JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 127, 93–102 (1987)

Stability for the Solutions of $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ with Varying p

PETER LINDQVIST

Institute of Mathematics, Helsinki University of Technology, 02150 Espoo 15, Helsinki, Finland

Submitted by V. Lakshmikantham

Received February 26, 1986

Elliptic partial differential equations with principal part $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ are applied in physics, e.g., for the description of phenomena in glaceology. The objective of our note is a natural question of stability for solutions, as p varies.

1. INTRODUCTION

Often problems for differential equations are motivated by physical interpretations and in our case the background is the sliding of glaciers. In his "Traité de Glaciologie," Lliboutry gives adequate nonlinear mathematical models, describing the physical phenomena involved. Currently the corresponding differential equations are mastered in several special cases. (Measurements on the Athabasca Glacier are said to agree with the numerical values calculated in the mathematical model.) See [14] for these and other related facts.

Especially, the minimization of the potential energy

$$\frac{1}{p}\int_G |\nabla u|^p \, dm - \int_G u\,dm + \frac{C}{\pi}\int_\Gamma |u|^\pi \, dS \qquad (\Gamma \subset \partial G) \tag{1.1}$$

is discussed in [14], where detailed information about the quantities involved in (1.1) is given. A point of interest is that

$$p = 1 + \frac{1}{g}, \qquad \pi = 1 + \frac{2}{g+1}$$

g being the exponent of Glen. Here $0 < g < \infty$ and thus $1 < p < \infty$, but for

[1] $\int |\nabla u_{p_k}|^q \, dm \geq \int |\nabla u|^q \, dm + q\int |\nabla u|^{q-2}\nabla u \cdot \nabla(u_{p_k} - u)\, dm.$

0022-247X/87 $3.00

all real glaciers $g \geqslant 3$. One usually takes $g=3$ $(p=\frac{4}{3})$. This choice, based mainly on empirical considerations, leads in a natural way to the following kind of questions.

PROBLEM. Suppose that u_p minimizes the potential energy (1.1) among all "admissible functions." Does $u_p \to u_{4/3}$ or $\nabla u_p \to \nabla u_{4/3}$ in some reasonable sense, as $p \to 4/3$?

The Euler–Lagrange equation corresponding to (1.1) is

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = -1$$

with certain nonlinear boundary conditions for the solution(s).

A similar stability problem is involved in the numerical analysis of a related problem in a domain $G \subset \mathbb{R}^n$,

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f, \qquad u \mid \partial G = \varphi \mid \partial G, \tag{1.2}$$

considered in [2, Chap. 5, Sect. 3, pp. 173–186, Chap. 3, Sect. 6.3, pp. 128–131]. Namely, for augmented Lagrangian methods explicit knowledge of the varying of the solution with p is desirable. See [2, p. 184], where a conjecture is based on the solution

$$u(x) = \frac{p-1}{p}\left(\frac{1}{n}\right)^{1/(p-1)} [R^{p/(p-1)} - |x|^{p/(p-1)}]$$

for (1.2), when $f \equiv 1$, $\varphi \equiv 0$, and G is the ball $|x| < R$.

The existence, regularity, and qualitative behavior of the solutions of (1.2) are now known to a great extent, cf. [9, 13, 4, 16], and corresponding numerical algorithms can be accurately performed on existing computers, cf. [6 or 7]. The "f-stability" is a simple question, but so far as we know, the following stability phenomenon has not been successfully investigated in literature:

1.3. THEOREM. *Suppose that $f \in L^\infty(G)$ and $\varphi \in C^\infty(\mathbb{R}^n)$ are given, G being a bounded domain in $\mathbb{R}^n$. If $u_p \in C(G) \cap W_p^1(G)$ minimizes the energy*

$$\mathscr{E}_p(u) = \frac{1}{p}\int_G |\nabla u|^p \, dm + \int_G fu \, dm \qquad (1 < p < \infty) \tag{1.4}$$

among all similar u with boundary values $u - \varphi \in W_{p,0}^1(G)$ then

$$\lim_{\substack{p \to q \\ p < q}} \int_G |\nabla u_p - \nabla u_q|^p \, dm = 0 \qquad (1 < q < \infty). \tag{1.5}$$

If, in addition, $\int_G |\nabla u_q|^{q+\varepsilon}\,dm<\infty$ *for some* $\varepsilon>0$, *then*

$$\lim_{\substack{p\to q\\ p>q}} \int_G |\nabla u_p-\nabla u_q|^p\,dm=0 \qquad (1<q<\infty). \tag{1.6}$$

The rôle of the higher integrability, when p approaches q from above, is illuminated in Remark 4.2. We do not even know whether

$$\lim_{\substack{p\to q\\ p>q}} \int_G |\nabla u_p-\nabla u_q|^q\,dm=0$$

or not, if (eventually) $\int |\nabla u_q|^{q+\varepsilon}\,dm=\infty$ for each $\varepsilon>0$.

The paper is organized as follows. After some preliminaries in Section 2, the case $p<q$ is treated in Section 3 and the case $p>q$ in Section 4. A brief discussion of the corresponding local stability theory is included in Section 5, although we have left this section incomplete.

We use standard notation. The abbreviation $\int_G\cdots dm=\int\cdots dm$ is frequently used for Lebesgue's integral.

2. Preliminaries

Suppose that G is a bounded domain in $\mathbb{R}^n$ and fix $\varphi\in C^\infty(\mathbb{R}^n)$, $f\in L^\infty(G)$. Define the energy $\mathscr{E}_p(u)$ by (1.4). Consider the variational problem of minimizing $\mathscr{E}_p(u)$ among all functions in the class

$$\mathscr{F}_p=\{u\in C(G)\cap W_p^1(G)\,|\,u-\varphi\in W_{p,0}^1(G)\}.$$

(If the boundary ∂G is sufficiently regular, we may require that $u\in C(\bar{G})\cap W_p^1(G)$ and $u\,|\,\partial G=\varphi\,|\,\partial G$.) The function u_p in the following well-known theorem is said to minimize $\mathscr{E}_p$.

2.1. Theorem. *There is a unique* u_p *in* $\mathscr{F}_p$ *such that* $\mathscr{E}_p(u_p)\leqslant\mathscr{E}_p(u)$ *for all* $u\in\mathscr{F}_p$.

Proof. The existence of a function $u_p\in W_p^1(G)$ with $u_p-\varphi\in W_{p,0}^1(G)$ such that $\mathscr{E}_p(u_p)\leqslant\mathscr{E}_p(u)$ for all admissible u is established via a minimizing sequence. The uniqueness of u_p is an easy consequence of the strict convexity of $\mathscr{E}_p(u)$. The continuity of u_p is proved in [4, Theorem 3.1].

2.2. *Remark.* If $u\in W_p^1(G)$, $u-\varphi\in W_{p,0}^1(G)$, and $\mathscr{E}_p(u)\leqslant\mathscr{E}_p(u_p)$, then $u=u_p$. (Any a priori knowledge of the continuity of u is not needed to reach this conclusion.)

2.3 THEOREM. *A function $u_p \in \mathscr{F}_p$ minimizes $\mathscr{E}_p$ if and only if*

$$\int_G [|\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\eta + f\eta]\, dm = 0$$

for all test-functions $\eta \in C_0^\infty(G)$.

Proof. The convexity of the functional $u \to \mathscr{E}_p(u)$ and a well-known device, credited to Lagrange, give this simple result.

2.4. *Remark.* It is known that u_p is so regular that the equation in Theorem 2.3 is equivalent to the Euler–Lagrange equation $\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = f$, where the second derivatives are interpreted in Sobolev's sense, cf. [16].

2.5. LEMMA. *Suppose that $u_p \in \mathscr{F}_p$ minimizes $\mathscr{E}_p$. Then there is a continuous function $C(p)$ of p, $1 < p < \infty$, such that*

$$\int |\nabla u_p|^p\, dm \leqslant C(p) < \infty.$$

Proof. Using Poincaré's inequality [5, Eq. (7.44)]

$$\int |\eta|^p\, dm \leqslant \left(\frac{\text{mes } G}{\omega_n}\right)^{p/n} \int |\nabla\eta|^p\, dm$$

for $\eta = u_p - \varphi \in W^1_{p,0}(G)$, we obtain

$$\begin{aligned}\left|\int f u_p\, dm\right| &\leqslant \left|\int f\varphi\, dm\right| + \left|\int f(u_p - \varphi)\, dm\right| \\ &\leqslant \int |f\varphi|\, dm + \frac{\varepsilon^p}{p}\int |u_p - \varphi|^p\, dm + \frac{\varepsilon^{-p'}}{p'}\int |f|^{p'}\, dm \\ &\leqslant \int |f\varphi|\, dm + \frac{\varepsilon^p}{p}\left(\frac{\text{mes } G}{\omega_n}\right)^{p/n}\int |\nabla u_p - \nabla\varphi|^p\, dm \\ &\quad + \frac{\varepsilon^{-p'}}{p'}\int |f|^{p'}\, dm\end{aligned}$$

for any $\varepsilon > 0$; $p' = p/(p-1)$. Here Young's inequality [9, Eq. (1.3)] has been used. Because $\mathscr{E}_p(u_p) \leqslant \mathscr{E}_p(\varphi)$ and $\int |\nabla u_p - \nabla\varphi|^p\, dm \leqslant 2^p \int |\nabla u_p|^p\, dm + 2^p \int |\nabla\varphi|^p\, dm$, we arrive at

$$\frac{1}{p}\int |\nabla u_p|^p\,dm = \mathscr{E}_p(u_p) - \int f u_p\,dm$$

$$\leqslant \mathscr{E}_p(\varphi) + \int |f\varphi|\,dm + \frac{(2\varepsilon)^p}{p}\left(\frac{\text{mes } G}{\omega_n}\right)^{p/n}\int |\nabla u_p|^p\,dm$$

$$+\frac{(2\varepsilon)^p}{p}\left(\frac{\text{mes } G}{\omega_n}\right)^{p/n}\int |\nabla\varphi|^p\,dm + \varepsilon^{-p'}\int |f|^{p'}\,dm.$$

Fixing $\varepsilon=\varepsilon(p)$ so that $(2\varepsilon)^p(\text{mes } G/\omega_n)^{p/n}=\frac{1}{2}$, we clearly achieve an estimate of the desired type.

3. The Case $p \to q-0$

For the proof of (1.5) we need the bound

$$\overline{\lim_{p\to q-0}}\int |\nabla u_p|^p\,dm \leqslant C(q)$$

given in Lemma 2.5. By Poincaré's lemma

$$\overline{\lim_{p\to q-0}}\int |u_p|^p\,dm \leqslant 2^{q-1}\int |\varphi|^q\,dm$$

$$+4^{q-1}\left(\frac{\text{mes } G}{\omega_n}\right)^{q/n}\left\{C(q)+\int |\nabla\varphi|^q\,dm\right\},$$

where some simple arrangements have been made. These two bounds, Hölder's inequality, and a standard diagonalization process enables us to find a function $u\in W^1_{q-\varepsilon}(G)$ for all $\varepsilon>0$ $(\varepsilon<q-1)$ and to construct indices $p_1<p_2<\cdots$, $q=\lim p_k$, such that (1) $\nabla u_{p_k}\to\nabla u$ weakly in each $L^{q-\varepsilon}(G)$, $\varepsilon>0$ and (2) $u_{p_k}\to u$ strongly in each $L^{q-\varepsilon}(G)$, $\varepsilon>0$. In particular, $\int |\nabla u|^{q-\varepsilon}\,dm \leqslant \underline{\lim}\int |\nabla u_{p_k}|^{q-\varepsilon}\,dm \leqslant (\text{mes } G)^{\varepsilon/q}\,C(q)^{(q-\varepsilon)/q}$ and hence $\overline{\lim}_{\varepsilon\to 0}\int |\nabla u|^{q-\varepsilon}\,dm\leqslant C(q)$. Essentially by Lebesgue's convergence theorem this implies that $\int |\nabla u|^q\,dm\leqslant C(q)$. A similar reasoning also shows that $\int |u|^q\,dm<\infty$.

Thus we have proved that $u\in W^1_q(G)$. Now it follows easily that $u-\varphi\in W^1_{q,0}(G)$. Hölders' inequality and the inequality $\int |\nabla u|^{q-\varepsilon}\,dm\leqslant \underline{\lim}\int |\nabla u_{p_k}|^{q-\varepsilon}\,dm$ yields

$$\int |\nabla u|^{q-\varepsilon}\,dm \leqslant \left(\underline{\lim}\int |\nabla u_{p_k}|^{p_k}\,dm\right)^{(q-\varepsilon)/q}(mes\ G)^{\varepsilon/q}$$

and hence

$$\int |\nabla u|^q \, dm \leqslant \varliminf_{k \to \infty} \int |\nabla u_{p_k}|^{p_k} \, dm. \tag{3.1}$$

Since u_p is minimizing, we have

$$\begin{aligned} \frac{1}{p_k} \int |\nabla u_{p_k}|^{p_k} \, dm &= \mathscr{E}_{p_k}(u_{p_k}) - \int f u_{p_k} \, dm \\ &\leqslant \mathscr{E}_{p_k}(u) - \int f u_{p_k} \, dm \end{aligned}$$

and thus

$$\varlimsup_{k \to \infty} \frac{1}{p_k} \int |\nabla u_{p_k}|^{p_k} \, dm \leqslant \mathscr{E}_q(u) - \int f u \, dm = \frac{1}{q} \int |\nabla u|^q \, dm.$$

By (3.1) and the above estimate

$$\int |\nabla u|^q \, dm = \lim_{k \to \infty} \int |\nabla u_{p_k}|^{p_k} \, dm. \tag{3.2}$$

According to Clarkson's inequalities [1, Theorem 2.28, p. 37] we have (1) for $p_k \geqslant 2$,

$$\begin{aligned} &\int \left| \frac{\nabla u + \nabla u_{p_k}}{2} \right|^{p_k} dm + \int \left| \frac{\nabla u - \nabla u_{p_k}}{2} \right|^{p_k} dm \\ &\qquad \leqslant \frac{1}{2} \int |\nabla u|^{p_k} \, dm + \frac{1}{2} \int |\nabla u_{p_k}|^{p_k} \, dm \end{aligned}$$

and (2) for $1 < p_k \leqslant 2$,

$$\begin{aligned} &\left(\int \left| \frac{\nabla u + \nabla u_{p_k}}{2} \right|^{p_k} dm \right)^{1/(p_k - 1)} + \left(\int \left| \frac{\nabla u - \nabla u_{p_k}}{2} \right|^{p_k} dm \right)^{1/(p_k - 1)} \\ &\qquad \leqslant \left(\frac{1}{2} \int |\nabla u|^{p_k} \, dm + \frac{1}{2} \int |\nabla u_{p_k}|^{p_k} \, dm \right)^{1/(p_k - 1)}. \end{aligned}$$

According to (3.2) the right-hand members of the above inequalities approach $\int |\nabla u|^q \, dm$ and $(\int |\nabla u|^q \, dm)^{1/(q-1)}$, respectively, as $p_k \to q$. A similar reasoning as the one leading to (3.1) shows that

$$\int |\nabla u|^q \, dm \leqslant \varliminf \int \left| \frac{\nabla u + \nabla u_{p_k}}{2} \right|^{p_k} dm.$$

Thus the aforementioned inequalities of Clarkson imply that

$$\lim_{k\to\infty}\int\left|\frac{\nabla u-\nabla u_{p_k}}{2}\right|^{p_k} dm=0. \tag{3.3}$$

We claim that $u=u_q$. By (3.3) $\lim \mathscr{E}_{p_k}(u_{p_k})=\mathscr{E}_q(u)$ and, since $\mathscr{E}_{p_k}(u_{p_k})\leqslant \mathscr{E}_{p_k}(u_q)$, $\lim \mathscr{E}_{p_k}(u_{p_k})\leqslant \mathscr{E}_q(u_q)$. This means that $\mathscr{E}_q(u)\leqslant \mathscr{E}_q(u_q)$ and hence $u=u_q$ (a.e. in G) by Remark 2.2.

We have obtained the result

$$\lim_{k\to\infty}\int|\nabla u_q-\nabla u_{p_k}|^{p_k}\,dm=0. \tag{3.3$'$}$$

In order to arrive at (1.5) we fix an arbitrary sequence $p_1', p_2', ..., p_k'<q$, $\lim p_k'=q$. By the above method we can extract an increasing subsequence, say $p_1, p_2, ...$ such that (3.3′) holds. Since the limit ∇u_q does not depend on the particular choice of $p_1', p_2', ...$, we deduce that (1.5) holds (reductio ad absurdum).

4. The Case $p\to q+0$

The crucial difficulty here is that the eventual possibility that

$$\int_G|\nabla u_q|^p\,dm=\infty$$

for each $p>q$ hinders us from reaching conclusions like $\lim_p\int|\nabla u_p|^p\,dm=\int|\nabla u_q|^q\,dm$ or even $\lim_p\int|\nabla u_p|^q\,dm = \int|\nabla u_q|^q\,dm$. Therefore we have assumed that

$$\int_G|\nabla u_q|^{q+\varepsilon}\,dm<\infty \tag{4.1}$$

for some $\varepsilon>0$.

4.2. *Remark.* According to an advanced theory of higher integrability, initiated by Bojarskii (1957) and developed by Gehring (1973) and Meyers [12], the local result

$$\int_K|\nabla u_q|^{q+\varepsilon}\,dm<\infty \qquad (K\Subset G)$$

always holds for some $\varepsilon>0$ depending mainly on K, q, and φ. However, if the boundary ∂G is regular enough (remember that φ is smooth), then (4.1) is valid. (For example, cubes and smooth domains are regular enough.) This fact, justifying our assumption, can be proved by the method in [8]. See also [3, Proposition 5.1].

Let us prove (1.6). Using Lemma 2.5 we can construct indices $p_1 > p_2 > \dots$, $q = \lim p_k$, and find a function $u \in W_q^1(G)$ such that $u - \varphi \in W_{q,0}^1(G)$ and $\nabla u_{p_k} \to \nabla u$ weakly in $L^q(G)$ and $u_{p_k} \to u$ strongly in $L^q(G)$. Especially,[1] $\int |\nabla u|^q \, dm \leqslant \underline{\lim} \int |\nabla u_{p_k}|^q \, dm$, a fact that together with Hölder's inequality yields

$$\int |\nabla u|^q \, dm \leqslant \underline{\lim} \int |\nabla u_{p_k}|^{p_k} \, dm. \tag{4.3}$$

We claim that $u = u_q$. According to the uniqueness (Remark 2.2) it is sufficient to establish that $\mathscr{E}_q(u) \leqslant \mathscr{E}_q(u_q)$. To this end, note that $\mathscr{E}_{p_k}(u_{p_k}) \leqslant \mathscr{E}_{p_k}(u_q)$ for $q < p_k \leqslant q + \varepsilon$. Hence $\overline{\lim}\, \mathscr{E}_{p_k}(u_{p_k}) \leqslant \mathscr{E}_q(u_q)$. By (4.3) we have

$$\mathscr{E}_q(u) = \frac{1}{q} \int |\nabla u|^q \, dm + \int fu \, dm$$

$$\leqslant \underline{\lim} \frac{1}{p_k} \int |\nabla u_{p_k}|^{p_k} \, dm + \lim \int f u_{p_k} \, dm = \underline{\lim}\, \mathscr{E}_{p_k}(u_{p_k}).$$

Collecting the results we arrive at $\mathscr{E}_q(u) \leqslant \mathscr{E}_q(u_q)$, as desired.

Since $u = u_q$, the above reasoning also shows that

$$\lim \mathscr{E}_{p_k}(u_{p_k}) = \mathscr{E}_q(u_q).$$

Because $\lim \int f u_{p_k} \, dm = \int fu \, dm = \int f u_q \, dm$, this implies that

$$\lim \int |\nabla u_{p_k}|^{p_k} \, dm = \int |\nabla u_q|^q \, dm. \tag{4.4}$$

Clarkson's inequalities in Section 3 are valid for $q < p_k \leqslant q + \varepsilon$. They again imply, in virtue of (4.4), that

$$\lim \int \left| \frac{\nabla u_{p_k} - \nabla u_q}{2} \right|^{p_k} dm = 0$$

by a similar argument as that in Section 3. From this we again obtain that $\lim_{p \to q+0} \int |\nabla u_p - \nabla u_q|^p \, dm = 0$. This concludes our proof.

5. On the Local Case

The actual boundary conditions, for example, in the glaceological model in Section 1 being so involved that they seem to require the introduction of an arsenal of trace spaces with an imbedding theory for these, it is desirable to treat the stability problem *locally*. Introducing the following well-known concept, we get rid of all boundary conditions:

We say that $u \in C(G) \cap W^1_{p,\mathrm{loc}}(G)$ minimizes $\mathscr{E}_p$ locally, if

$$\int_G |\nabla u|^{p-2}\,\nabla u \cdot \nabla \eta\, dm = \int_G f\eta\, dm$$

whenever $\eta \in C_0^\infty(G)$.

In view of Remark 4.2 the assumption that $\int_G |\nabla u_q|^{q+\varepsilon}\, dm < \infty$ for some $\varepsilon > 0$ is superfluous, the corresponding local result being proved in [12].

5.1. Theorem. *Suppose that u_{p_k} minimizes $\mathscr{E}_{p_k}$ locally for $k = 1, 2, 3, \ldots$, $q = \lim p_k$, and that $\overline{\lim} \int_K |\nabla p_k|^{p_k}\, dm < \infty$, whenever $K \Subset G$. Then there are indices $k_1 < k_2 < \cdots$ and a function u_q, minimizing $\mathscr{E}_q$ locally, such that*

$$\lim_{p \to q} \int |\eta(\nabla u_p - \nabla u_q)|^p\, dm = 0 \qquad (1 < q < \infty)$$

where p approaches q via the values $p_{k_1}, p_{k_2}, \ldots$ and $\eta \in C_0^\infty(G)$ is arbitrary.

About the Proof. The proof is a local variant of the methods in the global case and contains no essentially new ideas beyond those in the global proof. However, unessential technical complications makes the proof lengthy. We omit this proof.

References

1. R. Adams, "Sobolev Spaces," Academic Press, New York/San Francisco/London, 1975.
2. M. Fortin and R. Glowinski, "Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-Value Problems," North-Holland, Amsterdam/New York/Oxford, 1983.
3. M. Giaquinta and G. Modica, Regularity results for some classes of higher order non-linear elliptic systems, *J. Reine Angew. Math.* **311/312** (1979), 145–169.
4. M. Giaquinta and E. Giusti, On the regularity of the minima of variational integrals, *Acta Math.* **148** (1982), 31–46.
5. D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.

6. R. Glowinski and A. Marocco, On the solutions of a class of non-linear Dirichlet problems by a penalty-duality method and finite elements of order one, *in* "Optimization Techniques, IFIP Technical Conference" (G. I. Marchouk, Ed.), Lecture Notes in Computer Sciences **27**, pp. 327–333, Springer-Verlag, Berlin, 1975.
7. R. Glowinski and A. Marocco, Sur l'approximation par éléments finis d'ordre 1, et la résolution par penalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires, *C. R. Acad. Sci. Paris Sér. A* **278** (1984), 1649–1652.
8. S. Granlund, An L^p-estimate for the gradient of extremals, *Math. Scand.* **50** (1982), 66–72.
9. O. Ladyzhenskaya and N. Ural'tseva, "Linear and Quasi-Linear Elliptic Equations," Academic Press, New York/London, 1968.
10. E. Landesman and A. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.* **19** (1970), 609–623.
11. L. Lliboutry, "Traité de Glaciologie," Masson & Cie, Paris, (I) 1964 et (II) 1965.
12. N. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 189–206.
13. C. Morrey, "Multiple Integrals in the Calculus of Variations," Springer-Verlag, Berlin/Heidelberg/New York, 1966.
14. M. C. Pelissier, "Sur Quelques Problèmes Non Lineaires en Glaciologie, Thèse, Université de Paris-Sud; Publ. Math. Orsay, 1975.
15. M. C. Pélissier and L. Reynaud, Étude d'un modèle mathématique d'écoulement de glacier, *C. R. Acad. Sci. Paris Sér. A* **279** (1979), 531–534.
16. P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* **51** (1984), 126–150.