DISCRETE MATHEMATICS

Largest parallelotopes contained in simplices ☆

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Abstract

We establish in this paper a theorem for the volume of the largest parallelotope contained in a given simplex. Applying this theorem, we prove some inequalities for unions of parallelotopes in a given simplex and some spanning theorems for inscribed simplices. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Relative to a given n-polytope P in R^n, an n-simplex Ω in P is *largest* if it has maximum volume among all n-simplices contained in P. It is interesting to find a largest simplex contained in a given polytope. Of equal interest, and perhaps of even greater algorithmic difficulty, is the problem of finding a smallest n-simplex containing a given n-polytope. Quite recently, the researches for these two problems have been done extensively [2–6,8,10]. In [5], Klee has established the following interesting theorem.

Theorem 1. *If Ω is a smallest simplex containing a given polytope P, then the centroid of each facet of Ω belongs to P.*

Relative to a given n-simplex Ω, an n-parallelotope Π is *largest* if it has maximum volume among all n-parallelotopes contained in Ω. Applying Theorem 1, we in this

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paper establish a formula for the volume of the largest n-parallelotope contained in a given n-simplex. We apply this result to obtain some inequalities concerning the parallelotope-stack in a given simplex.

The other main purpose of this paper is to show that the paralleotopes which are spanned by some special simplices inscribed in a given simplex Ω are also contained in Ω. As applications, two well-known inequalities are derived.

We use the following notation throughout this paper. The sets of points with which we deal will be subsets of real n-dimensional Euclidean space, R^n. A set is called n-dimensional if it is a subset of R^n and not a subset of any hyperplane in R^n. We denote by $V(K)$ the n-dimensional volume of set K if it is n-dimensional and by $S(Q)$ the $(n-1)$-dimensional volume of set Q if it is a facet of K. Denote by $\mathrm{conv}(K)$ the convex hull of K. Let $\Omega = \langle A_0, A_1, \ldots, A_n \rangle$ denote the n-simplex in R^n with vertices $A_0, A_1, \ldots, A_n$.

1. A formula for the volume of the largest parallelotopes

Theorem 2. *Suppose that Π is an n-parallelotope contained in a given n-simplex Ω, then*

$$\max_{\Omega \supset \Pi} V(\Pi) = \frac{n!}{n^n} V(\Omega). \tag{1.1}$$

(After our paper was finished, we learned that formula (1.1) has been got by Lassak [7]. We found the result independently and our line of argument leads to extensions not covered in his paper.)

It is clear that Theorem 2 can be replaced equivalently by the following statement.

Theorem 2′. *Suppose that Ω is an n-simplex containing a given n-parallelotope Π, then*

$$\min_{\Omega \supset \Pi} V(\Omega) = \frac{n^n}{n!} V(\Pi). \tag{1.2}$$

Proof. We prove (1.2) by induction on n.

In the case of $n=2$ (1.2) is a well-known result (see [9, Section XIV. 25]). Let us assume that (1.2) holds in R^{n-1} and prove that (1.2) is true in R^n.

By an affine transformation we may assume that Π is an n-cube with volume a^n. Also we may assume that Ω lies in a ball, concentric with Π of radius $n^n a$, since otherwise Ω has a vertex A_j outside this ball, and the convex hull of $\{A_j\} \cup \Pi$, a subset of Ω, already has volume greater than $(n^n/n!)a^n$. By compactness the continuous function $V(\cdot)$, defined on the set of such simplex Ω containing Π, attains its infimum. Thus we may assume that $V(\Omega)$ is minimal, so that in particular

$$V(\Omega) \leqslant \frac{n^n}{n!} a^n. \tag{1.3}$$

Let G_i be the centroid of facet $\Omega^{[i]}=\langle A_0,\ldots,A_{i-1},A_{i+1},\ldots,A_n\rangle$ of Ω ($i=0,1,\ldots,n$), Ω^* denotes a simplex with vertex set $\{G_0,G_1,\ldots,G_n\}$, i.e. $\Omega^*=\langle G_0,G_1,\ldots,G_n\rangle$. Then the facets $\Omega^{*[i]}$ of Ω^* are parallel to the facets $\Omega^{[i]}$ of Ω and we have

$$V(\Omega^*)=\frac{1}{n^n}V(\Omega)\leqslant\frac{1}{n!}a^n. \tag{1.4}$$

By Theorem 1, all $n+1$ vertices of Ω^* belong to the cube Π.

If there is a facet $\Omega^{[i]}$ of Ω which meets Π in only a single point P, then P must be a vertex of Π and must also be the centroid of $\Omega^{[i]}$. So all $n+1$ vertices of the simplex Ω^* are vertices of Π. Now we choose a Cartesian coordinate system in R^n such that its origin is a vertex of Π and its axes are n edge-vectors from this vertex. So each coordinate of any vertex of Ω^* is either 0 or a. Let A be the $(n+1)\times n$ matrix whose rows list the coordinates of the vertices of Ω^* and let M be the $(n+1)\times(n+1)$ matrix formed from A by appending a columm of 1's. Then $\det M$ must be the integer multiple of a^n. This implies

$$0\neq V(\Omega^*)=\frac{1}{n!}\,|\det M|\geqslant\frac{a^n}{n!}. \tag{1.5}$$

From (1.4) and (1.5), we obtain $V(\Omega^*)=a^n/n!$. It follows that a facet of Ω^* lies in a facet of the cube Π. Hence that opposite facet of the cube Π lies a facet of Ω, which contradicts the hypothesis.

The above contradiction shows that at least one facet $\Omega^{[i]}$,

$$\Omega^{[i]}=\langle A_0,\ldots,A_{i-1},A_{i+1},\ldots,A_n\rangle,$$

(the base) of Ω, of $(n-1)$-dimesional volume $S(\Omega^{[i]})$, contains an edge of Π and makes a dihedral angle θ $(0<\theta<\pi/2)$ with a facet of Π. Making a hyperplane H parallel to $\Omega^{[i]}$ at distance $a\cos\theta$ from $\Omega^{[i]}$ meet Π in an $(n-1)$-parallelotope, of $(n-1)$-dimensional volume $a^{n-1}/\cos\theta$ and meets Ω in an $(n-1)$-simplex $\bar{\Omega}$. By induction hypothesis we have

$$S(\bar{\Omega})\geqslant\frac{(n-1)^{n-1}}{(n-1)!}\,\frac{a^{n-1}}{\cos\theta}. \tag{1.6}$$

Let x be the distance from the vertex A_i to hyperplane H. Then the altitude from the vertex A_i of Ω is of length $x+a\cos\theta$. Noticing that $\Omega^{[i]}$ and $\bar{\Omega}$ are similar, we obtain

$$\frac{S(\Omega^{[i]})}{S(\bar{\Omega})}=\frac{(x+a\cos\theta)^{n-1}}{x^{n-1}}. \tag{1.7}$$

From (1.6) and (1.7), we have

$$\begin{aligned} V(\Omega) &= \frac{1}{n}S(\Omega^{[i]})(x+a\cos\theta)\\ &= \frac{1}{n}S(\bar{\Omega})\frac{(x+a\cos\theta)^n}{x^{n-1}}\\ &\geqslant \frac{1}{n}\frac{(n-1)^{n-1}}{(n-1)!}a^{n-1}\frac{(x+a\cos\theta)^n}{x^{n-1}\cos\theta}. \end{aligned} \tag{1.8}$$

Setting $x = y\cos\theta$ and applying the arithmetic-geometry mean inequality, we infer

$$\begin{aligned}\frac{(x+a\cos\theta)^n}{x^{n-1}\cos\theta} &= \frac{(y+a)^n}{y^{n-1}}\\ &= \frac{(y+a)^n\cdot(n-1)a^{n+1}}{((n-1)a^2)(ay)^{n-1}}\\ &\geqslant \frac{(y+a)^n(n-1)a^{n+1}}{((n-1)ay+(n-1)a^2)^n/n^n}\\ &= \frac{n^n}{(n-1)^{n-1}}a. \qquad (1.9)\end{aligned}$$

Combining (1.8) and (1.9), we get

$$V(\Omega)\geqslant\frac{n^n}{n!}a^n \qquad (1.10)$$

with equality only if $y=(n-1)a$ and the equality of (1.6) holds.

(1.2) follows from (1.3) and (1.10). This completes the proof. □

Remark. Since ratios of volumes are invariant under nonsingular affine transformations, and simplices and parallelotopes are both preserved by such transformations, Theorem $2'$ could also be described as determining the volume of a smallest simplex containing a unit cube. That is interesting in view of the fact that finding the largest simplex contained in unit cube is a very difficult (and largely unsolved) problem. Even its most tractable case, that in which the dimension is congruent to 3 (mod 4), subsumes the famous problem concerning the existence of Hadamard matrices. Hudelson et al. [4] have studied this problem and they connect largest simplices and the Hadamard problem.

2. Parallelotope-stack in simplex

Definition 1. Let Π_1 and Π_2 be two n-parallelotopes, the union $\Pi_1\cup\Pi_2$ is called a parallelotope-stack of Π_1 and Π_2 if there is a hyperplane H (called the coupled hyperplane) such that the two parallelotopes lie in opposite halfspaces bounded by H and H contains a facet of each.

It should be noted that not all the parallelotope-stacks are convex.

For the parallelotope-stack in a given simplex, we establish the following inequality.

Theorem 3. *Suppose that $\Pi_1\cup\Pi_2$ is the parallelotope-stack in a given n-simplex Ω, and the coupled hyperplane H of $\Pi_1\cup\Pi_2$ is parallel to one of facets of Ω. Then*

$$V(\Pi_1\cup\Pi_2)\leqslant\frac{n!}{n^n(1-(n-1)^{(n-1)}/n^n)^{n-1}}V(\Omega). \qquad (2.1)$$

Proof. Let $\Pi_2^{[k]}$ and $\Pi_1^{[k]}$ be the facets of Π_2 and Π_1 which are parallel to H but do not lie in H and let $\Pi_2^{[l]}$ be the facet of Π_2 which lie in H, H' the hyperplane spanned by $\Pi_1^{[k]}$, $\boldsymbol{e}$ the normal vector of H. Without loss of generality we assume that the facet $\Pi_2^{[k]}$ lies in the facet $\Omega^{[k]}$ of Ω. Let h_2, h_1 and h be the altitudes of Π_2, Π_1 and Ω in the direction $\boldsymbol{e}$, respectively. Since $H \cap \Omega$ and $H' \cap \Omega$ are two $(n-1)$-dimensional simplices, and

$$\Pi_2^{[l]} \subset H \cap \Omega, \quad \Pi_1^{[k]} \subset H' \cap \Omega,$$

by applying Theorem 2, we obtain

$$V(\Pi_2^{[l]}) \leqslant \frac{(n-1)!}{(n-1)^{n-1}} S(H \cap \Omega) \tag{2.2}$$

and

$$V(\Pi_1^{[k]}) \leqslant \frac{(n-1)!}{(n-1)^{n-1}} S(H' \cap \Omega). \tag{2.3}$$

On the other hand, both simplices $\Omega_1 = \mathrm{conv}(\{A_k\} \cup (H \cap \Omega))$ and $\Omega_2 = \mathrm{conv}(\{A_k\} \cup (H' \cap \Omega))$ are similar to Ω, so we have

$$\frac{S(H' \cap \Omega)}{h_0^{n-1}} = \frac{S(H \cap \Omega)}{(h_0 + h_1)^{n-1}} = \frac{S_k}{h^{n-1}}, \tag{2.4}$$

where $h_0 = h - (h_1 + h_2)$ and $S_k = S(\Omega^{[k]})$. Combining (2.2)–(2.4), we infer

$$\begin{aligned} V(\Pi_1 \cup \Pi_2) &= V(\Pi_1) + V(\Pi_2) \\ &\leqslant \frac{(n-1)!}{(n-1)^{n-1}} [S(H' \cap \Omega) h_1 + S(H \cap \Omega) h_2] \\ &\leqslant \frac{(n-1)!}{(n-1)^{n-1}} \frac{S_k}{h^{n-1}} (h_0^{n-1} h_1 + (h_0 + h_1)^{n-1} h_2). \end{aligned} \tag{2.5}$$

Set

$$f(h_0, h_1, h_3) = h_0^{n-1} h_1 + (h_0 + h_1)^{n-1} h_2.$$

Now we must maximize $f(h_0, h_1, h_2)$ in the domain

$$D = \{(h_0, h_1, h_2)\colon\ h_0 + h_1 + h_2 = h,\ h_i > 0,\ i = 0, 1, 2\}.$$

Let λ be the Lagrange multiplier. We seek the extreme values of the function L with respect to h_i $(i = 0, 1, 2)$, where

$$L = h_0^{n-1} h_1 + (h_0 + h_1)^{n-1} h_2 + \lambda(h_0 + h_1 + h_2 - h).$$

Hence h_i $(i = 0, 1, 2)$ must satisfy

$$\begin{aligned} \frac{\partial L}{\partial h_0} &= (n-1) h_0^{n-2} h_1 + (n-1)(h_0 + h_1)^{n-2} h_2 + \lambda = 0, \\ \frac{\partial L}{\partial h_1} &= h_0^{n-1} + (n-1)(h_0 + h_1)^{n-2} h_2 + \lambda = 0, \\ \frac{\partial L}{\partial h_2} &= (h_0 + h_1)^{n-1} + \lambda = 0. \end{aligned} \tag{2.6}$$

From three equalities in (2.6), we can find the unique extreme point of $f(h_0,h_1,h_2)$ as follows:

$$h_0=\frac{(n-1)^2}{n^2(1-(n-1)^{n-1}/n^n)}h,\qquad h_1=\frac{(n-1)}{n^2(1-(n-1)^{n-1}/n^n)}h$$

and

$$h_2=\frac{n^{n-1}-(n-1)^{n-1}}{n^n-(n-1)^{n-1}}h.$$

This must be the maximum point of $f(h_0,h_1,h_2)$ (obviously, $f(h_0,h_1,h_2)$ has minimum 0).

$$\max_{(h_0,h_1,h_2)\in D} f(h_0,h_1,h_2)=\frac{(n-1)^{n-1}}{n^n(1-(n-1)^{n-1}/n^n)^{n-1}}h^n. \tag{2.7}$$

From (2.5) and (2.7), we get

$$\begin{aligned}V(\Pi_1\cup\Pi_2)&\leqslant\frac{(n-1)!}{(n-1)^{n-1}}\frac{S_k}{h^{n-1}}\max_{(h_0,h_1,h_2)\in D} f(h_0,h_1,h_2)\\&=\frac{(n-1)!}{h^n(1-(n-1)^{n-1}/n^n)^{n-1}}S_kh\\&=\frac{n!}{n^n(1-(n-1)^{n-1}/n^n)^{n-1}}V(\Omega).\end{aligned} \tag{2.8}$$

Noticing that the equalities in (2.2) and (2.3) hold at the same time, we know that the equality in (2.1) holds as desired. □

Definition 2. Let $\Pi_1,\Pi_2,\ldots,\Pi_m$ be n-parallelotopes in R^n. Then the union $\bigcup_{k=1}^m \Pi_k$ is called the parallelotope-stack of $\Pi_1,\Pi_2,\ldots,\Pi_m$ if there are $m-1$ mutually parallel hyperplanes $H_1,H_2,\ldots,H_{m-1}$ (whenever $i\leqslant j<k$, H_j is between H_i and H_k) such that H_i is the coupled hyperplane of $\Pi_i\cup\Pi_{i+1}$ $(i=1,2,\ldots,m-1)$. We also define $V(\bigcup_{i=1}^m \Pi_i)=\sum_{i=1}^m V(\Pi_i)$.

Theorem 4. *Suppose that $\bigcup_{k=1}^m \Pi_k$ is the parallelotope-stack of m parallelotopes $\Pi_1,\Pi_2,\ldots,\Pi_m$ in a given n-simplex Ω, and all coupled hyperplane H_i $(i=1,2,\ldots,m-1)$ are parallel to one of facets of Ω, then*

$$V\left(\bigcup_{k=1}^m \Pi_k\right)\leqslant\frac{n!}{(n-1)^{n-1}}C_mV(\Omega), \tag{2.9}$$

where C_m is the mth term of the series $\{C_k\}$ defined by recurrent equation

$$C_0=0,\quad C_k=\frac{(n-1)^{n-1}}{n^n}\frac{1}{(1-C_{k-1})^{n-1}}\quad (k\geqslant 1).$$

Proof. Let S_i be the $(n-1)$-dimensional volume of the facet $\Omega^{[i]}$ of Ω which is parallel to the coupled hyperplanes, h_i and h be the altitudes of Π_i and Ω in the direction of the normal vector of $\Omega^{[i]}$.

By the method similar to that used to obtain (2.1), we infer

$$V\left(\bigcup_{i=1}^{m}\Pi_i\right) \leqslant \frac{(n-1)!}{(n-1)^{n-1}}\frac{S_i}{h^{n-1}}(h_0^{n-1}h_1+(h_0+h_1)^{n-1}h_2 + \cdots + (h_0+h_1+\cdots+h_{m-1})^{n-1}h_m), \qquad (2.10)$$

where $h_0 = h - \sum_{i=1}^{m} h_i$.

Let $f(h_0,h_1,\dots,h_m) = h_0^{n-1}h_1+(h_0+h_1)^{n-1}h_2+\cdots+(h_0+h_1+\cdots+h_{m-1})^{n-1}h_m$. To establish (2.9) it only remains to find the maximum of $f(h_0,h_1,\dots,h_m)$ subject to the condition $h_0+h_1+\cdots+h_m = h$. But it is difficult to compute the extreme value by using Lagrange's method of multipliers as in the proof of Theorem 3.

Now we use induction for m to show

$$f(h_0,h_1,\dots,h_m) \leqslant C_m h^n \quad (m \geqslant 2). \qquad (2.11)$$

According (2.7), (2.11) is true when $m=2$. Let us assume that (2.11) holds for $m-1$, and we shall prove it holds true for m.

Let $x = h - h_m$, then $0 \leqslant x \leqslant h$. Applying the inductive hypothesis and the arithmetic-geometric mean inequality, we have

$$\begin{aligned} f(h_0,h_1,\dots,h_m) &= f(h_0,\dots,h_{m-1}) + (h-h_m)^{n-1}h_m \\ &\leqslant C_{m-1}(h-h_m)^n + (h-h_m)^{n-1}h_m \\ &= x^{n-1}(h-(1-C_{m-1})x) \\ &= \left(\frac{n-1}{1-C_{m-1}}\right)^{n-1}\left(\frac{1-C_{m-1}}{n-1}x\right)^{n-1}(h-(1-C_{m-1})x) \\ &\leqslant \left(\frac{n-1}{1-C_{m-1}}\right)^{n-1}\left(\frac{(n-1)(\frac{1-C_{m-1}}{n-1})x + (h-(1-C_{m-1})x)}{n}\right)^{n} \\ &= C_m h^n. \end{aligned}$$

So (2.11) is proved. Combining (2.10) and(2.11), we obtain the desired (2.9). □

It is easy to see that Theorem 4 is a generalization of Theorem 3. Since only for $m=2$ we can obtain the determinate extreme points of the problem, we indepedently list Theorem 3 and its proof.

Similar to Definition 2, we can define the parallelotope-stack $\bigcup_{i=1}^{\infty}\Pi_i$ of infinite parallelotopes and $V(\bigcup_{i=1}^{\infty}\Pi_i)$. For a given simplex Ω, if $\bigcup_{i=1}^{\infty}\Pi_i \subset \Omega$, then $\sum_{i=1}^{\infty}V(\Pi_i) < +\infty$. It is easy to prove $0 < C_m \leqslant 1/n$ by induction for m. Hence from Theorem 4 it follows that

$$\sum_{i=1}^{m} V(\Pi_i) \leqslant \frac{(n-1)!}{(n-1)^{n-1}} V(\Omega).$$

Therefore

$$\sum_{i=1}^{\infty} V(\Pi_i) \leqslant \frac{(n-1)!}{(n-1)^{n-1}} V(\Omega).$$

So we prove the following theorem.

Theorem 5. *Suppose that $\bigcup_{i=1}^{\infty} \Pi_i$ is the parallelotope-stack of Π_i $(i=0,1,\ldots)$ in a given n-simplex Ω, and its infinite coupled hyperplanes are all parallel to the same one of the facets of Ω. Then we have*

$$V\left(\bigcup_{i=1}^{\infty} \Pi_i\right) \leqslant \frac{(n-1)!}{(n-1)^{n-1}} V(\Omega)$$

and the upper bound can be attained.

Taking $n=3$ in Theorem 5, we find an interesting fact that the sum of volumes of such infinite parallelepipeds in a given tetrahedron T has always an upper bound $1/2V(T)$.

3. Spanning theorems and applications

Definition 3. Let $\Omega = \langle A_0, A_1, \ldots, A_n \rangle$ be an n-simplex in R^n. If an n-parallelotope is spanned by n edge-vectors from vertex A_i of Ω, then it is called the spanning parallelotope of Ω, denoted by $\Pi_i(\Omega)$.

Obviously, for a given simplex Ω, there exist $n+1$ spanning parallelotopes of Ω.

Let P be an interior point of n-simplex $\Omega = \langle A_0, \ldots, A_n \rangle$, for every $i \in \{0, 1, \ldots, n\}$, let B_i be the intersection of the line A_iP with the facet $\Omega^{[i]} = \langle A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \rangle$. Then the simplex J with the points $B_0, \ldots, B_n$ as vertices is called the Ceva simplex of P with respect to Ω.

Theorem 6 (The spanning theorem for Ceva simplex). *Let P be an interior point of an n-simplex $\Omega = \langle A_0, \ldots, A_n \rangle$, J the Ceva simplex of P with respect to Ω. Then there is $k \in \{0, 1, \ldots, n\}$ such that $\Pi_k(J) \subset \Omega$.*

Since $V(\Pi_k(J)) = n!V(J)$, from Theorems 6 and 2, we immediately get

Corollary 1. *Let P be an interior point of an n-simplex $\Omega = \langle A_0, \ldots, A_n \rangle$, J the Ceva simplex of P with respect to Ω. Then*

$$V(J) \leqslant \frac{1}{n^n} V(\Omega)$$

and equality holds if and only if $P = G$ while G is the centroid of Ω.

The results of Corollary 1 are well known; they were proved by M.S. Klamkin (see [9, Section XVIII. 2.46]. To prove Theorem 6, we need the following two lemmas.

Lemma 1. *Let $\Omega=\langle A_0,\dots,A_n\rangle$ be the coordinate simplex in R^n, let P and Q be points in R^n with barycentric coordinates $(\alpha_0,\alpha_1,\dots,\alpha_n)$ and $(\beta_0,\beta_1,\dots,\beta_n)$ respectively, and let point M lie in the line PQ and satify*

$$\frac{\overline{PM}}{\overline{MQ}}=k.$$

If $(\gamma_0,\gamma_1,\dots,\gamma_n)$ is the barycentric coordintes of M, then

$$\gamma_j=\frac{\alpha_j+k\beta_j}{1+k},\quad j=0,1,\dots,n \tag{3.1}$$

Proof. Lemma 1 immediately follows from the coordinate formulae for the point of division in rectangular coordinates system. □

Lemma 2. *Let Π be an m-dimensional parallelotope spanned by m vectors $\boldsymbol{OA}_i$, $i=1,2,\dots,m$, OB the diagonal of Π, Σ the $(m-1)$-dimensional simplex with vertex set $\{A_1,A_2,\dots,A_m\}$, M the intersection of OB with Σ. Then M must be the centroid of Σ and satisfy*

$$\frac{|OM|}{|MB|}=\frac{1}{m-1}. \tag{3.2}$$

Proof. By the following well-known facts

$$\boldsymbol{OB}=\sum_{i=1}^{m}\boldsymbol{OA}_i,$$

$$\boldsymbol{OG}=\frac{1}{m}\sum_{i=1}^{m}\boldsymbol{OA}_i,$$

where G is the centroid of Σ, we have

$$\boldsymbol{OB}=m\boldsymbol{OG}. \tag{3.3}$$

Hence, from (3.3), it follows that O, G and B are collinear. Therefore $M=G$. (3.2) also follows from (3.3). □

Proof of Theorem 6. Choose $\Omega=\langle A_0,A_1,\dots,A_n\rangle$ as the coordinate simplex in R^n. Let $(\lambda_0,\lambda_1,\dots,\lambda_n)$ be the barycentric coordinates of point P. Without loss of generality, we may assume that $\lambda_0=\min\{\lambda_0,\lambda_1,\dots,\lambda_n\}$. Since P is the interior point of Ω, then $\lambda_0>0$. Thus, the barycentric coordinates of B_i are $(\lambda_0/(1-\lambda_i),\dots,\lambda_{i-1}/(1-\lambda_i),0,\lambda_{i+1}/(1-\lambda_i),\dots,\lambda_n/(1-\lambda_i))$, $i=0,1,\dots,n$. Consider the spanning parallelotope $\Pi_0(J)$ of J. We can show $\Pi_0(J)\subset\Omega$. Hence it is only necessary to prove that all vertices of $\Pi_0(J)$ are either the interior points of Ω or boundary point of Ω. Let B be the vertex of $\Pi_0(J)$ which is different from B_i, $i=0,1,\dots,n$, with barycentric coordinates $(\beta_0,\beta_1,\dots,\beta_n)$. Hence we need only to prove $\beta_i \geqslant 0$, $i=0,1,\dots,n$.

Indeed, since $\Pi_0(J)$ is parallelotope spanned by $\boldsymbol{B_0B_i}, i=1,2,\dots,n$, there are $\{i_1,i_2,\dots,i_m\}\subset\{1,2,\dots,n\}$ $(2\leqslant m\leqslant n)$ such that B_0B is the diagonal of the m-dimensional parallelotope spanned by the m vectors $\boldsymbol{B_0B_{i_1}},\boldsymbol{B_0B_{i_2}},\dots,\boldsymbol{B_0B_{i_m}}$.

Let M be the centroid of $(m-1)$-dimensional simplex $\chi=\langle B_{i_1},B_{i_2},\dots,B_{i_m}\rangle$ with barycentric coordinates $(\delta_0,\delta_1,\dots,\delta_n)$. Then

$$\delta_j=\begin{cases}\dfrac{\lambda_j}{m}\displaystyle\sum_{k=1}^{m}\frac{1}{1-\lambda_{i_k}} & \text{for } j\neq i_1,i_2,\dots,i_m,\\[2ex] \dfrac{\lambda_{i_t}}{m}\displaystyle\sum_{k=1,k\neq t}^{m}\frac{1}{1-\lambda_{i_k}} & \text{for } j=i_t,\ t=1,2,\dots,m.\end{cases}$$

On the other hand, the barycentric coordinate of B_0 is

$$(0,\lambda_1/(1-\lambda_0),\dots,\lambda_n/(1-\lambda_0)).$$

Noting (3.2) and using Lemma 1, we have

$$\beta_0=m\delta_0-(m-1)0=\lambda_0\sum_{k=1}^{m}\frac{1}{1-\lambda_{i_k}}>0.$$

When $j=i_t$, we get

$$\begin{aligned}\beta_{i_t}&=m\delta_{i_t}-(m-1)\frac{\lambda_{i_t}}{1-\lambda_0}\\&=\lambda_{i_t}\left(\sum_{k=1,k\neq t}^{m}\frac{1}{1-\lambda_{i_k}}-\frac{m-1}{1-\lambda_0}\right)\\&\geqslant\lambda_{i_t}\left(\frac{m-1}{1-\lambda_0}-\frac{m-1}{1-\lambda_0}\right)\\&=0.\end{aligned}$$

When $j\neq i_t$ $(t=1,2,\dots,m)$ and $j\neq 0$, we have

$$\begin{aligned}\beta_j&=m\delta_j-(m-1)\frac{\lambda_j}{1-\lambda_0}\\&=\lambda_j\sum_{k=1}^{m}\frac{1}{1-\lambda_{i_k}}-\frac{(m-1)\lambda_j}{1-\lambda_0}\\&\geqslant\lambda_j\left(\frac{m}{1-\lambda_0}-\frac{m-1}{1-\lambda_0}\right)\\&=\lambda_j\frac{1}{1-\lambda_0}>0.\end{aligned}$$

Hence, for all $i \in \{1,2,\ldots,n\}$, we obtain $\beta_i \geqslant 0$. Therefore, $B \in \Omega$. This complets the proof. □

Given an interior point P of n-simplex Ω, drop perpendicular PH_i from P to the facet $\Omega^{[i]}$ at H_i $(i=0,1,\ldots,n)$. Then the simplex $\Omega_{[p]} = \langle H_0, H_1, \ldots, H_n \rangle$ is called the pedal simplex of P with respect to Ω. For the pedal simplex of an n-simplex in R^n, a natural problem is whether one can get the result similar to Theorem 6. We study this difficult problem in the case R^2, which is also nontrivial.

Theorem 7 (The spanning theorem for pedal triangles). *Let P be an interior point of a triangle $\triangle(A_1A_2A_3)$ and let H_i lie in the segment $A_{i-1}A_{i+1}$ (subscript module 3), $\triangle_{[P]} = \triangle H_1H_2H_3$ the pedal triangle of P with respect to $\triangle(A_1A_2A_3)$. Then there is $k \in \{1,2,3\}$ such that the parallelogram $\Pi_k(\triangle_{[P]}) \subset \triangle(A_1A_2A_3)$.*

Proof. It is easy to see the following facts:

(a) $\angle H_2H_1A_3 \geqslant \angle H_1H_2H_3$ is equivalent to $\angle A_1A_3A_2 + \angle PA_1H_3 \leqslant \pi/2$;

(b) If $\angle H_2H_1A_3 \geqslant \angle H_1H_2H_3$ and $\angle H_1H_2A_3 \geqslant \angle H_2H_1H_3$, then the parallelogram $\Pi_3(\triangle_{[P]}) \subset \triangle(A_1A_2A_3)$.

Let O, I, H be the circumcenter, incenter and orthocentre of $\triangle(A_1A_2A_3)$, θ the angle between HA_1 and A_1A_3, then A_1O and A_1H are symmetrical with respect to A_1I and $0 < \theta < \pi/2$.

Without loss of generality, we assume that P is the interior point of the triangle $\triangle(A_1OA_2)$. Then we have

$$\begin{aligned} \angle A_1A_3A_2 + \angle PA_1H_3 &\leqslant \angle A_1A_3A_2 + \angle OA_1A_2 \\ &= \angle A_1A_3A_2 + \theta \\ &= \pi/2. \end{aligned}$$

By (a), we have

$$\angle H_2H_1A_3 \geqslant \angle H_1H_2H_3.$$

Similarly, we get

$$\angle H_1H_2A_3 \geqslant \angle H_2H_1H_3.$$

From (b), we derive that the parallelogram $\Pi_3(\triangle_{[P]}) \subset \triangle(A_1A_2A_3)$, as desired. □

From Theorem 7, a well-known inequality follows:

Corollary 2 (Mitrinovic et al. [9]). *Let P be an interior point of a triangle $\triangle(A_1A_2A_3)$, $\triangle_{[P]}$ the pedal triangle of P with respect to $\triangle(A_1A_2A_3)$. Then*

$$\text{Area}(\triangle_{[P]}) \leqslant \tfrac{1}{4}\text{Area}(\triangle(A_1A_2A_3)). \tag{3.4}$$

A generalization to several dimensions of (3.4) may be seen in [12].

By modifying the method of the proof of Theorem 7, we can establish the spanning theorem for general inscribed triangles. To state this result, we need other notations. Let $\triangle(D_1D_2D_3)$ be an inscribed triangle of a triangle $\triangle(A_1A_2A_3)$, l_i the perpendicular of $A_{i-1}A_{i+1}$ passing point D_i (subscript module 3); let $l_2 \cap l_3 = A_1'$, $l_1 \cap l_3 = A_2'$, $l_1 \cap l_2 = A_3'$. Let O be circumcenter of $\triangle(A_1A_2A_3)$, $a_i = A_iO (i = 1,2,3)$. For $i \neq j$, denote by $H(a_i, A_j)$ the halfplane bounded by the line a_i, which implies that $A_j \in H(a_i, A_j)$.

Theorem 8 (The spanning theorem for inscribed triangles). *Let $\triangle(D_1D_2D_3)$ be an inscribed triangle of a triangle $\triangle(A_1A_2A_3)$ with D_i in the segment $A_{i-1}A_{i+1}$ (subscript module 3). Then, for $k \in \{1,2,3\}$, there is a parallelogram $\Pi_k(\triangle(D_1D_2D_3)) \subset \triangle(A_1A_2A_3)$ if and only if $A_i' \in H(a_i, A_j)$ and $A_j' \in H(a_j, A_i)$ where $\{i,j\} = \{1,2,3\} \setminus \{k\}$.*

Corollary 3. *Let $\triangle(D_1D_2D_3)$ be an inscribed triangle of a triangle $\triangle(A_1A_2A_3)$ and let D_i lie in the segment $A_{i-1}A_{i+1}$. If there is $\{i,j\} \subset \{1,2,3\}$ such that $A_i' \in H(a_i, A_j)$ and $A_j' \in H(a_j, A_i)$, then*

$$\text{Area}(\triangle(D_1D_2D_3)) \leqslant \tfrac{1}{4}\text{Area}(\triangle(A_1A_2A_3)).$$

For further reading

The following references are also of interest to the reader: [1,11].

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