A NEW UPPER BOUND FOR THE LIST CHROMATIC NUMBER

B. BOLLOBÁS and H.R. HIND†
Dept. of Pure Mathematics, Univ. of Cambridge, 16 Mill Lane, Cambridge, U.K. CB2 1SB

For large values of Δ, it is shown that all Δ-regular finite simple graphs possess a non-trivial vertex partition. This is then used to show that for finite simple graphs of maximal degree Δ(G) = Δ, the list chromatic number is bounded by \( \chi'_l(G) \leq 7\Delta/4 + o(\Delta) \).

Given a graph G and for each edge of G a set (list) of colours, we call an assignment of a single colour to each edge of G a list colouring if the colour assigned to an edge is in the list of colours associated with that edge and no two adjacent edges are assigned the same colour. The list chromatic number of the graph G, \( \chi'_l(G) \), is defined to be the minimal positive integer such that if for each edge of G, the list of colours associated with that edge has size at least \( \chi'_l(G) \), then there exists a list colouring for G for any choice of lists.

The list colouring conjecture, which has been attributed to various sources, states \( \chi'_l(G) \leq \chi'(G) \). Little progress has been made towards establishing this conjecture, even for special classes of graphs. Bollobás and Harris [2] have shown that for graphs with maximal degree \( \Delta(G) = \Delta \), \( \chi'_l(G) \leq 11\Delta/6 + o(\Delta) \) and Chetwynd and Häggkvist [3] have shown that for triangle free graphs \( \chi'_l(G) \leq 9\Delta/5 \). In this paper we combine the approaches used in the above two papers and the existence of a specific partition for regular graphs to obtain an improved upper bound for the list chromatic number. The term graph will be used to mean a finite simple graph.

Before proving the intended result, we need a few definitions. Given a graph G, let \( \Lambda : E(G) \rightarrow P(\mathbb{N}) \) be an arbitrary function. Then we define a \( \Lambda \)-colouring of graph G to be a function \( \phi \) such that

\[
\phi : E(G) \rightarrow \mathbb{N}
\]

where

\[
\phi(e) \in \Lambda(e) \quad \forall e \in E(G),
\]

and

\[
\phi(e) \neq \phi(e') \quad \text{if edges } e \text{ and } e' \text{ are adjacent.}
\]

† Supported by ORS grant ORS/84120 and CSIR grant 9/8/1-2019.
The more general name list colouring is applied to such a colouring where the function \( \Lambda \) is not specifically mentioned. The function \( \Lambda \) can be thought of as assigning a list of acceptable colours, \( \Lambda(e) \), for the edge \( e \). We refer to the function \( \Lambda \) as the list function for the graph. For a more detailed discussion of list colourings, see [2].

For simplicity we let \( \Lambda_i \) denote a function

\[
\Lambda_i : E(G) \to \mathbb{N}^{(i)}.
\]

For a given list function, \( \Lambda \), we define a partial list colouring of \( G \) to be a function, \( \psi \) say, which has an associated edge of \( G \), \( e^* \) say, such that

\[
\psi : E(G) \setminus \{e^*\} \to \mathbb{N}
\]

where

\[
\forall e \in E(G) \setminus \{e^*\}, \quad \psi(e) \in \Lambda(e)
\]

and

\[
\psi(e) \neq \psi(e') \text{ if edges } e \text{ and } e' \text{ are adjacent.}
\]

If we think of the function \( \phi \) as a colouring of the edges of the graph \( G \), then \( \psi \) is a colouring of all but one of the edges of \( G \).

Let \( E(v) \) be the set of edges incident to vertex \( v \) and for a partial list colouring \( \psi \), with associated edge \( e^* \), define

\[
\tilde{\psi} : V(G) \to \bigcup_{i=1}^{\Lambda} \mathbb{N}^{(i)}
\]

where

\[
\tilde{\psi}(v) = \{ \psi(e) : e \in E(v) \setminus \{e^*\} \}.
\]

The definitions given below were first given in [3], but are restated here for convenience. Let \( G \) be a graph with list function \( \Lambda \) and a partial list colouring \( \psi \). Let the associated edge for \( \psi \) be \( e^* \). An edge \( uv \) (where \( uv \neq e^* \)) is said to be a floppy edge if

\[
|\Lambda(uv) \setminus (\tilde{\psi}(u) \cup \tilde{\psi}(v))| \geq 1
\]

i.e. if there exists a partial list colouring, \( \psi^* \) say, with the same associated un coloured edge, \( e^* \), as \( \psi \), such that

\[
\psi^*(e) = \psi(e) \quad \forall e \neq uv \text{ or } e^*
\]

and

\[
\psi^*(uv) \in \Lambda(uv) \setminus \{ \psi(uv) \}.
\]

We call the colour \( \psi^*(uv) \) an escape colour for edge \( uv \). Later, with slight abuse of notation, we refer to the escape colour of a floppy edge; here having assigned a partial list colouring to the graph under consideration, we chose one.
(of the one or more) escape colours and assign this fixed colour as the escape colour of the edge.

By considering partitions of \( \Delta \)-regular graphs, we obtain an upper bound on the list chromatic number. We recall the Erdős–Lovász Theorem (see [1], page 22).

**Theorem 1.** Let \( A_1, A_2, \ldots, A_m \) be events with dependence graph \( F \). If \( F \) has maximal degree \( \Delta \geq 3 \) and

\[
P(A_i) \leq \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^\Delta}
\]

then

\[
P\left( \bigcap_{j=1}^m \overline{A_j} \right) > 0.
\]

We use this theorem to show that \( \Delta \)-regular graphs are vertex partitionable with vertex disjoint classes \( V_1, V_2, \ldots, V_r \) such that for \( x \in V_i \) there are non-trivial upper and lower bounds on the cardinality of \( \Gamma(x) \cap V_j \) (denoted by \( d_{V_j}(x) \)) for all \( j \neq i \).

As is customary, we let \( S_{n,p} \) be the sum of \( n \) independent Bernoulli random variables, with value one or zero, where each Bernoulli random variable has value one with probability \( p \). Then we see

\[
P(S_{n,p} = k) = \binom{n}{k}p^k(1-p)^{n-k}.
\]

**Definition.** For a given \( \Delta \) and \( r \), we call a set of ordered pairs \( D = \{ (\delta_{ij}, \Delta_{ij})_{1 \leq i, j \leq r, i \neq j} \} \) a \((\Delta, 0)\)-acceptable set if

(i) \( 0 \leq \delta_{ij} \leq \Delta_{ij} \leq \Delta \) for \( 1 \leq i, j \leq r \) and \( i \neq j \), and

(ii) there exists a set \( P = \{ p_1, \ldots, p_r \} \) (called the set of probabilities) with

\[
\sum_{i=1}^r p_i = 1, \quad \sum_{i=1}^r p_i = 1,
\]

such that

\[
\sum_{i=1}^r \sum_{j=1, j \neq i}^r p_i (P(S_{\Delta, p_i} < \delta_{ij}) + P(S_{\Delta, p_i} > \Delta_{ij})) \leq \frac{(\Delta^2)^{\Delta^2}}{(\Delta^2 + 1)^{\Delta^2 + 1}}.
\]

The theorem below follows from Theorem 1.

**Theorem 2.** Let \( D = \{ (\delta_{ij}, \Delta_{ij})_{1 \leq i, j \leq r, i \neq j} \} \) be a \((\Delta, 0)\)-acceptable set. Then every \( \Delta \)-regular graph has a partition, \( V(G) = \bigcup_{i=1}^r V_i \) with \( V_i \cap V_j = \emptyset \) if \( i \neq j \) and every vertex in \( V_i \) is adjacent to at least \( \delta_{ij} \) and at most \( \Delta_{ij} \) vertices in \( V_j \).

**Proof.** Let \( G \) be a \( \Delta \)-regular graph and \( D = \{ (\delta_{ij}, \Delta_{ij})_{1 \leq i, j \leq r, i \neq j} \} \) be a \((\Delta, 0)\)-acceptable set. Let \( P = \{ p_1, \ldots, p_r \} \) be the associated set of probabilities for \( D \).
Take a random partition of the vertex set \( V(G) \) into disjoint classes \( V_1, \ldots, V_r \), by putting a vertex into class \( V_i \) with probability \( p_i \).

Let \( A_x \) be the event that the condition of the theorem is violated for vertex \( x \), i.e. there exist \( i, j \) \((i \neq j)\) such that

\[
x \in V_i \text{ and } d_{V_i}(x) < \delta_{ij} \text{ or } d_{V_i}(x) > \Delta_{ij}.
\]

We note that event \( A_x \) is independent of the system \( \{\mathcal{A}_y : y \in V(G)\} \), that is to say the set of events \( \mathcal{A}_y \) such that \( x \) and \( y \) are not adjacent and have no common neighbours. The event \( A_x \) is thus independent of the system of all events \( \{\mathcal{A}_y : y \in V(G)\} \) from which at most \( \Delta^2 + 1 \) events have been omitted.

From the choice of event \( A_x \) we get

\[
P(A_x) = \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} p_i (P(S_{\Delta,p_i} < \delta_{ij}) + P(S_{\Delta,p_i} > \Delta_{ij})),
\]

but \( D \) is a \((\Delta, 0)\)-acceptable set with set of probabilities \( P \), so by condition (ii) of the definition,

\[
P(A_x) \leq \frac{(\Delta^2)\Delta^2}{(\Delta^2 + 1)\Delta^2 + 1}.
\]

It now follows from the Erdős–Lovász Theorem that

\[
P\left( \bigcap \overline{A_x} \right) > 0.
\]

**Definition.** For a given \( \Delta \) and \( r \), we call a set of ordered pairs \( D = \{(\delta_{ij}, \Delta_{ij})_{1 \leq i, j \leq r, i \neq j}\} \) a \((\Delta, \Delta/2)\)-acceptable set if

(i) \( 0 \leq \delta_{ij} \leq \Delta_{ij} \leq \Delta \) for \( 1 \leq i, j \leq r \) and \( i \neq j \), and

(ii) there exists a set \( P = \{p_1, \ldots, p_r\} \) (called the set of probabilities) with \( 0 < p_i < 1 \), \( \sum_{i=1}^{r} p_i = 1 \), such that

\[
\sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} p_i (P(S_{\Delta,p_i} < \delta_{ij}) + P(S_{\Delta,p_i} > \Delta_{ij}) + (1 - p_i)^{\Delta^2}) \leq \frac{(\Delta^2)\Delta^2}{(\Delta^2 + 1)\Delta^2 + 1}.
\]

Using a similar proof to that for Theorem 2, we get

**Corollary 3.** Let \( D = \{(\delta_{ij}, \Delta_{ij})_{1 \leq i, j \leq r, i \neq j}\} \) be a \((\Delta, \Delta/2)\)-acceptable set and \( G \) be a \( \Delta \)-regular graph. For each vertex \( x \in V(G) \) let \( U(x) \) be a set of at least \( \Delta/2 \) neighbours of \( x \). Then there exists a partition of the vertex set \( V(G) \) such that \( V(G) = \bigcup_{i=1}^{r} V_i \) with \( V_i \cap V_j = \emptyset \) for \( i \neq j \) and for every vertex \( x \in V_i \) there are at least \( \delta_{ij} \) and at most \( \Delta_{ij} \) vertices adjacent to \( x \) in \( V_j \). Also at least one vertex in \( U(x) \) is in \( V_i \) for each \( j \in \{1, 2, \ldots, r\} \setminus \{i\} \).

**Proof.** The proof follows that of Theorem 3, but with a new event \( \Lambda'_x \) defined to include the possibility for \( x \in V_i \) there is a \( V_j \) containing no vertex in \( U(x) \). □
In applying Corollary 3, we need a technical lemma.

**Lemma 4.** For $\Delta$ sufficiently large and $r = 2$, the set $D = \{(1, 100 \log \Delta); (1, \Delta)\}$ is a $(\Delta, \Delta/2)$-acceptable set.

**Proof.** Condition (i) of the definition of $(\Delta, \Delta/2)$-acceptability is clearly satisfied by the set $D$. Let $P$, the associated set of probabilities for $D$, be chosen to be the set $P = \{\Delta - 8 \log \Delta/\Delta, 8 \log \Delta/\Delta\}$.

We now seek to establish the inequality

$$
\sum_{i=1}^{2} \sum_{j=1, j \neq i}^{2} p_i(P(S_{\Delta, p_i} < \delta_j) + P(S_{\Delta, p_i} \geq \Delta_j) + (1 - p_i)^{\Delta/2}) \leq \frac{(\Delta^2)^{\Delta^2}}{(\Delta^2 + 1)^{\Delta^2+1^2}}.
$$

Consider the righthand side of the proposed inequality; for $\Delta \geq 3$,

$$
\frac{(\Delta^2)^{\Delta^2}}{(\Delta^2 + 1)^{\Delta^2+1}} \geq \frac{1}{\Delta^3}.
$$

We have chosen $p_1 = 1 - 8 \log \Delta/\Delta$, $p_2 = 8 \log \Delta/\Delta$. Thus with $\delta_{12} = 1$, $\delta_{21} = 1$, $\Delta_{12} = 100 \log \Delta$ and $\Delta_{21} = \Delta$, all terms except $P(S_{\Delta, p_2} > 100 \log \Delta)$, in the left-hand side of the proposed inequality are easily seen to be $o(\Delta^{-3})$. An upper bound for $P(S_{\Delta, p_2} > 100 \log \Delta)$ is available (see [1], page 14), namely

$$
P(S_{\Delta, p_2} > 100 \log \Delta) < \left(\frac{e}{u}\right)^{100 \log \Delta}
$$

and with $u = 12.5$, it follows that

$$
P(S_{\Delta, p_2} > 100 \log \Delta) < \left(\frac{e}{12.5}\right)^{100 \log \Delta}
$$

$$
= \Delta^{100 \log (e/12.5)}
$$

$$
= o(\Delta^{-3})
$$

Thus we have that

$$
\sum_{i=1}^{2} \sum_{j=1, j \neq i}^{2} p_i(P(S_{\Delta, p_i} < \delta_j) + P(S_{\Delta, p_i} \geq \Delta_j) + (1 - p_i)^{\Delta/2}) = o(\Delta^{-3})
$$

and

$$
\frac{(\Delta^2)^{\Delta^2}}{(\Delta^2 + 1)^{\Delta^2+1}} = O(\Delta^{-3})
$$

So for sufficiently large $\Delta$, the inequality has been established.

Before turning to the main result of the paper, we make two preliminary observations.
Remark 1. In the proof below which produces a list colouring of a graph $G$, for a given list function $\Lambda$, we may assume that a partial list colouring exists.

Suppose $G$ does not have a partial list colouring. Let $E_1 \subseteq E(G)$ be a maximally sized edge set such that for each edge $e \in E_1$ we can choose a colour, $\theta(e)$ say, with $\theta(e) \in \Lambda(e)$ such that $\theta(e) \neq \theta(e')$ if $e$ and $e'$ are distinct adjacent edges contained in the set $E_1$. Thus $G[E_1]$ is the largest edge induced subgraph of $G$ for which a $\Lambda$-colouring exists. Then choose an edge $e' \in E(G) \setminus E_1$ and define a new list function $\Lambda'_i : E(G) \rightarrow \mathcal{P}(\mathbb{N})$ such that $\Lambda'_i(f) = \Lambda_i(f)$ for all $f \in E_1 \cup \{e'\}$, and for $e \in E(G) \setminus (E_1 \cup \{e'\})$, the sets $\Lambda'_i(e)$ are disjoint and do not intersect the set $\bigcup_{f \in E_1 \cup \{e'\}} \Lambda(f)$.

Clearly $G$ has a partial list colouring for the list function $\Lambda'_i$. Therefore if we show that a list colouring exists for $G$ with list function $\Lambda'_i$, this implies that our choice of $E_1$ was not maximal. Thus without loss of generality we may suppose that $G$ has a partial list colouring for list function $\Lambda_i$.

Remark 2. In the proof below when showing the existence of a list colouring for a graph $G$, we may assume $G$ is $A$-regular.

If $G$ is not $A$-regular we may create a new $A$-regular graph $G'$ where $G \subseteq G'$, by adding vertices and edges to the graph $G$. We define a new function $\Lambda'_i : E(G') \rightarrow \mathbb{N}^{[l]}$ such that $\Lambda'_i|_{E(G)} = \Lambda_i$ and for each $e \in E(G') \setminus E(G)$ we choose $l$-sets, $\Lambda'_i(e)$, such that $\Lambda'_i(e) \cap \Lambda'_i(e') = \emptyset$ for all $e' \in E(G') \setminus \{e\}$. Then we obtain a $A$-regular graph $G'$ such that $G$ is list colourable with list function $\Lambda_i$ iff $G'$ is list colourable with list function $\Lambda'_i$.

Theorem 5. If $A$ is sufficiently large, then for every graph $G$ with maximum degree $\Delta(G) = \Delta$, we have

$$\chi''(G) \leq \frac{7A}{4} + [25 \log A].$$

Proof. Suppose $G$ is a graph of maximal degree $\Delta$, that $l > 7A/4 + [25 \log A]$ and $\Lambda_i : E(G) \rightarrow \mathbb{N}^{[l]}$ is an arbitrary function. We shall show that there exists a $\Lambda_i$-colouring for $G$.

From the remarks above, we may assume $G$ is $A$-regular and there exists a partial list colouring for $G$. We choose $\psi_0$ to be a partial list colouring with a maximal number of floppy edges. Let the associated uncoloured edge be denoted $u_0v_0$.

For each vertex $x \in V(G)$, let $\{y_1, y_2, \ldots, y_{m(x)}\}$ be the set of those vertices $y$ adjacent to the vertex $x$ such that $xy$ is not a floppy edge of the partial list colouring $\psi_0$. If $m(x) \geq \Delta/2$ define $U(x) = \{y_1, y_2, \ldots, y_{\lceil \Delta/2 \rceil}\}$ and if $m(x) < \Delta/2$ let $U(x) = \{y_1, \ldots, y_{m(x)}\} \cup \{z_{m(x)+1}, \ldots, z_{\lceil \Delta/2 \rceil}\}$, where $z_{m(x)+1}, \ldots, z_{\lceil \Delta/2 \rceil}$ are any other vertices adjacent to the vertex $x$.
For a set $W \subseteq V(G)$ and a vertex $x$, let $d_W(x)$ be the number of neighbours of $x$ in the set $W$. By Corollary 3 and Lemma 4 above, there exists a partition of $V(G)$ into $V_1$ and $V_2$ such that for each $x \in V_1$,

$$1 \leq d_{V_1}(x) \leq 100 \log \Delta,$$

and for each $x \in V_2$,

$$1 \leq d_{V_2}(x) \leq \Delta,$$

and for each $x \in V_2$,

$$V_2 \cap U(x) \neq \emptyset.$$ 

We distinguish three cases depending on the location of the endvertices of the uncoloured edge $a_0b_0$.

**Case (1).** $a_0 \in V_1$ and $b_0 \in V_1$.

We create a sequence of partial list colourings $\psi_0$, $\psi_1$, ..., and their associated uncoloured edges $a_0b_0$, $a_1b_1$, ... Having defined $\psi_i$ and edge $a_{i-1}b_{i-1}$, choose $\psi_{i+1}$ and the associated uncoloured edge $a_ib_i$ such that

1. $\psi_i$ is a partial list colouring,
2. $a_i = b_{i-1}$,
3. $b_i \in V_1$,
4. $\psi_i(e) = \psi_{i-1}(e)$ if $e \in E(G) \setminus \{a_{i-1}b_{i-1}, a_ib_i\}$, and $\psi_i(a_{i-1}b_{i-1}) = \psi_{i-1}(a_ib_i)$,
5. $\psi_i(a_{i-1}b_{i-1}) \notin \bigcup_{j<i} \psi_j(a_{j-1})$,
6. $\psi_i(a_{i-1}b_{i-1})$ is not the escape colour of a floppy edge incident to $a_{i-1}$, and
7. $a_ib_i \neq a_jb_j$ for any $j < i$.

Condition (6) ensures that the number of floppy edges in $\psi_i$ is not less than that in $\psi_{i-1}$ and since $\psi_0$ is chosen so that the number of floppy edges is maximal, equality must hold. Condition (7) implies that for a finite graph the sequence of partial list colourings constructed in this way is finite, say it ends with $\psi_i$. To simplify the notation let $ab = a_ib_i$ be the associated uncoloured edge. Condition (7) also ensures that if a colour is not in $\psi_0(V)$ but is in $\psi_i(v)$ for $i > 0$, then it is in $\tilde{\psi}_j(v)$ for all $j \geq i$.

We note that since $\psi_0$ has a maximal number of floppy edges, it is sufficient to show that either there is a list colouring of the graph $G$ or there is a partial list colouring $\psi_i$ with more floppy edges than $\psi_0$. Suppose neither case applies.

Given $G$ with a partial list colouring $\psi_i$, if $e$ is a floppy edge we let $\psi_i^*(e)$ denote the escape colour assigned to edge $e$. Then for the partial list colouring $\psi_i$ and a vertex $v \in V(G)$, let $F(\psi_i, v)$ be the set of colours assigned to floppy edges incident to vertex $v$, $H(\psi_i, v)$ be the set of escape colours assigned to floppy edges incident to vertex $v$, $K(\psi_i, v)$ be the set of new colours used to colour edges incident to $v$, $S(\psi_i, v)$ be the set of colours always used to colour an edge incident to $v$, and $R(\psi_i, v)$ be the set of colours originally used to colour an edge incident to $v$ but which have since been removed. Finally let $S'(\psi_i, v)$ be the set
of colours used to colour one edge incident to vertex \( v \) in partial list colouring \( \psi_0 \) and a different edge incident to \( v \) in partial list colouring \( \psi_i \). More precisely,

\[
F(\psi_i, v) = \{ \psi_i(\mathcal{A}(uv) \setminus (\bar{\psi}_0(v) \cup \bar{\psi}_i(v))) \mid \mathcal{A}(uv) \setminus (\bar{\psi}_0(v) \cup \bar{\psi}_i(v)) \geq 1 \},
\]

\[
H(\psi_i, v) = \{ \psi_i^2(\mathcal{A}(uv) \setminus (\bar{\psi}_0(v) \cup \bar{\psi}_i(v))) \mid \mathcal{A}(uv) \setminus (\bar{\psi}_0(v) \cup \bar{\psi}_i(v)) \geq 2 \}
\]

and

\[
K(\psi_i, v) = \bar{\psi}_i(v) \setminus \bar{\psi}_0(v),
\]

\[
S(\psi_i, v) = \bar{\psi}_i(v) \cap \bar{\psi}_0(v),
\]

\[
R(\psi_i, v) = \bar{\psi}_0(v) \setminus \bar{\psi}_i(v),
\]

\[
S'(\psi_i, v) = \{ \psi_i(\mathcal{A}(uv)) : \psi_i(\mathcal{A}(uv)) \in \bar{\psi}_0(v) \text{ and } \psi_i(\mathcal{A}(uv)) \notin \bar{\psi}_0(v) \}.
\]

For simplicity the partial list colouring \( \psi_i \) is omitted from the notation for the above sets where no ambiguity can arise. Further we let \( f(v) = f(\psi_i, v) = \mid F(\psi_i, v) \mid, h(v) = h(\psi_i, v) = \mid H(\psi_i, v) \mid \) and so on.

Since we cannot extend the sequence of partial list colourings beyond \( \psi_s \) it follows that all the colours in \( \Lambda_r(ab) \) must be in the union of

(i) \( \bar{\psi}_0(a) \) or \( \bar{\psi}_i(a) \),

(ii) the set of colours assigned to edges \( a_ib_j \) with \( j < s \), which are incident to vertex \( b \),

(iii) the set of escape colours of floppy edges incident to vertex \( a \), and

(iv) the set of colours assigned to edges of the form \( bx \) where \( x \in V_2 \).

Writing \( D(b) \) for this last set of colours and \( d(b) \) for its cardinality (from the choice of the partition, it follows that \( d(b) \leq 100 \log \Delta \)), it follows that

\[
\Lambda_r(ab) \subseteq (R(a) \cup S(a) \cup K(a)) \cup (K(b) \cup S'(b)) \cup H(a) \cup D(b),
\]

Setting \( K'(b) = K(b) \cap \Lambda_r(ab) \), we get

\[
\Lambda_r(ab) \subseteq (R(a) \cup S(a) \cup K(a)) \cup (K'(b) \cup S'(b)) \cup H(a) \cup D(b)
\]

since \( K'(b) \subseteq (R(a) \cup S(a)) = \bar{\psi}_0(a) \): the edge \( ab \) would have been defined as a floppy edge if a colour in \( \Lambda_r(ab) \) was not the colour of an edge incident to either vertex \( a \) or vertex \( b \) in the partial list colouring \( \psi_0 \). We also note that \( |S'(b)| = |K(b)| = k(b) \) since a colour is added to the set \( K(b) \) every time the vertex \( b \) occurs as the endvertex \( a_i \) of an edge \( a_ib_j \) in the sequence of uncoloured edges and a colour already used to colour an edge incident to the vertex \( b \) is used to colour a new edge incident to vertex \( b \) (i.e. added to \( S'(b) \)) every time \( b \) occurs as the endvertex \( b_j \) of such an edge \( a_ib_j \). These events occur in pairs. Thus we get

\[
l \leq r(a) + s(a) + k(a) + k(b) + h(a) + d(b) \quad (5.1)
\]

Clearly

\[
r(a) + s(a) \leq \Delta, \quad (5.2)
\]
New upper bound for list chromatic number 73

and each colour in \( L_i(ab) \) must be assigned by \( \psi_i \) to a non-floppy edge incident to vertex \( a \) or vertex \( b \), so

\[
f(a) + f(b) \leq 2\Delta - l.
\]

Since the number of escape colours assigned to floppy edges incident to vertex \( a \) is at most the number of colours assigned to those floppy edges, a crude upper bound for \( h(a) \) is

\[
h(a) \leq 2\Delta - l.
\] (5.3)

It remains to obtain bounds for \( k(a) \) and \( k(b) \).

With \( f(a) \) bounded above by \( 2\Delta - l < \Delta/2 \), our choice of partition for the graph \( G \), ensures that there exists a non-floppy edge \( ac \) such that \( c \in V_2 \). At most \( \Delta \) of the colours in \( L_i(ac) \) are in \( \psi_0(c) \) and further \( \psi_0(c) = \psi_1(c) \) so at least \( l - \Delta \) colours in \( L_i(ac) \) must be in \( \psi_0(a) \). Since these colours must also be in \( \psi_i(a) \) for each \( 0 \leq j \leq s \) (or we obtain a new floppy edge \( ac \)), it follows that

\[
s(a) \geq l - \Delta,
\]

and (with a similar argument for vertex \( b \))

\[
k(a) \leq 2\Delta - l
\]

\[
k(b) \leq 2\Delta - 1.
\] (5.4)

Recalling that \( d(b) \leq 100 \log \Delta = o(\Delta) \) and substituting the bounds (5.2), (5.3) and (5.4) into inequality (5.1) we get

\[
l \leq \Delta + (2\Delta - l) + (2\Delta - l) + (2\Delta - l) + 100 \log \Delta
\]

\[
= 7\Delta - 3l + 100 \log \Delta
\]

or

\[
l \leq \frac{7\Delta}{4} + 25 \log \Delta,
\]

which is a contradiction. Since \( \psi_0 \) was chosen to be a partial list colouring for \( G \) with a maximal number of floppy edges, this contradiction shows that there is a \( \Lambda_i \)-colouring of \( G \) for list function \( \Lambda_i \).

**Case 2.** \( a_0 \in V_1 \) and \( b_0 \in V_2 \).

We choose an edge \( a_0b_0 \) such that \( a_0' = a_0, b_0 \in V_1 \) and function

\[
\psi'_0: E(G) \setminus \{a_0b_0\} \to \mathbb{N}
\]

such that \( \psi'_0(e) = \psi_0(e) \) for all \( e \in E(G) \setminus \{a_0b_0, a_0'b_0\} \) and \( \psi'_0(a_0b_0) = \psi_0(a_0b_0) \) and such that \( \psi'_0 \) is a partial list colouring with at least as many floppy edges as for partial list colouring \( \psi_0 \). This case then reduces to Case 1.

We need only show that such an edge \( a_0b_0 \) exists. We define the sets \( F(\psi_0, v) \) and \( H(\psi_0, v) \) and the cardinalities \( f(\psi_0, v) \) and \( h(\psi_0, v) \) as in Case 1. Suppose
such an edge \( a_0'b_0 \) does not exist, then
\[
\Lambda_i(a_0'b_0) \subseteq \bar{\psi}_0(b_0) \cup H(\psi_0, b_0) \cup D(a_0)
\]
so
\[
l \leq \Delta + h(\psi_0, b_0) + d(a_0)
\]
or
\[
h(\psi_0, b_0) \geq l - \Delta - o(\Delta).
\]
Since the number of floppy edges incident to vertex \( b_0 \) must be at least as large as the number of escape colour assigned to these edges,
\[
f(\psi_0, b_0) \geq (l - \Delta - o(\Delta)).
\]
Then for sufficiently large \( \Delta \), at most \( 2\Delta - (l - \Delta - o(\Delta)) = 3\Delta - l - o(\Delta) < l \) non-floppy edges are incident to vertices \( a_0 \) and \( b_0 \), so there exists a colour in \( \Lambda_i(a_0'b_0) \) assigned to either only floppy edges incident to vertices \( a_0 \) or \( b_0 \), or not assigned to an edge incident to vertex \( a_0 \) or vertex \( b_0 \). This gives us an immediate list colouring for \( G \).

**Case 3.** \( a_0 \in V_2 \) and \( b_0 \in V_2 \).

Here we note that there exists a non-floppy edge from \( u \in V_2 \) to some \( v \in V_1 \) for every \( u \in V_2 \) such that the number of floppy edges incident to vertex \( u \) is at most \( \Delta/2 \).

We define a sequence of partial list colourings \( \psi_0, \psi_1, \ldots \) and associated edges \( a_0b_0, a_1b_1, \ldots \), using the conditions listed below. Having \( \psi_{i-1} \) and \( a_{i-1}b_{i-1} \) define \( \psi_i \) and \( a_ib_i \) such that
1. \( \psi_i \) is a partial list colouring,
2. \( a_i = b_{i-1} \),
3. \( \psi_i(e) = \psi_{i-1}(e) \) if \( e \in E(G) \setminus \{a_{i-1}b_{i-1}, a_ib_i\} \), and \( \psi_i(a_{i-1}b_{i-1}) = \psi_{i-1}(a_ib_i) \),
4. \( \psi_i(a_{i-1}b_{i-1}) \notin \bigcup_{j<i} \bar{\psi}_j(a_{i-1}) \),
5. \( \psi_i(a_{i-1}b_{i-1}) \) is not the escape colour of a floppy edge incident to \( a_{i-1} \),
6. \( a_ib_i \neq a_jb_j \) for any \( j < i \), and
7. if possible, (without violating one of the above conditions) \( b_i \in V_1 \). If this occurs we obtain Case 2, and proceed as in that case.

We note that condition (6) ensures that the sequence of partial list colourings produced is finite, say ending in \( \psi_r \) with associated uncoloured edge \( a_rb_r \). Either condition (7) applies and we have obtained an uncoloured edge \( a_rb_r \) such that \( a_r \in V_2 \) and \( b_r \in V_1 \) in which case we proceed as in Case 2, or at no stage in defining the sequence of partial list colourings could we find an edge \( a_rb_r \) such that \( b_r \in V_1 \). In this case an argument similar to that given in Case 1, but with the roles of \( V_1 \) and \( V_2 \) interchanged gives us the result that
\[
l \leq \frac{7\Delta}{4}.
\]
The $25 \log \Delta$ term which appears in the inequality in Case 1 does not arise in this case since condition (7) implies that each of the colours in the set $D(b)$ is an escape colour of an edge incident to the vertex $a$, or is a colour assigned to be an edge incident to the vertex $a$, or is not in the set $\Lambda_r(ab)$. This completes the proof. □

It is possible to give a slightly improved value for the above upper bound for the list chromatic number (namely $\chi'_l(G) \leq 12\Delta/7 + o(\Delta)$). The improvement, however, is modest and has a proof which is more lengthy and does not add significantly to the proof techniques for bounding $\chi'_l$. For this reason this improved upper bound is not included in this paper.

References