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# Classes of Modules with the Exchange Property

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## 1. INTRODUCTION

In searching for isomorphic refinements of direct-sum decompositions of modules, Crawley and Jónsson introduced an exchange property for modules that is strongly reminiscent of Steinitz' Exchange Lemma for vector spaces. An $R$-module $M$ (where $R$ is an associative ring with identity) is said to have the (finite) exchange property if $M$ can be fitted into any (finite) sum grid as follows: Whenever $M$ occurs as a direct summand of a (finite) direct sum $A = \bigoplus_{i \in I} A_i$, then $A = M \oplus \bigoplus_{i \in I} C_i$ for suitable submodules $C_i$ of the $A_i$ [2]. This concept led to very general extensions of the classical refinement theorems (see, e.g., [2, 3, 17]).

Many of the problems suggested by Crawley and Jónsson in their pioneering paper are still open. The most salient one: Does the finite exchange property imply the unrestricted exchange property? We show that the answer is yes for modules which possess decompositions into indecomposable summands. In the background of this observation stands the following:

THEOREM. *Let $(M_j)_{j \in J}$ be a semi-T-nilpotent family of modules (i.e., for each sequence $M_{j_1} \to^{f_1} M_{j_2} \to^{f_2} M_{j_3} \to^{f_3} \cdots$ of nonisomorphisms, where the indices $j_n \in J$ are pairwise different, and for each $x \in M_{j_1}$ there exists a $p$ such that $f_p \circ \cdots \circ f_2 \circ f_1(x) = 0$). Moreover, suppose that each $M_j$ has the exchange property. Then $\bigoplus_{j \in J} M_j$ has the exchange property in either of the following two cases*:



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(I) *$M_j$ and $M_k$ have no nontrivial isomorphic direct summand for $j \neq k$.*

(II) *$M_j \cong M_k$ for all $j, k \in J$.*

In particular, the theorem provides the missing link for a complete description of those modules with decompositions into indecomposable summands which enjoy the exchange property (Corollary 5). Only a few special cases had been completely covered by the predecessors of this result (see [8–11, 21–23]). As a consequence, we rediscover the fact that each projective right $R$-module over a right perfect ring has the exchange property (see [9, 22]). Furthermore, the theorem is applicable to certain direct sums of modules which cannot be refined to decompositions into indecomposable summands (see Corollary 8).

Barring a few interlopers, the following are all major classes of modules with the exchange property which were known (of course, finite sums of any of the candidates below again have the exchange property by [2, p. 812]):

(1) The modules with local endomorphism rings [17] and certain infinite direct sums of these (cf. Corollary 5).

(2) Each finitely generated module, whose endomorphism ring has liftable idempotents and is von Neumann regular modulo its radical [20].

(3) The quasi-injective modules [4, 18].

(4) The $\mathbb{Z}$-adically complete (= algebraically compact and reduced) abelian groups [19, Theorem 3].

(5) The torsion-complete primary abelian groups [2, p. 847]. A $p$-group is called torsion-complete if it is the $p$-torsion subgroup of some $p$-adically complete group.

In Theorem 10 we establish a class of modules having the exchange property which includes the modules under (3), (4), (5). Namely: Each strongly invariant submodule of an arbitrary algebraically compact module has the exchange property. (We call a submodule $M$ of $X$ strongly invariant if $f(M) \subset M$ for all $f \in \mathrm{Hom}_R(M, X)$.) In addition, this class contains the linearly compact modules over commutative rings. Thus, in the commutative case, we answer in the positive the question of Crawley and Jónsson whether each artinian module has the exchange property [2, p. 855].

Throughout, the unadorned term "module" stands for "right $R$-module."

Recall that an $R$-module $M$ is called algebraically compact if each system of linear equations $\sum_{i \in I} X_i a_{ij} = m_j$ $(j \in J)$ with a column-finite $R$-matrix $(a_{ij})$ and $m_j \in M$, such that each finite subsystem is solvable, has a global solution. Moreover, $M$ is called pure-injective provided that homomorphisms $A \to M$ can be extended to modules $B$ containing $A$ as a pure submodule. By [16, Theorem 2], algebraic compactness is the same as pure-injectivity.

*Prerequisites.* The following elementary devices will be used repeatedly:

LEMMA 1 (SEE [1, PROPOSITION 5.5]). *Suppose that $M$ has a decomposition $M = U \oplus V$ with corresponding projection $p: M \to V$. Furthermore, let $W$ be an arbitrary submodule of $M$. Then $M = U \oplus W$ precisely if the restriction of $p$ to $W$ is an isomorphism $W \to V$.*

LEMMA 2 [2, p. 812]. *If $M$ has the exchange property and*

$$A = M \oplus B \oplus E = \bigoplus_{i \in I} A_i \oplus E,$$

*then there exist submodules $C_i$ of $A_i$ such that*

$$A = M \oplus \bigoplus_{i \in I} C_i \oplus E.$$

## 2. THE EXCHANGE PROPERTY OF $M$ CAN BE TESTED IN DIRECT SUMS OF COPIES OF $M$

Pursuing work of Warfield [20] and Monk [13], Nicholson showed in [14, Theorem 2.1] that a module $M$ with endomorphism ring $S$ has the *finite* exchange property if and only if, given any endomorphism $f$ of $M$, there exists an idempotent $e \in Sf$ with $(1-e) \in S(1-f)$; this was observed independently by Goodearl (cf. [6, p. 617]). The characterization of the unrestricted exchange property via an analogous "approximability of endomorphisms by idempotents," which we will give below, was inspired by this result.

Given two $R$-modules $U$ and $V$, call a family $(f_i)_{i \in I}$ of homomorphisms $U \to V$ summable if, for each $u \in U$, we have $f_i(u) = 0$ for all but a finite number of $i \in I$. Write $\sum_{i \in I} f_i$ for the obvious "sum" in that case. (Clearly, the summability thus defined just amounts to convergence of the series $\sum_{i \in I} f_i$ in the finite topology of $\mathrm{Hom}_R(U, V)$.)

Following Crawley and Jónsson, we say that $M$ has the $\aleph$-exchange property ($\aleph$ being a cardinal number) if $M$ can be exchanged in direct sums with at most $\aleph$ summands.

PROPOSITION 3. *For a module $M$ with endomorphism ring $S$ and for any cardinal number $\aleph$ the following statements are equivalent:*

(1) *$M$ has the $\aleph$-exchange property.*

(2) *Whenever M occurs as a direct summand of a direct sum of at most* $\aleph$ *copies of* $M$, *say,*

$$M \oplus B = \bigoplus_{i \in I} A_i \qquad \text{with} \quad A_i \cong M \quad \text{for all } i \in I$$

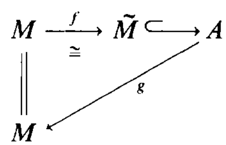
*and* $|I| \leqslant \aleph$, *then there exist submodules* $C_i \subset A_i$ *such that*

$$M \oplus \bigoplus_{i \in I} C_i = \bigoplus_{i \in I} A_i.$$

(3) *For each summable family* $(f_i)_{i \in I}$ *in* $S$ *with* $\sum_{i \in I} f_i = 1$ *and* $|I| \leqslant \aleph$ *there are pairwise orthogonal idempotents* $e_i \in Sf_i$ *such that* $\sum_{i \in I} e_i = 1$.

*Proof.* (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Suppose (2) is satisfied and let $(f_i)_{i \in I}$ be a summable family of endomorphisms of $M$ such that $\sum_{i \in I} f_i = 1$ and $|I| \leqslant \aleph$. If we define $A = \bigoplus_{i \in I} A_i$ with $A_i = M$ for all $i$ and $\tilde{M} = \{(f_i(m))_{i \in I}: m \in M\}$, then $\tilde{M} \cong M$ is a direct summand of $A$. This follows from the commutativity of the diagram




where $f(m) = (f_i(m))_{i \in I}$ and $g((m_i)_{i \in I}) = \sum_{i \in I} m_i$. Hence, the $A_i$ can be decomposed by hypothesis, say, $A_i = B_i \oplus C_i$, such that $A = \tilde{M} \oplus \bigoplus_{i \in I} C_i = \bigoplus_{i \in I} B_i \oplus \bigoplus_{i \in I} C_i$. By Lemma 1, this equality means that the projection $p: A \to \bigoplus_{i \in I} B_i$ along $\bigoplus_{i \in I} C_i$ induces an isomorphism $\tau = p|_{\tilde{M}}: \tilde{M} \to \bigoplus_{i \in I} B_i$. By $\tau_j$ denote the composite map $p_j \circ \tau: \tilde{M} \to \bigoplus_{i \in I} B_i \to B_j$, where $p_j$ is the canonical projection.

We will show that the definition $e_i := g\tau^{-1}\tau_i f \in S$ meets our wishes. First, note that $e_i e_j = g\tau^{-1}p_i\tau_j f \in S$ is equal to $e_i$ if $i = j$ and to zero otherwise. Next, observe that, denoting by $\rho_i$ the projection $A_i \to B_i$ along $C_i$, we have $\tau_i f = \rho_i f_i$ and consequently $e_i \in Sf_i$. In particular, the family $(e_i)_{i \in I}$ is again summable; the equality $\sum_{i \in I} e_i = 1$ follows immediately from our construction.

(3) $\Rightarrow$ (1): Start with a situation

$$A = M \oplus B = \bigoplus_{i \in I} A_i \qquad \text{with} \quad |I| \leqslant \aleph$$

and denote by $\pi_i: A \to A_i$ and $p: A \to M$ the corresponding projections. The family $(f_i)_{i \in I}$ with $f_i = p\pi_i|_M$ is then clearly summable and adds up to the identity. By (3) we can therefore find orthogonal idempotents $e_i = s_i f_i \in Sf_i$ with $\sum_{i \in I} e_i = 1$.

We claim that, setting $\varphi_i = e_i s_i p \pi_i: A \to M$, we obtain

$$A = M \oplus \bigoplus_{i \in I} (A_i \cap \operatorname{Ker}(\varphi_i))$$

which shows that $M$ can be fitted into $\bigoplus_{i \in I} A_i$ as desired. First note that the family $(\varphi_i)_{i \in I}$ is summable and write $\varphi$ for its sum. Next, observe that $\varphi_i|_M = e_i$ and infer that $\varphi_i \varphi_j = \delta_{ij} \varphi_i$ and $\varphi|_M = 1_M$. Since this implies $\varphi^2 = \varphi$, all that is left to be done is to check that $\operatorname{Ker}(\varphi)$ equals $\bigoplus_{i \in I}(A_i \cap \operatorname{Ker}(\varphi_i))$. ∎

Nicholson's result is retrieved as a special case of (1) ⇔ (3):

COROLLARY 4 [14, 2.1 AND 1.11]. *For a module $M$ with endomorphism ring $S$, the following conditions are equivalent*:

(1) *$M$ has the finite exchange property.*

(2) *For each finite number $f_1, ..., f_n$ of elements of $S$ with $\sum_{i=1}^n f_i = 1$, there are orthogonal idempotents $e_i \in Sf_i$ such that $\sum_{i=1}^n e_i = 1$.*

(3) *For each $f \in S$ there is an idempotent $e \in Sf$ such that $(1 - e) \in S(1 - f)$.*

## 3. Direct Sums of Modules with the Exchange Property

It is easy to verify that finite direct sums of modules with the exchange property inherit this asset (see [2, p. 812]). That this is no longer true for infinite sums, in general, was already observed by Crawley and Jónsson in their initial paper: The direct sum $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/(p^n)$ of cyclic abelian groups, for instance, fails to have the exchange property by [2, p. 852]. As a consequence, the problem arises: For which families of modules with the exchange property does the direct sum retain this property?

An interesting special case of this problem is the following question: Which modules possessing decompositions into indecomposable summands have the exchange property? Since, for any indecomposable module the exchange property is tantamount to a local endomorphism ring by [17], we may start with a module $M = \bigoplus_{j \in J} M_j$ such that each $M_j$ has a local endomorphism ring. In a long list of papers [8–11, 21–23], the following three conditions were compared with each other:

(1) $M$ has the exchange property.

(2) $M$ has the finite exchange property.

(3) The family $(M_j)_{j\in J}$ is semi-$T$-nilpotent, meaning that for each sequence $M_{j_1} \to^{f_1} M_{j_2} \to^{f_2} M_{j_3} \to^{f_3} \cdots$ of nonisomorphisms, where $j_k \in J$ and $j_k \neq j_l$ for $k \neq l$, and for each $x \in M_{j_1}$ there exists a natural number $p$ such that $f_p \circ f_{p-1} \circ \cdots \circ f_1(x) = 0$.

In the papers listed above the equivalence of (2) $\Leftrightarrow$ (3) is proved (for (2) $\Rightarrow$ (3) see [10, Lemma 9] and [23, Theorem 1], for (3) $\Rightarrow$ (2) see [23, Theorem 1]); moreover, the implication (3) $\Rightarrow$ (1) is established in the special cases where all $M_j$'s are injective, resp. all $M_j$'s are isomorphic (see [21], [22] and [9]).

The following theorem will yield, as an immediate consequence, the equivalence of (1)–(3) in general. Our method of proof is completely different from the one used in the above-mentioned articles; in particular, it is free of category techniques and is rather conceptual.

THEOREM 5. *Let $(M_j)_{j\in J}$ be a semi-T-nilpotent family of modules (not necessarily indecomposable) with the exchange property. Then $\bigoplus_{j\in J} M_j$ has the exchange property in either of the following cases:*

(I) *$M_j$ and $M_k$ have no nontrivial isomorphic direct summand for $j \neq k$.*

(II) *$M_j \cong M_k$ for all $j, k \in J$.*

*Remark.* In Case II, the condition of semi-$T$-nilpotence is particularly strong: it forces all $M_j$'s to be indecomposable. Even though this case has been previously settled in [9], we include a particularly brief argument based on Proposition 3 and an idea of J. Stock [15, Satz 5.2].

*Proof of Theorem 5*

*Case* I. We start by well-ordering the index set $J$. For simplicity we assume that $J = \{\alpha\colon \alpha \text{ ordinal}, \alpha \leqslant \rho\}$ for some ordinal number $\rho$.

In view of Proposition 3, we may reduce the test of the exchange property of $M = \bigoplus_{\alpha\leqslant\rho} M_\alpha$ to the test in direct sums of copies of $M$: Suppose

$$A = \bigoplus_{\alpha\leqslant\rho} M_\alpha \oplus B = \bigoplus_{\substack{i\in I\\ \alpha\leqslant\rho}} A_{\alpha i}$$

with $A_{\alpha i} \cong M_\alpha$ for each $i$.

Our goal is to successively insert the $M_\alpha$'s on the right-hand side of the above equality by discarding certain summands of the $A_{\alpha i}$'s in order to make room for the $M_\alpha$'s. More precisely, we claim the existence of families

$(C_\alpha)_{\alpha \leqslant \rho}$ and $(D_\alpha)_{\alpha \leqslant \rho}$ of submodules of $A$ with $C_\alpha = \bigoplus_{i \in I} A'_{\alpha i}$ and $D_\alpha = \bigoplus_{i \in I} A''_{\alpha i}$ such that we have

$$(1) \qquad A_{\alpha i} = A'_{\alpha i} \oplus A''_{\alpha i} \qquad \text{for all } \alpha \text{ and } i$$

and

$$(2) \quad A = \bigoplus_{\alpha \leqslant \beta} M_\alpha \oplus \bigoplus_{\alpha \leqslant \beta} C_\alpha \oplus \bigoplus_{\substack{i \in I \\ \alpha > \beta}} A_{\alpha i} = \bigoplus_{\alpha < \beta} M_\alpha \oplus \bigoplus_{\alpha < \beta} C_\alpha \oplus \bigoplus_{\substack{i \in I \\ \alpha \geqslant \beta}} A_{\alpha i}$$

for each $\beta \leqslant \rho$.

Once such families are established, the special choice $\beta = \rho$ completes the proof.

The required families $(C_\alpha)_{\alpha \leqslant \rho}$ and $(D_\alpha)_{\alpha \leqslant \rho}$ are constructed by transfinite induction. Suppose that for some $\gamma \leqslant \rho$ we already have $(C_\alpha)_{\alpha < \gamma}$ and $(D_\alpha)_{\alpha < \gamma}$ with (1) and such that (2) holds for each $\beta < \gamma$.

First suppose that $\gamma$ is a successor ordinal, say, $\gamma = \beta + 1$. In view of Lemma 2, we derive from

$$A = \bigoplus_{\alpha \leqslant \beta} M_\alpha \oplus \bigoplus_{\alpha \leqslant \beta} C_\alpha \oplus \bigoplus_{\substack{i \in I \\ \alpha > \beta}} A_{\alpha i} = \bigoplus_{\alpha \leqslant \beta} M_\alpha \oplus \bigoplus_{\substack{i \in I \\ \alpha \leqslant \beta}} A'_{\alpha i} \oplus \bigoplus_{\substack{i \in I \\ \alpha > \beta}} A_{\alpha i}$$

and the exchange property of $\bar{M}_\gamma = M_{\beta+1}$ the existence of decompositions $A'_{\alpha i} = \tilde{A}_{\alpha i} \oplus \tilde{\tilde{A}}_{\alpha i}$ for $\alpha \leqslant \beta$ and $A_{\alpha i} = \tilde{A}_{\alpha i} \oplus \tilde{\tilde{A}}_{\alpha i}$ for $\alpha > \beta$ such that

$$A = \bigoplus_{\alpha \leqslant \beta} M_\alpha \oplus M_{\beta+1} \oplus \bigoplus_{\substack{\alpha \in I \\ \alpha \leqslant \rho}} \tilde{A}_{\alpha i}.$$

We infer

$$M_{\beta+1} \cong \bigoplus_{\substack{i \in I \\ \alpha \leqslant \rho}} \tilde{\tilde{A}}_{\alpha i}$$

and hence, by hypothesis, $\tilde{\tilde{A}}_{\alpha i} = 0$ for all $\alpha \neq \beta + 1$. Set $A'_{\beta+1,i} := \tilde{A}_{\beta+1,i}$ and $A''_{\beta+1,i} := \tilde{\tilde{A}}_{\beta+1,i}$. Defining, moreover, $C_{\beta+1} := \bigoplus_{i \in I} A'_{\beta+1,i}$ and $D_{\beta+1} = \bigoplus_{i \in I} A''_{\beta+1,i}$, we have

$$A = \bigoplus_{\alpha \leqslant \beta+1} M_\alpha \oplus \bigoplus_{\alpha \leqslant \beta+1} C_\alpha \oplus \bigoplus_{\substack{i \in I \\ \alpha > \beta+1}} A_{\alpha i}$$

as desired.

Now let $\gamma$ be a limit ordinal. In a first step we establish the equation

$$A = \bigoplus_{\alpha < \gamma} M_\alpha \oplus \bigoplus_{\alpha < \gamma} C_\alpha \oplus \bigoplus_{\substack{i \in I \\ \alpha \geqslant \gamma}} A_{\alpha i}.$$

Observe that

$$A = \bigoplus_{\alpha<\gamma} C_\alpha \oplus \bigoplus_{\alpha<\gamma} D_\alpha \oplus \bigoplus_{\substack{i\in I\\ \alpha\geqslant\gamma}} A_{\alpha i},$$

because we have $C_\alpha = \bigoplus_{i\in I} A'_{\alpha i}$, $D_\alpha = \bigoplus_{i\in I} A''_{\alpha i}$ and $A_{\alpha i} = A'_{\alpha i} \oplus A''_{\alpha i}$ for all $\alpha < \gamma$. Let $p_\gamma \colon \bigoplus_{\alpha<\gamma} M_\alpha \to \bigoplus_{\alpha<\gamma} D_\alpha$ be the restriction of the projection $\tilde{p}_\gamma \colon A \to \bigoplus_{\alpha<\gamma} D_\alpha$ along

$$\bigoplus_{\alpha<\gamma} C_\alpha \oplus \bigoplus_{\substack{i\in I\\ \alpha\geqslant\gamma}} A_{\alpha i};$$

for $\beta < \gamma$ define $p_\beta$ and $\tilde{p}_\beta$ analogously. By Lemma 1, the equality at which we are aiming holds if and only if $p_\gamma$ is an isomorphism.

Since, by induction hypothesis, $p_\beta$ is an isomorphism for each $\beta < \gamma$, it is clear that $p_\gamma$ is injective.

Assume that $p_\gamma$ is not surjective. Then there exists some $\beta_1 < \gamma$ and some $x_1 \in D_{\beta_1}$ such that $x_1$ does not belong to the image of $p_\gamma$. Using again the fact that $p_\beta$ is an isomorphism for all $\beta < \gamma$, we can define

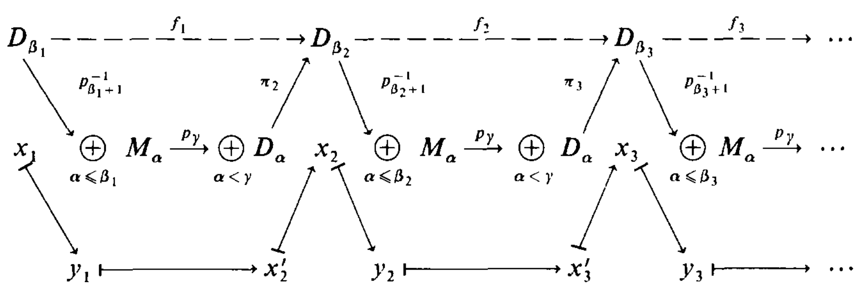
$$x'_2 := p_\gamma \circ p^{-1}_{\beta_1+1}(x_1) \in \bigoplus_{\alpha<\gamma} D_\alpha.$$

We claim that there exists an ordinal number $\beta_2$ with $\beta_1 < \beta_2 < \gamma$ such that the $\beta_2$-component of $x'_2$ in $\bigoplus_{\alpha<\gamma} D_\alpha$ lies outside $\operatorname{Im}(p_\gamma)$: Set $z = x'_2 - \widetilde{p_{\beta_1+1}}(x'_2)$ and observe that $z \in \bigoplus_{\beta_1<\alpha<\gamma} D_\alpha$. In view of $p_{\beta_1+1} = \widetilde{p_{\beta_1+1}} \circ p_\gamma$ we obtain, moreover,

$$z = x'_2 - p_{\beta_1+1} \circ p^{-1}_{\beta_1+1}(x_1) = p_\gamma(p^{-1}_{\beta_1+1}(x_1)) - x_1,$$

and since $x_1$ does not belong to $\operatorname{Im}(p_\gamma)$, neither does $z$. Therefore $\beta_2$ exists as required. Having picked such a $\beta_2$, let $x_2$ be the $\beta_2$-component of $x'_2$ in $\bigoplus_{\alpha<\gamma} D_\alpha$.

Now proceed with $x_2$ instead of $x_1$. Inductively, our process yields a diagram as follows:



Here $y_n = p_{\beta_n+1}^{-1}(x_n)$, and $\pi_n$ denotes the obvious projection. That is, we obtain an increasing sequence $\beta_1 < \beta_2 < \beta_3 < \cdots$ of ordinal numbers below $\gamma$ and a sequence $x_1, x_2, x_3,\ldots$ of elements $x_n \in D_{\beta_n} \backslash \mathrm{Im}(p_\gamma)$, respectively. Furthermore, setting $f_n = \pi_n \circ p_\gamma \circ p_{\beta_n+1}^{-1}$, we have $x_{n+1} = f_n(x_n)$.

Once it becomes clear that $D_{\beta_n} \cong M_{\beta_n}$ for all $n$, this situation is seen to be incompatible with the semi-$T$-nilpotence of $(M_\alpha)_{\alpha \leqslant \rho}$; in fact, the hypothesis entails that all of the maps $f_n$ are nonisomorphisms. But, keeping in mind that $\bigoplus_{i\in I} A_{\beta_n i} = C_{\beta_n} \oplus D_{\beta_n}$, we deduce $D_{\beta_n} \cong M_{\beta_n}$ from the following equality which is part of the induction hypothesis:

$$\bigoplus_{\alpha<\beta_n} M_\alpha \oplus \bigoplus_{\alpha<\beta_n} C_\alpha \oplus \bigoplus_{\substack{i\in I\\ \alpha\geqslant\beta_n}} A_{\alpha i} = \bigoplus_{\alpha\leqslant\beta_n} M_\alpha \oplus \bigoplus_{\alpha\leqslant\beta_n} C_\alpha \oplus \bigoplus_{\substack{i\in I\\ \alpha>\beta_n}} A_{\alpha i}.$$

The modules $C_\gamma$ and $D_\gamma$ can now be found exactly as for a successor ordinal, and the induction is complete.

*Case* II. Once more, Proposition 3 permits us to focus on a situation where $M = \bigoplus_{j\in J} M_j$ is a summand of a direct sum of copies of $M$:

$$A = M \oplus B = \bigoplus_{i\in I} A_i$$

with $A_i \cong M_j$ for all $i \in I$ and $j \in J$. Clearly, we may assume that $J$ is infinite.

Choose a subset $L \subset I$ which is maximal with respect to the following properties:

(1) $M \cap \bigoplus_{l\in L} A_l = 0$.

(2) Each finite subsum of $\bigoplus_{j\in J} M_j \oplus \bigoplus_{l\in L} A_l$ is a direct summand of $A$.

Now suppose that $M \oplus \bigoplus_{l\in L} A_l$ is not all of $A$. We will construct a sequence $(i_n)_{n\in\mathbb{N}}$ of elements of $I$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of nonzero elements $x_n$ of $A_{i_n}$, respectively, such that, for some nonisomorphisms $f_n: A_{i_n} \to A_{i_{n+1}}$, we have $x_{n+1} = f_n \circ \cdots \circ f_1(x_1)$. However, in view of $A_l \cong M_j$, the existence of such sequences contradicts the semi-$T$-nilpotence of $(M_j)_{j\in J}$ (since all $M_j$'s and $A_i$'s are isomorphic, the requirement of distinctness of the indices $i_n$ becomes irrelevant).

Pick $i_1 \in I$ together with $x_1 \in A_{i_1}$ such that $x_1 \notin M \oplus \bigoplus_{l\in L} A_l$. By the maximality of $L$, there exists a finite subsum $F = \bigoplus_{\mathrm{fin}} M_j \oplus \bigoplus_{\mathrm{fin}} A_l$ of $\bigoplus_J M_j \oplus \bigoplus_L A_l$ such that either $F \cap A_{i_1} \neq 0$ or $F + A_{i_1}$ is not a direct summand of $A$. On the other hand, the hypothesis tells us that $F$ has the exchange property, and therefore we can find a subset $I'$ of $I$ with

$$A = F \oplus \bigoplus_{I'} A_i.$$

(Note that our hypotheses force the $M_j$, and hence also the $A_i$, to be indecomposable in Case II.)

For $i \in I'$, let $p_i$ be the projection $A \to A_i$ corresponding to this decomposition. By the choice of $x_1$ there is some $i_2 \in I'$ with $p_{i_2}(x_1) \notin M \oplus \bigoplus_{l \in L} A_l$. Moreover, the restriction $f_1: A_{i_1} \to A_{i_2}$ of $p_{i_2}$ to $A_{i_1}$ is not an isomorphism, because otherwise we would have $A = F \oplus A_{i_1} \oplus \bigoplus_{i \in I' \setminus \{i_2\}} A_i$, which we have excluded by our choice of $F$. Now iterate the procedure with $x_2 = f_1(x_1)$. An obvious induction completes the proof. ∎

Example. Showing that the method employed in Case I of the preceding theorem fails, in general, if the $M_j$'s are allowed to have common direct summands, even in the case of vector spaces. More precisely, the transfinite induction of Case I may then collapse as we reach the first limit ordinal.

Suppose $A$ is a vectorspace of countably infinite dimension over some field not of characteristic 3, say, $A = \bigoplus_{i \in \mathbb{N}} \langle x_i \rangle \oplus \bigoplus_{i \in \mathbb{N}} \langle y_i \rangle$, where the $\langle x_i \rangle$, $\langle y_i \rangle$ are one-dimensional subspaces. If we set $M_j = \langle x_j + 3y_{j+1} \rangle$, we clearly have

$$A = \bigoplus_{j \in \mathbb{N}} M_j \oplus \bigoplus_{i \in \mathbb{N}} \langle y_i \rangle = \bigoplus_{i \in \mathbb{N}} \langle x_i \rangle \oplus \bigoplus_{i \in \mathbb{N}} \langle y_i + 3x_{i+1} \rangle.$$

Observe that we can successively insert the $M_j$'s on the right-hand side by throwing out $\langle x_j \rangle$, respectively, to make room for $M_j$, i.e.,

$$A = \bigoplus_{j \leqslant n} M_j \oplus \bigoplus_{i > n} \langle x_i \rangle \oplus \bigoplus_{i \in \mathbb{N}} \langle y_i + 3x_{i+1} \rangle,$$

whereas $A \neq \bigoplus_{j \in \mathbb{N}} M_j \oplus \bigoplus_{i \in \mathbb{N}} \langle y_i + 3x_{i+1} \rangle$. ∎

Corollary 6. *Suppose $M = \bigoplus_{j \in J} M_j$, where each $M_j$ is indecomposable. Then the following conditions are equivalent:*

(1) *$M$ has the exchange property.*

(2) *$M$ has the finite exchange property.*

(3) *All the summands $M_j$ have local endomorphism rings, and the family $(M_j)_{j \in J}$ is semi-T-nilpotent.*

*Proof.* (3) ⇒ (1): Single out a set of representatives $(M_k)_{k \in K}$ of the isomorphism classes of the $M_j$, $j \in J$. Next, collect all those $M_j$'s which are isomorphic to a fixed $M_k$ and denote their direct sum by $N_k$. Then each $N_k$ has the exchange property by Case II of the preceding theorem. Moreover, König's Graph Lemma guarantees that the family $(N_k)_{k \in K}$ is again semi-$T$-nilpotent. The exchange property of $M = \bigoplus_{k \in K} N_k$ is therefore a consequence of Case I of Theorem 5.

(1) ⇒ (2) is clear.

(2) ⇒ (3): For the convenience of the reader we include a compact

proof which is an amalgamation of arguments due to Harada and Sai [10, Lemma 9] and Yamagata [23, Theorem 1]:

The following technical remark, referred to as $*$, will be employed repeatedly. Suppose $C = P \oplus Q = \bigoplus_{i \in I} C_i$ is a module whose summands $C_i$ all have local endomorphism rings. If $I'$ is a finite subset of $I$ with $P \cap \bigoplus_{i \in I'} C_i \neq 0$, then there exists an index $k \in I'$ such that $C = C_k \oplus P' \oplus Q$ for a certain submodule $P'$ of $P$.

To see this, invest the exchange property of $\bigoplus_{i \in I'} C_i$ to arrive at an equation

$$C = \bigoplus_{i \in I'} C_i \oplus P_1 \oplus Q_1$$

where $P = P_1 \oplus P_2$, $Q = Q_1 \oplus Q_2$. By Lemma 1, this means that the projection $p\colon C \to P_2 \oplus Q_2$ along $P_1 \oplus Q_1$ induces an isomorphism

$$\bigoplus_{i \in I'} C_i \to P_2 \oplus Q_2.$$

Observe that necessarily $P_2 \neq 0$ by our choice of $I'$. According to Azumaya's Theorem (see, e.g., [1, p. 144]), $P_2$ has in turn a direct summand $P_{21}$ with local endomorphism ring, $P_2 = P_{21} \oplus P_{22}$, say, and there exists $k \in I'$ so that the following composition of maps is an isomorphism

$$C_k \xrightarrow{p|_{C_k}} P_2 \oplus Q_2 \xrightarrow{q} P_{21};$$

here $q$ denotes the projection along $P_{22} \oplus Q_2$. Apply Lemma 1 once again to deduce the equality

$$C = C_k \oplus P_{22} \oplus P_1 \oplus Q,$$

which completes the proof of $*$.

Now assume (2), deduce that all $M_j$ have local endomorphism rings, and start with a sequence $j_1, j_2, j_3, \ldots$ of pairwise different elements of $J$ and a sequence of nonisomorphisms $M_{j_1} \to^{f_1} M_{j_2} \to^{f_2} M_{j_3} \to^{f_3} \cdots$. We may obviously assume that $J = \{j_n : n \in \mathbb{N}\}$, and we write $j_n = n$ for simplicity. Furthermore, we may assume that either all of the maps $f_n$ are monomorphisms or all of them are nonmonomorphisms: For if infinitely many $f_n$'s are not injective, say, $f_{m_1}, f_{m_2}, f_{m_3}, \ldots$ with $m_1 < m_2 < m_3 < \cdots$, then replace the set $J$ by the set $\{m_i : i \in \mathbb{N}\}$ and consider the nonmonomorphisms $f_{m_{i+1}-1} \circ \cdots \circ f_{m_i}\colon M_{m_i} \to M_{m_{i+1}}$.

Next define modules $M'_n \cong M_n$ by $M'_n = \{x + f_n(x) : x \in M_n\}$. Clearly, we have

$$M = M'_1 \oplus M_2 \oplus M'_3 \oplus M_4 \oplus \cdots = M_1 \oplus M'_2 \oplus M_3 \oplus M'_4 \oplus \cdots.$$

Since, by hypothesis, $\bigoplus_{n\in\mathbb{N}} M'_{2n-1}$ has the finite exchange property, we obtain

$$M = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus A \oplus B \qquad \text{with} \quad A \subset \bigoplus_{n\in\mathbb{N}} M_{2n-1} \text{ and } B \subset \bigoplus_{n\in\mathbb{N}} M'_{2n}.$$

Suppose first that all of the maps $f_n$ are nonmonomorphisms. Then we have $A = 0$, since otherwise $*$ (with $P = A$ and $Q = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus B$) would yield the existence of an index $k$ such that $M = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus M_{2k-1} \oplus A' \oplus B$. But this is impossible since $\mathrm{Ker}(f_{2k-1}) \subset M'_{2k-1} \cap M_{2k-1}$. Picking any $x \in M_1$, we deduce

$$x \in \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus B \subset \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus \bigoplus_{n\in\mathbb{N}} M'_{2n},$$

which in turn implies the existence of some $p$ with $f_p \circ f_{p-1} \circ \cdots \circ f_1(x) = 0$.

Finally, consider the case where all the $f_n$ are monomorphisms and consequently not surjective. We show first that

$$M = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus A \oplus M'_2 \oplus B_2 \qquad \text{for some } B_2 \subseteq B.$$

Set $X = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus A$ and let $p_X\colon M \to X$, respectively $p_B\colon M \to B$, be the projections for the decomposition $M = X \oplus B$. Moreover, denote by $p_{2n}$ the projection $M \to M'_{2n}$ with respect to the decomposition $M = \bigoplus_{n\in\mathbb{N}} M'_{2n} \oplus \bigoplus_{n\in\mathbb{N}} M_{2n-1}$ and by $in_{2n}$ the canonical injection $M'_{2n} \to M$. Invest the fact that $f_1$ is not surjective to see that $p_2 p_X$ is not surjective either; in fact, we have $p_2(X) \subseteq M'_2 \cap (\mathrm{Im}(f_1) \oplus M_3) \subsetneq M'_2$. From the fact that $p_2(p_X + p_B)\, in_2$ is the identity in the local ring $\mathrm{End}_R(M'_2)$ we consequently deduce that $p_2 p_B\, in_2$ is an isomorphism in $\mathrm{End}_R(M'_2)$. But the latter means that the projection $p_B$ induces an isomorphism from $M'_2$ onto a direct summand of $B$, say, $B'_2$. If $B = B'_2 \oplus B_2$, then Lemma 1 yields the desired equality.

The game can be repeated for $M'_4$ where $B_2$ now plays the rôle of $B$ and $\bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus A \oplus M'_2$ the rôle of $X$. The result is a decomposition

$$M = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus A \oplus M'_2 \oplus M'_4 \oplus B_4 \qquad \text{with} \quad B_4 \subseteq B_2.$$

The process continues in an obvious finite induction. In particular, we see that the sum $\bigoplus_{n\in\mathbb{N}} M'_{2n-1} + \bigoplus_{n=1}^{m} M'_{2n} + A$ is direct for all $m \in \mathbb{N}$, which entails

$$M = \bigoplus_{n\in\mathbb{N}} M'_{2n-1} \oplus \bigoplus_{n\in\mathbb{N}} M'_{2n} \oplus A$$

since $B \subseteq \bigoplus_{n\in\mathbb{N}} M'_{2n}$.

If $A=0$, we are done as in the preceding case. Otherwise, we have $A \cap \bigoplus_{I'} M_{2n-1} \neq 0$ for some finite set $I' \subset \mathbb{N}$, and another application of $*$ shows

$$M = \bigoplus_{n \in \mathbb{N}} M'_{2n-1} \oplus \bigoplus_{n \in \mathbb{N}} M'_{2n} \oplus M_{2k-1} \oplus A'$$

for some $2k-1 \in I'$ and some $A' \subset A$. We must have $A'=0$ at this stage since otherwise a repetition of the previous argument would yield an index $l$ different from $k$ such that

$$M = \bigoplus_{n \in \mathbb{N}} M'_{2n-1} \oplus \bigoplus_{n \in \mathbb{N}} M'_{2n} \oplus M_{2k-1} \oplus M_{2l-1} \oplus A'';$$

if $k<l$, the inclusion $M_{2k-1} \subset M'_{2k-1} \oplus M'_{2k} \oplus \cdots \oplus M'_{2l-2} \oplus M_{2l-1}$ is incompatible with this decomposition of $M$; symmetrically, $l<k$ leads to a contradiction.

Now pick $m>k$. From $M_{2m-1} \subset \bigoplus_{n \in \mathbb{N}} M'_{2n-1} \oplus \bigoplus_{n \in \mathbb{N}} M'_{2n} \oplus M_{2k-1}$ we again deduce the existence of some index $p$ with $Ke(f_p \circ f_{p-1} \circ \cdots \circ f_{2m-1}) \neq 0$. Because all the maps $f_n$ are monomorphisms, this last case clearly does not occur. ∎

Corollary 7 [9, 22]. *If the identity of $R$ is a finite sum of orthogonal primitive idempotents, then the following conditions are equivalent:*

(1) *Each projective right $R$-module has the exchange property.*

(2) *The free right $R$-module $R^{(\mathbb{N})}$ has the finite exchange property.*

(3) *$R$ is right perfect.*

*Proof.* This is an immediate consequence of Corollary 6. ∎

*Remark.* If the overall hypothesis in Corollary 7 is removed, conditions (1)–(3) are no longer equivalent. Kutami and Oshiro were the first to exhibit a nonartinian Boolean ring which satisfies (1) in [12]. In [15], Stock supplemented the picture as follows: (1) holds for any ring $R$ with right-$T$-nilpotent Jacobson radical and a von Neumann regular factor ring. Provided that all idempotents of $R$ are central (e.g., in the commutative case), the converse is also true, whereas, in general, (1) does not force the radical factor ring of $R$ to be regular.

Corollary 8. *Suppose that $(M_j)_{j \in J}$ is a family of modules, the lengths of which are uniformly bounded by some integer $N$. Then $\bigoplus_{j \in J} M_j$ has the exchange property.*

*Proof.* Since each $M_j$ is a finite direct sum of modules with local endomorphism rings, we may assume that the $M_j$'s are indecomposable to begin with. But by [10, Lemma 11], any composition of $2^N - 1$ nonisomorphisms between the $M_j$'s is then the zero map. Consequently the family $(M_j)_{j\in J}$ is semi-$T$-nilpotent, and our claim follows from Corollary 6.

*Remarks.* 1. As a consequence of Corollary 8 we obtain that each decomposition of $\bigoplus_{j\in J} M_j$ into indecomposable modules complements direct summands in the sense of [1, p. 141], provided that the lengths of the $M_j$'s are uniformly bounded. This is a mild extension of a result of Anderson and Fuller (see [1, 29.6]).

2. In Corollary 8, the common bound on the lengths of the $M_j$'s is not redundant. On the other hand, the existence of such a bound is not necessary for $\bigoplus_{j\in J} M_j$ to have the exchange property: Think of $\bigoplus_{p \text{ prime}} \mathbb{Z}/(p^p)$.

COROLLARY 9. *Choose $R = \mathbb{Z}$ and let $M_p$ be a $p$-adically complete abelian group for each prime $p$. Then $\bigoplus_{p \text{ prime}} M_p$ has the exchange property.*

*Proof.* In view of Theorem 5, it suffices to note that $\operatorname{Hom}_{\mathbb{Z}}(M_p, M_q) = 0$ for $p \neq q$. ∎

## 4. INVARIANT SUBMODULES OF ALGEBRAICALLY COMPACT MODULES

PROPOSITION 10. *For a module $M$ with the finite exchange property, the following statements are equivalent:*

(1) *$M$ has the exchange property.*

(2) *Whenever $M$ is a direct summand of a direct sum $A = \bigoplus_{i\in I} A_i$ with $A_i \cong M$ for all $i$, there exists a submodule $C$ of $A$ which is maximal with respect to the following properties:*

(a) *$C = \bigoplus_{i\in I} C_i$ with $C_i \subset A_i$.*

(b) *$C \cap M = 0$.*

(c) *The canonical embedding $M \to A/C$ splits.*

*Proof.* (1) ⇒ (2) is clear.

For the converse, suppose (2) is satisfied. To verify (1) we may, by Proposition 3, restrict our attention to the situation

$$M \oplus B = \bigoplus_{i\in I} A_i = A$$

with each $A_i \cong M$. Choose a maximal $C \subset A$ as in (2). Identifying the module $M$ with its image in $A/C = X$ and denoting $A_i/C_i$ by $X_i$, we have

$$M \oplus Y = \bigoplus_{i\in I} X_i = X,$$

where $Y$ is some complement of $M$ in $X$. The maximality of $C$ guarantees that, for each nonzero submodule $Z_i$ of some $X_i$ with $M \cap Z_i = 0$, the sum $M \oplus Z_i$ is not a direct summand of $X$.

Our aim is to show $Y = 0$, which means $A = M \oplus \bigoplus_{i \in I} C_i$. For this purpose it clearly suffices to check that $Y \cap \bigoplus_{i \in I'} X_i = 0$ for each finite subset $I'$ of $I$. For simplicity of notation we suppose that the finite subset at which we are looking is of the form $I' = \{1, ..., n\}$. Moreover, we denote by $p: X \to Y$ the projection along $M$, by $e_k: X \to X_k$ the projection along $\bigoplus_{i \neq k} X_i$. Setting $e = e_1 + \cdots + e_n$ we may then identify the endomorphism ring $S$ of $\bigoplus_{k=1}^n X_k$ with $e(\mathrm{End}_R(X))\, e$.

All we have to show is that $epe_k \in \mathrm{Rad}(S)$ for $1 \leqslant k \leqslant n$ ($\mathrm{Rad}(S)$ standing for the Jacobson radical of $S$). For then we infer that $epe = epe_1 + \cdots + epe_n \in \mathrm{Rad}(S)$, and since $epe$ induces the identity on $Y \cap \bigoplus_{k=1}^n X_k$, we conclude that the latter intersection is zero.

First observe that each $X_k$ (and therefore $\bigoplus_{k=1}^n X_k$) has the finite exchange property. In fact: The finite exchange property of $M$ yields $X = M \oplus Z_k \oplus Z$ for some $Z_k \subset X_k$ and $Z \subset \bigoplus_{i \neq k} X_i$. By our construction, $Z_k = 0$, and consequently $X_k$ is isomorphic to a direct summand of $M$.

To see that $epe_1 \in \mathrm{Rad}(S)$, let $s \in S$ and apply Proposition 3 to $\bigoplus_{k=1}^n X_k$ together with the endomorphism $a = sepe_1$ and $e - a \in S$. This provides us with two orthogonal idempotents of $S$, say, $f = ra \in Sa$ and $e - f \in S(e - a)$. Clearly, we may assume $r = fr$. We claim $f = 0$, which implies $e \in S(e - a)$. Since $s \in S$ is arbitrary, the latter means $epe_1 \in \mathrm{Rad}(S)$.

In view of $f^2 = f = rsepe_1$, our claim will follow if we can show that $\varphi = e_1 rsep = 0$. But $r = fr$ guarantees that $\varphi$ is an idempotent of $\mathrm{End}_R(X)$, and hence $X$ is the direct sum of the kernel $\mathrm{Ker}(\varphi)$ and the image $\mathrm{Im}(\varphi)$ of $\varphi$. Now $M$ is contained in $\mathrm{Ker}(\varphi)$, and we arrive at an equality

$$X = M \oplus Z \oplus \mathrm{Im}(\varphi).$$

But since $\mathrm{Im}(\varphi)$ is contained in $X_1$, our construction forces $\mathrm{Im}(\varphi)$ to be zero, and the proof is complete. ∎

DEFINITION. We call a submodule $M$ of a module $X$ strongly invariant if $f(M) \subset M$ for each homomorphism $f \in \mathrm{Hom}_R(M, X)$.

Note that for any quasi-injective module $X$, "strong invariance" of $M$ is the same as "invariance" in the classical sense.

THEOREM 11. *Each strongly invariant submodule of any algebraically compact module has the exchange property.*

Before we give a proof, we list classes of examples covered by this statement. Several of them are new. On the other hand, numerous occurrences of the exchange property, which were previously established by various methods, are subsumed and thus seen from a unifying point of view. For torsion-complete abelian groups (special case of 5 below), the original proof of the exchange property in [2, p. 847] makes heavy use of techniques specific to abelian group theory.

EXAMPLES OF STRONGLY INVARIANT SUBMODULES OF ALGEBRAICALLY COMPACT MODULES.

1. *All quasi-injective modules.* Note that each quasi-injective module is strongly invariant in its injective envelope. That these modules enjoy the exchange property is Fuchs' theorem in [5], which was preceded by Warfield's analogous statement for injectives [18, p. 265].

2. *All algebraically compact modules.* (It was pointed out to the authors by H. Lenzing that an alternative proof of the exchange property in this case can be derived from Gruson and Jensen's [7, 1.2 and 3.2], combined with [18, p. 265].) Specializing to $R = \mathbb{Z}$, we use the well-known fact that "algebraically compact + reduced" is the same as "$\mathbb{Z}$-adically complete" (see [5, p. 163]) to rediscover Warfield's result that each $\mathbb{Z}$-adically complete abelian group has the exchange property [19, Theorem 3].

3. *All linearly compact modules over an arbitrary commutative ring.* ($M$ is called linearly compact if each system of congruences $X_i - m_i \in U_i$ $(i \in I)$ with $m_i \in M$ and $U_i$ a submodule of $M$, which admits a solution for each finite subsystem, admits a global solution.) Namely: By [16, Proposition 9], each linearly compact module over a commutative ring is algebraically compact.

4. *All artinian modules over an arbitrary commutative ring.* This is a special case of 3. Theorem 5 thus provides a partial answer to the question of Crawley and Jónsson, whether each artinian module has the exchange property. Note that, in this case, Theorem 5 can be rephrased as follows: Each indecomposable artinian module over a commutative ring has a local endomorphism ring.

5. *All torsion submodules of algebraically compact modules with respect to any hereditary torsion theory.* For instance: Given a multiplicatively closed subset $S$ of a commutative ring $R$ and an algebraically compact module $X$, then $T_S(X) = \{x \in X : xs = 0 \text{ for some } s \in S\}$ is among the above. In particular, the classical torsion submodule of any algebraically compact module over a commutative integral domain has the exchange property. Specializing still further to $R = \mathbb{Z}$, we obtain the exchange property for all torsion-complete abelian $p$-groups.

*Proof of Theorem* 11. Suppose that $X$ is an algebraically compact module and $M$ a strongly invariant submodule with endomorphism ring $S$.

First, we show that the ring $S$, viewed as a right module over itself, is algebraically compact: Note that $S_S = \mathrm{Hom}_R(M, M)_S \cong \mathrm{Hom}_R(M, X)_S$; to see that the latter $S$-module is algebraically compact (= pure-injective), start with a pure inclusion $U_S \to V_S$ and observe that the pure-injectivity of $X_R$ forces the upper row (and hence also the lower one) of the following commutative diagram to be an epimorphism.

$$\begin{array}{ccc} \mathrm{Hom}_R(U \otimes_S M, X) & \longrightarrow & \mathrm{Hom}_R(V \otimes_S M, X) \\ \| \wr & & \| \wr \\ \mathrm{Hom}_S(U, \mathrm{Hom}_R(M, X)) & \longrightarrow & \mathrm{Hom}_S(V, \mathrm{Hom}_R(M, X)) \end{array}$$

By [25, Theorem 9] we conclude that the ring $S$ is von Neumann regular modulo its radical and has the lifting property for idempotents modulo the radical. Now apply [20, Theorems 2 and 3] to see that $M$ has the *finite* exchange property.

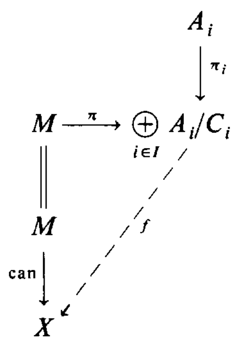
That $M$ even enjoys the unrestricted exchange property will follow from Proposition 10. Suppose $M$ is a direct summand of $A = \bigoplus_{i \in I} A_i$ with $A_i \cong M$ for each $i$. By Zorn's Lemma, choose $C \subset A$ maximal with the properties

(a) $C = \bigoplus_{i \in I} C_i$ with $C_i \subset A_i$,

(b) $C \cap M = 0$,

(c) the canonical embedding $\pi: M \to A/C$ is pure.

All we have to show is splitness of the latter embedding. To do this, consider the following commutative diagram:




where $\pi_i: A_i \to A_i/C_i$ denotes the canonical epimorphism. The existence of $f$ follows from the pure injectivity of $X$. In view of $A_i \cong M$, the hypothesis of strong invariance forces $f\pi_i(A_i)$ to be contained in $M$ for each $i \in I$, which is tantamount to $\mathrm{Im}(f) \subset M$. This completes the proof. ∎

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