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Topology and its Applications 80 (1997) 1–6

TOPOLOGY AND ITS APPLICATIONS

Editorial

P.S. Alexandroff and Topology: an introductory note

In an article, published in 1976 in "Uspekhi Matematicheskikh Nauk", on the occasion of P.S. Alexandroff's 80th birthday, I.M. Gelfand and V.A. Ponomarev wrote:

Dedicated to P.S. Alexandroff, who has done so much for development of general ideas in mathematics.

We think, this is a very precise evaluation.

When he was still in gymnasium, the main interest of Pavel Alexandroff was already directed at fundamental questions of mathematics—foundations of geometry and non-Euclidean geometry. At the age of 18 he became a student of N.N. Lusin at Moscow University.

In 1915 Alexandroff obtained his first result in mathematics: he proved a fundamental theorem on the cardinality of Borelian subsets: if uncountable, every such subset has the same cardinality as the set of real numbers. As we now know, this can be considered as almost the best possible ZFC-approximation to a solution of the continuum problem. The key step was to show that every uncountable Borelian set contains a copy of the Cantor set. The mechanism created by P.S. Alexandroff in order to prove the theorem, based on the notion of the A-operation (the name was given by M.Ya. Souslin to honour Alexandroff), has influenced very much the further development of set-theoretic methods in topology. Among those on whom Alexandroff's theorem made a very strong impression was Lebesgue, who also recognized a philosophical value of this result.

Another important early result of Alexandroff is his theorem that every absolute G_δ-set is homeomorphic to a complete metric space.

During Summer 1922, P.S. Alexandroff and P.S. Urysohn conducted one of the first systematic studies of compactness in a general setting; results were reported to the Moscow Mathematical Society. Their work was done independently from Vietoris, who published a paper in the same direction somewhat earlier. With the work of Alexandroff, Urysohn, and Vietoris, the most important part of General Topology—theory of compact spaces—came into existence. The famous memoir of Alexandroff and Urysohn greatly influenced further brilliant work on compactness done by Tychonoff, which included the theorems

PII S0166-8641(96)00180-0

on invariance of compactness under products and on embeddings of Tychonoff spaces into compacta, by M. Stone and E. Čech on Stone–Čech compactifications.

One of the most fruitful periods of Alexandroff's life was his Göttingen period 1923–1935. In this period he made his major discoveries in the theory of compact spaces and in dimension theory at this time. He founded homological dimension theory. He introduced the notion of an essential map onto the n-cube and characterized compacta which were at least n-dimensional as those having an essential map onto the n-cube. Later W. Hurewicz gave a nice reformulation of this result in terms of mappings to spheres. In this period, collaborating with H. Hopf, he proved that cohomological dimension agrees with covering dimension for finite-dimensional spaces. He defined the notion of the Alexandroff k-width $a_k X$ of a compact subset $X \subset \mathbb{R}^n$ and characterized k-dimensional subsets X as having positive $(k-1)$-width, whereas $a_k X = 0$. He proved many other results about compact subsets of Euclidean space. In particular, he invented his obstruction theory and proved that $\dim X \leqslant k$ for $X \subset \mathbb{R}^n$ if and only if X can be removed by an arbitrary small move from any $(n-k-1)$-dimensional polyhedron in $\mathbb{R}^n$. The culmination point of that period was the First International Topology Conference organized by P.S. Alexandroff in Moscow in August 1935. It was really the first topological conference and almost all topologists of that time attended it or presented their papers. We mention that Dirac was among the participants.

Despite the fact that in his Göttingen period Alexandroff spent most of his time out of Russia, he founded a topological seminar in Moscow in 1924. Among his students at that time were L.A. Tumarkin, A.N. Tychonoff, V.V. Nemytskij, N.B. Vedenisov, V.A. Efremovich and later L.S. Pontrjagin, A.G. Kurosh and G.S. Chogoshvili. Several generations of Moscow topologists grew up in that seminar. We cannot mention all the names. Here are only a few of them: M.F. Bockstein, Yu.M. Smirnov, K.A. Sitnikov, O.V. Lokutsievskij, V.G. Boltjanskij, I.A. Vainshtein, E.F. Mischenko, V.I. Ponomarev, R. Frum-Ketkov, I.K. Lifanov, B.A. Pasynkov, V.I. Kuzminov, E.G. Skljarenko, A.S. Mischenko, A.V. Chernavskij, B.T. Levshenko, A.V. Zarelua, I.A. Shvedov, M.A. Shtanko, A.A. Maltsev, M.M. Čoban, V.V. Fedorchuk, V.V. Filipov, E.V. Shchepin, S. Illiadis, A.V. Ivanov, V.P. Zolotarev, P.K. Osmatesku, S.J. Nedev, B.E. Shapirovskij, V.I. Malykhin, L.B. Shapiro, A.Ch. Chigogidze, S.A. Bogatyi, S.A. Antonjan, Yu. Lisitsa, A.P. Kombarov, A.P. Shostak, I. Guran, I.M. Leibo, S. Ageev, P. Semenov, V.V. Uspenskij, M.M. Zarichnyi, M.G. Tkachenko, D.B. Shakhmatov, V.V. Tkachuk, O.G. Okunev, V.G. Pestov. Among "corresponding members" of the seminar—I.I. Parovichenko, M.Ja. Antonovskij, N.V. Velichko, E.G. Pytkeev, S.P. Gul'ko, A.A. Gryzlov, L.G. Zambahidze. The seminar is still active at the Moscow State University and it bears Alexandroff's name.

During his life Alexandroff maintained a great friendship with many outstanding people. His closest friends and collaborators were P.S. Urysohn from 1922 to his accidental death in 1924, H. Hopf from 1926 to World War II, which separated them and prevented the completion of their work on the second volume of their book on topology, and A.N. Kolmogorov with whom he owned a house in Komarovka, a suburb of Moscow, which was a center of mathematical life for many Russian mathematicians.

Alexandroff's interest in fundamental notions and constructions in mathematics had nothing in common with a tendency to narrowness (as could be the case with a researcher of a lesser talent and intelligence). On the contrary, whatever was a direction of his work in topology, it always served him as a basis for better understanding mathematics as a whole, it had unifying features in it. In particular, the notion of compactness is now playing a fundamental role in many basic principles of mathematics, such as the Hahn–Banach theorem, the Krein–Milman theorem, the Gel'fand–Kolmogorov theorem, the Stone–Weierstrass theorem, and others.

One of the rare qualities of Alexandroff as a mathematician and a person was his ability to unify in his research and teaching set-theoretic topology and algebraic topology. In fact, he connected these two areas of topology by his discovery of the notion of the nerve of a covering of a topological space and of the notion of inverse spectrum of polyhedra and of spaces in general. Alexandroff's coauthor and collaborator H. Hopf was very much impressed by this discovery of his. It allows to apply such constructions from combinatorial topology as homology and cohomology to general spaces.

It may be appropriate to mention here that, being a protagonist of general ideas in mathematics, Alexandroff placed a value on a result not according to technical difficulty of its proof, but, first of all, according to its position in a mathematical theory, according to the new harmony and beauty which it brings into the theory. The majority of his results, notions he introduced and methods he developed in the proofs, are *natural* results, notions and methods, despite the fact that they were new! It took an outstanding mathematician to recognize and introduce them; often, what Alexandroff did was later considered as transparent or almost obvious, but that came *after*! To cite a few examples, it is enough to refer to the already mentioned notions of the nerve of a covering, of the Alexandroff–Čech homology, of an essential map, of a locally finite family of sets, of an inverse spectrum and its limit, and also the construction of a one-point compactification of a locally compact Hausdorff space, his description (in 1939) of the Stone–Čech compactification of a Tychonoff space (in terms of regular open filters), the notion of long line (Alexandroff line), the notion of strongly infinite-dimensional compacta. This list could be easily extended.

Alexandroff and Urysohn were also the first to consider the metrization problem. Their general criterion for metrizability of a topological space was published in 1923. Though now this solution of the metrization problem is not recognized as the most elegant, the criterion contained all the germs not only of the modern metrizability criteria, but also of the theory of paracompactness (in particular, the notion of a development and a version of star refinements). The work on the metrization problem continued for many years and made a great impact on the shape of general topology, especially on classification of spaces. Its most important byproduct was the theory of paracompact spaces, where the star result was obtained by A.H. Stone: he proved that every metrizable space is paracompact. This opened the doors for culmination: the general metrizability criterion proved independently by R.H. Bing, J. Nagata and Yu.M. Smirnov. Despite this achievement, the theme of metrizability and paracompactness continues to play an important role in general topology.

Alexandroff also introduced the notion of a dyadic compactum, which plays an important role in the theory of compact topological groups. The theory of dyadic compacta is now an important chapter of general topology, with excellent results, such as the theorem of a student of Alexandroff, A.S. Esenin-Vol'pin on metrizability of every first countable dyadic compactum. A. Gerlits, B.A. Efimov, L.B. Shapiro, E.V. Shchepin, M. Bell, V.I. Kuzminov and L.I. Ivanovskij have contributed greatly to this theory.

The ideas and results in the Alexandroff–Urysohn memoir on compactness gave birth to another important domain of general topology: the theory of cardinal invariants of topological spaces. Alexandroff and Urysohn introduced a powerful ramification method, and proved the first nontrivial result on cardinality of nonmetrizable compacta: the cardinality of every perfectly normal compactum does not exceed 2^{ω}. Cardinal invariants are now indispensable in the study of the fine structure of topological spaces, especially, of compacta. B.A. Efimov, Z. Balogh, B.E. Shapirovskij, M.G. Tkachenko, A. Ostaszevskij, V.V. Fedorchuk, V.I. Malykhin, D.B. Shakhmatov, A. Hajnal, I. Juhasz, A.A. Gryzlov, S. Todorchevich, M.E. Rudin, J. Roitman, R. Pol, E.A. Reznichenko, R.E. Hodel, D.P. Baturov and others have turned this subject into a flourishing branch of topology.

Alexandroff gave birth to and raised cohomological dimension theory. He himself proved many basic results there. Then his students L.S. Pontryagin, M.F. Bockstein, V.G. Boltyanskij, K.A. Sitnikov, E.G. Skljarenko, V.I. Kuzminov, I.A. Shvedov, A.V. Zarelua and other distinguished contributors such as K. Borsuk, Y. Kodama and E. Dyer continued the job. Now it is a great theory with many applications in different areas of mathematics.

Alexandroff is one of the founders of infinite-dimensional dimension theory. He introduced the notion of a weakly (and strongly) infinite-dimensional space. He involved L.A. Tumarkin in dimension theory. The deep results obtained by L.A. Tumarkin, Yu.M. Smirnov, D.W. Henderson, L. Rubin, R. Schori, J. Walsh and R. Pol make this branch of dimension theory very attractive. There are still many difficult open problems. One of them is formulated at the end of this note.

In 1953, P.S. Alexandroff was elected a full member of the Academy of Science of USSR. After this begins a new period of his activity as a Professor of Moscow University. One should mention here that this title was of the greatest value to Alexandroff. More than once, he would confess: "If somebody would awaken me in the midst of the night and ask: 'Who are you?', I would immediately reply: 'I am a Professor of Moscow University'!" During Summer 1954, he chaired a commission supervising entrance examinations to the mechanico–mathematical faculty of MGU. In the 1954/5 academic year, Alexandroff lectured in analytic geometry for undergraduate students of the first year.

In May 1955, he arranged for the first two meetings of a proseminar in topology for undergraduate students of the first two years. He was assisted in this by A.S. Parhomenko, Yu.M. Smirnov, O.V. Lokutsievskij, and V.A. Uspenskij. This proseminar was destined to gradually become a cradle of a whole new generation of topologists in the Soviet Union, specializing in set-theoretic topology, geometric topology, algebraic topology (homological dimension theory, first of all), and topological algebra.

In 1961 Alexandroff brought several of his new graduate students to the First Prague Symposium in General Topology. That was the beginning of the integration of his new school in topology into the world community. P.S. Alexandroff delivered an invited lecture at the symposium with what is now considered a program of mutual classification of spaces and mappings. He put it in the following way:

A. *Which spaces can be represented as images of "nice" (e.g., metric or zero-dimensional, etc.) spaces under "nice" continuous mappings?*

B. *Which spaces can be mapped onto "nice" spaces by "nice" mappings?*

In the years to follow, these two generic questions provided a subject for many deep investigations in Moscow and abroad. One might mention as examples V.I. Ponomarev's characterization of first countable spaces as open continuous images of metrizable spaces, the characterization of perfect preimages of metrizable spaces, a deep study of the class MOBI by Chaber. Deep results to the theory of mappings were contributed by E. Michael, H.H. Wicke and J. Worrell, M.M. Čoban, K. Morita, T. Hoshina, K. Nagami, V.V. Filippov, N.V. Velichko and E.G. Pytkeev.

Alexandroff himself proved a basic result in this direction in an earlier period: he established that every compact metrizable space can be represented as a continuous image of the Cantor set (which is zero-dimensional). He initiated a research on the behaviour of dimension under continuous open mappings with restrictions on preimages of points with the following result: dimension is preserved under countable-to-one continuous open mappings of metrizable compacta. In 1937 A.N. Kolmogorov constructed his example of the integer 2-adic group action on the Menger curve with the orbit space homeomorphic to a Pontryagin surface. So this was the first example of an open mapping between compacta raising dimension. Then L.V. Keldysh constructed an open continuous mapping with zero-dimensional preimages of a one-dimensional continuum onto the square. She (and independently R.D. Anderson) also produced an example of an open continuous mapping with connected preimages of points of a three-dimensional cube onto a cube of higher dimension. Later great contributions to this area were made by A.V. Chernavskij, D. Wilson and J. Walsh. This direction of research in general topology is still rich with difficult and natural questions and in view of recent results by S. Ferry on turning maps into fibrations, it builds one more bridge between general and algebraic topology.

In 1961, already 65, Alexandroff contributed again to metrization theory with the notion of uniform base. A research inspired by this brought forward several important concepts of a similar kind: of a regular base, which clarifies the mechanism of paracompactness of a metric space; of bases of a given rank, which serve well in dimension theory; of a base of countable order; of a Noetherian base, etc., which are important for the deeper classification of spaces. A fundamental result on metrizability of compacta with point countable bases was obtained by A.S. Mischenko.

Alexandroff possessed an important quality of a leader to formulate right problems. He posed several problems which influenced considerably the development of topology. Thus M.F. Bockstein solved Alexandroff's problem (posed in 1935 at the First International Topology Conference) on a countable basis of Abelian groups in cohomological dimension theory. That gave major progress in the theory. The above mentioned example by Kolmogorov was also initiated by Alexandroff's problem. It turned out that this example was related to the Hilbert–Smith conjecture on compact groups acting on manifolds, which is still open. Perhaps the most famous of his problems was the problem on the coincidence of the cohomological dimension and the covering dimension. The work on that problem in the 1930s by Pontrjagin, Hopf and Alexandroff himself influenced the development of big pieces of algebraic topology. Then the problem was abandoned for 40 years until in the 1970s R. Edwards discovered that the Alexandroff problem is equivalent to the cell-like mapping problem from Bing's topology school. The work on this problem of Alexandroff and the solution introduced new features to geometric topology. It is remarkable that a counterexample to the Alexandroff problem appeared recently as a boundary set of a Riemannian manifold which was a counterexample to the integral coarse Novikov conjecture for real operator algebras.

In the conclusion, we recall two old problems of Alexandroff which are still open, i.e., which are yet to bring a flow of new discoveries in topology:

Problem 1. Is the product of two weakly infinite-dimensional compacta weakly infinite-dimensional?

Problem 2. Let $D \subset \mathbb{R}^n$ be a topological n-disk, show that for Alexandroff's width, the equality $a_k(D) = a_k(\partial D)$ holds for $k < n - 1$.

We recall that for a subset $X \subset \mathbb{R}^n$ the Alexandroff width is defined as

$$a_k(X) = \inf\{\|f\| \mid f : X \to L \subset \mathbb{R}^n,\ \dim L = k,\ L \in PL\},$$
$$\|f\| = \max\{\|f(x) - x\| \mid x \in X\}.$$

For Alexandroff, research, teaching, his relationship with colleagues and students, all these were the most important in his personal life. His enthusiastic, deeply emotional reaction to first results of his students often left a trace on them, which would last and inspire them throughout all their life. Alexandroff also taught students to listen to good music, to go to the Moscow Conservatory, to read good literature, to swim, to ski, to make long ("topological") walks in the forests, to enjoy each other's company—in short, to be harmonious people of the Hellenic style.

He was an outstanding mathematician, a great teacher, and a great man.

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Guest Editors