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New examples of entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$

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Abstract

In this paper we obtain new explicit examples of complete and non-complete entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$. The existence of these entire maximal graphs shows that entire maximal graphs in this Lorentzian product space are not necessarily complete, on the contrary that in the Lorentz–Minkowski space. Moreover, in [A.L. Albujer, L.J. Alías, Calabi–Bernstein results for maximal surfaces in Lorentzian product spaces, Preprint, 2006], the author jointly with Alías gave a Calabi–Bernstein theorem for maximal surfaces immersed into the Lorentzian product space $M^2 \times \mathbb{R}_1$, where M^2 is a connected Riemannian surface of non-negative Gaussian curvature, and these examples show that the assumption on K_M is necessary.

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1. Introduction

The importance in General Relativity of maximal surfaces, that is, spacelike surfaces with zero mean curvature and, more generally, of spacelike surfaces with constant mean curvature, is well known (see, for instance, [9]). Here by spacelike we mean that the induced metric from the ambient Lorentzian metric is a Riemannian metric on the surface. The terminology *maximal* comes from the fact that these surfaces locally maximize area among all nearby surfaces having the same boundary [4,8].

One of the most relevant global results in Lorentzian geometry is the Calabi–Bernstein theorem for maximal surfaces in the Lorentz–Minkowski space $\mathbb{R}^3_1$. It states, in a parametric version, that the only complete maximal surfaces in $\mathbb{R}^3_1$ are the spacelike planes. This result was first proved by Calabi [5], and extended later to the general n-dimensional Lorentz–Minkowski space by Cheng and Yau [6].

An interesting remark on the topology of spacelike surfaces in the Lorentz–Minkowski space is that every complete spacelike surface in $\mathbb{R}^3_1$ can be seen as an entire graph. But, it is worth pointing out that the converse is not true in general. In fact, there exist examples of spacelike entire graphs in $\mathbb{R}^3_1$ which are not complete. This fact points out

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a curious difference between the behaviour of surfaces in Euclidean space $\mathbb{R}^3$ and that of surfaces in the Lorentz–Minkowski space. Actually, recall that every embedded surface in the Euclidean space which is a closed subset in $\mathbb{R}^3$ is necessarily complete; in particular every entire graph is complete (see, for instance, [2] for more details). However, in the case where the mean curvature is constant, every embedded spacelike surface in the Lorentz–Minkowski space which is a closed subset in $\mathbb{R}^3_1$ is necessarily complete, [6]. Therefore, the Calabi–Bernstein theorem can also be established in an equivalent non-parametric form, and it asserts that the only entire maximal graphs in $\mathbb{R}^3_1$ are the spacelike planes.

Calabi–Bernstein type properties have also been studied in other 3-dimensional Lorentzian ambient spaces. For instance, Latorre and Romero [7] solved new Calabi–Bernstein problems in Robertson–Walker spaces under certain assumptions on the warping function. The author, jointly with Alías, proved in [1] a Calabi–Bernstein type result for maximal surfaces immersed into a Lorentzian product space of the form $M \times \mathbb{R}_1$, where M is a connected Riemannian surface of non-negative Gaussian curvature, K_M, in both versions, the parametric and the non-parametric one. As a direct consequence of [3, Proposition 3.3], when M is simply connected the fact we pointed before that every complete spacelike surface in the Lorentz–Minkowski space can be seen as an entire graph is still true for spacelike surfaces in those Lorentzian product spaces. However, what it is not true in general is that every entire maximal graph in a Lorentzian product space is necessarily complete. Therefore, both versions of the theorem are independent. In [1] we also showed that the assumption on the non-negativity of K_M is necessary. To see it, we gave counterexamples of the Calabi–Bernstein theorem in both versions. In particular we showed the existence of complete and non-complete entire maximal graphs, different to slices, in the Lorentzian product $\mathbb{H}^2 \times \mathbb{R}_1$ via a duality result between minimal entire graphs in $\mathbb{H}^2 \times \mathbb{R}$ and maximal entire graphs in $\mathbb{H}^2 \times \mathbb{R}_1$, but we did not give the explicit expressions for these graphs.

Montaldo and Onnis gave in [10] explicit examples of entire minimal graphs in the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ (see also [11]). These graphs are necessarily complete since they are embedded in a Riemannian manifold. Motivated by that work, we provide new explicit examples of complete, Example 3.2, and non-complete, Examples 3.1 and 3.3, entire maximal graphs in the Lorentzian product space $\mathbb{H}^2 \times \mathbb{R}_1$, reason why it complements [1]. Moreover, the existence of these examples is also interesting because shows that the equivalence in the Lorentz–Minkowski space $\mathbb{R}^3_1$ between spacelike graphs of constant mean curvature and spacelike complete surfaces of constant mean curvature fails in $\mathbb{H}^2 \times \mathbb{R}_1$.

2. Preliminaries

Along this work, we will consider the upper half-plane model for the hyperbolic plane, that is,

$$\mathbb{H}^2 = \{x = (x_1, x_2) \in \mathbb{R}^2:\ x_2 > 0\}$$

endowed with the complete metric

$$\langle\, ,\rangle_{\mathbb{H}^2} = \frac{1}{x_2^2}(dx_1^2 + dx_2^2).$$

The hyperbolic plane is a Riemannian surface of constant Gaussian curvature -1. Let us consider now the product manifold $\mathbb{H}^2 \times \mathbb{R}$ endowed with the Lorentzian metric

$$\langle\, ,\rangle = \pi^*_{\mathbb{H}}(\langle\, ,\rangle_{\mathbb{H}^2}) - \pi^*_{\mathbb{R}}(dt^2),$$

where $\pi_{\mathbb{H}}$ and $\pi_{\mathbb{R}}$ denote the projections from $\mathbb{H}^2 \times \mathbb{R}$ onto each factor. For simplicity, we will write

$$\langle\, ,\rangle = \langle\, ,\rangle_{\mathbb{H}^2} - dt^2,$$

and we will denote by $\mathbb{H}^2 \times \mathbb{R}_1$ the 3-dimensional product manifold $\mathbb{H}^2 \times \mathbb{R}$ endowed with that Lorentzian metric.

Let $\Omega \subseteq \mathbb{H}^2$ be a connected domain. A graph over Ω is determined by a smooth function $u \in \mathcal{C}^\infty(\Omega)$ and it is given by

$$\Sigma(u) = \{(x, u(x)):\ x \in \Omega\} \subset \mathbb{H}^2 \times \mathbb{R}_1.$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma(u)$ is

$$\langle\, ,\rangle = \langle\, ,\rangle_{\mathbb{H}^2} - du^2. \tag{1}$$

The graph is said to be entire if $\Omega = \mathbb{H}^2$. It can be easily seen that a graph $\Sigma(u)$ is a spacelike surface if and only if $|Du|^2 < 1$ being Du the gradient of u in Ω and $|Du|$ its norm, both with respect to the hyperbolic metric $\langle\, ,\rangle_{\mathbb{H}^2}$ in Ω.

If $\Sigma(u)$ is a spacelike graph over a domain Ω, then it is not difficult to see that the vector field

$$N(x) = \frac{1}{\sqrt{1-|Du(x)|^2}}\big(Du(x) + \partial_t|_{(x,u(x))}\big), \quad x \in \Omega,$$

defines the future-pointing Gauss map of $\Sigma(u)$. That is, N is the unique unitary normal vector field globally defined on $\Sigma(u)$ which is in the same time-orientation as ∂_t, so that

$$\langle N, \partial_t\rangle \leqslant -1 < 0 \quad \text{on } \Sigma.$$

The shape operator (or second fundamental form) of $\Sigma(u)$ with respect to N, $A : T\Sigma(u) \to T\Sigma(u)$ is given by

$$AX = -\frac{1}{\sqrt{1-|Du|^2}} D_X Du - \frac{\langle D_X Du, Du\rangle_{\mathbb{H}^2}}{(1-|Du|^2)^{3/2}} Du,$$

for every tangent vector field X on Ω, where D denotes the Levi–Civita connection in Ω with respect to the metric $\langle\, ,\rangle_{\mathbb{H}^2}$. It follows from here that the mean curvature of a spacelike graph $\Sigma(u)$, $H(u) = -(1/2)\operatorname{tr}(A)$, is given by

$$2H(u) = \operatorname{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)$$

where Div stands for the divergence operator on Ω with respect to the metric $\langle\, ,\rangle_{\mathbb{H}^2}$. Consequently, $\Sigma(u)$ is a maximal graph if and only if the function u satisfies the following partial differential equation on the domain Ω,

$$\operatorname{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du|^2 < 1. \tag{2}$$

We will refer to this equation as the maximal surface equation.

The hyperbolic metric $\langle\, ,\rangle_{\mathbb{H}^2}$ is conformal to the flat Euclidean metric, which allows us to express (2) in terms of the Euclidean differential operators. For a given smooth function $u = u(x) \in \mathcal{C}^\infty(\mathbb{H}^2)$, its hyperbolic gradient Du and its Euclidean gradient $D_o u$ in $\mathbb{R}^2$ are related by

$$Du(x) = x_2^2 D_o u(x), \quad x = (x_1, x_2),$$

and then

$$\big|Du(x)\big|^2 = x_2^2 \big|D_o u(x)\big|_o^2,$$

where $|\cdot|$ and $|\cdot|_o$ denote, respectively, the norm of a vector field in $\mathbb{H}^2$ and in $\mathbb{R}^2$. In particular,

$$\frac{Du(x)}{\sqrt{1+|Du(x)|^2}} = \frac{x_2^2 D_o u(x)}{\sqrt{1+x_2^2|D_o u(x)|_o^2}}. \tag{3}$$

The divergence Div of the hyperbolic metric and the divergence Div_o of the Euclidean metric are related by

$$\operatorname{Div} = \operatorname{Div}_o - \frac{2}{x_2}\, dx_2. \tag{4}$$

Finally, from (3) and (4), the maximal surface equation is equivalent to

$$\big(1 - x_2^2|D_o u|_o^2\big)x_2^2 \Delta_o u + x_2^2\big(x_2 u_{x_2}|D_o u|_o^2 + x_2^2 Q(u)\big) = 0 \tag{5}$$

and

$$x_2^2|D_o u|_o^2 < 1, \tag{6}$$

where Δ_o stands for the Euclidean Laplacian, and

$$Q(u) = u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2}.$$

3. Examples of maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$

Let us show some explicit entire solutions to the maximal surface equation (2). To get this, we will solve Eqs. (5) and (6) for particular types of smooth functions.

Example 3.1. First, we will look for solutions depending only on one variable. If we consider solutions of type $u(x_1, x_2) = u(x_1)$, (5) reduces to $u''(x_1) = 0$, so u must be of the form $u(x_1, x_2) = ax_1 + b$, $a, b \in \mathbb{R}$. By (6), $x_2^2 u'(x_1)^2 = a^2 x_2^2 < 1$ so, as we are looking for entire graphs, the inequality is only satisfied in $\mathbb{H}^2$ for $a = 0$. Consequently, $u(x_1, x_2) = b$, $b \in \mathbb{R}$, which correspond to slices. Moreover, these are the only spacelike planes in $\mathbb{H}^2 \times \mathbb{R}_1$.

Now, let $u(x_1, x_2) = u(x_2)$ depend only on the second variable, then u will determine a maximal surface when it verifies

$$u''(x_2) + x_2 u'(x_2)^3 = 0 \tag{7}$$

and

$$x_2^2 u'(x_2)^2 < 1. \tag{8}$$

From (7) by integration we get

$$u(x_1, x_2) = u(x_2) = \ln\bigl(x_2 + \sqrt{a + x_2^2}\bigr) + b, \quad a, b \in \mathbb{R},\ a \geqslant 0.$$

Observe that we can assume $b = 0$ unless a translation, which is an isometry in the ambient space. From (8), u must satisfy

$$x_2^2 u'(x_2)^2 = \frac{x_2^2}{a + x_2^2} < 1,$$

which holds if and only if $a > 0$. We have obtained in that way a family of entire maximal graphs, $\Sigma_a(u)$ for all $a > 0$, over $\mathbb{H}^2 \times \mathbb{R}_1$, providing explicit examples which show that the assumption $K_M \geqslant 0$ in [1, Theorem 4.3] is necessary. However, these graphs are not complete, in the sense that the Riemannian metric induced on the graph from the Lorentzian metric on $\mathbb{H}^2 \times \mathbb{R}_1$ is not complete. To see it, let $\alpha : (1, \infty) \to \Sigma_a(u)$ be the divergent curve in $\Sigma_a(u)$ given by

$$\alpha(s) = \bigl(0, s, u(s)\bigr).$$

Then $\alpha'(s) = (0, 1, u'(s))$ and

$$\|\alpha'(s)\|^2 = \frac{1}{s^2} - \frac{1}{a + s^2} = \frac{a}{s^2(a + s^2)},$$

where $\|\cdot\|$ denotes the norm of a vector field on $\Sigma_a(u)$, so α has finite length because

$$\int_1^\infty \|\alpha'(s)\|\, ds = \int_1^\infty \sqrt{\frac{a}{s^2(a + s^2)}}\, ds = \operatorname{arcsinh}(\sqrt{a}\,),$$

being $\Sigma_a(u)$ non-complete.

Example 3.2. We can also obtain new interesting examples of entire graphs in $\mathbb{H}^2 \times \mathbb{R}_1$ considering radial solutions to the maximal surface equation, that is, solutions of type

$$u(x_1, x_2) = f(x_1^2 + x_2^2).$$

In this case, (5) and (6) become

$$f'(z) + z f''(z) = 0, \tag{9}$$

and

$$4x_2^2 z f'(z)^2 < 1 \tag{10}$$

where $z = x_1^2 + x_2^2 > 0$. From (9),

$$f(z) = a \ln z + b, \quad a, b \in \mathbb{R},$$

and then (10) results

$$\frac{4a^2 x_2^2}{x_1^2 + x_2^2} < 1$$

in $\mathbb{H}^2$, so $a^2 < 1/4$. Therefore, the functions

$$u(x_1, x_2) = a \ln(x_1^2 + x_2^2) + b, \quad a, b \in \mathbb{R}, \ -1/2 < a < 1/2$$

define another family of entire maximal graphs. As in Example 3.1, we can assume $b = 0$.

In contrast to Example 3.1, these graphs are complete. To see it, observe that for every tangent vector field X on $\mathbb{H}^2$, we obtain from (1) using Cauchy–Schwarz inequality

$$\langle X, X\rangle = \langle X, X\rangle_{\mathbb{H}^2} - \langle X, Du\rangle_{\mathbb{H}^2}^2 \geqslant \langle X, X\rangle_{\mathbb{H}^2}\big(1 - |Du|^2\big). \tag{11}$$

With a direct computation,

$$|Du|^2 = x_2^2 |D_o u|_o^2 = \frac{4a^2 x_2^2}{x_1^2 + x_2^2}$$

so, as $a^2 < 1/4$, it follows that

$$1 - |Du|^2 = 1 - \frac{4a^2 x_2^2}{x_1^2 + x_2^2} = \frac{4a^2 x_1^2}{x_1^2 + x_2^2} + (1 - 4a^2) \geqslant 1 - 4a^2 > 0,$$

which jointly with (11) yields that

$$\langle X, X\rangle \geqslant (1 - 4a^2)\langle X, X\rangle_{\mathbb{H}^2}.$$

Therefore, $L \geqslant \sqrt{1 - 4a^2} L_{\mathbb{H}^2}$ where L and $L_{\mathbb{H}^2}$ stand for the length of a curve on $\mathbb{H}^2$ with respect to the Riemannian metrics $\langle\, ,\rangle$ and $\langle\, ,\rangle_{\mathbb{H}^2}$ respectively. As the hyperbolic metric $\langle\, ,\rangle_{\mathbb{H}^2}$ is a complete metric, $\langle\, ,\rangle$ is also complete.

These examples show that the assumption $K_M \geqslant 0$ is also necessary in [1, Theorem 3.3].

Example 3.3. Given a smooth real function f, consider solutions to the maximal surface equation of type

$$u(x_1, x_2) = f\left(\frac{x_2}{x_1^2 + x_2^2}\right).$$

In this case,

$$u_{x_1}(x_1, x_2) = -\frac{2x_1 x_2 f'(z)}{(x_1^2 + x_2^2)^2} \quad \text{and} \quad u_{x_2}(x_1, x_2) = -\frac{(x_2^2 - x_1^2) f'(z)}{(x_1^2 + x_2^2)^2}$$

where $z = \frac{x_2}{x_1^2 + x_2^2}$. The maximal surface equation results

$$z f'(z)^3 + f''(z) = 0 \tag{12}$$

and

$$x_2^2 |D_o u|_o^2 = \frac{x_2^2 f'(z)^2}{(x_1^2 + x_2^2)^2} < 1. \tag{13}$$

By integration, the solutions of (12) are

$$f(z) = \ln\big(z + \sqrt{z^2 + a}\big) + b, \quad a, b \in \mathbb{R}, \ a \geqslant 0$$

and (13) becomes

$$\frac{x_2^2}{x_2^2 + a(x_1^2 + x_2^2)^2} < 1$$

which holds for $a > 0$. Summing up, considering again $b = 0$, we have obtained the family of entire graphs $\Sigma_a(u)$ over $\mathbb{H}^2 \times \mathbb{R}_1$ determined by

$$u(x_1, x_2) = \ln\left(\frac{x_2}{x_1^2 + x_2^2} + \sqrt{\frac{x_2^2}{(x_1^2 + x_2^2)^2} + a}\right), \quad a \in \mathbb{R},\ a > 0.$$

As in the first example, these entire maximal graphs are not complete surfaces. To see it, consider the divergent curve $\alpha : (0, 1) \to \Sigma_a(u)$ given by

$$\alpha(s) = \big(0, s, u(0, s)\big).$$

Then $\alpha'(s) = (0, 1, u_{x_2}(0, s))$, so $\|\alpha'(s)\|^2 = \frac{1}{s^2} - u_{x_2}(0, s)^2$. As we have

$$u_{x_2}(x_1, x_2) = \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)\sqrt{x_2^2 + a(x_1^2 + x_2^2)^2}},$$

then

$$\left\|\alpha'(s)\right\|^2 = \frac{1}{s^2} - \frac{1}{s^2(1 + as^2)} = \frac{a}{1 + as^2}$$

and finally we get

$$\int_0^1 \left\|\alpha'(s)\right\| ds = \sqrt{a} \int_0^1 \frac{1}{\sqrt{1 + as^2}}\, ds = \operatorname{arcsinh}(\sqrt{a}),$$

so α has finite length and the entire graphs $\Sigma_a(u)$ are not complete maximal surfaces.

In fact, when $a = 1$, it can be easily seen that $\Sigma_1(u)$ defines, except for an isometry, the entire maximal graph that we obtained in [1, Example 5.3] via a duality result between entire minimal graphs in the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ and entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$, [1, Theorem 5.1].

Finally, it can be seen that there are not entire maximal graphs determined by functions of type $u(x_1, x_2) = f\big(\frac{x_1}{x_1^2 + x_2^2}\big)$ because in that case, conditions (5) and (6) are incompatible.

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