\lambda_i},$$

where $w_1,\dots,w_m$ satisfy the Bethe Ansatz equations (6.11)*, are given by the ${}^LG$-oper*

$$\mu_{(z_i),(w_j),\infty}(\overline{\nabla})\in \operatorname{Op}^{\mathrm{RS}}_{{}^LG}\bigl(\mathbb{P}^1\bigr)^{(\lambda_i)}_{(z_i);\pi(-\chi)},$$

where $\overline{\nabla}=\partial_t+\lambda(t)$ with $\lambda(t)$ given by formula (6.7).

Proof. According to the proof of Theorem 6.5, $\phi(w_1^{i_1},\dots,w_m^{i_m})$ is an eigenvector of $\mathfrak{z}(\widehat{\mathfrak{g}})_u=\operatorname{Fun}\operatorname{Op}_{{}^LG}(D_u)$ and its eigenvalue is the ${}^LG$-oper on $\mathbb{P}^1$ whose restriction to D_u is the Miura transformation of $\overline{\nabla}|_{D_u}$. But then this oper is nothing but $\mu_{(z_i),(w_j),\infty}(\overline{\nabla})$. $\square$

6.5. Bethe Ansatz for finite-dimensional modules

Now we specialize to the case when all $\mathfrak{g}$-modules M_i are irreducible and finite-dimensional. Thus, $M_i=V_{\lambda_i}$ for some dominant integral highest weight λ_i. In this case the oper $\mu_{(z_i),(w_j),\infty}(\overline{\nabla})$ automatically has no monodromy around the points $z_1,\dots,z_N$. Indeed, its restriction to $D_{z_i}^\times$ is the ${}^LG$-oper which is the gauge equivalence class of

$$\partial_t+p_{-1}-\frac{\lambda_i}{t-z_i}+\mathbf{u}(t),\qquad \mathbf{u}(t)\in\mathfrak{h}^*[\![t-z_i]\!].$$

Applying the gauge transformation with $\lambda(t-z_i)^{-1}$, we obtain the operator

$$\partial_t+\sum_{j=1}^{\ell}t^{\langle\lambda_i,\check{\alpha}_j\rangle}f_j+\mathbf{u}(t),\qquad \mathbf{u}(t)\in\mathfrak{h}^*[\![t-z_i]\!],$$

which is regular at $t=z_i$. Hence this ${}^LG$-oper has trivial monodromy around z_i for each $i=1,\dots,N$.

Now consider the Bethe vector $\phi(w_1^{i_1},\dots,w_m^{i_m})\in\bigotimes_{i=1}^N M_{\lambda_i}$. Let $\overline{\phi}(w_1^{i_1},\dots,w_m^{i_m})$ be its projection onto $\bigotimes_{i=1}^N V_{\lambda_i}$. Then we find from Theorem 6.7 that the eigenvalues of the Gaudin algebra on it are given by the ${}^LG$-oper

$$\mu_{(z_i),(w_j),\infty}(\overline{\nabla})\in\operatorname{Op}_{{}^LG}\bigl(\mathbb{P}^1\bigr)^{(\lambda_i)}_{(z_i);\pi(-\chi)}\subset\operatorname{Op}^{\mathrm{RS}}_{{}^LG}\bigl(\mathbb{P}^1\bigr)^{(\lambda_i)}_{(z_i);\pi(-\chi)},$$

where $\operatorname{Op}_{{}^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\chi)}$ stands for the monodromy-free locus in $\operatorname{Op}^{\mathrm{RS}}_{{}^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\chi)}$.

This is compatible with the statement of Theorem 5.11 which states that the eigenvalue on *any* eigenvector in $\bigotimes_{i=1}^N V_{\lambda_i}$ belongs to this monodromy-free locus.

An interesting problem is the *completeness of the Bethe Ansatz*: is it true that the Bethe eigenvectors gives us a basis of $\bigotimes_{i=1}^N V_{\lambda_i}$ for generic $z_1,\dots,z_N$ and χ? If this is so, this would mean that there is a basis of $\bigotimes_{i=1}^N V_{\lambda_i}$ labeled by solutions of the Bethe Ansatz equations (6.11).

In the case of the ordinary Gaudin model (corresponding to $\chi = 0$) this problem has been investigated in great detail. The completeness of the Bethe Ansatz has been proved in some special cases in [40,48,49]. However, examples constructed in [41] show that Bethe Ansatz may be incomplete for some fixed highest weights λ_i and all possible values of z_i's.

Completeness of the Bethe Ansatz in the ordinary Gaudin model is discussed in detail [22, Section 5]. According to Theorem 2.7(3) [22], which is recalled in Theorem 5.9, the joint eigenvalues of the Gaudin algebra are realized as a subset of the set

$$\mathrm{Op}_{^LG}\big(\mathbb{P}^1\big)_{(z_i),\infty;(\lambda_i),\lambda_\infty} = \mathrm{Op}_{^LG}\big(\mathbb{P}^1\big)^{(\lambda_i)}_{(z_i);\pi(0)}$$

of monodromy-free opers with $\chi = 0$, so that they have regular singularity at ∞. It was conjectured in [22] that this inclusion is actually a bijection. We can try to prove this by associating to an oper in $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(0)}$ an eigenvector of the Gaudin algebra in $\bigotimes_{i=1}^N V_{\lambda_i}$. This can be done by using the *Miura opers*, as explained in [22]. The problem is caused by the *degenerate* opers in $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(0)}$, in the terminology of [21,22].

Each non-degenerate oper gives rise to a Bethe vector of the form $\bar{\phi}(w_1^{i_1}, \dots, w_m^{i_m})$. If all opers in $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(0)}$ are non-degenerate and the corresponding Bethe vectors are non-zero, then the Bethe Ansatz is complete (see [21, Proposition 4.10] and [22, Proposition 5.5]). However, a degenerate oper does not give rise to a Bethe vector. Thus, we may have a bijection between $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(0)}$ and the set of joint eigenvalues of the Gaudin algebra on $\bigotimes_{i=1}^N V_{\lambda_i}$ (as conjectured in [22]), but the Bethe Ansatz may still be incomplete because of the presence of degenerate opers, to which we cannot attach Bethe vectors (we believe that this is the reason behind the counterexample of [41]). However, as explained in [22, Section 5.5], the construction of the Bethe vectors presented above may be generalized so as to enable us to attach eigenvectors (of a slightly different form) even to degenerate opers. This gives us a way to construct an eigenbasis of the Gaudin algebra in $\bigotimes_{i=1}^N V_{\lambda_i}$ even if there are degenerate opers in $\mathrm{Op}_{^LG}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(0)}$.[10]

The approach of [22] may be generalized to the case of the Gaudin model with regular semi-simple χ. However, the notion of Miura oper becomes more subtle here because the oper connection has irregular singularity at ∞. We hope to discuss this question in more detail elsewhere.

6.6. The case of $\mathcal{A}_\chi$

Now we specialize the above results to the case of the Gaudin algebra $\mathcal{A}_\chi = \mathcal{A}^{1,1}_{0,\infty}(\mathfrak{g})_{0,\chi}$, corresponding to $N = 1$ with $z_1 = 0$ and $\lambda_1 = \lambda$. We fix a regular semi-simple $\chi \in \mathfrak{g}^*$.

In this case the Bethe vectors in the Verma module M_λ have the form

$$\phi\big(w_1^{i_1}, \dots, w_m^{i_m}\big) = \sum_{\sigma \in S_m} \frac{f_{i_{\sigma(1)}} f_{i_{\sigma(2)}} \dots f_{i_{\sigma(m)}}}{(w_{\sigma(1)} - w_{\sigma(2)})(w_{\sigma(2)} - w_{\sigma(3)}) \dots (w_{\sigma(m-1)} - w_{\sigma(m)}) w_{\sigma(m)}} v_\lambda, \tag{6.15}$$

[10] We note that for $\mathfrak{g} = \mathfrak{sl}_n$ and generic χ it follows from [45] that the action of $\mathcal{A}^{(1),1}_{(z_i),\infty}(\mathfrak{g})_{(0),\chi}$ on $\bigotimes_{i=1}^N V_{\lambda_i}$ is diagonalizable and has simple spectrum.

where the sum is over all permutations on m letters. This vector has the weight

$$\lambda - \sum_{j=1}^{m} \alpha_{i_j}.$$

The corresponding Bethe Ansatz equations (6.11) have the form

$$\frac{\langle \check{\alpha}_{i_j}, \lambda \rangle}{w_j} - \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_j}, \alpha_{i_s} \rangle}{w_j - w_s} = \langle \check{\alpha}_{i_j}, \chi \rangle, \quad j = 1, \dots, m. \tag{6.16}$$

If the Bethe Ansatz for an irreducible $\mathfrak{g}$-module V_λ is complete for a particular $\chi \in \mathfrak{g}^*$, then the Bethe vectors (6.15) give us a basis of V_λ labeled by the solutions of Eqs. (6.16). We remark that in the case of $\mathfrak{g} = \mathfrak{sl}_n$ it was shown in [45] that $\mathcal{A}_\chi$ is closely related to the Gelfand–Zetlin algebra. This suggests that for a general simple Lie algebra $\mathfrak{g}$ the Gaudin algebra $\mathcal{A}_\chi$ may give us a new powerful tool for analyzing finite-dimensional representations.

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