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On the Lupaş q-transform

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ABSTRACT

The *Lupaş q-transform* emerges in the study of the *limit q-Lupaş operator*. The latter comes out naturally as a limit for a sequence of the Lupaş q-analogues of the Bernstein operator. Lately, it has been studied by several authors from different perspectives in mathematical analysis and approximation theory. This operator is closely related to the q-deformed Poisson probability distribution, which is used widely in the q-boson operator calculus.

Given $q \in (0, 1)$, $f \in C[0, 1]$, the *q-Lupaş transform* of f is defined by:

$$(\Lambda_q f)(z) := \frac{1}{(-z; q)_\infty} \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k)q^{k(k-1)/2}}{(q; q)_k} z^k.$$

In this paper, we study some analytic properties of $(\Lambda_q f)(z)$. In particular, we examine the conditions under which $\Lambda_q f$ can either be an entire function, or a rational one.

1. Introduction

The importance of the Bernstein polynomials opened the gates to the discovery of their numerous generalizations and applications in various mathematical disciplines. As an example, recent generalizations based on the q-integers have emerged due to the speedy development of the q-calculus. Alexandru Lupaş was the person who pioneered the work on the q-versions of the Bernstein polynomials. In 1987, he introduced a q-analogue of the Bernstein operator, and investigated its approximation and shape-preserving properties (cf. [1]). Since then, the study of the q-analogue has been in progress. See [2–6], where different convergent properties of the q-analogue have been investigated.

It should be mentioned here that, lately, another generalization of Bernstein polynomials based on the q-integers, called the *q-Bernstein polynomials* has been studied by many authors (cf. [7] and the references therein). So far, nevertheless, the Lupaş q-analogues have remained in shadows. However, they possess an advantage of being *positive* linear operators for all $q > 0$, while the q-Bernstein polynomials yield positive linear operators only for $q \in (0, 1]$.

In what follows, we use the following standard notation (cf. [8], Ch. 10, Section 10.2):

$$(z; q)_0 := 1; \qquad (z; q)_n := \prod_{k=0}^{n-1}(1 - zq^k); \qquad (z; q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k).$$

Let $q > 0$. For any $n = 0, 1, 2, \ldots$ the *q-integer* $[n]_q$ is defined by:

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad (n = 1, 2, \ldots), \qquad [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by:

$$[n]_q! := [1]_q[2]_q \ldots [n]_q \quad (n = 1, 2, \ldots), \qquad [0]_q! := 1.$$

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For integers $0 \le k \le n$, the *q-binomial* coefficient is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-binomial coefficients appear in the q-binomial theorem. We refer to the following particular cases of this theorem; see [8], Ch. 10, Section 10.2, Corollary 10.2.2. The first one is an extension of Newton's binomial formula:

$$(-z;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} z^k. \tag{1.1}$$

The other needed expansions are Euler's Identities:

$$(-z;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} z^k, \quad |q| < 1; \tag{1.2}$$

and

$$\frac{1}{(-z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k} z^k, \quad |q| < 1,\ |z| < 1. \tag{1.3}$$

Following Lupaş, we denote, for $n \in \mathbb{N}$,

$$b_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+qx)\cdots(1-x+q^{n-1}x)}, \quad k = 0, \ldots, n. \tag{1.4}$$

Equality (1.1) implies that

$$\sum_{k=0}^{n} b_{nk}(q;x) = 1 \quad \text{for } x \notin \left\{ \frac{1}{1-q^j} \right\}_{j=1}^{\infty}.$$

Definition 1.1. Let $q > 0,\ f : [0,1] \to \mathbb{C}$. The *Lupaş q-analogue of the Bernstein operator* is defined by:

$$(R_{n,q}f)(x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{nk}(q;x), \quad x \in [0,1],\ n \in \mathbb{N}, \tag{1.5}$$

where $b_{nk}(q;x)$ are given by (1.4). The function $(R_{n,q}f)(x)$ was introduced in [1], where it was called a *q-analogue of the Bernstein operator*. We call it the Lupaş q-analogue to emphasize the role of Lupaş in studying the first of the several presently known q-versions of the Bernstein polynomials.

Clearly, if $q = 1$, then (1.5) reduces to the classical Bernstein polynomials. In the case $q \neq 1$, $R_{n,q}$ are *rational functions* rather than polynomials. It follows directly from (1.5) that $R_{n,q}$ generate *positive* linear operators on $C[0,1]$ possessing the *end-point interpolation* property:

$$(R_{n,q}f)(0) = f(0), \qquad (R_{n,q}f)(1) = f(1).$$

Besides, the linear functions are fixed points of these operators. The convergence of $\{(R_{n,q}f)(x)\}$ for $q \neq 1$ being constant has been studied in [3,4,6]. To present the relevant results, we first mention the following duality formula (see [4], Theorem 3):

With $g(x) := f(1-x)$, we have, for any $q > 0$,

$$R_{n,q}(f;x) = R_{n,1/q}(g;1-x) \quad \text{for } x \in [0,1].$$

This equality allows us to reduce the case $q > 1$ in the study of the Lupaş q-analogue of the Bernstein operator to the case $0 < q < 1$. Therefore, in all further considerations, we restrict our attention only to the case $0 < q < 1$.

The next operator turns out to be helpful in the investigation of convergence.

Definition 1.2. Let $0 < q < 1$. The *limit q-Lupaş operator* on $C[0,1]$ is given by:

$$(R_{\infty,q}f)(x) := \begin{cases} \dfrac{1}{(-x/(1-x);q)_\infty} \cdot \displaystyle\sum_{k=0}^{\infty} f(1-q^k) l_{\infty,k}\left(q; \frac{x}{1-x}\right) & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1 \end{cases} \tag{1.6}$$

where

$$l_{\infty,k}(q;t) := \frac{q^{k(k-1)/2} t^k}{(q;q)_k}.$$

As it has been proved in [4], for any $f \in C[0,1]$, the sequence $\{(R_{n,q}f)(x)\}$ converges uniformly on $[0,1]$ to $(R_{\infty,q}f)(x)$, where $R_{\infty,q}f = f$ if and only if f is a linear function. It should be emphasized that $R_{n,q}$, despite being positive linear operators on $C[0, 1]$, do not satisfy the conditions of Korovkin's Theorem, and that the limit operator $R_{\infty,q}$ is not the identity operator. However, operators $R_{n,q}$ satisfy the conditions of Wang's Korovkin-type Theorem (cf. [9], Theorem 2), and serve as an example for it. Not only does this theorem guarantee the existence of the limit operator $R_{\infty,q}$, but it also provides an estimate for the rate of convergence via the second modulus of smoothness. A probabilistic approach to this operator has been developed in [10], whose results imply that $R_{\infty,q}$ is closely related to the q-deformed Poisson distribution. The latter plays an important role in the q-boson operator calculus (cf., e.g. [11,12]), where it is used to describe the distribution of energy in a q-analogue of the coherent state.

In this paper, the interest is on the function $(R_{\infty,q}f)(z/(z+1))$. Since $x = z/(z+1)$, this means that we use the substitution $z = x/(1-x)$ in (1.6). With the help of the power series and the infinite product involved in (1.6), we may extend our considerations to the complex plane. To be specific, we employ the following definition.

Definition 1.3. Let $0 < q < 1$, $f \in C[0, 1]$. The *Lupaş q-transform* of f is given by:

$$(\Lambda_q f)\,(z) := \frac{1}{(-z;q)_\infty} \cdot \sum_{k=0}^{\infty} f(1-q^k)\,\frac{q^{k(k-1)/2}}{(q;q)_k}\,z^k, \quad z \in \mathbb{C} \setminus \{-q^k\}_{k\in\mathbb{Z}_+}.$$

If we restrict Λ_q to the space $C[0, 1]$ over the reals, we obtain a positive linear operator of $C[0, 1]$ into itself with norm 1. Generally speaking, $\Lambda_q f$ is evidently a meromorphic function whose simple poles are contained in the set

$$J_q := \{-q^k\}_{k\in\mathbb{Z}_+}.$$

However we notice that, although $\Lambda_q f$ is generally considered to be a meromorphic function, in some special cases it may be an entire or a rational one. For example,

$$(\Lambda_q 1)\,(z) = 1 \quad \text{and} \quad (\Lambda_q t)\,(z) = \frac{z}{1+z}.$$

In this paper, we supply necessary and sufficient conditions for $\Lambda_q f$ to be either an entire or a rational function, as mentioned above.

2. Statement of results

Within this section and the forthcoming one, we assume $0 < q < 1$ to be fixed.

We notice that $\Lambda_q f$ is uniquely determined by the values of f on $\{1-q^k\}$ and, conversely, a function $\Lambda_q f$ defines the values $\{f(1-q^k)\}_{k=0}^{\infty}$. It is natural, therefore, to consider the following equivalence relation on $C[0, 1]$:

$$f \sim g \Leftrightarrow f(1-q^k) = g(1-q^k) \quad \text{for all } k \in \mathbb{Z}_+.$$

Obviously, $\Lambda_q f = \Lambda_q g$ if and only if $f \sim g$.

Before our results are presented, let us mention that operators with a structure similar to that of the limit q-Lupaş have been studied from different angles in [13–15,10,16,17]. Unlike the problems addressed in those articles, the present paper aims to examine questions of an entirely different nature. Specifically, it deals with some analytic properties of a meromorphic function $(\Lambda_q f)(z)$.

The main results of the paper are the following statements.

Theorem 2.1. *The function $(\Lambda_q f)(z)$ is entire if and only if $f \sim C = const$.*

The theorem implies that all singularities of $(\Lambda_q f)(z)$ are removable if and only if $\Lambda_q f$ is constant.

Suppose now that $(\Lambda_q f)(z)$ has a finite number of poles. The next theorems shows that in this case $(\Lambda_q f)(z)$ is a rational function.

We denote by $\mathcal{P}_m$ the set of all polynomials of degree $\le m$ with complex coefficients.

Theorem 2.2. *The function $(\Lambda_q f)(z)$ has a finite number of poles, say, $-q^{-i_1}, \ldots, -q^{-i_p}$ $(i_1 < \cdots < i_p)$ if and only if $f \sim g \in \mathcal{P}_m$ with $m = 1 + i_p$.*

Remark 2.1. It is worth pointing out that the degree of g is determined by the position of the pole farthest from 0 rather than by the quantity of the poles.

Corollary 2.3. *A function $(\Lambda_q f)$ has a finite number of poles if and only if it is rational.*

3. Proofs of the theorems

Our reasonings are based on the following auxiliary lemmas which are of interest for their own sake.

Lemma 3.1. *For all $j \in \mathbb{Z}_+$, the following equalities hold:*

$$(\Lambda_q (1-t)^j)\,(z) = \frac{1}{(-z;q)_j}. \tag{3.1}$$

Proof. Applying the definition, we write:

$$(\Lambda_q(1-t)^j)\,(z) = \frac{1}{(-z;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^j z)^k}{(q;q)_k}.$$

By virtue of Euler's Identity (1.2), we have:

$$\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^j z)^k}{(q;q)_k} = (-q^j z;q)_\infty = \frac{(-z;q)_\infty}{(-z;q)_j}.$$

The statement now follows. □

Lemma 3.2. *The function $\Lambda_q f$ has the form*

$$(\Lambda_q f)(z) = \frac{p(z)}{(-z;q)_m} \quad \text{with } p \in \mathcal{P}_m$$

if and only if $f \sim g \in \mathcal{P}_m$. In particular, $(\Lambda_q f)(z) = C = const$ if and only if $f \sim C$.

Proof. If $f \sim g \in \mathcal{P}_m$, then $g(t) = \sum_{j=0}^{m} a_j(1-t)^j$ and Lemma 3.1 implies that

$$(\Lambda_q f)\,(z) = (\Lambda_q g)\,(z) = \frac{p_f(z)}{(-z;q)_m}, \quad \text{where } p_f \in \mathcal{P}_m.$$

Now, we consider the linear operator $T_m : \mathcal{P}_m \to \mathcal{P}_m$ given by $T_m(f) = p_f$. Clearly, $f \in \mathrm{Ker}(T_m)$ if and only if $f(1-q^k) = 0$ for all $k \in \mathbb{Z}_+$, whence $\mathrm{Ker}(T_m) = \{0\}$. Hence T_m is invertible on $\mathcal{P}_m$ and for any $p \in \mathcal{P}_m$ there exists $f \in \mathcal{P}_m$ so that

$$(\Lambda_q f)(z) = \frac{p(z)}{(-z;q)_m}. \quad \square$$

In what follows, we denote by letter C (possibly with indices) a positive constant whose value does not need to be specified. The indices on C may be either a numbering (that is, if more that one constant is involved) or an indicator of the dependence on certain parameters. Having said so, we write $f(x) \asymp g(x)$ if $C_1 f(x) \le g(x) \le C_2 f(x)$ for some C_1 and C_2.

Proof of Theorem 2.1. If $f \sim C$, then Lemma 3.1 implies immediately that $(\Lambda_q f)(z) \equiv C$ and the statement of the theorem is obvious.

For $f \in C[0,1]$, we denote:

$$\rho_f(z) := \sum_{k=0}^{\infty}\frac{f(1-q^k)q^{k(k-1)/2}}{(q;q)_k} z^k. \tag{3.2}$$

Then, for any $f \in C[0,1]$, ρ_f is an entire function satisfying

$$|\rho_f(z)| \le \|f\| \cdot (-|z|;q)_\infty.$$

The growth estimates for $(-|z|;q)_\infty$ have been found, for example, in [18], formula (2.6):

$$(-r;q)_\infty \asymp \exp\left\{\frac{\ln^2(r/\sqrt{q})}{2\ln(1/q)}\right\}, \quad r > 0, \tag{3.3}$$

whence

$$|\rho_f(z)| \le C_1\|f\| \cdot \exp\left\{\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\}. \tag{3.4}$$

Now, assume that $(\Lambda_q f)$ is an entire function. Then $(\Lambda_q f)(z)$ is a quotient of the entire functions ρ_f and $(-z;q)_\infty$ whose growth estimates are given by (3.4) and (3.3), respectively. It has been known that if a quotient $\varphi = f_1/f_2$ of entire functions f_1 and f_2 is entire, then (see, e.g. [19], Chapter 2, Section 2.4) the following estimate holds:

$$\ln M(r;\varphi) \le 3\ln M(2r;f_1) + 3\ln M(2r;f_2) + O(1), \tag{3.5}$$

where

$$M(r;f) := \max_{|z|\le r}|f(z)|.$$

Hence if $\Lambda_q f$ is entire, its growth is estimated by

$$M(r;\Lambda_q f) \le C_2\exp\{C_3\ln^2 r\}.$$

Apart from that, we have:

$$|(\Lambda_q f)(x)| \leq \|f\| \quad \text{for } x > 0.$$

Using the Phragmén–Lindelöf Theorem (cf. [19], Chapter 6, Section 6.1) we conclude that

$$|L(f, q; z)| \leq \|f\| \quad \text{for all } z \in \mathbb{C}.$$

Therefore, by Liouville's Theorem $\Lambda_q f$ is constant. Applying Lemma 3.2, we get the required statement. □

Proof of Theorem 2.2. If $f \sim g \in \mathcal{P}_m$, then, by Lemma 3.2, $(\Lambda_q f)(z)$ is a rational function with a simple pole at $-q^{m-1}$ and, possibly, poles contained in $\{-q^{-1}, \ldots, -q^{m-2}\}$, as stated.

Assume that $(\Lambda_q f)(z)$ has a finite number of poles, say $-q^{-i_1}, \ldots, -q^{-i_p}$ $(i_1 < \cdots < i_p)$. We set

$$P(z) := \prod_{s=1}^{p}(1 + zq^{i_s}).$$

Then $(\Lambda_q f)(z) \cdot P(z) =: h(z)$ is an entire function. We write:

$$h(z) = \frac{\rho_f(z)}{(-z; q)_\infty / P(z)}, \tag{3.6}$$

where ρ_f is defined by (3.2). Both the numerator and denominator of (3.6) are entire functions whose growth estimates are given by (3.4) and (3.3). As $h(z)$ is an entire function, we apply (3.5) to estimate its growth and we conclude that

$$M(r; h) \leq C_1 \exp\{C_2 \ln^2 r\}.$$

In addition,

$$|h(x)| \leq \|f\| \cdot |P(x)| \leq C_3 + C_4 x^p \quad \text{for } x > 0.$$

From the Phragmén–Lindelöf Theorem, it follows that

$$|h(z)| \leq C_5 + C_6 |z|^p, \quad z \in \mathbb{C},$$

whence by Liouville's Theorem, $h(z) \in \mathcal{P}_p$.

We set $m = i_p + 1$. Then

$$(\Lambda_q f)(z) = \frac{h(z)}{P(z)} = \frac{h_1(z)}{(-z; q)_m},$$

where

$$h_1(z) = h(z) \cdot \prod_{0 \leq k \leq m,\ k \neq j_s} (1 + q^k z) \in \mathcal{P}_m.$$

The statement now follows from Lemma 3.2. □

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