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Matchings from a set below to a set above

Paul Erdős*
Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Réaltanoda u. 13–17, H-1053 Hungary

Jean A. Larson**
Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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Abstract

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One way to represent a matching in a graph of a set A with a set B is with a one-to-one function $m : A \to B$ for which each pair $\{a, m(a)\}$ is an edge of the graph. If the underlying set of vertices of the graph is linearly ordered and every element of A is less than every element of B, then such a matching is a *down-up* matching. In this paper we investigate graphs on well-ordered sets of type α and in many circumstances find either large independent sets of type β or down-up matchings with the initial set of some prescribed size γ. In this case we write $\alpha \to (\beta, \gamma\text{-matching})$.

0. Introduction

In their triple paper on set mappings [8] Erdős, Hajnal and Milner investigated the partition relation

$$\alpha \to (\beta, \text{infinite path})^2.$$

To see if this relation holds for some ordinals α and β, one checks to see if every graph on a set of vertices of type α either has an independent set of type β or has an infinite path. It is easy to see that a successor ordinal $\alpha = \beta + 1$ does not satisfy the relation $\alpha \to (\alpha, \text{infinite path})$, since the graph which joins the first β points to

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the last point and has no other edges is a graph which has no independent set of type α and no path of length longer than two edges (paths are not allowed to use the same vertex more than once). Limit ordinals offer more challenge. Erdős, Hajnal and Milner [8] proved that limit ordinals α less than $\omega_1^{\omega+2}$ satisfy the partition relation $\alpha \rightarrow (\alpha$, infinite path). Unfortunately this pleasing situation does not continue to hold for larger ordinals. Under the assumption of the Diamond Principle, Baumgartner and Larson [5] have shown that ordinals α at least as large as $\omega_1^{\omega+2}$ and of cardinality ω_1 satisfy the negative partition relation $\alpha \nrightarrow (\omega_1^{\omega+2}$, infinite path). By way of contrast, under the assumption of Martin's Axiom for ω_1 and the assumption that the continuum hypothesis fails, Larson [10] has shown that cofinally many ordinals α less than ω_2 satisfy the positive partition relation $\alpha \rightarrow (\alpha$, infinite path).

One of our goals in this paper is to find a partition relation which is true 'most' of the time that it makes sense. In the current paper we continue to investigate graphs whose underlying vertex set is linearly ordered, but we propose as an alternative configuration to the infinite path the *down-up·matching*, which is a matching of a set A with a set B where $A < B$, that is, where every element $a \in A$ is less than every element $b \in B$. If every graph on a set of vertices of type α either has an independent set of type β or a down-up matching with initial set A of cardinality κ, then we write

$$\alpha \rightarrow (\beta, \kappa\text{-matching}).$$

We were inspired to look at matchings because of long term interest in matchings in graphs which dates back in particular to König's Duality Theorem on matchings and covers. We have also long been interested in its generalization, Menger's Theorem. Many people worked on criteria for matchability and the related question of the existence of transversals. For countable graphs see for example [7–13]. Aharoni, Nash-Williams and Shelah [4] developed general criteria for the existence of matchings in infinite graphs, not just countable ones. For more on these matters see the Springer volume by Holz, Podewski and Steffens [9]. The criterion of Podewski and Steffens [12] from 1976 combines with work of Brualdi [6] to give a proof of König's Duality Theorem in the countable case. In 1983–84 Aharoni generalized König's Duality Theorem to all infinite graphs (see [1–2]). The proof even in countable graphs has been shown to be necessarily quite difficult (see Aharoni, Magidor and Shore [3]).

We have looked at two different kinds of problems. The first problem seeks to classify the set of ordinals α of cardinality κ which satisfy $\alpha \rightarrow (\alpha, \kappa$-matching). In solution to this problem we have proved the following theorem whose proof is in Section 1.

Theorem 1.4. *For every cardinal κ and every ordinal $\theta < \kappa^+$, $\kappa \cdot \theta \rightarrow (\theta, \kappa$-matching).*

The second problem we consider is a variation on the first where we ask for more in the way of the down-up matching between a set A and a set B with $A < B$, namely that A have a prescribed order type γ.

Thus by $\alpha \rightarrow (\beta, \gamma\text{-matching})$ we mean that every graph with vertex set α either has an independent set of type β or a down-up matching $m : A \rightarrow B$ with A of type γ. We have only started an exploration of this partition relation, by looking at the case for countably infinite indecomposable ordinals. We develop positive results in Section 2; we construct counter-examples in Section 3 and draw conclusions in Section 4. We have one general result which is uniformly true.

Theorem 4.1. *If ω^b and $\gamma \geq \omega^2$ are countably infinite indecomposable ordinals, then*

$$\gamma^b \cdot \omega \rightarrow (\omega^b, \gamma\text{-matching}) \quad \text{and} \quad \gamma^b \not\rightarrow (\omega^b + 1, \gamma\text{-matching}).$$

The next theorem lists the cases in which our results are optimal.

Theorem 4.2. *Suppose ω^b and $\gamma \geq \omega^2$ are countably infinite indecomposable ordinals.*

(1) *If b is a limit ordinal, then $\gamma^b \rightarrow (\omega^b, \gamma\text{-matching})$, and for all $\alpha < \gamma^b$, $\alpha \not\rightarrow (\omega^b, \gamma\text{-matching})$.*

(2) *If b is finite or if b is a successor ordinal and $\gamma \geq \omega^\omega$, then $\gamma^b \cdot \omega \rightarrow (\omega^b, \gamma\text{-matching})$ and $\gamma^b \not\rightarrow (\omega^b, \gamma\text{-matching})$.*

We conclude the paper in Section 4 with directions for further study.

1. Matchings with regular cardinals

In this section we prove that for all cardinals κ and all sufficiently large ordinals α of power κ, every graph on α either has a matching of a set A of size κ with a set B for which $A < B$ or has a large independent set. We start with three lemmas on bipartite graphs. To simplify the statements, let us call a set $C \subseteq A$ *small* if it has cardinality less than κ where κ is the size of A, and let us say C is *almost all* of A if $A - C$ is small.

Bipartite Lemma 1.1. *Suppose $G \subseteq A \times B$ is any bipartite graph and A has cardinality κ. Then either G has a κ-matching or there is a small subset set $A' \subseteq A$ and a subset $B' \subseteq B$ which is almost all of B so that every edge from A into B' arises from A' and every point of A' is joined to κ many points of B'.*

Proof. Define by recursion as long as possible sequence $\langle q_\alpha \rangle$ of elements of B and a partial matching r as follows: at stage α, let $q_\alpha \in B$ be an element b which is different from q_β for $\beta < \alpha$ and which is joined to some element $a = r(q_\alpha)$ of

$A - \{r(q_\beta) \mid \beta < \alpha\}$. If the recursive definition continues for all κ steps, then r is the desired matching, or r^{-1} if one prefers a matching from a subset of A into B. So suppose that the recursion stops at δ. Let A' be the range of the matching defined up to δ, $A' = \{r(q_\beta) \mid \beta < \delta\}$ and let B' be the difference of B and the domain of the matching, $B' = B - \{q_\beta \mid \beta < \delta\}$. Clearly A' is small since it has the same cardinality as δ. Similarly B' is large, since it omits a set of the same cardinality as δ. Since the recursion does not continue, all edges from points of B' go to points of A', and the lemma follows. □

High Valence Points Lemma 1.2. *Suppose $G \subseteq A \times B$ is any bipartite graph and A has cardinality κ. If for all regular $\lambda \leqslant \kappa$, the set A_λ has cardinality at least λ, where A_λ is the set of all a in A with degree at least λ, then G has a κ-matching.*

Proof. Define the matching m by recursion on $\alpha < \kappa$. At stage α, assume $\langle p_\beta \mid \beta < \alpha\rangle$ and $\langle m(p_\beta) \mid \beta < \alpha\rangle$ have been defined with each p_β in A and each $m(p_\beta)$ in B. Let $\lambda(\alpha)$ be $|\alpha|^+$. Since $\alpha < \kappa$, it follows that $\lambda(\alpha)$ is a regular cardinal with $\lambda(\alpha) \leqslant \kappa$. By hypothesis, $A_{\lambda(\alpha)}$ has cardinality at least $\lambda(\alpha)$, so there is an element p_α in $A_{\lambda(\alpha)}$ which is different from p_β for $\beta < \alpha$. Since the degree of p_α is at least $\lambda(\alpha)$, there is some point $m(p_\alpha)$ in B different from $m(p_\beta)$ for $\beta < \alpha$ to which p_α is joined. After κ steps this recursion defines the desired matching m. □

The above lemmas are used to prove the next lemma which is a variant of the Bipartite Lemma.

Matchless Lemma 1.3. *Suppose $G \subseteq A \times B$ is any bipartite graph and A has cardinality κ. If G has no κ-matching then for all regular cardinals $\lambda \leqslant \kappa$, there are sets $C(\lambda) \subseteq A$ of power less than κ and $D(\lambda) \subseteq B$ which is almost all of B so that all edges from A into $D(\lambda)$ actually come from $C(\lambda)$ and every element of $C(\lambda)$ is joined to at least λ points of $D(\lambda)$.*

Proof. Start by using the Bipartite Lemma 1.1 to get sets $A' \subseteq A$ of power less than κ and $B' \subseteq B$ which is almost all of B so that all edges in G which end in B' arise in A'. For notational convenience let G' be the restriction of G to $A' \times B'$.

Since G' has no κ-matching, by the High Valence Points Lemma 1.2, there is some regular cardinal $\mu \leqslant \kappa$ so that A_μ has power less than μ where, for any regular cardinal ν, A_ν is the set of elements in A' having degree at least ν in G'.

Let ν be the maximum of μ, λ and $|A'|^+$. Since all these cardinals are regular and less than or equal to κ, it follows for ν also. Let $C(\lambda)$ be A_ν. Since $\mu \leqslant \nu$, the set A_ν is a subset of A_μ. Let $E(\lambda)$ be the set of all points in B' joined to some element of $A' - C(\lambda)$. Since each point of $A' - C(\lambda)$ is joined to less than ν points of B' and there are less than ν points in $A' - C(\lambda)$, the set $E(\lambda)$ has less than ν elements. Thus $D(\lambda) = B' - E(\lambda)$ is almost all of B' and hence almost all of B. Moreover, every element in $C(\lambda)$ is joined to at least ν elements of $D(\lambda)$ as required. $\square$

With these lemmas in hand, we are ready to prove the main theorem on cardinal matchings.

Theorem 1.4 (cardinal matching). *For all infinite cardinals κ and all ordinals $\theta < \kappa^+$, $\kappa \cdot \theta \rightarrow (\theta, \kappa\text{-matching})$.*

Proof. We prove the theorem for a fixed κ by induction on θ. For $\theta = 1$, the theorem is trivially true, since any point is an independent set.

Suppose θ is greater than 1 and the theorem is true up to θ. If θ is decomposable, set $\nu = 2$ and decompose θ into the sum of smaller ordinals as $\theta = \theta(0) + \theta(1)$. If θ is indecomposable, let ν be the cofinality of θ and express θ as the sum over $\alpha < \nu$ of a nondecreasing sequence $\theta(\alpha)$ of indecomposable ordinals. Consider a graph G on $\kappa \cdot \theta$. If it has a down-up κ-matching, we are done so we may assume that G has no such κ-matching. Express the set $\kappa \cdot \theta$ as the sum over $\alpha < \nu$ of sets $X(\alpha)$ of type $\kappa \cdot \theta(\alpha)$. For α with $\alpha + 1 < \nu$, let $\text{Tail}(\alpha)$ be the union of $X(\beta)$ for β with $\alpha < \beta < \nu$. Use the High Valence Points Lemma 1.2 on each G restricted to $X(\alpha) \times \text{Tail}(\alpha)$ to get a regular cardinal $\lambda(\alpha) \leq \kappa$ so that the set $A(\alpha)$ of points of $X(\alpha)$ of degree at least $\lambda(\alpha)$ in the subgraph has cardinality less than $\lambda(\alpha)$. If ν is a regular cardinal, let I be a subset of ν of power ν on which the function λ which assigns to each α the value $\lambda(\alpha)$ is nondecreasing. Since the union over α in I of $X(\alpha)$ is order-isomorphic to $\kappa \cdot \theta$, without loss of generality, we may assume that I is all of ν, and thereby simplify our notation. Otherwise $\nu = 2$ and only $\lambda(0)$ has been defined. For notational uniformity define $\lambda(1) = \lambda(0)$, so in both cases λ is nondecreasing and defined on all of ν.

For each α satisfying $0 < \alpha < \nu$, let $\text{Init}(\alpha)$ be the union over $\beta < \alpha$ of $X(\beta)$. For each such α, express $X(\alpha) - A(\alpha)$ as the sum of $Y(\alpha, \zeta)$ for $\zeta < \theta(\alpha)$, where each $Y(\alpha, \zeta)$ has type κ. Apply the Matchless Lemma 1.3 to each $Y(\alpha, \zeta)$ for the graph G restricted to $\text{Init}(\alpha) \times Y(\alpha, \zeta)$ for the cardinal $\lambda(\alpha)$ to get a small set $C(\alpha, \zeta) \subseteq \text{Init}(\alpha)$ and a subset $D(\alpha, \zeta) \subseteq Y(\alpha, \zeta)$ which is almost all of it, so that all edges from $\text{Init}(\alpha)$ into $D(\alpha, \zeta)$ arise from $C(\alpha, \zeta)$ and every element of $C(\alpha, \zeta)$ is joined to at least $\lambda(\alpha)$ elements of $D(\alpha, \zeta)$. It follows that $C(\alpha, \zeta)$ is a subset of the union of the sets $A(\gamma)$ for $\gamma < \alpha$. Let $U(0)$ be the difference $X(0) - A(0)$ and for α with $0 < \alpha < \nu$, let $U(\alpha)$ be the union over $\zeta < \theta(\alpha)$ of $D(\alpha, \zeta)$. Since $A(0)$ is small and each $D(\alpha, \zeta)$ is almost all of $Y(\alpha, \zeta)$, for all $\alpha < \nu$, the set $U(\alpha)$ has type $\kappa \cdot \theta(\alpha)$. Furthermore, if $\alpha < \beta$, then there are no edges from $U(\alpha)$ to $U(\beta)$ since any such edge must arise from points of $A(\alpha)$ and all of those have been omitted. Apply the induction hypothesis to each $U(\alpha)$ to get an independent set $W(\alpha)$ of type $\theta(\alpha)$. Finally W, the union of these sets, is the desired independent set of type θ.

2. Positive results for countable ordinals

In this section we prove that if b and γ are countable ordinals and γ is infinite and indecomposable, then $\gamma^b \cdot \omega \to (\omega^b, \gamma\text{-matching})$. We can sharpen the result if b is a limit or if b is infinite and $\gamma = \omega$ to $\gamma^b \to (\omega^b, \gamma\text{-matching})$. The first step is a lemma on bipartite graphs reminiscent of the lemmas in the previous section.

Countable Bipartite Lemma 2.1. *Let α, β be countable indecomposable ordinals with α infinite. If A has type $\alpha \cdot \beta$, B is infinite and $G \subseteq A \times B$ is any bipartite graph, then either there is a matching from a subset of A of type $\omega \cdot \beta$ with an infinite subset of B, or there is a set $C \subseteq A$ of type α and a finite subset $D \subseteq B$ so that $G \restriction C \times B = C \times D$.*

Proof. Since A has type $\alpha \cdot \beta$, it is the sum over $\delta < \beta$ of sets $A(\delta)$ of type α. Let $\langle \delta_n \mid n < \omega \rangle$ be a sequence of elements less than β chosen so that every $\delta < \beta$ is repeated infinitely often. Define by recursion as long as possible a matching m on a sequence of elements $p_n \in A(\delta_n)$ with $m(p_n) \in B \backslash \{m(p_k) \mid k < n\}$. If the recursive definition continues for ω steps, then the matching m is defined on $P = \{p_n \mid n < \omega\}$ and has range $Q = \{m(p_n) \mid n < \omega\}$. Since P has infinitely many elements from each $A(\delta)$, it has order type at least $\omega \cdot \beta$ as required. Otherwise, for some $n = n_0$, there is no choice of p_n from $A(\delta_n)$ that can be paired with some $m(p_n)$ in $Q_n := B \backslash \{m(p_k) \mid k < \omega\}$. That is, all edges from points of $A(\delta)$ go to points of Q_n. Since the type of $A(\delta)$ is indecomposable and Q_n is finite, there are a subset $C \subseteq A(\delta)$ of type α and a finite set $D \subseteq Q_n$ so that every point of C is joined to every point of D, and furthermore, no point of C is joined to any point of $B \backslash D$. The sets C and D are the sets required for the second alternative of the lemma. □

The second lemma is a recursion lemma that allows us to paste together various matchings.

Recursion Lemma 2.2. *Suppose that γ is an infinite indecomposable countable ordinal, and $\langle \alpha_n \mid n < \omega \rangle$ and $\langle \beta_n \mid n < \omega \rangle$ are nondecreasing sequences of countable indecomposable ordinals. Let*

$$\alpha := \sum_{n<\omega} \alpha_n \cdot \gamma \quad \text{and} \quad \beta := \sum_{n<\omega} \beta_n.$$

If $\alpha_n \to (\beta_n, \gamma\text{-matching})$ for all $n < \omega$, then $\alpha \to (\beta, \gamma\text{-matching})$.

Proof. First write α as the sum of sets $A(n)$ of type $\alpha_n \cdot \gamma$ for $n < \omega$ and suppose that G is a graph on α. Let $B(n) := \bigcup \{A(k) \mid n < k < \omega\}$ and let $G(n) := G \cap A(n) \times B(n)$. Apply the Countable Bipartite Lemma to each of the graphs $G(n)$. If some application yields a matching of a set of type γ with a countable set completely above it, then we are done. Otherwise, for each $n < \omega$, we get a set $C(n) \subseteq A(n)$ of type α_n and a finite set $D(n) \subseteq B(n)$ so that all edges from $C(n)$ to $B(n)$ actually go to $D(n)$. By the hypothesis on each α_n, either there is a matching of the desired kind, or there is a set $E(n) \subseteq C(n)$ of type β_n which is independent for the graph G. By recursion define an infinite subset $N \subseteq \omega$ so that if $n < m$ are both in N, then $E(m)$ has no point of $D(n)$. Let E be the union over $n \in N$ of the sets $E(n)$. Clearly E is independent for G. Moreover, since $\langle \beta_n \mid n < \omega \rangle$ is nondecreasing, the order type of $E := \sum_{n \in N} \beta_n$ is β as desired. □

Basis Lemma 2.3. *For all infinite countable indecomposable ordinals γ, $\gamma \cdot \omega \to (\omega, \gamma\text{-matching})$.*

Proof. In a trivial way $1 \to (1, \gamma\text{-matching})$. Thus with $\alpha_n = \beta_n = 1$, the Basis Lemma follows from the Recursion Lemma. □

Lemma 2.4 (ω^k-matching). *Let α and $\beta = \omega^b$ be countable indecomposable ordinals with $b = 0$ or b a limit ordinal, and let $k \geqslant 2$ be a positive integer. If $\alpha \to (\beta, \omega^k\text{-matching})$, then for any positive integer j,*

$$\alpha \cdot \omega^{jk+1} \to (\beta \cdot \omega^j, \omega^k\text{-matching}).$$

Thus in particular,

$$\omega^{jk+1} \to (\omega^j, \omega^k\text{-matching}).$$

Proof. The second partition relation follows from the implication by choosing $\alpha = \beta = 1$, since in a trivial way $1 \to (1, \omega^k\text{-matching})$. The proof of the implication is by induction on j. The basis step is $j = 1$. Let $\varepsilon_n := \alpha$ for all $n < \omega$. Then $\varepsilon := \sum_{n<\omega} \varepsilon_n \cdot \omega^k = \alpha \cdot \omega^{k+1}$. Let $\theta_n := \beta$ for all $n < \omega$. By hypothesis, $\varepsilon_n \to (\theta_n, \omega^k\text{-matching})$ for all $n < \omega$. Hence by the Recursion Lemma, $\varepsilon = \alpha \cdot \omega^{k+1} \to (\beta \cdot \omega, \omega^k\text{-matching})$.

For the induction step, suppose that $j > 1$ and that the lemma is true for $j - 1$. Further suppose that G is a graph on $\alpha \cdot \omega^{jk+1}$. Write $\alpha \cdot \omega^{jk+1}$ as the sum over $n < \omega$ of sets $A(n)$ of type $\alpha \cdot \omega^{jk}$, let $B(n) := \bigcup \{A(i) \mid n < i < \omega\}$ and let $G(n) := G \cap A(n) \times B(n)$. Notice that ω^{jk} can be expressed as $\omega^{(j-1)\cdot k+1} \cdot \omega^{k-1}$. For each $n < \omega$, write $A(n)$ as the sum over $\xi < \omega^{k-1}$ of $A(n, \xi)$ where $A(n, \xi)$ has type $\alpha \cdot \omega^{(j-1)\cdot k+1}$. For each $n < \omega$ and $\xi < \omega^{k-1}$, let $G(n, \xi) := G \cap A(n, \xi) \times B(n)$. Apply the Countable Bipartite Lemma to each of the graphs $G(n, \xi)$ and ask either for a matching of a set $P(n, \xi) \subseteq A(n, \xi)$ of type $\omega \cdot 1$ with some subset of $B(n)$ via a mapping $m_{n,\xi}$, or for a subset $U(n, \xi) \subseteq A(n, \xi)$ of type $\alpha \cdot \omega^{(j-1)\cdot k+1}$ and a finite set $D(n, \xi) \subseteq B(n)$ so that all edges of $G(n, \xi)$ from $U(n, \xi)$ to $B(n)$ go to $D(n, \xi)$.

Suppose for some $n < \omega$ and every $\xi < \omega^{k-1}$, application yields a matching. By recursion, we can thin each of the sets $P(n, \xi)$ out to an infinite subset $R(n, \xi)$ so that the various matchings have disjoint ranges on the thinned sets. The union $R = \bigcup\{R(n, \xi) \mid \xi < \omega^{k-1}\}$ is a subset of $A(n)$ of type ω^k matched via $m = \bigcup\{m_{n,\xi} \mid \xi < \omega^{k-1}\}$ with an infinite subset of $B(n)$, hence we are done.

Thus we may suppose that for every $n < \omega$, there is some $\xi(n)$ so that the Bipartite Lemma yields sets $U(n) = U(n, \xi(n))$ and $D(n) = D(n, \xi(n))$. For each $n < \omega$, by the induction hypothesis applied to $j-1$, either there is a matching of the desired kind inside $U(n)$ and we are done, or there is a set $V(n) \subseteq U(n)$ of type $\beta \cdot \omega^{j-1}$ which is independent for the graph G. By recursion on $n < \omega$, define an infinite subset $N \subseteq \omega$ so that if $n < p$ are both in N, then $V(p)$ has no point of $D(n)$. Let V be the union over $n \in N$ of the sets $V(n)$. Clearly V is independent for G. Moreover, V has type $\beta \cdot \omega^{j-1} \cdot \omega = \beta \cdot \omega^j$ as desired.

Therefore by induction on j, the lemma follows. □

Theorem 2.5. *Suppose that $\omega^b > \omega$ and γ are countably infinite indecomposable ordinals. Then $\gamma^b \cdot \omega \to (\omega^b, \gamma\text{-matching})$ and if b is a limit or b is infinite and $\gamma = \omega$, then $\gamma^b \to (\omega^b, \gamma\text{-matching})$.*

Proof. The case for $\gamma = \omega$ is special so we treat it first. If b is finite, then the theorem is true by the Cardinal Matching Theorem with $\kappa = \omega$, since $\gamma^b \cdot \omega = \omega^b \cdot \omega = \omega^{b+1}$. If b is infinite, then we again use the Cardinal Matching Theorem, but now use the computation that $\omega \cdot \omega^b = \omega^b$.

The remainder of the proof is by induction on b. The Basis Lemma says that the theorem is true for $b = 1$ and any suitable choice for γ.

The induction step when b is a successor ordinal is divided into cases according to the choice of γ. We have treated the case $\gamma = \omega$. Next suppose $\gamma = \omega^k$ where $k > 1$ is a positive integer and write $b = d + j$ where d is a limit ordinal and j is a positive integer. Note that $\omega^d \to (\omega^d, \omega^k\text{-matching})$ either trivially when $\omega^d = 1$ or by the induction hypothesis. Thus by Lemma 2.4, $\gamma^d = (\omega^k)^d \to (\omega^d, \omega^k\text{-matching})$ and the induction follows in this case. Finally suppose that $\gamma = \omega^e$ for some infinite ordinal e and write $b = c + 1$. From the induction hypothesis and the fact that $\gamma^{c+1} \geq \gamma^c \cdot \omega$, we conclude that $\gamma^{c+1} \to (\omega^c, \gamma\text{-matching})$. For all $n < \omega$, let $\alpha_n = \gamma^c$ and let $\delta_n = \omega^c$. Then $\gamma^{c+1} \cdot \omega$ is the sum over $n < \omega$ of $\alpha_n \cdot \gamma$ and ω^{c+1} is the sum over $n < \omega$ of δ_n. Hence by the Recursion Lemma, $\gamma^{c+1} \cdot \omega \to (\omega^{c+1}, \gamma\text{-matching})$. This last case completes the induction step for successors.

Finally consider the induction step when b is a limit ordinal. Let $\langle b(n) \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals whose limit is b. Then $\alpha = \gamma^b$ is the sum over $n < \omega$ of $\gamma^{b(n)}$. Since the sequence of $b(n)$'s is strictly increasing and $\gamma \geq \omega$, γ^b is also the sum over $n < \omega$ of $\alpha_n := \gamma^{b(n)} \cdot \omega$ and $\alpha_n \cdot \gamma$. Let $\beta_n := \omega^{b(n)}$. Then β is the sum over $n < \omega$ of β_n. By the induction hypothesis and the fact that α_n is greater than or equal to $\gamma^{b(n)}$, we conclude that $\alpha_n \to (\beta_n, \gamma\text{-matching})$. Hence by the Recursion Lemma, $\gamma \to (\omega^b, \gamma\text{-matching})$, which completes this induction step.

Therefore by induction on b, the theorem is true for all b and γ. □

3. Counterexamples for countable ordinals

In this section we present some arguments which give limitations on when the partition relation $\alpha \to (\beta, \gamma\text{-matching})$ can hold. Before we go into the main constructions, we present some simple results which indicate why we have focused on α indecomposable. The first is a simple monotonicity result.

Monotonicity Lemma 3.1. *Suppose α, β, γ, α'; β' and γ' are all ordinals.*

(1) *If $\alpha \to (\beta, \gamma\text{-matching})$ and $\alpha' > \alpha$, then $\alpha' \to (\beta, \gamma\text{-matching})$.*

(2) *If $\alpha \to (\beta, \gamma\text{-matching})$ and $\beta' < \beta$, then $\alpha \to (\beta'\ \gamma\text{-matching})$.*

(3) *If $\alpha \to (\beta, \gamma\text{-matching})$ and $\gamma' < \gamma$, then $\alpha \to (\beta, \gamma'\text{-matching})$.*

Lemma 3.2 (on decomposable ordinals). *Suppose $\alpha = \alpha_0 + \alpha_1$ is decomposable with $\alpha_1 \leqslant \alpha_0 < \alpha$. If β and γ are both indecomposable, then $\alpha \to (\beta, \gamma\text{-matching})$ if and only if either $\alpha_0 \to (\beta, \gamma\text{-matching})$ or $\alpha_1 \to (\beta, \gamma\text{-matching})$.*

Proof. One direction follows from the Monotonicity Lemma. To prove the contrapositive of the other direction assume one has witnesses to G_0 and G_1 to the negations of the two partition relations for α_0 and α_1 respectively. Then since α is the sum, one can define G on α as the required counterexample essentially as the union of G_0 and G_1. $\square$

For the purpose of smoothing inductive arguments, we extend the definition of the partition relation to include a case in which the desired independent set is bounded in our original set. To that end, write $\alpha \not\to (\text{bounded } \beta, \gamma\text{-matching})$ to mean that there is no bounded independent set of type β and no γ-matching. Clearly the above monotonicity results extend to this partition relation.

Lemma 3.3 (negative lifting). *Suppose that α, β and γ are all countably infinite indecomposable ordinals and that $\gamma \geqslant \omega^2$.*

(1) (Small) *If $\alpha \not\to (\beta, \gamma\text{-matching})$ then $\alpha \cdot \omega \not\to (\text{bounded } \beta, \gamma\text{-matching})$.*

(2) (Medium) *Let δ be an indecomposable ordinal with $\gamma = \omega \cdot \delta$. If $\alpha \not\to (\text{bounded } \beta, \gamma\text{-matching})$, then $\alpha \cdot \delta \not\to (\beta + 1, \gamma\text{-matching})$.*

Proof. To start the proof, let $G \subseteq [\alpha]^2_<$ be a graph which has no γ-matching and for the first item no independent set of type β and for the second item no bounded independent set of type β. Let δ be as given in the second item with $\gamma = \omega \cdot \delta$ and let it be ω for the first item. The set $\delta \times \alpha$ ordered lexicographically has type $\alpha \cdot \delta$ Let $\text{INDEX}: \delta \times \alpha \to \omega$ be a one-to-one onto mapping, and let $\text{BLOCK}: \delta \times \alpha \to \omega$ be a mapping with the properties that for each $c < \delta$, the function $\text{BLOCK}(c, \cdot)$ is monotonic and increases to infinity and if $c \neq c'$, then the ranges of $\text{BLOCK}(c, \cdot)$ and $\text{BLOCK}(c' \cdot)$ are disjoint. Note that INDEX simply gives the index associated with each element relative to some enumeration of the set $\delta \times \alpha$. The function BLOCK is more complicated. It reflects the structure of $\alpha \cdot \delta$ as a sum over $c < \delta$ of sets of type α and of each these sets of type α as a sum of ω many pieces. These small pieces are what we think of as 'blocks' of the partition.

Let $F \subseteq [\delta \times \alpha]^2$ be the graph which has an edge between $\langle c, a\rangle$ and $\langle c', a'\rangle$ if and only if either $c = c'$ and a and a' are joined in G or $c < c'$ and $\mathrm{BLOCK}(c, a) > \mathrm{INDEX}(c', a')$. Call an edge *red* if it satisfies the first clause of this definition and *blue* if it satisfies the second clause.

To prove the lemma, it is enough to show that F has no γ-matching and no bounded independent set of type β for the first item, and no independent set of type $\beta + 1$ for the second item.

First consider the matchings. Assume by way of contradiction that a set A of type γ is matched with a set B via a mapping m and every element of A is below every element of B. Since any joined pair is joined either by a red edge or by a blue edge and since γ is indecomposable, we may assume without loss of generality that all edges of the matching have the same color. Let $\langle w, z\rangle$ be the least element of B.

If $\langle c, a\rangle$ is joined to $m(\langle c, a\rangle) = \langle c', a'\rangle$ by a red edge, then $c = c' \leqslant w \leqslant c'$. Thus, if all pairs from A are matched by red edges, then all pairs begin with the same element w. Since F restricted to the set of sequences that begin with w is isomorphic to G, and G has no γ-matching, it follows that all pairs from A are matched by blue edges.

If for some b the set A intersects the 'block' mapped to b in an infinite set $V(b)$, then we have the contradiction that an infinite set $V(b)$ is matched into a finite set, namely the set of all elements $\langle c', a'\rangle$ with $\mathrm{INDEX}(c', a') < b$. Thus for each b, A intersects the 'block' mapped to b in a finite set. Consequently, A intersects each $\{c\} \times \alpha$ in a set of type at most ω. Since every element of A is less than $\langle w, z\rangle$, the least element of B, it follows that the set A is a subset of $(w + 1) \times \alpha$. Therefore A has type at most $\omega \cdot (w + 1) < \omega \cdot \delta$ which contradicts our assumption that A has type γ. This contradiction shows that F has no γ-matching.

Next we consider independent sets. Assume by way of contradiction that X is an independent set of type β which is bounded by $\langle t, \tau\rangle$ for the first item and remains independent with the addition of $\langle t, \tau\rangle$ for the second item.

For the first item, t is finite. Since β is indecomposable and X is a subset of the union over $c \leqslant t$ of $\{c\} \times \alpha$, for some $s \leqslant t$, X intersects $\{s\} \times \alpha$ in a set of type β. However, on $\{s\} \times \alpha$, the graph F is isomorphic to the graph G, and G has no independent sets of type β. This contradiction completes the proof of the first item of the lemma.

Finally consider the second item. As in the previous case, for each c, the graph F on $\{c\} \times \alpha$ is isomorphic to the graph G, which now has no bounded independent sets of type β. Thus for each b, the set X intersects the 'block' mapped to b in a set $W(b) \subseteq \{c\} \times \alpha$ for some c where $W(b)$ has type less than β

since it is bounded in $\{c\} \times \alpha$. Since X has type β and β is indecomposable, the set B of b for which $W(b)$ is non-empty must be infinite. Let b in B be greater than INDEX(t, τ) and let $\langle c, a\rangle$ be in $W(b)$. Then $\langle c, a\rangle$ and $\langle t, \tau\rangle$ are joined by a blue edge contradicting the assumption that $X \cup \{\langle t, \tau\rangle\}$ is independent. This contradiction proves the second item of the lemma and completes the proof. □

For the first corollary we combine the parts of the above lemma in various ways with the monotonicity lemma.

Corollary 3.4 (more lifting). *Suppose that α, β and γ are all countably infinite indecomposable ordinals and that $\gamma \geqslant \omega^2$.*

(1) (Large) *If $\alpha \not\rightarrow (\beta, \gamma$-matching) then $\alpha \cdot \gamma \not\rightarrow (\beta + 1, \gamma$-matching).*

(2) (Alternate) *If $\alpha \not\rightarrow (\beta + 1, \gamma$-matching), then $\alpha \cdot \gamma \not\rightarrow (\beta \cdot \omega + 1, \gamma$-matching).*

Proof. The large lifting follows from small and medium liftings. The alternate lifting follows from monotonicity followed by small and medium liftings. □

The next corollary shows that for matchings of finite powers of ω in graphs defined on finite powers of ω, the positive results are sharp.

Corollary 3.5. *For all positive integers j, k with $k \geqslant 2$, $\omega^{jk} \not\rightarrow (\omega^{j-1} + 1, \omega^k$-matching).*

Proof. The proof is by induction on j. The basis of the induction is $j = 1$. In this case, the complete graph is the required example. For $j > 1$ first use the induction hypothesis which gives $\omega^{(j-1)k} \not\rightarrow (\omega^{j-2} + 1, \omega^k$-matching) and then use Corollary 3.5 (alternate lifting) to complete the induction step. □

The next corollary in which the desired size matching is again a finite power of ω is a bit of a disappointment since there is a gap between the counterexample it provides and the positive result of the previous section.

Corollary 3.6. *For any positive integers j, k with $k \geqslant 2$ and any countable indecomposable ordinal β of the form $\beta = \omega^b$ for some limit ordinal b,*

$$\beta \cdot \omega^{jk-1} \not\rightarrow (\beta \cdot \omega^{j-1} + 1, \omega^k\text{-matching}).$$

Proof. The proof is by induction on j. For the basis $(j = 1)$ start with the counterexample given by the empty graph which witnesses $\beta \not\rightarrow$ (bounded β, ω^k-matching). Apply Lemma 3.3 (medium lifting) to conclude that $\beta \cdot \omega^{k-1} \not\rightarrow$ $(\beta + 1, \omega^k$-matching) as desired. For the induction step, apply Corollary 3.5 (alternate lifting). □

The final lifting lemma is designed for the limit stages of the theorem to follow which proves some partition relations of the form $\alpha \not\to (\beta, \gamma\text{-matching})$ by induction on β.

Bounded Set Lifting Lemma 3.7. *Suppose β and γ are countable indecomposable ordinals. If $\alpha_n \not\to (\beta, \gamma\text{-matching})$ for a nondecreasing sequence of indecomposable ordinals $\langle \alpha_n \mid n < \omega \rangle$ and $\alpha = \sum_{n<\omega} \alpha_n$,* then *$\alpha \not\to$ (bounded β, γ-matching) and consequently $\alpha \not\to (\beta + 1, \gamma\text{-matching})$.*

Proof. Write α as the sum over $n < \omega$ of sets $A(n)$ where $A(n)$ has type α_n, and let $G(n)$ be a graph on $A(n)$ witnessing $\alpha_n \not\to (\beta, \gamma\text{-matching})$. Let G be the union over $n < \omega$ of the graphs $G(n)$. Since every edge of G lies entirely inside some $A(n)$, there can be no γ-matching.

Let I be an independent set for G. For each $n < \omega$ let $I(n)$ be the intersection of I with $A(n)$. By the hypothesis on each α_n, the sets $I(n)$ all have type less than β. Since β is indecomposable, for each $m < \omega$, the union of the sets $I(n)$ for $n < m$ also has type less than β. Hence I is not a bounded independent set of type β and it has type at most $\beta < \beta + 1$.

Since G has no bounded independent set of type β, no independent set of type $\beta + 1$ nor any γ-matching, it is the example needed to prove the lemma. □

Next we consider γ-matchings where γ is an indecomposable ordinal of infinite exponent.

Theorem 3.8. *Suppose that $\beta = \omega^b$ and $\gamma \geq \omega^2$ are countably infinite indecomposable ordinals.*

(1) *If b is a limit, then $\gamma^b \not\to$ (bounded ω^b, γ-matching).*

(2) *If b is finite or if b is a successor and $\gamma \geq \omega^\omega$, then $\gamma^b \not\to (\omega^{b-1} + 1, \gamma\text{-matching})$.*

(3) *If b is an infinite successor, $\gamma < \omega^\omega$ and $\gamma^b = \omega^\delta$, then $\omega^{\delta-1} \not\to (\omega^{b-1} + 1, \gamma\text{-matching})$.*

Proof. First suppose that γ is less than ω^ω, namely that γ is ω^k for some finite $k \geq 2$. If b is a limit ordinal, the γ^b is the same as ω^b, so the empty graph is the desired counterexample. If b is finite, then the theorem follows from Corollary 3.5. If b is an infinite successor ordinal, $b = d + j$ where d is a limit ordinal and j is finite, then γ^b is $\omega^d \cdot \omega^{jk}$ and ω^{b-1} is $\omega^d \cdot \omega^{j-1}$, so this case follows by application of Corollary 3.6.

Fix $\gamma \geq \omega^\omega$ and prove the theorem induction on b. Notice that the empty graph shows $\gamma \not\to (2, \gamma\text{-matching})$, and that $\omega^0 + 1 = 2$. Thus this graph starts the induction for $b = 1$. If $b = d + 1$ is a successor either of 0 or another successor ordinal, use Corollary 3.4 (alternate lifting). If $b = d + 1$ is the successor of a limit ordinal d, then use Lemma 3.3 (medium lifting). Finally if b is a limit ordinal, first use monotonicity and the induction hypothesis to conclude for all $d < b$ that $\gamma^d \nrightarrow (\omega^b, \gamma\text{-matching})$ and then use the Bounded Set Lifting Lemma 3.7 to reach the desired conclusion. Thus by induction, the theorem holds for all b. □

The final theorem in this section is a corollary of the previous one.

Theorem 3.9. *For all infinite indecomposable ordinals ω^b and $\gamma \geq \omega^2$, $\gamma^b \nrightarrow (\omega^b + 1, \gamma\text{-matching})$.*

Proof. If b is finite or a limit ordinal or if it is a successor ordinal and $\gamma \geq \omega^\omega$, then Theorem 3.9 follows from Theorem 3.8 by monotonicity. Suppose that b is an infinite successor ordinal and $\gamma < \omega^\omega$. Further suppose that $\gamma^b = \omega^\delta$ for some ordinal δ. Notice in the proof of Theorem 3.8 that δ has the form $d + jk$ where $\gamma = \omega^k$, so that subtraction makes sense, and use that theorem to get $\omega^{\delta-1} \nrightarrow (\omega^{b-1} + 1, \gamma\text{-matching})$. By monotonicity, conclude that $\omega^{\delta-1} \nrightarrow (\omega^b, \gamma\text{-matching})$. Next apply Lemma 3.3 (small lifting) to get $\omega^\delta \nrightarrow (\text{bounded } \omega^b, \gamma\text{-matching})$. Finally use monotonicity to conclude the theorem. □

4. Conclusions and questions

In the final section we pull together the results of the previous sections to see what remains to be done. The section on cardinal matchings was satisfyingly complete and requires no further discussion here. The next theorem puts together a positive result from Section 2 (Theorem 2.5) and a counter-example from Section 3 (Theorem 3.9) to give a general statement true for all countably infinite indecomposable ordinals and nontrivial ordinal matchings.

Theorem 4.1. *If ω^b and $\gamma \geq \omega^2$ are countably infinite indecomposable ordinals, then*

$$\gamma^b \cdot \omega \rightarrow (\omega^b, \gamma\text{-matching})) \quad \textit{and} \quad \gamma^b \nrightarrow (\omega^b + 1, \gamma\text{-matching}).$$

The next theorem lists the cases in which our results are optimal.

Theorem 4.2. *Suppose ω^b and $\gamma \geq \omega^2$ are countably infinite indecomposable ordinals.*

(1) *If b is a limit ordinal, then $\gamma^b \rightarrow (\omega^b, \gamma\text{-matching})$, and for all $\alpha < \gamma^b$, $\alpha \nrightarrow (\omega^b, \gamma\text{-matching})$.*

(2) *If b is finite or if b is a successor ordinal and $\gamma \geq \omega^\omega$, then $\gamma^b \cdot \omega \rightarrow (\omega^b, \gamma\text{-matching})$ and $\gamma^b \nrightarrow (\omega^b, \gamma\text{-matching})$.*

Proof. The second item follows from Theorem 2.5 and Theorem 3.8 with some help from the Monotonicity Lemma.

The first item starts with a part of Theorem 2.5. To begin the proof of the remainder, notice that if b is a limit ordinal and $\langle b(n) \mid n<\omega\rangle$ is a cofinal sequence, then $\langle \gamma^{b(n)} \mid n<\omega\rangle$ is cofinal in γ^b. By Theorem 3.9, $\gamma^{b(n)} \nrightarrow (\omega^{b(n)}+1, \gamma\text{-matching})$. Since $\omega^{b(n)}+1$ is less than ω^b, the remainder of the first item follows by monotonicity. □

The above theorem gives a general result true for all countably infinite indecomposable ordinals ω^b and $\gamma>2$. It also lists the case in which we have results which are optimal of this form. The following result obtained from Theorem 2.5 and Theorem 3.8 leaves an obvious open question.

Proposition 4.3. *Suppose that j and k are positive integers with $k \geqslant 2$ and η is a limit ordinal. Then $\omega^{\eta+jk+1} \to (\omega^{\eta+j}, \omega^k\text{-matching})$ and $\omega^{\eta+jk-1} \nrightarrow (\omega^{\eta+j}, \omega^k\text{-matching})$.*

Question 4.4. Suppose that j and k are positive integers with $k \geqslant 2$ and η is a limit ordinal. Is it true that $\omega^{\eta+jk} \to (\omega^{\eta+j}, \omega^k\text{-matching})$?

Another direction for study is the case for uncountable ordinals and uncountable ordinal matchings, but we have not yet had time to investigate it.

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