Monochromatic infinite paths

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Abstract
Suppose the edges of the complete graph on the natural numbers are colored with 2 colors. We show that (1) there is a monochromatic infinite path whose vertex set has upper density ≥2/3, and (2) there is a monochromatic infinite path such that, for infinitely many n, the set \{1, \ldots, n\} contains at least the first .21n vertices of the path. We also consider the analogous problems for colorings with 3 or more colors.

1. Definitions and examples

For the most part we follow the terminology and notation of [1]; however, a graph may be finite or infinite. \(K_n\) is the complete graph on the set \(\mathbb{N} = \{1, 2, 3, \ldots\}\) of natural numbers. The upper density of a subgraph \(G\) of \(K_n\) is
\[
\limsup_{n \to \infty} \frac{|V(G) \cap \{1, \ldots, n\}|}{n},
\]
the upper density of the vertex set \(V(G)\). By a path we mean either a finite path or a one-way-infinite path. Now suppose \(P = (x_1, x_2, x_3, \ldots)\) is an infinite path in \(K_n\); i.e., \(V(P)\) consists of the distinct natural numbers \(x_i\) \((i \in \mathbb{N})\), and \(E(P) = \{(x_i, x_{i+1}) : i \in \mathbb{N}\}\). The strong upper density of \(P\) is
\[
\limsup_{n \to \infty} \frac{f(n)}{n}\quad \text{where } f(n) = \sup\{m : \{x_1, \ldots, x_m\} \subseteq \{1, \ldots, n\}\};
\]
equivalently, it is
\[
\limsup_{m \to \infty} \frac{m}{\max\{x_1, \ldots, x_m\}}.
\]

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Clearly, the strong upper density of $P$ is less than or equal to the upper density; for an increasing path the two are equal, but the paths considered here are generally not increasing. If $P = (x_1, x_2, \ldots, x_n)$ is a finite path in $K_m$, the upper density and strong upper density of $P$ are 0; the ratio $n/\max\{x_1, \ldots, x_n\}$ will be called the local density of $P$.

Suppose the edges of $K_m$ are colored with finitely many colors. By Ramsey's theorem [8, Theorem A] there is a monochromatic infinite complete subgraph, and so of course there are monochromatic infinite paths in $K_m$. It is easy to see that, even with only 2 colors, the coloring can be chosen so that any monochromatic increasing path must have density 0; indeed, the vertices of such a path can be forced to grow arbitrarily fast [2, Theorem 3.1]. On the other hand, if the path does not have to be increasing, there will always be a monochromatic path with positive upper density. (Nothing can be said about the lower density, which may still be 0; see Theorem 1.1.) The easiest way to see this is by observing that, if the edges of $K_m$ are colored with $n$ colors, then vertices can be covered with $n$ monochromatic paths, one of each color (Theorem 2.1); it follows that one of these monochromatic paths must have upper density $\approx 1/n$ (Corollary 2.2). This result is probably not best possible (certainly not for $n = 2$), but it is off by at most a factor of 2: the edges of $K_m$ can be colored with $n$ colors so that every monochromatic path has upper density $\approx 2/n$ (Corollary 1.3). Even if $n$ is large and, instead of a monochromatic path, we merely want an $(n - 1)$-colored path (i.e., a path having edges of at most $n - 1$ different colors), we still cannot get a path with upper density 1 (Theorem 1.5). For $n = 2$, the gap between our positive and negative results can be narrowed somewhat: there is always a monochromatic path with upper density $\approx 2/3$ (Theorem 3.5), and there is a coloring for which every monochromatic path has upper density $\approx 8/9$ (Corollary 1.6).

Monochromatic paths with positive strong upper density need not exist if more than 2 colors are used (Theorem 1.7). If the edges of $K_m$ are colored with 2 colors, then there is a monochromatic infinite path with strong upper density $\approx 1/(3 + \sqrt{5})$ (Theorem 4.3). (A weaker version of this result, with 1/24 instead of $1/(3 + \sqrt{5})$, was stated without proof in our earlier paper [2].) On the other hand, the edges of $K_m$ can be colored with 2 colors so that every monochromatic infinite path has strong upper density $\approx 2/3$ (Corollary 1.9).

**Theorem 1.1.** The edges of $K_m$ can be partitioned into disjoint sets $E_i$ ($i \in \mathbb{N}$) so that for any subgraph $G$ of $K_m$ having bounded degree and no isolated vertices, if $E(G) \cap E_i = \emptyset$ for some $i$, then $V(G)$ has lower density 0.

**Proof.** Choose positive integers $a_1, a_2, a_3, \ldots$ so that

$$\lim_{n \to \infty} \frac{(a_1 + \cdots + a_{n-1})/a_n}{a_n} = 0.$$ 

Partition $\mathbb{N}$ into intervals $A_1 < A_2 < \cdots$ with $|A_n| = a_n$; also, partition $\mathbb{N}$
into disjoint infinite sets $N_i (i \in \mathbb{N})$. For each $i \in \mathbb{N}$, define
\[ E_i = \bigcup_{n \in \mathbb{N}} \{ (x, y) : x \in A_n, y \in \mathbb{N}, x < y \}. \]

Let $G$ be a subgraph of $K_m$ having maximum degree $\Delta$ and no isolated vertices. If $E(G) \cap E_j = \emptyset$, then $|V(G) \cap A_n| \leq \Delta(a_1 + \cdots + a_{n-1})$ for every $n \in \mathbb{N}$; it follows that $V(G)$ has lower density 0. \qed

**Theorem 1.2.** Let $n$ be a positive integer. The edges of $K_m$ can be colored with $n$ colors so that every $m$-colored path $(m = 0, 1, \ldots, n)$ has upper density $\leq 2m/n$.

**Proof.** Partition $\mathbb{N}$ into disjoint sets $A_1, \ldots, A_n$, each having density $1/n$, and let
\[ E_i = \{ (x, y) \in E(K_m) : \min\{x, y\} \in A_i \}. \]
Then $\{ E_1, \ldots, E_n \}$ is a partition (i.e., a coloring) of $E(K_m)$. Now let $P = (x_1, x_2, \ldots, x_n, \ldots)$ be any $m$-colored infinite path in $K_m$; say $E(P) \subseteq \bigcup \{ E_i : i \in M \}$ where $M \subseteq \{ 1, \ldots, n \}$, $|M| = m$. Let $A = \bigcup \{ A_i : i \in M \}$. For each $j \in \mathbb{N}$, if $x_j \notin A$, then $x_j > x_{j+1} \in A$. Hence, for every $k \in \mathbb{N}$, $|V(P) \cap \{ 1, \ldots, k \}| \leq 2|A \cap \{ 1, \ldots, k \}|$. Since $A$ has density $m/n$, it follows that $P$ has upper density $\leq 2m/n$. \qed

**Corollary 1.3.** Let $n$ be a positive integer. The edges of $K_m$ can be colored with $n$ colors so that every monochromatic path has upper density $\leq 2/n$.

**Lemma 1.4.** Let $a_1, b_1, \ldots, a_m, b_m, \ldots$ be an infinite sequence of positive (with the possible exception $a_1 = 0$) integers such that $a_m \leq b_m$ for each $m$. Partition $\mathbb{N}$ into intervals $A_1 < B_1 < A_2 < B_2 < \cdots$, $|A_m| = a_m$, $|B_m| = b_m$. Let $A = \bigcup \{ A_m : m \in \mathbb{N} \}$, and let $G$ be the graph with $V(G) = \mathbb{N}$ and $E(G) = \{ (x, y) \in E(K_m) : \min\{x, y\} \in A \}$. Then the maximum upper density of a path in $G$ is $\alpha$, where
\[ \alpha = \limsup_{m \to \infty} \frac{2(a_1 + a_2 + \cdots + a_m)}{(a_1 + b_1 + \cdots + a_{m-1} + b_{m-1}) + 2a_m}. \]

**Proof.** First, note that the path $a_1 + 1, 1, a_1 + 2, 2, \ldots, 2a_1, a_1, a_1 + b_1 + a_2 + 1, a_1 + b_1 + 1, \ldots, a_1 + b_1 + 2a_2, a_1 + b_1 + a_2 + a_3 + a_3 + 1, \ldots$ has upper density $\alpha$.

Now let $P$ be any infinite path in $G$; we have to show that $P$ has upper density $\leq \alpha$. Let $s_m = a_1 + b_1 + \cdots + a_m + b_m$. For any positive integer $k$, determine $m$ so that $s_{m-1} < k \leq s_m$, and define
\[ f(k) = 2(a_1 + \cdots + a_{m-1}) + \min\{2a_m, k - s_{m-1}\}. \]
It is easy to see that $|V(P) \cap \{ 1, \ldots, k \}| \leq f(k)$, whence $P$ has upper density $\leq \limsup_{k \to \infty} f(k)/k$. Since the maximum value of $f(k)/k$ on the interval $s_{m-1} < k \leq s_m$ is attained when $k = s_{m-1} + 2a_m$, it follows that
\[ \limsup_{k \to \infty} f(k)/k = \limsup_{m \to \infty} f(s_{m-1} + 2a_m)/(s_{m-1} + 2a_m) = \alpha. \]
Theorem 1.5. Let $n$ be a positive integer. The edges of $K_n$ can be colored with $n$ colors so that every $(n-1)$-colored path has upper density $\leq 1 - (2^n - 1)^{-2}$.

Proof. Partition $\mathbb{N}$ into intervals $V_1 < V_2 < \cdots$ with $|V_i| = 2^{i-1}$. For each $k \in \{1, \ldots, n\}$, define

$$W_k = V_k \cup V_{k+n} \cup V_{k+2n} \cup \cdots$$

and

$$E_k = \{(x, y) \in E(K_n) : \min\{x, y\} \in W_k\}.$$ 

Now consider a fixed $k \in \{1, \ldots, n\}$, and let $G_k$ be the graph on $\mathbb{N}$ consisting of all edges not of color $k$, i.e., $E(G_k) = E(K_n) \setminus E_k$. We can use Lemma 1.4 to determine $\alpha_k$, the maximum upper density of a path in $G_k$. Setting

$$B_m = V_{k+(m-1)n}, \quad b_m = 2^{k+(m-1)n-1},$$

$$a_1 = 2^k - 1, \quad \text{and} \quad a_m = (2^n - 1)2^{k+(m-2)n} \text{ for } m \geq 2,$$

an elementary calculation shows that $\alpha_k = 1 - (2^n - 1)^{-2}$. $\square$

Corollary 1.6. The edges of $K_n$ can be colored with 2 colors so that every monochromatic path has upper density $\leq 8/9$.

Theorem 1.7. The edges of $K_n$ can be colored with 3 colors so that every monochromatic path has strong upper density 0.

Proof. Choose positive integers $a_1 < a_2 < \cdots$ so that $\lim_{k \to \infty} a_k / a_{k+1} = 0$. For $x \in \mathbb{N}$, define $f(x) = \min\{k : x \leq a_k\}$. For $x, y \in \mathbb{N}, x < y$, color the edge $\{x, y\}$ red if $f(y) - f(x) \geq 2$; otherwise, color $\{x, y\}$ blue if $f(x)$ is odd, green if $f(x)$ is even.

Clearly, an infinite monochromatic path can only be red. Let $x_1, x_2, x_3, \ldots$ be any infinite red path. Note that, if $\max\{x_1, \ldots, x_n\} = y, f(y) = k + 2$, then $y > a_{k+1}$ and $n \leq 2a_k + 1$, i.e., $n / \max\{x_1, \ldots, x_n\} < (2a_k + 1) / a_{k+1}$. It follows that

$$\lim_{n \to \infty} n / \max\{x_1, \ldots, x_n\} = 0. \quad \square$$ 

Theorem 1.8. Let $n$ be a positive integer. The edges of $K_n$ can be colored with $n$ colors so that every $(n-1)$-colored path has strong upper density $\leq 1 - (2^n - 1)^{-1}$.

Proof. Partition $\mathbb{N}$ into disjoint sets $A_1, \ldots, A_n$, where $A_i$ has density $\alpha_i = 2^{-1}/(2^n - 1)$. Define the coloring $\tilde{f} : E(K_n) \to \{1, \ldots, n\}$ by setting $\tilde{f}(\{x, y\}) = \min\{i : x \in A_i, y \in A_i\}$. Note that, if $P$ is a finite path with no edges of color $i$, and if $V(P) \subseteq \{1, \ldots, k\}$, then

$$|V(P) \cap A_i| \leq |(A_1 \cup \cdots \cup A_{i-1}) \cap \{1, \ldots, k\}| + 1.$$

It follows that an infinite path with no edges of color $i$ has strong upper density

$$\leq (1 - \alpha_i) + (\alpha_1 + \cdots + \alpha_{i-1}) = 1 - (2^n - 1)^{-1}. \quad \square$$
Corollary 1.9. The edges of \( K_n \) can be colored with 2 colors so that every monochromatic path has strong upper density \( \approx 2/3 \).

2. Vertex coverings

In our earlier paper [2] we stated without proof that, for any coloring of \( E(K_n) \) with \( n \) colors, there is a monochromatic path with upper density \( \approx 1/n \). That remark was based on the following observation about covering the vertices with monochromatic paths. (A monochromatic path may consist of a single vertex.)

**Theorem 2.1.** Let \( n \) be a positive integer, and suppose the edges of \( K_n \) are colored with the \( n \) colors \( 1, 2, \ldots, n \). Then the vertices of \( K_n \) can be covered with vertex-disjoint paths \( R_1, R_2, \ldots, R_n \) such that each \( R_i \) is monochromatic of color \( i \).

**Corollary 2.2.** Let \( n \) be a positive integer. If the edges of \( K_n \) are colored with \( n \) colors, then there is a monochromatic path with upper density \( \approx 1/n \).

In order to get the analogous result for \( m \)-colored paths, we will prove the following generalized version of Theorem 2.1, which reduces to Theorem 2.1 when \( m = 1 \). (Another generalization of Theorem 2.1, for a graph which is directed and only 'approximately complete', has been given by Rado [7]; finite versions of Theorem 2.1 have been considered by Gyárfás [5].)

**Theorem 2.3.** Let \( m \) and \( n \) be integers, \( 1 \leq m \leq n \). Suppose \( E_1, \ldots, E_n \subseteq E(K_n) \) are such that each edge of \( K_n \) belongs to at least \( m \) of the \( E_i \)'s. Then there are paths \( R_1, \ldots, R_n \) in \( K_n \) such that \( E(R_i) \subseteq E_i \) for each \( i \in \{1, \ldots, n\} \), and each vertex of \( K_n \) is on exactly \( m \) of the \( R_i \)'s.

**Proof.** Define \( f : E(K_n) \to \{1, \ldots, n\} \) so that \( f((x, y)) \in \{i: (x, y) \in E_i\} \) and \( |f((x, y))| = m \). Define \( I_1, I_2, \ldots \) inductively so that, for each \( t \in \mathbb{N} \), there are infinitely many \( y \in \mathbb{N} \) such that \( f((x, y)) = I_x \) for all \( x \leq t \). For each \( i \in \{1, \ldots, n\} \), let \( A_i = \{x \in \mathbb{N}: i \in I_x\} \). Thus, if \( x \in A_i \) and \( u \in A_j \), then there are infinitely many \( y \) such that both \( \{x, y\} \) and \( \{u, y\} \) belong to \( E_i \).

We define finite paths \( R_{i,k} \), for \( i \in \{1, \ldots, n\} \) and \( k \in \mathbb{N} \), by induction on \( k \). First, choose \( a_1, \ldots, a_n \in \mathbb{N} \) so that \( a_i \in A_i \) if \( A_i \neq \emptyset \), while \( a_i \notin \{a_j: j \neq i\} \) if \( A_i = \emptyset \). Let \( R_{i,1} \) be the trivial path consisting of the single vertex \( a_i \). Next, let \( k \in \mathbb{N} \), and suppose \( R_{1,k}, \ldots, R_{n,k} \) have been defined so that, if \( A_i \neq \emptyset \), then \( R_{i,k} \) is an \( a_i - u_i \) path for some \( u_i \in A_i \). Let \( x \) be the least positive integer such that \( |\{i: x \in V(R_{i,k})\}| < m \), and choose \( j \in I_x \setminus \{i: x \in V(R_{i,k})\} \). Then \( x \in A_j \) and \( u_j \in A_j \); hence we can choose \( y \) so that \( \{x, y\} \in E_j \), \( \{u_j, y\} \in E_j \), and \( y \notin V(R_{j,k}) \). Let \( R_{j,k+1} \) be the path obtained by adding to \( R_{j,k} \) the vertices \( y \) and \( x \) and the edges \( \{u_j, y\} \) and \( \{y, x\} \); and let \( R_{j,k+1} = R_{i,k} \) for \( i \neq j \). Finally, let \( R_i = \bigcup_{k=1}^{\infty} R_{i,k} \). It is easy to see that \( R_1, \ldots, R_n \) have the desired properties. \( \Box \)
Corollary 2.4. Let \( m \) and \( n \) be integers, \( 1 \leq m \leq n \), and let \( \alpha_1, \ldots, \alpha_n \) be real numbers such that \( \alpha_1 + \cdots + \alpha_n = m \). Suppose \( E_1, \ldots, E_n \subseteq E(K_n) \) are such that each edge of \( K_n \) belongs to at least \( m \) of the \( E_i \)'s. Then at least one of the following statements holds:

1. for each \( i \in \{1, \ldots, n\} \), there is a path \( P \) such that \( E(P) \subseteq E_i \), and \( P \) has upper density \( \alpha_i \);
2. for some \( i \in \{1, \ldots, n\} \), there is a path \( P \) such that \( E(P) \subseteq E_i \), and \( P \) has upper density greater than \( \alpha_i \).

Proof. Choose paths \( R_1, \ldots, R_n \) as in Theorem 2.3, and let \( \beta_i \) be the upper density of \( R_i \). Then \( \beta_1 + \cdots + \beta_n > m = \alpha_1 + \cdots + \alpha_n \); hence, either \( \beta_i = \alpha_i \) for every \( i \), or else \( \beta_i > \alpha_i \) for some \( i \). \( \square \)

The case \( m = 1 \) of Corollary 2.4 is an asymmetrical version of Corollary 2.2:

Corollary 2.5. Let \( n \) be a positive integer, and let \( \alpha_1, \ldots, \alpha_n \) be real numbers such that \( \alpha_1 + \cdots + \alpha_n = 1 \). If the edges of \( K_n \) are colored with \( n \) colors \( 1, 2, \ldots, n \) then either, for each \( i \), there is a monochromatic path of color \( i \) with upper density \( \alpha_i \), or else, for some \( i \), there is a monochromatic path of color \( i \) with upper density \( > \alpha_i \).

Corollary 2.6. Let \( m \) and \( n \) be integers, \( 1 \leq m \leq n \). If the edges of \( K_n \) are colored with \( n \) colors, then there is an \( m \)-colored path with upper density \( \geq m/n \).

Proof. Let a coloring \( f : E(K_n) \to \{1, \ldots, n\} \) be given. For each \( i \in \{1, \ldots, n\} \), let \( \alpha_i = m/n \) and let \( E_i = \bigcup_{j=1}^{m} \{x, y\} : f(x, y) = i + j \text{ (mod} n) \rangle \). By Corollary 2.4 there is a path \( P_i \) with upper density \( \geq m/n \), such that \( E(P_i) \subseteq E_i \) for some \( i \). \( \square \)

The results of the next section will show that, at least in the cases where \( n/2 < m < n \), Corollary 2.6 does not give the best possible result (Theorems 3.5 and 3.9). Limitations on possible improvements of Corollary 2.6 are given by Theorem 1.2 (if \( 1 \leq m < n/2 \)) and Theorem 1.5 (if \( n/2 \leq m < n \)).

3. Monochromatic path-forests

The aim of this section is to improve on the results given by Corollary 2.6 for \( n/2 \leq m < n \); in particular, it will be shown that, if the edges of \( K_n \) are colored with 2 colors, then there is a monochromatic path with upper density \( \geq 2/3 \). To this end, we need to consider a rather special problem in Ramsey theory for finite graphs.
A finite graph is a path-forest if each of its components is a path. The external vertices of a path-forest are the endpoints of its component paths, i.e., the vertices of degree 0 or 1. Let \( G \) be a (finite or infinite) complete graph: by a monochromatic path-forest (abbreviated MPF) for a given coloring of the elements (edges and vertices) of \( G \), we mean a finite subgraph \( H \) of \( G \) such that \( H \) is a path-forest, and all edges and external vertices of \( H \) have the same color.

Given a positive integer \( n \), let \( \varphi(n) \) be the greatest integer \( m \) for which the following statement is true: if \( G \) is a complete graph of order \( n \) then, for any coloring of the elements of \( G \) with 2 colors, there is a monochromatic path-forest in \( G \) of order \( \geq m \).

**Lemma 3.1.** \( \varphi(n) \leq \varphi(n + 1) \leq \varphi(n) + 1 \).

**Proof.** The first inequality is obvious; we prove the second. Let \( G \) be a complete graph of order \( n + 1 \). Choose \( x_0 \in V(G) \), and color the elements of \( G - x_0 \) with 2 colors so that every MPF in \( G - x_0 \) has order \( \leq \varphi(n) \). Extend the coloring to \( G \) by giving \( \{x, x_0\} \) the same color as \( x \) for each \( x \in V(G - x_0) \); it does not matter which color we give to \( x_0 \). Now, if \( H \) is an MPF in \( G \), and if \( x_0 \in V(H) \), then \( H - x_0 \) is an MPF in \( G - x_0 \); hence \( |V(H)| \leq \varphi(n) + 1 \). \( \square \)

**Lemma 3.2.** \( \varphi(3n + 2) \leq 2n + 1 \) for every integer \( n \geq 0 \).

**Proof.** Consider a complete graph of order \( 3n + 2 \). Color \( 2n + 1 \) of the vertices red, and color the remaining \( n + 1 \) vertices blue; color an edge red if both endpoints are red, and blue otherwise. Then a red MPF has no blue vertices, while a blue MPF has at most \( n \) red vertices. \( \square \)

**Lemma 3.3.** \( \varphi(3n + 1) \geq 2n + 1 \) for every integer \( n \geq 2 \).

**Proof.** Let the elements of a complete graph of order \( 3n + 1 \) be colored with 2 colors, blue and red, and assume for a contradiction that there is no MPF of order \( 2n + 1 \). Then there must be at least \( n + 1 \) vertices of each color; we may assume that there are \( n + b \) blue vertices and \( n + r \) red vertices, where \( 1 < b < n \), \( 1 < r < n \), and \( b + r < n + 1 \).

Let \( E \) be the set of all edges joining a blue vertex to a red one, and let \( E_i \) be the set of all blue edges and \( E_2 \) the set of all red edges in \( E \); thus \( |E_i| + |E_2| = |E| = (n + b)(n + r) \). For a blue vertex \( v \) define \( \text{rd}(v) \), the red degree of \( v \), as the number of edges in \( E_2 \) that are incident with \( v \), i.e., the number of red edges joining \( v \) to red vertices.

It is easy to see that, if \( v_1, \ldots, v_k \) are distinct blue vertices with \( \text{rd}(v_i) > i \) for each \( i \), then there is a red MPF of order \( n + r + k \). Since there is no MPF of order \( 2n + 1 = n + r + b \), there do not exist \( b \) distinct blue vertices \( v_1, \ldots, v_b \) such that \( \text{rd}(v_i) > i \) for each \( i \). This means that, for some \( x \in \)
{1, \ldots, b}, there are \( b - x \) blue vertices with red degree \( x \). Since any blue vertex has red degree \( \leq n + r \), it follows that
\[
|E| \leq (b - x)(n + r) + (n + x)x = x^2 - rx + b(n + r).
\]
Let us define
\[
R(x) = x^2 - rx + b(n + r)
\]
and
\[
R = \max\{R(x) : 1 \leq x \leq b\} = \max\{R(1), R(b)\};
\]
then \( |E| \leq R \). An analogous argument shows that \( |E| \leq B = \max\{B(1), B(r)\} \) where \( B(x) = x^2 - bx + r(n + b) \). It follows that
\[
(n + b)(n + r) = |E| + |E| \leq B + R.
\]
We will get a contradiction by showing that \( (n + b)(n + r) > B + R \). Without loss of generality, we may assume that \( b \leq r \). Now, if \( b = r \), then \( n = 2r - 1 \), \( R = B = B(r) = r(3r - 1) \), and so
\[
(n + b)(n + r) - B - R = (3r - 1)(r - 1) > 0;
\]
on the other hand, if \( b < r \), then
\[
B = B(r) = r(n + r), \quad R = R(1) = 1 - r + b(n + r),
\]
and so
\[
(n + b)(n + r) - B - R = n^2 - r^2 + r - 1 \geq r - 1 > 0. \quad \Box
\]

Lemma 3.4. \( \varphi(n) = \lceil(2n + 1)/3 \rceil \) for \( n \neq 4 \).

Proof. For \( k \geq 2 \), it follows from Lemmas 3.1, 3.2, and 3.3 that \( \varphi(3k + 1) = \varphi(3k + 2) = 2k + 1 \); also, since \( \varphi(3k + 2) = 2k + 1 \) and \( \varphi(3k + 4) = 2k + 3 \), it follows by Lemma 3.1 that \( \varphi(3k + 3) = 2k + 2 \). Thus \( \varphi(n) = \lceil(2n + 1)/3 \rceil \) for all \( n \geq 1 \). Now, it is easy to see that \( \varphi(1) = \varphi(2) = 1 \) and \( \varphi(3) = \varphi(4) = 2 \). Since \( \varphi(4) = 2 \) and \( \varphi(7) = 5 \), it follows by Lemma 3.1 that \( \varphi(5) = 3 \) and \( \varphi(6) = 4 \). \( \Box \)

Theorem 3.5. If the edges of \( K_n \) are colored with 2 colors, then there is a monochromatic path with upper density \( > 2/3 \).

Proof. Let a coloring of the edges of \( K_n \) with 2 colors, blue and red, be given. Color the vertices blue and red so that every blue (red) vertex is incident with infinitely many blue (red) edges. Choose positive integers \( a_1 < a_2 < \cdots \) so that \( \lim_{n \to \infty} a_n/(a_1 + \cdots + a_n) = 1 \), and partition \( \mathbb{N} \) into intervals \( A_1 < A_2 < \cdots \) with \( |A_n| = a_n \). For each \( n \in \mathbb{N} \), let \( H_n \) be an MPF with \( V(H_n) \subseteq A_n \) and \( |V(H_n)| \geq \varphi(a_n) \). Without loss of generality, we assume that there is an infinite sequence \( n_1 < n_2 < \cdots \) such that each \( H_{n_i} \) is a red MPF. We consider 2 cases:
Case 1: There exist a finite set \( F \subseteq \mathbb{N} \) and red vertices \( x_0, y_0 \in \mathbb{N} \setminus F \) such that there is no red path in \( K_n - F \) connecting \( x_0 \) to \( y_0 \).

Partition \( \mathbb{N} \setminus F \) into disjoint sets \( X \) and \( Y \), where \( x \in X \) if and only if there is a red path in \( K_n - F \) connecting \( x_0 \) to \( x \). Then \( X \) and \( Y \) are infinite sets, since both \( x_0 \) and \( y_0 \) are incident with infinitely many red edges. Since all edges between \( X \) and \( Y \) are blue, there is a blue path \( P \) with \( V(P) = X \cup Y = \mathbb{N} \setminus F \); i.e., there is a monochromatic path with upper density 1.

Case 2: For every finite set \( F \subseteq \mathbb{N} \), any 2 vertices not in \( F \) can be connected by a red path which is disjoint from \( F \).

Then one can easily construct a red path \( P \) which contains infinitely many of the \( H_n \)'s as subgraphs. By Lemma 3.4, we have
\[
\lim_{n \to \infty} \frac{\varphi(n)}{n} = \frac{2}{3};
\]
hence
\[
\lim_{n \to \infty} \frac{\varphi(a_n)}{(a_1 + \cdots + a_n)} = \frac{2}{3}
\]
and \( P \) has upper density \( \geq \frac{2}{3} \). \( \square \)

An \( m \)-colored path-forest, for a given coloring of the elements of a complete graph \( G \), is a finite subgraph of \( G \) which is a path-forest and has edges and external vertices of at most \( m \) different colors. For positive integers \( m \leq n \) and \( r \), let \( \varphi_{n,m,r} \) be the greatest integer \( s \) for which the following statement is true: if \( G \) is a complete graph of order \( r \), and if the elements of \( G \) are colored with \( n \) colors, then there is an \( m \)-colored path-forest in \( G \) of order \( \geq s \). In particular, then, \( \varphi_{2,1}(r) = \varphi(r) \). We also define \( \varphi_{0,0}(r) = 0 \).

Lemma 3.6. \( \varphi_{n,m,r}(r) \geq \frac{rm}{n} \).

**Proof.** Use a path-forest consisting of isolated vertices. \( \square \)

Lemma 3.7. Let \( n, d, \) and \( r \) be positive integers, \( 2d \leq n \); then
\[
\varphi_{n,n-2d}(r) \geq (\varphi_{n,n-2d}(r) + 2r - 2)/3.
\]

**Proof.** Note first that, by Lemma 3.4, we have \( \varphi(m) \geq [2m/3] \geq 2(m - 1)/3 \) for every positive integer \( m \).

Let \( G \) be a complete graph of order \( r \), and let a coloring \( f: V(G) \cup E(G) \to \{1, \ldots, n\} \) be given. Choose an \((n-2d)\)-colored path-forest \( H \) with \( |V(H)| = s \geq \varphi_{n,n-2d}(r) \); we may assume that \( s < r \), so that \( G' = G - V(H) \) is a complete graph of order \( r - s \). (If \( 2d = n \), then \( H \) is the 'pointless graph', \( s = 0 \), and \( G' = G \).) Suppose the edges and external vertices of \( H \) are colored with the colors \( 1, 2, \ldots, n - 2d \). Define a coloring \( f': V(G') \cup E(G') \to \{1, 2\} \) so that \( f'(x) = 1 \) if \( 1 \leq f(x) \leq n - d \), and \( f'(x) = 2 \) if \( n - d < f(x) \leq n \). Let \( H' \) be a path-forest in
$G'$ which is monochromatic for the coloring $f'$ and has order $|V(H')| \geq \varphi(r-s) \geq 2(r-s-1)/3$. Then $H \cup H'$ is an $(n-d)$-colored path-forest for $f$, and

$$|V(H \cup H')| \geq (s + 2r - 2)/3 \geq (\varphi_{n-d}(r) + 2r - 2)/3.$$

\[ \square \]

**Lemma 3.8.** Let $m$, $n$, and $k$ be positive integers such that $1 - 2^{-k} \leq m/n < 1$, and let $r$ be any positive integer; then $q_{n,m}(r) \geq \alpha r$ where $\alpha = 1 - (1 - m/n)(2/3)^k$.

**Proof.** Note that $m < n$ and $(n-m)^2 \leq n$. For $i = 0, 1, \ldots, k$, let

$$d_i = (n-m)2^{k-i}, \quad s_i = \varphi_{n,m-d_i}(r), \quad \text{and} \quad t_i = r - s_i.$$

By Lemma 3.6 we have $s_i \geq r(n-d_i)/n$, i.e., $t_i \leq r(n-d_i)/n = r(1-\alpha)^3$. By Lemma 3.7, for $i = 1, \ldots, k$ we have $s_i \geq (s_{i-1} + 2r - 2)/3$, i.e., $t_i \leq (t_{i-1} + 2)/3$. It follows that

$$t_k \leq t_{k-1} + 2/3 + 2/9 + \cdots + 2/3^k < t_0/3^k + 1 \leq r(1-\alpha) + 1,$$

i.e., $q_{n,m}(r) = s_i > \alpha r - 1$. \[ \square \]

**Theorem 3.9.** Let $m$ and $n$ be positive integers and let $k$ be a nonnegative integer such that $1 - 2^{-k} \leq m/n < 1$. If the edges of $K_{n,m}$ are colored with $n$ colors, then there is an $m$-colored path with upper density $\geq 1 - (1 - m/n)(2/3)^k$.

**Proof.** If $k = 0$, this is just Corollary 2.6. If $k > 0$, imitate the proof of Theorem 3.5, using Lemma 3.8 instead of Lemma 3.4. \[ \square \]

4. Strong upper density

In this section we show that, for any coloring of $E(K_{n,m})$ with 2 colors, there is a monochromatic path with positive strong upper density. We use 2 well-known results from Ramsey theory for finite graphs. Lemma 4.1 is due independently to Rosta [9] and to Faudree and Schelp [3]; Lemma 4.2 is due independently to Gyárfás and Lehel [6] and to Faudree and Schelp [4].

**Lemma 4.1.** For $n \geq 2$, if the edges of the complete graph $K_{n,m}$ are colored with 2 colors, then there is a monochromatic cycle of length $2n$.

**Lemma 4.2.** For $n \geq 1$, if the edges of the complete bipartite graph $K(2n-1, 2n-1)$ are colored with 2 colors, then there is a monochromatic path of order $2n$.

**Theorem 4.3.** If the edges of $K_{n,m}$ are colored with 2 colors, then there is a monochromatic path with strong upper density $\geq 1/(3 + \sqrt{3})$. 
Proof. Let a coloring of the edges of $K_n$ with 2 colors, blue and red, be given. If $G$ is a monochromatic nonempty subgraph of $K_n$, let $c(G)$ denote the common color of all edges of $G$.

Choose integers $2 \leq a_1 \leq a_2 \leq \cdots$ so that $\lim_{n \to \infty} a_{n+1}/(a_1 + \cdots + a_n) = \sqrt{3}$.

Define

$$s_n = a_1 + \cdots + a_n, \quad \alpha_n = (a_n + 1)/3s_n,$$

$$\beta_n = (2a_n - 1)/(3s_n + a_{n+1} + 2a_n), \quad \gamma_n = \min\{\alpha_n, \alpha_{n+1}, \beta_n\};$$

then

$$\lim_{n \to \infty} \alpha_n - \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n - 1/(3 + \sqrt{3}).$$

Partition $\mathbb{N}$ into intervals $A_1 < A_2 < \cdots$ with $|A_n| = 3a_n$. By Lemma 4.1, for each $n$ we can choose a monochromatic cycle $F_n$ of length $2a_n$ with $V(F_n) \subset A_n$.

For each $n > 1$, we let $W_n$ be the set consisting of the $2a_{n-1}$ smallest elements of $V(F_n)$; thus $\max W_n = 3s_{n-1} + a_n + 2a_{n-1}$. For each $n$, $V(F_n)$ and $W_{n+1}$ are disjoint sets of cardinality $2a_n$; hence, by Lemma 4.2, we can choose a monochromatic path $G_n$ of order $2a_n$ whose vertices alternate between $V(F_n)$ and $W_{n+1}$.

Let $M = \{n \in \mathbb{N} : c(F_n) \neq c(G_n) \text{ or } c(F_{n-1}) \neq c(G_n)\}$. We consider 2 cases:

Case 1: $M$ is finite.

Choose $n_0 \in \mathbb{N}$ so that $c(F_n) = c(G_n) = c(F_{n+1})$ for all $n \geq n_0$. For each $n \geq n_0$, choose an edge $e_n = \{x_n, y_n\} \in E(G_n)$ with $x_n \in V(F_n)$, $y_n \in V(F_{n+1})$, and $x_n \neq y_{n-1}$, if $n > n_0$. For each $n > n_0$ let $H_n$ be a $y_{n-1} - x_n$ path in $F_n$ of length $\geq a_n$. Putting together the $H_n$'s and $e_n$'s we get a monochromatic infinite path with strong upper density $\geq 1/3$.

Case 2: $M$ is infinite.

For each $n \in M$, choose $k(n) \in \{n, n + 1\}$ so that $c(F_k(n)) \neq c(G_n)$. The path $G_n$ has one endpoint in $F_n$ and one in $F_{n+1}$; deleting from $G_n$ the endpoint not in $F_k(n)$, we get a path $G'_n$ of order $2a_n - 1$ which has both of its endpoints, call them $x_n$ and $y_n$, in $F_k(n)$. Let $H_n$ be an $x_n - y_n$ path in $F_k(n)$ of length $\geq a_k(n)$.

Now, for each $n \in \mathbb{N}$, we have monochromatic $x_n - y_n$ paths $G'_n$ and $H_n$, with $c(G'_n) \neq c(H_n)$ and $V(G'_n) \cup V(H_n) \subseteq A_n \cup A_{n+1}$.

By Ramsey's theorem, there is an infinite sequence $n_1 < n_2 < \cdots$ in $M$ such that $n_{i+1} - n_i \geq 1$ and the edges $e_i = \{y_{n_i}, x_{n_i}\}$ all have the same color. Call this red. For each $i \in \mathbb{N}$, there is a red path $J_i \in \{G'_n, H_n\}$. Thus there is an infinite red path $P$ with $E(P) = E(J_1) \cup \{e_i\} \cup E(J_2) \cup \{e_i\} \cup \cdots$.

Note that $H_n$ has local density $\geq \alpha_{k(n)}$, and $G'_n$ has local density $\geq \beta_n$; hence $J_i$ has local density $\geq \gamma_n$. Since $\lim_{n \to \infty} \gamma_n = 1/(3 + \sqrt{3})$, it is easy to see that $P$ has strong upper density $\geq 1/(3 + \sqrt{3})$. □
References