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Semigroup Crossed Products and the Toeplitz Algebras of Nonabelian Groups*

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We consider the quasi-lattice ordered groups (G, P) recently introduced by Nica. We realise their universal Toeplitz algebra as a crossed product $B_P \rtimes P$ by a semigroup of

Toeplitz–Cuntz algebras as special cases.

INTRODUCTION

Ever since Coburn proved that all non-unitary isometries generate isomorphic C^*-algebras [4], mathematicians have been proving interesting generalisations involving isometric representations of semigroups (Coburn's Theorem can be viewed as discussing representations of the semigroup $\mathbb{N}$). Thus, in particular, Douglas extended Coburn's Theorem to representations of positive cones of ordered subgroups of $\mathbb{R}$ by non-unitary isometries [9], and Murphy to representations of the positive subsemigroups of totally ordered groups [10]. More recently, Nica studied this problem for a large class of partially ordered groups (G, P), which he called *quasi-lattice ordered groups*, and a family of isometric representations which he called *covariant representations* [14]. Nica showed that analogues of Coburn's Theorem hold for some highly nonabelian groups; his most striking example is the subsemigroup $P = \mathbb{N} * \mathbb{N} * \cdots * \mathbb{N}$ of the free group $G = \mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$, for which Cuntz's results in [6] show that all covariant isometric representations generate the (essentially unique) Toeplitz–Cuntz algebra.

Nica associates two C^*-algebras to each quasi-lattice ordered group (G, P): the *Wiener–Hopf* or *Toeplitz* algebra $\mathscr{T}(G, P)$ is the C^*-subalgebra

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of $B(\ell^2(P))$ generated by the translation operators $T_x: \varepsilon_y \mapsto \varepsilon_{xy}$, and $C^*(G, P)$ is the universal C^*-algebra whose representations are given by the covariant isometric representations of P. He shows that both $C^*(G, P)$ and $\mathscr{T}(G, P)$ have "crossed-product"-like structures, and that one can profitably view $\mathscr{T}(G, P)$ as the reduced crossed product associated to the full crossed product $C^*(G, P)$. If the Toeplitz representation of $C^*(G, P)$ on $\ell^2(P)$ is faithful, he says (G, P) is amenable, and deduces from [6] that $(\mathbb{F}_n, \mathbb{N} * \cdots * \mathbb{N})$ is amenable in this sense.

Here we make two main points, both of which confirm that Nica has had an important insight. First, we show how Nica's quasi-lattice orders provide a tailor-made setting for the key arguments of Cuntz [5]. Thus, for any amenable quasi-lattice order (G, P), there is an elegant analogue of the Coburn–Douglas–Murphy uniqueness theorem (Corollary 3.9). Second, we show directly that a number of interesting examples are amenable (Theorem 4.4). These include the semigroups $(\mathbb{F}_n, \mathbb{N} * \cdots * \mathbb{N})$ described above, and more general free product partial orders; thus, for example, Cuntz's uniqueness theorem and its generalisation due to Dinh [8] emerge as corollaries of our work.

Our framework is that of crossed products of C^*-algebras by semigroups of endomorphisms, as discussed (though primarily for abelian groups) in recent work of Stacey, Murphy and others [20, 12, 11, 3, 1, 2]. Although we shall not develop the general theory to any great extent, we shall do enough to show that the "crossed-product" structure observed by Nica in $C^*(G, P)$ is precisely that: there is a natural action of P on a C^*-subalgebra B_P of $\ell^\infty(P)$ such that the crossed product $B_P \rtimes P$ has the universal property characterising $C^*(G, P)$ (Corollary 2.4). From there, it is relatively easy to see as in [3, 2] that, for amenable (G, P), a representation of $C^*(G, P)$ will be faithful when it is faithful on B_P and satisfies a certain norm estimate. We establish this norm estimate for any quasi-lattice ordered group by modifying the arguments of Cuntz [5]. Thus we obtain our Coburn-style uniqueness theorem for any amenable (G, P).

To prove that free products of abelian quasi-lattice orders are automatically amenable, we have to show that a canonical expectation Φ of $B_P \rtimes P$ onto B_P is faithful (see Definition 3.4). We tackle this in two stages. The canonical map of the free product onto the (abelian) direct product $\mathscr{G}$ gives a dual action θ of $\hat{\mathscr{G}}$ on $B_P \rtimes P$. The fixed-point algebra $(B_P \rtimes P)^\theta$ can then be faithfully represented on a Hilbert space $\mathscr{H}$ in such a way that Φ is the composition of averaging over θ and compressing to the diagonal in $B(\mathscr{H})$. Since both are faithful, the amenability follows.

We begin with a brief discussion of quasi-lattice ordered groups (G, P), their covariant isometric representations, and their Toeplitz algebras. We introduce the subalgebra B_P of $\ell^\infty(P)$, show how covariant isometric representations W of P give representations π_W of B_P, and give criteria which ensure π_W is faithful. In Section 2, we consider general semigroup dynamical systems, their covariant representation theory, and their crossed products. We then show how the correspondence $W \mapsto (\pi_W, W)$ induces an isomorphism of Nica's $C^*(G, P)$ onto $B_P \rtimes P$. In Section 3, we prove our analogue of Coburn's Theorem—assuming amenablity of (G, P)—and in Section 4 show that free product of abelian orders are amenable. We then show in Section 5 how our results imply those of Cuntz and Dinh, and discuss some new examples.

In our last section, we confess that our results in Section 4 have been sanitised: they are actually true for free products of amenable quasi-lattice ordered groups, not just abelian ones. To prove this, we have to use a canonical dual coaction of the group $\mathcal{G}$ rather than the action θ of the dual group $\hat{\mathcal{G}}$. However, we can make the argument of Section 4 go through using only relatively elementary properties of the discrete coactions of [13, 17].

We wish to thank A. Nica for bringing his work to our attention. Part of our work was done while we were visiting the Mathematics Department of the University of Colorado at Boulder, and we are grateful for their hospitality.

2. Quasi-Lattice Ordered Groups

Let P be a subsemigroup of a group G such that $P \cap P^{-1} = \{e\}$. There is a partial order on G defined by $x \leqslant y$ if $x^{-1}y \in P$, which is left invariant in the sense that $x \leqslant y$ implies $zx \leqslant zy$ for every $z \in G$. In general two elements may lack a common upper bound, and even if they have one there may not be a smallest one.

Definition 1.1. The partially ordered group (G, P) is *quasi-lattice ordered* if every finite subset of G with an upper bound in P has a *least* upper bound in P.

Equivalently, (G, P) is quasi-lattice ordered if and only if every element of G having an upper bound in P has a least such, and every two elements in P having a common upper bound have a least common upper bound, [14, Sect. 2.1].

Notation. If x and y have a common upper bound in P, their least common upper bound will be denoted by $x \vee y$, and if $A \subset G$ has an upper bound in P, then σA will denote the smallest such upper bound for A. For notational purposes it will be convenient to introduce a symbol ∞ and say $x \vee y = \infty$ whenever x and y have no common upper bound in P.

Totally ordered groups and lattice orders are quasi-lattice ordered, as are their direct and semidirect products. A free product can be given a quasi-lattice order as follows: suppose $\{(G_i, P_i)\}_{i \in I}$ is a family of quasi-lattice ordered groups and denote by G the free product $\prod_i^* G_i$ of the G_i. The subsemigroup P generated by the images in G of all the positive cones P_i gives rise to a quasi-lattice order structure on G. The simplest example is $\mathbb{F}_2$, the free group on two generators a and b, viewed as $\mathbb{Z} * \mathbb{Z}$, together with the unital semigroup $\mathbb{F}_2^+ = \mathbb{N} * \mathbb{N}$ generated by a and b. A reduced word w in a and b precedes another one z in the corresponding order if and only if w is an initial segment (or subword) of z.

The case in which G_i is isomorphic to a countable dense subgroup of the reals for each $i = 1, 2, ..., n$ is related to the discrete product systems of [8], and will be discussed in some detail in Section 5. For a discussion of the elementary properties of quasi-lattice orders, and an extensive list of examples, see [14, Sect. 2].

The condition of quasi-lattice order has an interesting algebraic consequence in terms of projections in $\ell^\infty(P)$. Denote by 1_x the function on P defined by

$$1_x(y) = \begin{cases} 1 & \text{if} \quad y \geqslant x, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The quasi-lattice condition gives

$$1_x 1_y = \begin{cases} 1_{x \vee y} & \text{if } x, y \text{ have a common upper bound,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

because $\{z \in P : z \geqslant x\} \cap \{z \in P : z \geqslant y\} = \{z \in P : z \geqslant x \vee y\}$. Thus $B_P = \overline{\operatorname{span}}\{1_x : x \in P\}$ is an abelian C^*-algebra.

Definition 1.2. A representation of the unital semigroup P by isometries on a Hilbert space is a map $W: P \to B(H)$ such that $W_e = 1$ and $W_x W_y = W_{xy}$ for $x, y \in P$. A representation is *covariant* if it satisfies

$$W_x W_x^* W_y W_y^* = \begin{cases} W_{x \vee y} W_{x \vee y}^* & \text{if } x, y \text{ have a common upper bound,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

To simplify the notation we use the convention $1_\infty = 0$ and $W_\infty = 0$ for any covariant representation W of P; thus we can write $1_x 1_y = 1_{x \vee y}$ and $W_x W_x^* W_y W_y^* = W_{x \vee y} W_{x \vee y}^*$ always. With this convention, the covariance condition (1.3) is equivalent to

$$W_x^* W_y = W_{x^{-1}(x \vee y)} W_{y^{-1}(x \vee y)}^* \qquad \text{for all} \quad x, y \in P. \tag{1.4}$$

The motivation for this covariance condition is that the range projections of such representations of P give rise to representations of the algebra B_P; see [14, Lemma 3.9] and Proposition 1.3 below. The main example of an isometric covariant representation of P is the Toeplitz representation on $\ell^2(P)$ (the Wiener–Hopf representation in [14]). It is given by

$$T_x \varepsilon_z = \varepsilon_{xz} \qquad x, z \in P, \tag{1.5}$$

where $\{\varepsilon_z : z \in P\}$ is the canonical orthonormal basis for $\ell^2(P)$. The covariance condition is satisfied because the operator $T_x T_x^*$ is just multiplication by 1_x. The C^*-algebra generated by these isometries will be called the Toeplitz C^*-algebra and denoted by $\mathscr{T}(G, P)$.

It is easy to exhibit an isometric representation which is *not* covariant: e.g., for $(G, P) = (\mathbb{Z}^2, \mathbb{N}^2)$ let $V_{(1,0)} = V_{(0,1)}$ be the unilateral shift on $\ell^2(\mathbb{N})$.

The universal C^*-algebra for covariant isometric representations of P, denoted by $C^*(G, P)$, is the C^*-algebra generated by a canonical covariant isometric representation $i: P \to C^*(G, P)$ with the following property: if V is a covariant isometric representation of P then there is a homomorphism $\theta_V: C^*(G, P) \to C^*(\{V_x : x \in P\})$ such that $\theta_V(i(x)) = V_x$. The C^*-algebra $C^*(G, P)$ exists by [14, Sect. 4.1] or Section 2 below, where we exhibit a particular semigroup crossed product having the universal property, and uniqueness up to isomorphism is easy to verify.

We need a few basic facts about representations of B_P. It is easy to see that the set $P \cup \{\infty\}$ is an abelian semigroup under the binary operation $\vee$. The following proposition shows that representations of B_P on Hilbert space are in one to one correspondence with representations of $(P \cup \{\infty\}, \vee)$ by projections on the same Hilbert space. This is essentially [14, Lemma 3.9], with the additional characterization of faithfulness given in part (3) which will be needed in Section 2 to characterize faithful representations of $C^*(G, P)$.

Proposition 1.3. *Let (G, P) be a quasi-lattice ordered group.*

(1) *If $\pi: B_P \to B(H)$ is a nondegenerate representation of B_P, then $L_x = \pi(1_x)$ is a family of projections in $B(H)$ satisfying*

$$L_e = 1 \quad \textit{and} \quad L_x L_y = L_{x \vee y} \qquad \textit{for} \quad x, y \in P. \tag{1.6}$$

(For convenience of notation, we take $L_\infty = 0$.)

(2) *Conversely, if $\{L_x : x \in P\}$ satisfies* (1.6), *then there is a representation $\pi_L: B_P \to B(H)$ such that $\pi_L(1_x) = L_x$.*

(3) *For L as above, π_L is faithful if and only if L satisfies*

$$\prod_{i=1}^{n} (L_a - L_{z_i}) \neq 0 \quad \textit{whenever } a < z_i \qquad \textit{for} \quad i = 1, 2, ..., n. \tag{1.7}$$

The proof of (1) is immediate. Since (1.6) implies that the map $1_x \mapsto L_x$ is multiplicative, to prove (2) it suffices to show that if F is a finite subset of P, and $\lambda_x \in \mathbb{C}$ for $x \in F$, then

$$\left\| \sum_{x \in F} \lambda_x L_x \right\| \leqslant \left\| \sum_{x \in F} \lambda_x 1_x \right\| : \tag{1.8}$$

in this case the map has a well-defined contractive extension to linear combinations, which further extends to their closure. This extension is faithful if and only if equality holds.

To prove the inequality (1.8) we need an expression for the norm of linear combinations of the form $\sum_{x \in F} \lambda_x L_x$, which we obtain in the following lemma, using a convenient set of mutually orthogonal projections.

LEMMA 1.4. *Suppose $L: P \to B(H)$ is a family of projections satisfying $L_e = 1$ and $L_x L_y = L_{x \vee y}$ for $x, y \in P$. Let F be a finite subset of P and $\lambda_x \in \mathbb{C}$ for each $x \in F$. For every nonempty proper subset A of F let $Q_A^L = \prod_{x \in F \setminus A} (L_{\sigma A} - L_{\sigma A \vee x})$. Also let $Q_{\varnothing}^L = \prod_{x \in F} (1 - L_x)$ and $Q_F^L = \prod_{x \in F} L_x = L_{\sigma F}$. Then $\{Q_A^L : A \subset F\}$ is a decomposition of the identity into mutually orthogonal projections such that*

$$\sum_{x \in F} \lambda_x L_x = \sum_{A \subset F} \left(\sum_{x \in A} \lambda_x \right) Q_A^L; \qquad \textit{and} \tag{1.9}$$

$$\left\| \sum_{x \in F} \lambda_x L_x \right\| = \max \left\{ \left| \sum_{x \in A} \lambda_x \right| : A \subset F \textit{ and } Q_A^L \neq 0 \right\}. \tag{1.10}$$

Proof. Since the projections L_x commute,

$$\begin{aligned} 1 = \prod_{x \in F} (L_x + (1 - L_x)) &= \sum_{A \subset F} \prod_{x \in A} L_x \prod_{x \in F \setminus A} (1 - L_x) \\ &= \sum_{A \subset F} L_{\sigma A} \prod_{x \in F \setminus A} (1 - L_x) = \sum_{A \subset F} Q_A^L, \end{aligned}$$

which is the desired decomposition of the identity. If A and B are different subsets of F, suppose without loss of generality that $z \in B \setminus A$. Then

$$Q_A^L Q_B^L = Q_A^L (L_{\sigma A} - L_{\sigma A \vee z}) L_{\sigma B} Q_B^L = 0$$

because $\sigma A \vee \sigma B = \sigma A \vee z \vee \sigma B$.

Multiplying $\sum_{x \in F} \lambda_x L_x$ by this decomposition,

$$\sum_{x \in F} \lambda_x L_x = \sum_{x \in F} \lambda_x L_x \left(\sum_{A \subset F} Q_A^L \right)$$
$$= \sum_{x \in F} \sum_{A \subset F} \lambda_x \prod_{y \in F \setminus A} (L_x L_{\sigma A} - L_x L_{\sigma A \vee y})$$
$$= \sum_{x \in F} \sum_{A \subset F} \lambda_x \prod_{y \in F \setminus A} (L_{\sigma A \vee x} - L_{\sigma A \vee y \vee x}).$$

If $x \notin A$ the product vanishes because the factor corresponding to $y = x$ is zero. If $x \in A$ then $\sigma A \vee x = \sigma A$ and $\sigma A \vee y \vee x = \sigma A \vee y$, hence the projection multiplying λ_x is $\prod_{y \in F \setminus A} (L_{\sigma A} - L_{\sigma A \vee y}) = Q_A^L$ for every $x \in A$, proving (1.9). Since the Q_A^L are mutually orthogonal, (1.10) follows easily from (1.9). ∎

Remark 1.5. Let $\{Q_A : A \subset F\}$ denote the decomposition of the identity corresponding to $\{1_x : x \in F\}$. If $x \leqslant \sigma A$ for some $x \in F \setminus A$, then $\sigma A = \sigma A \vee x$ and $1_{\sigma A} - 1_{\sigma A \vee x} = 0$ so Q_A vanishes. Conversely, if $Q_A = 0$, evaluation at $a = \sigma A$ gives

$$\prod_{x \in F \setminus A} (1_a(a) - 1_{a \vee x}(a)) = Q_A(a) = 0.$$

Thus there is at least one $x_0 \in F \setminus A$ with $1_{a \vee x_0}(a) = 1$, i.e., with $a \vee x_0 = a$. This shows that $Q_A \neq 0$ if and only if A is an initial segment of F, in the sense that $A = \{x \in F : x \leqslant \sigma A\}$.

We are now ready to finish the proof of Proposition 1.3. By (1.10), in order to prove the inequality (1.8) and hence (2), it is enough to show that $Q_A \neq 0$ whenever $Q_A^L \neq 0$. Assume $Q_A = 0$. By Remark 1.5 there exists $x_0 \in F \setminus A$ with $x_0 \leqslant \sigma A$, and this implies $Q_A^L = 0$ because the factor $L_a - L_{a \vee x_0}$ vanishes.

A slight sharpening of the preceding argument proves (3) as well. Suppose π_L is faithful; if $a < z_i$ for $i = 1, 2, ..., n$, then evaluation at a shows that $\prod_{i=1}^n (1_a - 1_{z_i}) \neq 0$, hence

$$\prod_{i=1}^{n} (L_a - L_{z_i}) = \pi_L \left(\prod_{i=1}^{n} (1_a - 1_{z_i}) \right) \neq 0.$$

Suppose now (1.7) holds. By (1.10), to prove π_L is isometric it is enough to show that $Q_A^L \neq 0$ whenever $Q_A \neq 0$. Assume $Q_A \neq 0$ and let $\{x_i : i = 1, 2, ..., n\}$ be a listing of $F \setminus A$. By Remark 1.5 $\sigma A < x_i \vee \sigma A$ for all $i = 1, 2, ..., n$ and (1.7) with $a = \sigma A$ and $z_i = x_i \vee \sigma A$ yields $Q_A^L \neq 0$, finishing the proof of Proposition 1.3.

2. Semigroup Dynamical Systems

A semigroup dynamical system is a triple (A, P, α) in which α is an action of the semigroup P by endomorphisms of the unital C^*-algebra A. We do not insist that the endomorphisms be unital; indeed our main example consists of nonunital endomorphisms. Following [20, 2] we define a *covariant representation* of the dynamical system to be a pair (π, V) in which π is a unital representation of A on a Hilbert space H and V is an isometric representation of P on H such that $\pi(\alpha_x(a)) = V_x \pi(a) V_x^*$ for $x \in P$ and $a \in A$. We shall refer to these as *covariant pairs* to stress that there are two components, as opposed to the notion of covariant isometric representations given in Definition 1.2.

Proposition 2.1. *Suppose (A, P, α) is a semigroup dynamical system which has a covariant pair. Then there is a triple (B, i_A, i_P) consisting of a C^*-algebra B, a unital homomorphism $i_A\colon A \to B$ and a semigroup homomorphism i_P of P into the isometries in B such that*:

(1) $i_A(\alpha_x(a)) = i_P(x)\, i_A(a)\, i_P(x)^*$ *for $x \in P$ and $a \in A$*;

(2) *for every covariant representation (π, V) of (A, P, α) there is a unital representation $\pi \times V$ of B with $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_P = V$; and*

(3) *the C^*-algebra B is generated by $\{i_A(a)\colon a \in A\} \cup \{i_P(x)\colon x \in P\}$.*

The triple (B, i_A, i_P) is unique up to isomorphism.

Proof. Consider the collection of all covariant pairs (π_l, V_l) on a Hilbert space H_l such that $C^*(\pi_l(A) \cup V_l(P))$ acts cyclically on H_l. The pair $i_A = \bigoplus_l \pi_l$ and $i_P = \bigoplus_l V_l$ is covariant, and nontrivial because of the assumed existence of a covariant pair. Let B be the C^*-algebra generated by $i_A(A) \cup i_P(P)$. Since any covariant pair (π, V) decomposes as a direct sum of cyclic pairs because the C^*-algebra $C^*(\pi(A) \cup V(P))$ does, the pair (i_A, i_P) has the universal property.

Suppose (B, i_A, i_P) and (B', i'_A, i'_P) are triples satisfying conditions (1), (2) and (3). Since (B, i_A, i_P) satisfies (2) there is a homomorphism $i'_A \times i'_P\colon B \to B'$ which is onto by (3), and it is easy to see that it has an inverse given by $i_A \times i_P\colon B' \to B$, proving that (B, i_A, i_P) is unique up to isomorphisms. ∎

Definition 2.2. The *crossed product* of A by the action of P is the unital C^*-algebra B together with the universal generating covariant pair (i_A, i_P), and is denoted $A \rtimes_\alpha P$, or $A \rtimes P$ if there is no ambiguity about the action.

It is important to point out that there are semigroup dynamical systems which admit no nontrivial covariant pairs, such as the backwards shift on c_0 [20, Example 2.1a].

Next we introduce a system naturally associated with a quasi-lattice order and suitable for studying covariant isometric representations. The algebra is $B_P = \overline{\operatorname{span}}\{1_x : x \in P\} \subset \ell^\infty(P)$ and the action α of P on B_P consists of left translation on $\ell^\infty(P)$ restricted to B_P. This canonical action α of P by endomorphisms of B_P is determined by $\alpha_s(1_x) = 1_{sx}$ for $s, x \in P$.

To obtain a specific covariant pair for (B_P, P, α), let each $f \in B_P$ act as the multiplication operator M_f on $\ell^2(P)$ and T_x be the isometry on $\ell^2(P)$ defined by $T_x(\varepsilon_z) = \varepsilon_{xz}$. Since $M_{\alpha_x(1_y)} = M_{1_{xy}} = T_{xy} T^*_{xy} = T_x T_y T^*_y T^*_x = T_x M_{1_y} T^*_x$, the pair (M, T) is covariant. Thus the crossed product $B_P \rtimes P$ is nontrivial.

Proposition 1.3 has an analogue which relates arbitrary covariant pairs of (B_P, P, α) to isometric representations of P which are covariant in the sense of [14] (Definition 1.2).

PROPOSITION 2.3. *Let (G, P) be a quasi-lattice ordered group.*

(1) *If (π, W) is a covariant representation of (B_P, P, α), then W is a covariant isometric representation of P, and $\pi(1_x) = W_x W^*_x$.*

(2) *If W is a covariant isometric representation of P, then there is a representation π_W of B_P such that $\pi_W(1_x) = W_x W^*_x$, and the pair (π_W, W) is covariant for (B_P, P, α).*

(3) *If W is as above, π_W is faithful if and only if*

$$\prod_{i=1}^{n} (I - W_{x_i} W^*_{x_i}) \neq 0 \qquad \textit{whenever} \quad x_1, x_2, ..., x_n \in P \setminus \{e\}. \tag{2.1}$$

Proof. If (π, W) is covariant then $W_x W^*_x = \pi(\alpha_x(1)) = \pi(1_x)$, thus W is covariant.

To prove (2), let $L_x = W_x W^*_x$; since W is covariant, L satisfies (1.6), and by Proposition 1.3(2), $\pi_L = \pi_W$ is a well defined representation of B_P. Since $\pi_W(\alpha_x(1_y)) = \pi_W(1_{xy}) = W_{xy} W^*_{xy} = W_x W_y W^*_y W^*_x = W_x \pi_W(1_y) W^*_x$, the pair (π_W, W) is covariant.

To prove (3), write $L_a - L_{a \vee z_i} = W_a(I - W_{x_i} W^*_{x_i}) W^*_a$ with $x_i = a^{-1}(a \vee z_i)$, then

$$\prod_{i=1}^{n} (L_a - L_{a \vee z_i}) = W_a \prod_{i=1}^{n} (I - W_{x_i} W^*_{x_i}) W^*_a$$

because W_a is an isometry. Hence (2.1) is equivalent to (1.7) and the proof is finished by an application of Proposition 1.3 (3). ∎

COROLLARY 2.4. *Let (G, P) be a quasi-lattice ordered group. Then*

(1) *the maps i_{B_P} and i_P are faithful,*

(2) *$(C^*(G, P), i) \cong (B_P \rtimes P, i_P)$, and*

(3) span$\{i_P(x)\, i_P(y)^* : x, y \in P\}$ *is dense in $B_P \rtimes P$.*

Proof. Both components of the covariant pair (M, T) corresponding to the Toeplitz representation are faithful. Thus the universal covariant pair (i_{B_P}, i_P) has faithful components. Parts (1) and (2) of Proposition 2.3 establish a one to one correspondence between covariant isometric representations of P and covariant pairs for the system (B_P, P, α). Since covariance implies $i_{B_P}(1_x) = i_P(x)\, i_P(x)^*$, elements of the form $i_P(x)$ generate the crossed product as a C^*-algebra. Thus $B_P \rtimes P$ together with the canonical embedding i_P of P is universal for covariant isometric representations of P and (2) follows by uniqueness.

In particular, i_P is a covariant isometric representation of P. Thus from (1.4) we have

$$i_P(x)\, i_P(y)^*\, i_P(u)\, i_P(v)^* = i_P(xy^{-1}(y \vee u))\, i_P(vu^{-1}(y \vee u))^*,$$

so the collection $\{i_P(x)\, i_P(y)^* : x, y \in P\}$ is closed under multiplication. Hence its linear span is a subalgebra of $B_P \rtimes P$ which contains $i_P(x) = i_P(x)\, i_P(e)^*$ for each $x \in P$, proving (3). ∎

From now on $C^*(G, P)$ will be identified with $B_P \rtimes P$, the canonical representation of P being i_P. Since i_{B_P} is faithful we will often abuse the notation and view B_P as the subalgebra of $C^*(G, P)$ generated by the elements $i_P(x)\, i_P(x)^*$.

3. FAITHFUL REPRESENTATIONS OF $B_P \rtimes P$

If (A, P, α) is a semigroup dynamical system, products of the form

$$X = i_P(x_1)^*\, i_A(a_1)\, i_P(y_1) \cdots i_P(x_n)^*\, i_A(a_n)\, i(y_n)$$

span a dense set in the associated crossed product. Moreover, for every covariant representation (π, V), the corresponding products $\pi \times V(X) = V_{x_1}^* \pi(a_1)\, V_{y_1} \cdots V_{x_n}^* \pi(a_n)\, V_{y_n}$ span a dense set of the range $C^*(\pi, V)$ of $\pi \times V$. Assume the correspondence

$$\Phi(X) = \begin{cases} X & \text{if } \ x_1^{-1} y_1 \cdots x_n^{-1} y_n = e \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

extends by linearity to a contraction defined on a dense set of $C^*(G, P)$, and assume as well that there is an analogous contraction ϕ on $C^*(\pi, V)$ taking $\pi \times V(X)$ to $\pi \times V(X)$ or 0.

Suppose further that $\Phi(bb^*) = 0$ only for $b = 0$, and consider the following diagram:

$$\begin{array}{ccc} A \rtimes_\alpha P & \xrightarrow{\pi \times V} & C^*(\pi, V) \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \phi} \\ \Phi(A \rtimes_\alpha P) & \xrightarrow{\pi \times V} & \phi(C^*(\pi, V)). \end{array} \tag{3.2}$$

The diagram commutes because Φ and ϕ are defined by analogous formulae on the spanning set. Thus if $\pi \times V(b^*b) = 0$ for some $b \in A \rtimes_\alpha P$, then

$$\pi \times V(\Phi(b^*b)) = \phi(\pi \times V(b^*b)) = 0.$$

If $\pi \times V$ is faithful on the range of Φ then $\Phi(b^*b) = 0$, hence $b = 0$. Thus, faithfulness of representations of the crossed product is reduced to faithfulness on the range of Φ. These considerations raise three key questions:

- when does the correspondence defined in (3.1) extend to a contraction?
- under what conditions is Φ faithful?
- for which covariant pairs is $\pi \times V$ faithful on the range of Φ?

In Section 6 we construct a canonical positive contractive conditional expectation Φ_δ from $A \rtimes_\alpha P$ onto the fixed point algebra of a coaction δ of the full group G. However, for the system (B_P, P, α) the crossed product is the closed linear span of products of the form $i(x)\, i(y)^*$, and (3.1) reduces to

$$\Phi\colon i(x)\, i(y)^* \mapsto \delta_{x, y} i(x)\, i(y)^*;$$

thus it is possible to give a relatively elementary definition of Φ, which we do here for the sake of keeping technicalities to a minimum. The next proposition shows that for any covariant pair (π, V), faithfulness of π suffices for the existence of a contraction ϕ from $\pi \times W(B_P \rtimes P)$ onto $\pi(B_P)$ determined by $\phi(V_x V_y^*) = \delta_{x, y} V_x V_y$. The proof is based on a well-known idea of Cuntz [5, Proposition 1.7]; in both [2, Theorem 2.4] and [14, Sect. 5.2] basically this same argument was applied to totally ordered abelian groups. The main point we make here is that this holds in general, without extra hypotheses, for every quasi-lattice ordered group.

PROPOSITION 3.1. *Suppose (G, P) is a quasi-lattice ordered group and (π, W) is a covariant pair for (B_P, P, α). If π is faithful, then the map*

$$\phi\colon \sum_{x,\, y \in F} \lambda_{x,\, y} W_x W_y^* \mapsto \sum_{x \in F} \lambda_{x,\, x} W_x W_x^*$$

is contractive, and hence extends to a contraction of $C^*(\{W_x : x \in P\}) = \pi \times W(B_P \rtimes_\alpha P)$ onto $C^*(\{W_x W_x^* : x \in P\}) = \pi(B_P)$.

Proof. The argument consists of, given a finite linear combination of the form $\sum_{x,\, y \in F} \lambda_{x,\, y} W_x W_y^*$, finding a projection Q such that

$$QW_x W_y^* Q = 0 \qquad \text{for} \quad x, y \in F,\ x \neq y \qquad \text{and} \tag{3.3}$$

$$\left\| Q \left(\sum_x \lambda_{x,\, x} W_x W_x^* \right) Q \right\| = \left\| \sum_x \lambda_{x,\, x} W_x W_x^* \right\|. \tag{3.4}$$

The construction of Q is deferred to the following lemma. If such a projection exists, then

$$\left\| \sum_x \lambda_{x,\, x} W_x W_x^* \right\| = \left\| Q \left(\sum_x \lambda_{x,\, x} W_x W_x^* \right) Q \right\| = \left\| Q \left(\sum_{x,\, y} \lambda_{x,\, y} W_x W_y^* \right) Q \right\|$$
$$\leqslant \left\| \sum_{x,\, y} \lambda_{x,\, y} W_x W_y^* \right\|,$$

proving that the map is contractive, hence well-defined on linear combinations, and that it extends to a projection of norm one from all of $C^*(\{W_x : x \in P\})$ onto $C^*(\{W_x W_x^* : x \in P\})$. ∎

LEMMA 3.2. *Suppose (π, W) is a covariant pair for (B_P, P, α) with π faithful. Let F be a finite subset of P and $\lambda_{x,\, y} \in \mathbb{C}$ for $x, y \in F$. There exists a projection $Q \in \pi(B_P)$ satisfying* (3.3) *and* (3.4).

Proof. The projections $L_x = W_x W_x^*$ satisfy (1.6), so by Lemma 1.4 there is a subset A of F such that

$$\left\| \sum_{x \in F} \lambda_{x,\, x} W_x W_x^* \right\| = \left| \sum_{x \in A} \lambda_{x,\, x} \right|,$$

and such that the projection $Q_A^L = \prod_{z \in F \setminus A} (W_a W_a^* - W_{a \vee z} W_{a \vee z}^*)$ is nonzero and satisfies

$$\left(\sum_{x \in F} \lambda_{x,\, x} W_x W_x^* \right) Q_A^L = \left(\sum_{x \in A} \lambda_{x,\, x} \right) Q_A^L. \tag{3.5}$$

Thus $Q_A \neq 0$ and, by Remark 1.5, $A = \{x \in F : x \leqslant a\}$, where $a = \sigma A$. Suppose $x, y \in A$ with $x \neq y$ and define

$$d_{x,y} = \begin{cases} (x^{-1}a)^{-1}(x^{-1}a \vee y^{-1}a) & \text{if } (x^{-1}a)^{-1}(x^{-1}a \vee y^{-1}a) \neq e, \\ (y^{-1}a)^{-1}(x^{-1}a \vee y^{-1}a) & \text{otherwise.} \end{cases} \tag{3.6}$$

Note that $d_{x,y}$ may be ∞, but it is never e, because if $(x^{-1}a)^{-1}(x^{-1}a \vee y^{-1}a) = e$, then $(x^{-1}a) = (x^{-1}a \vee y^{-1}a)$, hence $x^{-1}a \geqslant y^{-1}a$. But $x \neq y$, so $(x^{-1}a \vee y^{-1}a) > y^{-1}a$, forcing $(y^{-1}a)^{-1}(x^{-1}a \vee y^{-1}a) > e$. The operator

$$Q = \prod_{z \in F \setminus A} (W_a W_a^* - W_{a \vee z} W_{a \vee z}^*) \prod_{x \neq y,\, x,\, y \in A} (W_a W_a^* - W_{ad_{x,y}} W_{ad_{x,y}}^*),$$

is a projection because it is the image under $\pi \times V$ of a projection in B_P; since W_a is an isometry, Q can be written in the slightly different form

$$Q = W_a \left(\prod_{z \in F \setminus A} (I - W_{a^{-1}(a \vee z)} W_{a^{-1}(a \vee z)}^*) \prod_{x,\, y \in A,\, x \neq y} (I - W_{d_{x,y}} W_{d_{x,y}}^*) \right) W_a^*$$

using the covariance of W. Moreover, $z \in F \setminus A$ implies $a \vee z > a$, and $d_{x,y} > e$ for $x \neq y$ in A; thus, since π is assumed to be faithful, Q is not trivial by Proposition 2.3(3).

Next observe that the covariance condition implies

$$W_b^*(W_a W_a^*) = (W_{b^{-1}(a \vee b)} W_{b^{-1}(a \vee b)}^*) W_b^* \qquad a, b \in P, \tag{3.7}$$

which, taking adjoints, yields

$$(W_a W_a^*) W_b = W_b(W_{b^{-1}(a \vee b)} W_{b^{-1}(a \vee b)}^*) \qquad a, b \in P. \tag{3.8}$$

To prove (3.3), take $x, y \in F$ and assume $x \neq y$. If either x or y is not in A, for instance $y \notin A$, then the product $QW_x W_y^* Q$ contains a factor

$$W_y^*(W_a W_a^* - W_{(a \vee y)} W_{(a \vee y)}^*)$$

which by (3.7) is equal to

$$(W_{y^{-1}(a \vee y)} W_{y^{-1}(a \vee y)}^* - W_{y^{-1}(a \vee y \vee y)} W_{y^{-1}(a \vee y \vee y)}^*) W_y^* = 0.$$

So assume both x and y are in A, with $x \neq y$. We may as well suppose $d_{x,y} = (x^{-1}a)^{-1}(x^{-1}a \vee y^{-1}a)$ which we call d for simplicity of notation; taking adjoints exchanges x and y and reduces the other case to this one. The product $QW_x W_y^* Q$ contains a factor

$$(W_a W_a^* - W_{ad} W_{ad}^*) W_x W_y^* W_a W_a^*,$$

which by (3.7) and (3.8) is equal to

$$(W_x(W_{x^{-1}(a\vee x)}W^*_{x^{-1}(a\vee x)} \\ - W_{x^{-1}(ad\vee x)}W^*_{x^{-1}(ad\vee x)})\,W_{y^{-1}(a\vee y)}W^*_{y^{-1}(a\vee y)})\,W^*_y.$$

Since $a\vee x=a$, $a\vee y=a$ and $ad\vee x=ad$, the above can be rewritten as

$$\begin{aligned} &W_x(W_{x^{-1}a}W^*_{x^{-1}a}-W_{x^{-1}ad}W^*_{x^{-1}ad})\,W_{y^{-1}a}W^*_{y^{-1}a}W^*_y \\ &\quad = W_x(W_{x^{-1}a}W^*_{x^{-1}a}W_{y^{-1}a}W^*_{y^{-1}a}-W_{x^{-1}ad}W^*_{x^{-1}ad}W_{y^{-1}a}W^*_{y^{-1}a})\,W^*_y \\ &\quad = W_x(W_{x^{-1}a\vee y^{-1}a}W^*_{x^{-1}a\vee y^{-1}a}-W_{x^{-1}ad\vee y^{-1}a}W^*_{x^{-1}ad\vee y^{-1}a})\,W^*_y \\ &\quad =0, \end{aligned}$$

because $x^{-1}ad=x^{-1}a(x^{-1}a)^{-1}(x^{-1}a\vee y^{-1}a)=(x^{-1}a\vee y^{-1}a)$. Notice that in the case $d=\infty$ (i.e., $x^{-1}a\vee y^{-1}a=\infty$) the argument still applies using our convention that $W_\infty=0$. Thus $QW_xW^*_yQ=0$ finishing the proof of (3.3).

Since Q is a nonzero subprojection of Q^L_A, (3.4) follows easily from (3.5). ∎

Corollary 3.3. *The map*

$$\sum_{x,y}\lambda_{x,y}i_P(x)\,i_P(y)^*\mapsto\sum_x\lambda_{x,x}i_P(x)\,i_P(x)^*$$

extends to a contraction Φ of $B_P\rtimes P$ onto B_P.

Proof. Proposition 3.1 applies to the pair (i_P, i_{B_P}) because i_{B_P} is faithful. ∎

Definition 3.4. The quasi-lattice ordered group (G,P) is *amenable* if $\Phi\colon B_P\rtimes_\alpha P\to B_P$ is faithful on positive elements, in the sense that $\Phi(b^*b)=0$ for $b\in B_P\rtimes_\alpha P$ implies $b=0$.

Remark 3.5. In [14] (G,P) is defined to be amenable if the Toeplitz representation is faithful; our definition is equivalent by Corollary 3.8 below, and is easier to extend to general semigroup dynamical systems.

Remark 3.6. If G is abelian, the universal property gives a natural action θ of $\hat{G}$ by automorphisms of the crossed product $B_P\rtimes P$. One first notes that if γi_P denotes the map $g\mapsto\gamma(g)\,i_P(g)$, then $(B_P\rtimes P, i_{B_P}, \gamma i_P)$ is also a crossed product for (B_P, P, α), and the uniqueness of the crossed product gives an automorphism θ_γ of $B_P\rtimes P$ such that

$$\theta_\gamma(i_P(y))=\gamma(y)\,i_P(y)\qquad\text{for}\quad y\in P.$$

This is pointwise continuous on elements of the generating set $i_{B_P}(B_P) \cup i_P(P)$, and because automorphisms are automatically isometric, θ is a strongly continuous action of $\hat{G}$ on $B_P \rtimes P$.

The fixed point algebra $(B_P \rtimes P)^\theta$ of this dual action is precisely the C^*-algebra $\overline{\text{span}}\{i(x)\, i(x)^*: \ x \in P\}$ of Lemma 4.1, and the map Φ: $i(x)\, i(y)^* \mapsto \delta_{x,y} i(x)\, i(y)^*$ corresponds to the conditional expectation obtained by averaging over the compact $\hat{G}$-orbits. Thus Φ is faithful (e.g., see the proof of [3, Lemma 2.2]), and (G, P) is amenable (cf. [14, Sect. 4.3]).

We are now ready to state the main result of the section which characterizes the faithful representations of $B_P \rtimes P \cong C^*(G, P)$, for amenable (G, P), in terms of the associated isometric representations of P.

Theorem 3.7. *Let (G, P) be an* amenable *quasi-lattice ordered group and let $W: P \to B(H)$ be a covariant isometric representation. The representation $\pi_W \times W$ is an isomorphism of $B_P \rtimes P$ onto $C^*(\{W_x: x \in P\})$ if and only if*

$$\prod_{i=1}^{n} (I - W_{x_i} W_{x_i}^*) \neq 0 \qquad \textit{whenever} \quad x_1, x_2, ..., x_n \in P \backslash \{e\}. \tag{3.9}$$

Proof. First assume that (3.9) holds. To prove that $\pi_W \times W$ is faithful it suffices to verify the assumptions about the commutative diagram (3.2) discussed at the beginning of the section. The representation π_W is faithful by Proposition 2.3(3), so the existence of the contractive map ϕ which makes the diagram commute is given by Proposition 3.1. Since Φ is faithful on positive elements by assumption, it follows that $\pi_W \times W$ is faithful.

Conversely, if $\pi_W \times W$ is faithful, so is $\pi_W = (\pi_W \times W) \circ i_{B_P}$ and Proposition 2.3 implies (3.9). ∎

Corollary 3.8 [14, Sect. 4.2]. *The Toeplitz representation of $C^*(G, P)$ is faithful if and only if (G, P) is amenable.*

Proof. Let (M, T) be the covariant pair corresponding to the Toeplitz representation. Evaluation at ε_e shows that T satisfies (3.9), so amenability of (G, P) implies faithfulness of $M \times T$. By [14, Remark 3.6], the conditional expectation ϕ is given by a diagonal map, hence it is faithful on positive elements of the Toeplitz algebra, so if $C^*(G, P)$ is naturally isomorphic to $\mathcal{T}(G, P)$, then Φ must also be faithful on positive elements of $C^*(G, P)$, which is our definition of amenability. ∎

Corollary 3.9. *The quasi-lattice ordered group (G, P) is amenable if and only if whenever W and W' are two covariant representations of P*

satisfying (3.9), *the map* $W_x \mapsto W'_x$ *extends to an isomorphism of* $C^*(W_x\colon x \in P)$ *onto* $C^*(W'_x\colon x \in P)$.

Proof. If (G, P) is amenable and (3.9) holds, both W and W' give faithful representations of $B_P \rtimes_\alpha P$. Since $(\pi' \times W') \circ (\pi \times W)^{-1}$ maps W_x into W'_x, the algebras are canonically isomorphic. The converse is true because both the canonical embedding i_P and the Toeplitz representation T satisfy (3.9), thus their C^*-algebras are canonically isomorphic, so (G, P) is amenable by the preceding corollary. ▮

Remark 3.10. If G is totally ordered and W is any isometric representation of P, then $x \leqslant y$ implies $W_x W_x^* \geqslant W_y W_y^*$, and the covariance condition is automatically satisfied. Condition (3.9) can be simplified to involve only the smallest of the x_i, and it says that none of the isometries W_x is a unitary operator. If in addition G is abelian, (G, P) is amenable by Remark 3.6 and Corollary 3.9 reduces to [10, Theorem 2.9] and to [9, Theorem 1] when $G \subset \mathbb{R}$.

4. Amenability of Free Product Orders

The theory developed in the previous section is only as useful as the class of known amenable examples is large. From [14, Corollary 4.3.5] (see also Remark 3.6), (G, P) is amenable provided G is abelian. In this section we show how the existence of a certain abelian quotient of a quasi-lattice order can be used to prove that free product orders of abelian groups are amenable.

The *left regular* (or Toeplitz) representation of $C^*(G, P)$ is the representation $\lambda = M \times T$ on $\ell^2(P)$ where M is the representation of B_P as multiplication operators and $T_x(\varepsilon_z) = \varepsilon_{xz}$. Thus $\lambda(i_P(x)) = T_x$, and $\lambda(i_B(f)) = M_f$, the latter condition actually following from the former because $\lambda(i_B(1_x)) = \lambda(i_P(x)\, i_P(x)^*) = T_x T_x^* = M_{1_x}$.

The proof of the following lemma follows the argument in [7, Proposition 2.10], modified to accommodate nonabelian quasi-lattice orders.

Lemma 4.1. *Let* (G, P) *and* $(\mathcal{G}, \mathcal{P})$ *be quasi-lattice ordered groups and suppose there is an order preserving homomorphism* $\varphi\colon (G, P) \to (\mathcal{G}, \mathcal{P})$ *such that, whenever* $x, y \in P$ *have a common upper bound in* P,

$$\varphi(x) = \varphi(y) \qquad \textit{only if} \quad x = y, \qquad \textit{and} \tag{4.1}$$

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y). \tag{4.2}$$

Then $\overline{\operatorname{span}}\{i(x)\, i(y)^*\colon x, y \in P \textit{ with } \varphi(x) = \varphi(y) \in \mathcal{P}\}$ *is a* C^**-subalgebra of* $B_P \rtimes_\alpha P$ *on which the left regular representation is faithful.*

Proof. Let $\mathscr{F}$ denote the collection of all finite subsets $F \subset \mathscr{P}$ such that $x \vee y \in F$ whenever $x, y \in F$. For each $F \in \mathscr{F}$ let $\mathscr{K}_F = \overline{\operatorname{span}}\{i(x)\, i(y)^*: \varphi(x) = \varphi(y) \in F\}$, and write $\mathscr{K}_t$ instead of $\mathscr{K}_{\{t\}}$. Let $s, t \in \mathscr{P}$ and suppose $\varphi(u) = \varphi(v) = s$ and $\varphi(x) = \varphi(y) = t$. Since the inclusion i is covariant, $i(u)\, i(v)^*\, i(x)\, i(y)^* = i(uv^{-1}(v \vee x))\, i(yx^{-1}(v \vee x))^*$. If v and x have no common upper bound, the product is zero, while if $v \vee x \in P$, (4.2) implies

$$\varphi(uv^{-1}(v \vee x)) = \varphi(yx^{-1}(v \vee x)) = \varphi(v \vee x) = \varphi(v) \vee \varphi(x). \tag{4.3}$$

Thus the collection of products $i(x)\, i(y)^*$ with $\varphi(x) = \varphi(y)$ is closed under multiplication and $\overline{\operatorname{span}}\{i(x)\, i(y)^*: \varphi(x) = \varphi(y) \in \mathscr{P}\}$ is a C^*-subalgebra of $B_P \rtimes_\alpha P$.

The argument leading to (4.3) also implies that $\mathscr{K}_s \mathscr{K}_t \subset \mathscr{K}_{s \vee t}$ for $s, t \in \mathscr{P}$, and since every $F \in \mathscr{F}$ is closed under $\vee$, $\mathscr{K}_F$ is a C^*-subalgebra of $\overline{\operatorname{span}}\{i(x)\, i(y)^*: \varphi(x) = \varphi(y) \in \mathscr{P}\}$.

For any finite subset H of $\mathscr{P}$ the set $H^\vee = \{x \vee y \vee \cdots \vee z: x, y, ..., z \in H\}$ is finite, contains H and is closed under $\vee$, thus $\mathscr{F}$ is a directed set and $\{\mathscr{K}_F: F \in \mathscr{F}\}$ is an inductive system with limit $\overline{\bigcup_{F \in \mathscr{F}} \mathscr{K}_F} = \overline{\operatorname{span}}\{i(x)\, i(y)^*: \varphi(x) = \varphi(y)\}$. By [2, Lemma 1.3], to prove that λ is faithful on $\overline{\bigcup_{F \in \mathscr{F}} \mathscr{K}_F}$ it suffices to show that λ is faithful on $\mathscr{K}_F$ for each $F \in \mathscr{F}$.

For each $t \in \mathscr{P}$, let P_t denote the projection onto the subspace $H_t = \overline{\operatorname{span}}\{\varepsilon_z \in \ell^2(P) : \varphi(z) = t\}$, so that $\ell^2(P) = \bigoplus_t H_t$ and $\sum_t P_t = I$. If $\varphi(x) = \varphi(y)$ and $z \in P$, then

$$\begin{aligned}\lambda(i(x)\, i(y)^*)\, \varepsilon_z &= \lambda(i(x)\, i(y)^*\, i(z))\, \varepsilon_e \\ &= \lambda(i(x)\, i(y^{-1}(y \vee z))\, i(z^{-1}(y \vee z))^*)\, \varepsilon_e.\end{aligned}$$

Since $\lambda(i(z^{-1}(y \vee z)))^*\, \varepsilon_e = T^*_{z^{-1}(y \vee z)} \varepsilon_e = 0$ unless $y \vee z = z$, the left regular representation satisfies

$$\lambda(i(x)\, i(y)^*)\, \varepsilon_z = \begin{cases} \varepsilon_{xy^{-1}z} & \text{if } \; y \leqslant z, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

If $\varphi(x) = \varphi(y) \nleqslant \varphi(z)$ then $y \nleqslant z$, and by (4.4) $\lambda(i(x)\, i(y)^*)\, \varepsilon_z = 0$. Thus,

$$\lambda(\mathscr{K}_s)\, P_t = (0) \qquad \text{for} \quad s \nleqslant t. \tag{4.5}$$

If $\varphi(x) = \varphi(y) = \varphi(z) = s$, then either y and z have no common upper bound in P, in which case $\lambda(i(x)\, i(y)^*)\, \varepsilon_z = 0$ or, by (4.1), $y = z$, and (4.4) implies $\lambda(i(x)\, i(y)^*)\, \varepsilon_z = \varepsilon_x$. This shows that $\lambda(i(x)\, i(y)^*)$ is the rank-one operator $\langle \cdot, \varepsilon_y \rangle\, \varepsilon_x$ on H_s, so λ gives a correspondence between the system of matrix units $\{i(x)\, i(y)^*: \varphi(x) = \varphi(y) = s\}$ and the system of matrix units associated with the canonical orthonormal basis in H_s. As consequence λ

is an isomorphism between $\mathscr{K}_s$ and the closure of the finite rank operators on H_s. So λ is isometric on $\mathscr{K}_s$ for each $s \in \mathscr{P}$.

Note that $\varphi(P)$ may well be strictly contained in $\mathscr{P}$, so for some $s \in \mathscr{P}$ there may be no $x \in P$ with $\varphi(x) = s$, in which case $\mathscr{K}_s$ is the zero C^*-algebra and H_s the zero Hilbert space.

To show that λ is isometric on $\mathscr{K}_F$ for each $F \in \mathscr{F}$, suppose $T \in \mathscr{K}_F$, and write

$$T = \lim_n \sum_{s \in F} T_{n,s} \qquad \text{where} \quad T_{n,s} \in \mathscr{K}_s \quad \text{for } s \in F.$$

Since F is finite, it has a minimal element, i.e., an element s_0 such that $s \not\leq s_0$ for every $s \in F \setminus \{s_0\}$. By (4.5), $\lambda(T_{n,s}) P_{s_0} = 0$ for $s \in F \setminus \{s_0\}$, and hence

$$\lambda\left(\sum_{s \in F} T_{n,s}\right) P_{s_0} = \lambda(T_{n,s_0}) P_{s_0}.$$

If $\lambda(T) = 0$ then

$$\lambda\left(\sum_{s \in F} T_{n,s}\right) P_{s_0} = \lambda(T_{n,s_0}) P_{s_0} \to 0$$

and since $\|\lambda(T_{n,s_0}) P_{s_0}\| = \|T_{n,s_0}\|$, it follows that

$$T = \lim_n \sum_{s \in F \setminus \{s_0\}} T_{n,s}.$$

Now observe that $F \setminus \{s_0\}$ is still in $\mathscr{F}$ because $s_0 \neq x \vee y$ for $x, y \in F \setminus \{s_0\}$. Thus the same argument can be repeated, at most $|F|$ times, to conclude that $T = 0$. ∎

Proposition 4.2. *Suppose there is an order preserving homomorphism $\varphi\colon (G, P) \to (\mathscr{G}, \mathscr{P})$ such that, whenever $x, y \in P$ have a common upper bound in P,*

(i) *$\varphi(x) = \varphi(y)$ only if $x = y$, and*

(ii) *$\varphi(x) \vee \varphi(y) = \varphi(x \vee y)$.*

If $\mathscr{G}$ is abelian, then (G, P) is amenable.

Proof. As in Remark 3.6, the homomorphism φ induces an action θ of the dual group $\hat{\mathscr{G}}$ on $C^*(G, P)$ such that $\theta_\gamma(i(y)) = \gamma(\varphi(x))\, i(y)$ for $\gamma \in \hat{\mathscr{G}}$, with an associated positive faithful conditional expectation $\Phi_\theta\colon C^*(G, P) \to C^*(G, P)^\theta$ obtained by taking averages over the compact $\hat{\mathscr{G}}$-orbits and characterized by $i(x)\, i(y)^* \mapsto \delta_{\varphi(x), \varphi(y)}\, i(x)\, i(y)^*$. The range of

Φ_θ is the fixed point algebra $C^*(G, P)^\theta$ and coincides with the C^*-algebra $\overline{\operatorname{span}}\{i(x)\,i(y)^*\colon \varphi(x)=\varphi(y)\}$ of Lemma 4.1. By looking at the elements $i(x)\,i(y)^*$ we can see that $\Phi\colon B_P \rtimes_\alpha P \to B_P$ factors through Φ_θ, i.e., $\Phi = \Phi \circ \Phi_\theta$; thus it suffices to prove the faithfulness of Φ on the positive elements of $\overline{\operatorname{span}}\{i(x)\,i(y)^*\colon \varphi(x)=\varphi(y)\}$, which is the range of Φ_θ.

Let E_z denote the rank one projection onto $\varepsilon_z \in \ell^2(P)$; the diagonal map Δ defined by $\Delta(T) = \sum_{z\in P} E_z T E_z$ for $T \in B(\ell^2(P))$ is completely positive and faithful on positive elements. If $x, y \in P$ satisfy $\varphi(x) = \varphi(y)$, then

$$\begin{aligned}\Delta \circ \lambda(i(x)\,i(y)^*) &= \sum_{z\in P} \langle T_x T_y^* \varepsilon_z, \varepsilon_z\rangle\, E_z = \sum_{z\in P\,:\,y\leqslant z} \langle \varepsilon_{xy^{-1}z}, \varepsilon_z\rangle\, E_z\\ &= \delta_{x,y} \sum_{z\in P\,:\,y\leqslant z} E_z = \delta_{x,y} T_y T_y^* = \delta_{x,y} \lambda(i(y)\,i(y)^*)\\ &= \lambda \circ \Phi(i(x)\,i(y)^*).\end{aligned}$$

Since both Δ and Φ are bounded linear maps, it follows that $\Delta \circ \lambda = \lambda \circ \Phi$ on $\overline{\operatorname{span}}\{i(x)\,i(y)^*\colon \varphi(x)=\varphi(y)\}$. Since λ is injective on $(B_P \rtimes P)^\theta$ by Lemma 4.1, it follows that Φ is faithful on the positive elements of $(B_P \rtimes P)^\theta$. ∎

Whenever $\{(G_i, P_i)\colon i \in I\}$ is a collection of quasi-lattice ordered groups, their free product $G = \prod_i^* G_i$ has a natural quasi-lattice order with positive semigroup $P = \prod_i^* P_i$ generated by the images in G of the positive semigroups P_i. There is a natural homomorphism from the free product of a family onto its direct product, $\prod_i G_i$, with positive semigroup $\prod_i P_i$, obtained by decreeing that the G_i's commute. The following proposition shows that this homomorphism satisfies the hypotheses of Lemma 4.1.

Proposition 4.3. *Let (G, P) be the free product of the family $\{(G_i, P_i)\colon i \in I\}$ of quasi-lattice ordered groups, and let $(\mathcal{G}, \mathcal{P})$ be its direct product. If φ denotes the natural homomorphism of G onto $\mathcal{G}$, then $\varphi(P) \subseteq \mathcal{P}$, and whenever $x, y \in P$ have a common upper bound in P,*

$$\varphi(x) = \varphi(y) \qquad \textit{implies} \quad x = y, \qquad \textit{and} \tag{4.6}$$

$$\varphi(x) \vee \varphi(y) = \varphi(x \vee y). \tag{4.7}$$

Proof. The homomorphism φ sends each $x \in G_i$ in the free product to the same x in the direct product. This guarantees $\varphi(P) = \prod_i P_i$, so φ is an order preserving homomorphism onto $(\mathcal{G}, \mathcal{P})$.

If $z = x \vee y \in P$ then both x and y are initial segments of z. Suppose z is given in reduced form $z = z_{i_1} z_{i_2} \cdots z_{i_n}$, with $z_{i_k} \in P_{i_k}$ and $i_k \neq i_{k+1}$. There is a smallest index j, $1 \leqslant j \leqslant n$ such that $z_{i_1} z_{i_2} \cdots z_{i_j} \geqslant x$ in P. Then $x = z_{i_1} z_{i_2} \cdots z_{i_{j-1}} a$ where $0 < a \leqslant z_{i_j}$ in G_{i_j}. Similarly $y = z_{i_1} z_{i_2} \cdots z_{i_{k-1}} b$ where

$0 < b \leqslant z_{i_k}$ in G_{i_k}. Since $z_{i_1} z_{i_2} \cdots z_{i_{\max(j,k)}}$ is an upper bound for both x and y, $\max(j, k) = n$.

First suppose $j < n$. Then $a \leqslant z_{i_j} < z_{i_j} \cdots z_{i_{n-1}} b$, hence $x < y$. Thus in this case $x \vee y = y$, $\varphi(x) < \varphi(y)$, and (4.6) and (4.7) are trivially true. The case $k < n$ is similar.

Next suppose $j = k = n$. Then $x = z_{i_1} z_{i_2} \cdots z_{i_{n-1}} a$ and $y = z_{i_1} z_{i_2} \cdots z_{i_{n-1}} b$ for some $a, b \in P_{i_n}$ satisfying $a \vee b = z_{i_n}$. If $\varphi(x) = \varphi(y)$ then $\varphi(a) = \varphi(b)$; since both a and b are in P_{i_n}, $a = b$ and therefore $x = y$, proving (4.6). To prove (4.7) observe that $\varphi(a) \vee \varphi(b) = \varphi(z_{i_n})$ because a, b, z_{i_n} are in P_{i_n}, hence

$$\begin{aligned}\varphi(x \vee y) &= \varphi(z_{i_1} z_{i_2} \cdots z_{i_{n-1}})\, \varphi(z_{i_n}) = \varphi(z_{i_1} z_{i_2} \cdots z_{i_{n-1}})(\varphi(a) \vee \varphi(b)) \\ &= \varphi(z_{i_1} z_{i_2} \cdots z_{i_{n-1}})\, \varphi(a) \vee \varphi(z_{i_1} z_{i_2} \cdots z_{i_{n-1}})\, \varphi(b) \\ &= \varphi(x) \vee \varphi(y). \quad \blacksquare\end{aligned}$$

THEOREM 4.4. *The free product of a family $\{(G_i, P_i) : i \in I\}$ of abelian quasi-lattice ordered groups is amenable.*

Proof. By Proposition 4.3 the natural homomorphism onto the direct product satisfies (4.6) and (4.7). Since the direct product of the G_i's is abelian, the free product order is amenable by Proposition 4.2. ▮

5. EXAMPLES

Suppose $n \geqslant 2$ is an integer. Let $\mathbb{F}_n$ denote the free group on n generators $\{a_1, a_2, ..., a_n\}$, and let $\mathbb{F}_n^+$ be the unital subsemigroup generated by the nonnegative powers of the a_i's; the resulting partial order is called the free product order. Notice that Theorem 4.4 implies that $(\mathbb{F}_n, \mathbb{F}_n^+)$ is amenable.

As observed by Nica [14, Lemma 5.1.2], covariance of a representation means that the isometries corresponding to elements without a common upper bound have orthogonal ranges, and it suffices to require this for the generators. Thus n-tuples $(W_1, W_2, ..., W_n)$ of isometries with orthogonal ranges are in one to one correspondence with isometric covariant representations W of $\mathbb{F}_n^+$, via $W_{a_i} = W_i$; thus $C^*(\mathbb{F}_n, \mathbb{F}_n^+)$ is the universal C^*-algebra for n isometries with orthogonal ranges, i.e., the Toeplitz–Cuntz algebra $\mathcal{T}\mathcal{O}_n$. Clearly $\pi_W \times W(C^*(\mathbb{F}_n, \mathbb{F}_n^+)) = C^*(\{W_i : 1 \leqslant i \leqslant n\})$, which was shown in [6] to be independent of the choice of isometries provided that the range projections $W_i W_i^*$ add up to strictly less than 1. Instead of following Nica and deriving amenability of $(\mathbb{F}_n, \mathbb{F}_n^+)$ from this uniqueness of $\mathcal{T}\mathcal{O}_n$, we can use the amenability of $(\mathbb{F}_n, \mathbb{F}_n^+)$ to establish the uniqueness of $\mathcal{T}\mathcal{O}_n$.

COROLLARY 5.1 [6, Lemma 3.1]. *Suppose $W_1, W_2, ..., W_n$ are n isometries with orthogonal ranges. Then the corresponding representation of $\mathcal{T}\mathcal{O}_n$ is faithful if and only if $\sum_{i=1}^n W_i W_i^* < 1$.*

Proof. By Theorem 4.4, $(\mathbb{F}_n, \mathbb{F}_n^+)$ is amenable, so Theorem 3.7 applies. Thus $\pi_W \times W$ is faithful if and only if $\prod_{x \in F}(1 - W_x W_x^*) \neq 0$ for every finite subset F of $P \backslash \{e\}$. Each x in a given finite set is bounded below by exactly one a_i, namely, the first letter in the reduced word x, and $a_i \leqslant x$ implies $1 - W_{a_i} W_{a_i}^* \leqslant 1 - W_x W_x^*$. Thus for any F, we have

$$\prod_{i=1}^{n} (1 - W_{a_i} W_{a_i}^*) \leqslant \prod_{x \in F} (1 - W_x W_x^*).$$

Since a_i and a_j have no common upper bound unless $i = j$, the projections $W_{a_i} W_{a_i}^*$ are mutuallly orthogonal and

$$\prod_{i=1}^{n} (1 - W_{a_i} W_{a_i}^*) = 1 - \sum_{i=1}^{n} W_{a_i} W_{a_i}^*.$$

Thus $\prod_{x \in F}(1 - W_x W_x^*) \neq 0$ for every finite subset F of $P \backslash \{e\}$ if and only if Cuntz's condition $\sum_{i=1}^n W_{a_i} W_{a_i}^* < 1$ is satisfied. ∎

As in the finite case, there is a one to one correspondence between infinite sequences of isometries $\{W_i\}_{i=1}^{\infty}$ with orthogonal ranges and isometric covariant representations of $\mathbb{F}_\infty^+$. Hence $C^*(\mathbb{F}_\infty, \mathbb{F}_\infty^+)$ is the universal C^*-algebra for such sequences of isometries, namely, the Cuntz algebra $\mathcal{O}_\infty$. In contrast to the finite case, however, condition (3.9) is automatically satisfied by any covariant representation of $\mathbb{F}_\infty$.

COROLLARY 5.2 [5, Theorem 1.12]. *Suppose $\{W_i : i = 1, 2, ...\}$ are isometries with orthogonal ranges. Then the corresponding representation of $\mathcal{O}_\infty$ is faithful, and hence $\mathcal{O}_\infty$ is simple.*

Proof. As in the case of finitely many isometries, $(\mathbb{F}_\infty, \mathbb{F}_\infty^+)$ is amenable by Theorem 4.4 and $W_{a_i} = W_i$ defines a covariant isometric representation W of $\mathbb{F}_\infty^+$. By Theorem 3.7 $\pi_W \times W$ is faithful if and only if $\prod_{x \in F}(1 - W_x W_x^*) \neq 0$ for every finite subset F of $\mathbb{F}_\infty^+ \backslash \{e\}$.

If $\prod_{x \in F}(1 - W_x W_x^*) = 0$ for some finite $F \subset \mathbb{F}_\infty^+ \backslash \{e\}$, then $\prod_{i=1}^n (1 - W_{a_i} W_{a_i}^*) = 0$ for some n finite but large enough to include all the first letters of the words in F. Expanding the product as before gives $\sum_{i=1}^n W_{a_i} W_{a_i}^* = 1$, which contradicts the mutual orthogonality of the projections $W_{a_i} W_{a_i}^*$ for $i = 1, 2, ..., \infty$. Thus $\prod_{x \in F}(1 - W_x W_x^*) \neq 0$ for all finite subsets $F \subset \mathbb{F}_\infty^+ \backslash \{e\}$ and $\pi_W \times W$ is faithful. ∎

If Γ_i is a countable dense subgroup of $\mathbb{R}$, with $\Gamma_i^+ = \Gamma_i \cap \mathbb{R}^+$, the free product $(G, P) = (\prod_i^* \Gamma_i, \prod_i^* \Gamma_i^+)$ is amenable by Theorem 4.4. An isometric representation of P is covariant if and only if the Γ_i^+ are sent into mutually orthogonal semigroups of isometries. Thus $C^*(\prod_i^* \Gamma_i, \prod_i^* \Gamma_i^+)$ is the universal C^*-algebra generated by orthogonal representations of the Γ_i^+ by isometries. The case $\Gamma_i = \Gamma$ for all i for a fixed Γ was first studied by H. Dinh in [8] in connection with discrete product systems, as a generalisation of the Cuntz algebras; the following corollary extends Dinh's simplicity result [8, Theorem 1.1] to the C^*-algebra generated by *different* subsemigroups of $\mathbb{R}$.

COROLLARY 5.3. *Let $\{(\Gamma_i, \Gamma_i^+) : i \in \mathcal{I}\}$ be a family consisting of two or more countable ordered subgroups of $\mathbb{R}$, at least one of which is dense. Then the C^*-algebra $C^*(\prod_i^* \Gamma_i, \prod_i^* \Gamma_i^+)$ of their free product is simple.*

Proof. Since $(\prod_i^* \Gamma_i, \prod_i^* \Gamma_i^+)$ is amenable by Theorem 4.4, the proof reduces to showing that (3.9) is automatically satisfied, so that every representation is faithful.

If F is a finite subset of $P \backslash \{e\}$, then each reduced word $x \in F$ has its first letter $y(x)$ in some $\Gamma_i^+ \backslash \{e\}$ viewed as a subgroup of the free product, namely, there is a largest element $y(x) \in \Gamma_i^+ \subset \prod_i^* \Gamma_i^+$ such that $e < y(x) \leqslant x$. Since each Γ_i is totally ordered and F is finite, we can let $y_{\min}(x)$ be the smallest element of the set $\{y(z) : z \in F$, such that $y(z)$ and $y(x)$ are in the same $\Gamma_i\}$. Thus $H = \{y_{\min}(x) : x \in F\}$ consists of nonzero elements, at most one from each Γ_i^+. Since for every $x \in F$ there is a smaller $y_{\min}(x)$ in H, we have that

$$\prod_{y \in H} (1 - W_y W_y^*) \leqslant \prod_{x \in F} (1 - W_x W_x^*)$$

for any covariant isometric representation W of P. Thus, if $\prod_{x \in F} (1 - W_x W_x^*) = 0$, then $\prod_{y \in H} (1 - W_y W_y^*) = 0$. Since H intersects each Γ_i^+ at most in one point, the projections $W_y W_y^*$ with $y \in H$ are mutually orthogonal, and expanding the product gives

$$\sum_{y \in H} W_y W_y^* = 1. \tag{5.1}$$

It follows that H contains exactly one nonzero element $y = y_i$ from each Γ_i^+, otherwise (5.1) would leave no room for the semigroup of isometries corresponding to the missing Γ_i^+. It also follows that $W_{y_i} W_{y_i}^* W_z W_z^* = W_z W_z^*$ for every $z \in \Gamma_i^+ \backslash \{0\}$, which implies $y_i \leqslant y_i \vee z = z$ because W_x is not a unitary for any $x \neq 0$. This contradicts the assumption that at least one of the Γ_i is divisible, so no such finite F exists, and every representation is faithful by Theorem 3.7. ∎

6. Free Products of Amenable Groups

We begin by showing that every semigroup crossed product $A \rtimes_\alpha P$ carries a coaction of the full group G. This in turn gives rise to a conditional expectation onto a fixed point algebra, which in the case of $C^*(G, P) = B_P \rtimes_\alpha P$ coincides with the expectation of Section 3.

Proposition 6.1. *Suppose (A, P, α) is a semigroup dynamical system. There is an injective coaction*

$$\delta\colon A \underset{\alpha}{\rtimes} P \to (A \underset{\alpha}{\rtimes} P) \otimes_{\max} C^*(G)$$

such that

$$\delta \circ i_A(a) = i_A(a) \otimes 1 \qquad \textit{for} \quad a \in A \tag{6.1}$$

$$\delta \circ i_P(x) = i_P(x) \otimes i_G(x) \qquad \textit{for} \quad x \in P, \tag{6.2}$$

where i_G denotes the canonical injection of G into $C^*(G)$ as a generating set of unitary elements.

Proof. By representing $(A \rtimes_\alpha P) \otimes_{\max} C^*(G)$ on a Hilbert space $\mathscr{H}$, the pair $(i_A \otimes 1, i_P \otimes i_G)$ becomes a covariant representation of (A, P, α). Thus there is a nondegenerate representation $\delta = (i_A \otimes 1) \times (i_P \otimes i_G)$ of $A \rtimes_\alpha P$ into $B(\mathscr{H})$, which maps the generators $i_A(a)$ and $i_P(x)$ into $(A \rtimes_\alpha P) \otimes C^*(G)$, and hence takes values in $(A \rtimes_\alpha P) \otimes C^*(G)$. To see that δ is injective, it suffices to show that for any covariant representation (π, V) of (A, P, α) on $\mathscr{H}$, the corresponding representation $\pi \times V$ of $A \rtimes_\alpha P$ factors through δ.

Consider the trivial representation $\varepsilon\colon C^*(G) \to B(\mathscr{H})$ such that $\varepsilon(i_G(x)) = 1$ for all $x \in G$. The image of ε commutes with that of $\pi \times V$, so there is a representation $(\pi \times V) \otimes \varepsilon$ of $(A \rtimes_\alpha P) \otimes_{\max} C^*(G)$ on $\mathscr{H}$ such that

$$((\pi \times V) \otimes \varepsilon) \circ \delta \circ i_A(a) = (\pi \times V) \otimes \varepsilon(i_A(a) \otimes 1) = \pi(A), \quad \text{and}$$

$$((\pi \times V) \otimes \varepsilon) \circ \delta \circ i_P(x) = (\pi \times V) \otimes \varepsilon(i_P(x) \otimes i_G(x)) = V_x.$$

By definition of $\pi \times V$ it follows that $((\pi \times V) \otimes \varepsilon) \circ \delta = \pi \times V$.

To see that δ is coassociative, we compute:

$$(\delta \otimes \mathrm{id}) \circ \delta(i_A(a)) = i_A(a) \otimes 1 \otimes 1 = \mathrm{id} \otimes \delta_G(\delta(i_A(a))), \quad \text{and}$$

$$(\delta \otimes \mathrm{id}) \circ \delta(i_P(x)) = (i_P(x) \otimes i_G(x)) \otimes i_G(x) = \mathrm{id} \otimes \delta_G(\delta(i_P(x))).$$

Since both $(\delta \otimes \mathrm{id}) \circ \delta$ and $(\mathrm{id} \otimes \delta_G) \circ \delta$ are (automatically continuous) homomorphisms agreeing on generators, they must agree on all of $A \rtimes_\alpha P$. ∎

Remark 6.2. Our proof of injectivity of δ only works if we use the full group algebra $C^*(G)$ rather than the reduced one $C_r^*(G)$. It does not matter which C^*-tensor product we use: we could equally well consider the representation $(\pi \times V) \otimes_{\min} \varepsilon$ on $\mathscr{H} \otimes \mathbb{C} \cong \mathscr{H}$. (See [19] and [18] for discussions of these options.)

DEFINITION 6.3. Let $\delta: B \to B \otimes C^*(G)$ be a coaction of a discrete group G on a C^*-algebra B. The *fixed-point algebra* is $B^\delta := \{b \in B : \delta(b) = b \otimes 1\}$.

For any discrete group, the functional $f \mapsto f(e)$ on $\ell^1(G)$ extends to a state τ (in fact a trace) on $C^*(G)$. It is shown in [13, 2.3] or [17, Lemma 1.3] that $\Psi_\delta := (\mathrm{id} \otimes \tau) \circ \delta$ is an expectation of B onto the fixed-point algebra B^δ. When $B = B_P \rtimes_\alpha P$, and δ is the coaction of 6.1, we have

$$\Psi_\delta(i_P(x)\, i_P(y)^*) = \begin{cases} i_P(x)\, i_P(x)^* & \text{if } \ x = y \\ 0 & \text{otherwise,} \end{cases} \tag{6.3}$$

and hence Ψ_δ is just the expectation Φ considered in Section 3. Thus the system (B_P, P, α) is amenable in the following sense precisely when (G, P) is amenable.

DEFINITION 6.4. The system (A, P, α) is *amenable* if $\Phi_\delta: (A \rtimes_\alpha P) \to (A \rtimes_\alpha P)^\delta$ is faithful, in the sense that $\Phi_\delta(b^*b) = 0$ implies $b = 0$ for $b \in (A \rtimes_\alpha P)$.

The next lemma generalises [14, Propositions 4.4.1 and 4.4.2], and has an easier proof.

LEMMA 6.5. *If G is an amenable group then (A, P, α) is amenable.*

Proof. Suppose that $b \in A \rtimes_\alpha P$ satisfies $\Psi_\delta(b^*b) = 0$, and let f be an arbitrary state on $A \rtimes_\alpha P$. Then

$$0 = f(\Psi_\delta(b^*b)) = f \otimes \tau(\delta(b^*b)) = \tau(f \otimes \mathrm{id}(\delta(b^*b))).$$

Because G is amenable, the trace τ is faithful on $C^*(G) = C_r^*(G)$, and hence $f \otimes \mathrm{id}(\delta(b^*b)) = 0$. This in turn implies that $f \otimes g(\delta(b^*b)) = 0$ for all states f on $A \rtimes_\alpha P$ and g on $C^*(G)$, and hence that $\delta(b^*b) = 0$ in $(A \rtimes_\alpha P) \otimes_{\min} C^*(G)$. Thus it follows from the injectivity of δ that $b = 0$. (Either note that G amenable implies $C^*(G)$ nuclear and $(A \rtimes_\alpha P) \otimes_{\min} C^*(G) = (A \rtimes_\alpha P) \otimes_{\max} C^*(G)$, or use Remark 6.2 to see that δ is injective into $(A \rtimes_\alpha P) \otimes_{\min} C^*(G)$.) ∎

We can now follow the argument of Proposition 4.2, to obtain:

PROPOSITION 6.6. *Suppose there is an order homomorphism φ of one quasi-lattice ordered group (G, P) into another $(\mathcal{G}, \mathcal{P})$ such that, whenever $x, y \in P$ have a common upper bound in P,*

(i) $\varphi(x) = \varphi(y)$ *only if* $x = y$, *and*

(ii) $\varphi(x) \vee \varphi(y) = \varphi(x \vee y)$.

If $(\mathcal{G}, \mathcal{P})$ is amenable, then (G, P) is amenable.

THEOREM 6.7. *The free product of a family of quasi-lattice orders in which the underlying groups are amenable is amenable.*

Proof. Since the direct sum of amenable groups is amenable, the result follows from Proposition 6.6 and Proposition 4.3. ▮

Remark 6.8. It is natural to ask at this stage whether the free product of amenable quasi-lattice ordered groups is amenable. By the proof of the preceding theorem, it would suffice to prove amenability of their direct sum as a quasi-lattice order, but we have been unable to do this in general. However, Nica's results involving the approximation property for positive definite functions, Definition 2 [14, Sect. 4.5], allow us to conclude that if two quasi-lattice orders (G, P) and (H, P) have the approximation property then their free product is amenable. Indeed, if $\theta: G \to \mathbb{C}$ and $\phi: H \to \mathbb{C}$ are positive definite, then the function $\theta \otimes \phi: (x, y) \mapsto \theta(x)\, \phi(y)$ is positive definite on $G \times H$, because the Schur product of two positive definite matrices is positive definite. This implies that $(G \times H,\ P \times Q)$ has the approximation property too, and, by Proposition 2 of [14, Sect. 4.5], that it is amenable.

Remark 6.9. Classical results about group actions suggest there may be a relation between amenability of the pair (G, P) and nuclearity of the associated full and reduced Toeplitz C^*-algebras, but we have not been able to establish such a connection. For instance we do not know whether amenability of (G, P) implies nuclearity of $C^*(G, P)$.

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