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The Rohlin property for shifts of finite type

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Abstract

We show that an automorphism of a unital AF C^* -algebra with a certain approximate Rohlin property has the Rohlin property. This generalizes a result of Kishimoto. Using this we show that the shift automorphism on the bilateral C^* -algebra associated with an aperiodic irreducible shift of finite type has the Rohlin property.

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1. Introduction

The Rohlin Lemma is one of the foundations of ergodic theory. The simple idea of levels of a stack moving almost cyclically under a transformation has proved to be an important and powerful tool. In 1975, A. Connes proved a “noncommutative analog of the very useful tower theorem of Rohlin” for automorphisms of von Neumann algebras [5]. The notion of a Rohlin property for automorphisms of C^* -algebras is due to Herman and Ocneanu [7]. It is useful in the study of automorphisms up

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to inner automorphism (see, for example, [3,4,6,10–13]). Evans and Kishimoto showed in [6] that two automorphisms of an AF C^* -algebra having the Rohlin property are outer conjugate if they induce the same action on K_0 . At present, only a few classes of automorphisms of C^* -algebras are known to possess the Rohlin property.

This work deals with the bilateral C^* -algebras associated with shifts of finite type, studied in [15,19,20]. The shift map on the shift space induces an automorphism of the C^* -algebra. Our main result is that these automorphisms have the Rohlin property. It follows from the result of Evans and Kishimoto [6] that the crossed-product of the bilateral C^* -algebra with the \mathbb{Z} -action of the automorphism is determined up to isomorphism by its K -theory. Krieger showed [15] that K -theory for shifts of finite type is an invariant for shift equivalence, and thus crossed-product algebras arising from shift equivalent shifts of finite type are isomorphic.

There are several slightly different versions of the Rohlin property for automorphisms of C^* -algebras in the literature, and it is not clear to this author that they are all equivalent. The definition used here comes from Kishimoto [10]. Likewise, there are many sets of conditions that might be regarded as approximate versions of the Rohlin property. Some of these are discussed in [18]. We use Kishimoto's "approximate Rohlin property" for automorphisms of AF C^* -algebras, also from [10]. These definitions, and some of their variations, are discussed in Section 2. There, we also state a few lemmas about projections and partial isometries in C^* -algebras for use in the later sections.

In Section 3, we prove that for automorphisms of unital AF algebras the approximate Rohlin property implies the Rohlin property. This improves a result of Kishimoto in [10], where an additional hypothesis, that $K_0(A)$ is of finite rank and has no infinitesimals, is used. We need the more general result because the K_0 groups for the algebras we are considering usually have infinitesimals. We show that the Rohlin property can be deduced from significantly weaker hypotheses. These results are not restricted to AF C^* -algebras.

The construction and basic properties of the bilateral C^* -algebra and the automorphism associated with a shift of finite type are given in Section 4. This is the algebra associated with one of Krieger's K -theoretic invariants in [15], although the algebra itself is not discussed there. It is a special case of Ruelle's construction in [19]. Wagoner studied this and two other algebras in [20].

We discuss shift equivalence and the relationship with the automorphism in Section 5. We show that K_0 for a bilateral C^* -algebra has no infinitesimals if and only if the shift of finite type is shift equivalent to a full shift. Finally, in Section 6 we prove that the automorphism of the bilateral C^* -algebra associated with an irreducible aperiodic shift of finite type has the Rohlin property. (This may be deduced for full shifts from the result of Kishimoto in [14].) It follows that the crossed product algebra is no stronger an invariant than the K -theoretic invariants introduced in [15]. It is in fact weaker because one can easily show that the full 2 shift and the full 4 shift have isomorphic crossed product algebras. We still do not know whether the pair consisting of the bilateral C^* -algebra and the automorphism associated with a shift of finite type is a stronger invariant than shift equivalence.

2. Definitions and basic lemmas

Let A be a unital C^* -algebra and write 1 for the identity of A . We quote a definition from [10].

Definition 2.1. An automorphism α of a unital C^* -algebra A has the Rohlin property if for any $k \in \mathbb{N}$ there are positive integers $k_0, \dots, k_{m-1} \geq k$ satisfying the following condition: for any finite subset F of A and $\varepsilon > 0$ there are projections $e_{i,j}$, $i = 0, \dots, m - 1$, $j = 0, \dots, k_i - 1$ in A such that

- (1) $\sum_{i=0}^{m-1} \sum_{j=0}^{k_i-1} e_{i,j} = 1$,
- (2) $\|\alpha(e_{i,j}) - e_{i,j+1}\| < \varepsilon$,
- (3) $\|[x, e_{i,j}]\| < \varepsilon$

for $i = 0, \dots, m - 1$, $j = 0, \dots, k_i - 1$, and $x \in F$, where $e_{i,k_i} = e_{i,0}$.

Variations of this definition appear in the literature. It is known that one can always put $m = 2$, $k_0 = k$ and $k_1 = k + 1$. Sometimes the condition that $\|\alpha(e_{i,k_i-1}) - e_{i,0}\| < \varepsilon$ is omitted, as in Definition 1.5 of Osaka and Phillips [18]. Obviously, the definition here implies that one. We do not know whether the two are equivalent. By “the Rohlin property”, we shall always mean this a priori stronger definition.

Notation. A stack of height m for α is a family of orthogonal projections f_0, f_1, \dots, f_{m-1} such that $\alpha(f_i) = f_{i+1}$, $i = 0, 1, \dots, m - 2$. By an ε -approximate stack we mean a family of orthogonal projections f_0, f_1, \dots, f_{m-1} such that $\|\alpha(f_i) - f_{i+1}\| < \varepsilon$, $i = 0, 1, \dots, m - 2$. We call a stack [or an ε -approximate stack] f_0, f_1, \dots, f_{m-1} cyclic if $\alpha(f_{m-1}) = f_0$ [resp., if $\|\alpha(f_{m-1}) - f_0\| < \varepsilon$].

Remark 2.2. From a stack $f_0, f_1, \dots, f_{nm-1}$ of height nm we can construct a stack e_0, e_1, \dots, e_{m-1} of height m having the same sum by taking $e_j = \sum_{i=0}^{n-1} f_{im+j}$, and if $\alpha(f_{nm-1}) = f_0$ then $\alpha(e_{m-1}) = e_0$. See Figs. 1 and 2.

For a compact space X we write $C(X)$ for the C^* -algebra of complex-valued continuous functions on X under the supremum norm. The unital C^* -subalgebra generated by $a \in A$ is denoted $C^*(a)$. We shall use three elementary (and undoubtedly suboptimal) approximation results for C^* -algebras. They all follow easily from Lemmas 2.5.1 and 2.5.7 of Lin [16].

Lemma 2.3. If e, f are projections in A with $\|e - f\| < \frac{1}{2}$ then there is a unitary $u \in A$ with $\|1 - u\| \leq 4\|e - f\|$ and $u^*eu = f$.

Lemma 2.4. If e and f are projections in A and $p \in A$ is a partial isometry with $\|p^*p - e\| < \varepsilon < \frac{1}{2}$ and $\|pp^* - f\| < \varepsilon$ then there is a partial isometry $q \in A$ with $q^*q = e$ and $qq^* = f$ and $\|p - q\| < 8\varepsilon$.

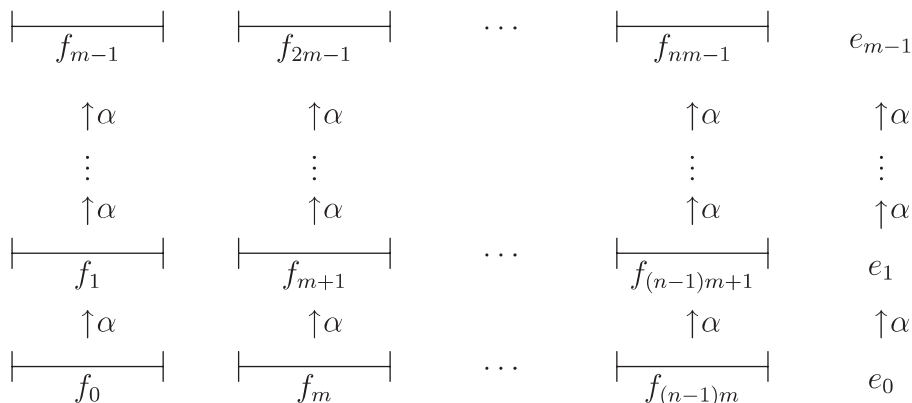


Fig. 1. The stack from Remark 2.2.

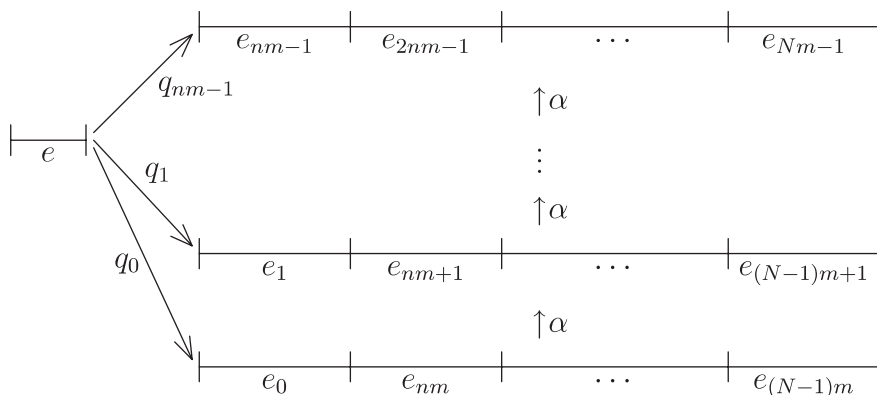


Fig. 2. In the proof of Theorem 3.1, q_k is a partial isometry with domain e and range $< \sum \{e_i : i \equiv k \pmod{nm}\}$. The automorphism α takes q_k close to q_{k+1} where the indices are taken modulo nm .

Lemma 2.5. *If $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ are families of orthogonal projections in A with $\|e_i - f_i\| < \varepsilon < \frac{1}{2n}$ for each i then there is a unitary $u \in A$ such that $\|1 - u\| < 8n\varepsilon$ and $u^*e_iu = f_i$ for each i .*

When $e, f \in A$ are equivalent projections we write $e \sim f$; i.e., $e \sim f$ if there exists $p \in A$ such that $p^*p = e$ and $pp^* = f$. We state two facts about equivalence classes and $K_0(A)$ and refer the reader to [1] for proofs.

Lemma 2.6. *If A has cancellation then for any $g \in K_0(A)_+$ with $g < [1]$ there is a projection $e \in A$ with $[e] = g$.*

Definition 2.7. When $K_0(A)$ is torsion-free, the rank of $K_0(A)$ is the dimension of the rational vector space $K_0(A) \otimes \mathbb{Q}$.

Lemma 2.8. Suppose A has cancellation. Write $[\cdot]$ for equivalence class in $K_0(A)$. If e and f are projections in A such that $[e] \leq [f]$ then there is a partial isometry $p \in A$ with $pp^* \leq f$ and $p^*p = e$, and $pp^* = f$ if $[e] = [f]$.

Many proofs that a certain automorphism has the Rohlin property use an intermediate concept similar to the following, which comes from [10].

Definition 2.9. An automorphism α of a unital AF algebra A has the approximate Rohlin property if for any $m, n \in \mathbb{N}$, any $\varepsilon > 0$, and any finite-dimensional C^* -subalgebra B of A there is an ε -approximate cyclic stack e_0, e_1, \dots, e_{m-1} in $A \cap B'$ such that

- (1) $[e_0] = [e_1] = \dots = [e_{m-1}]$ and
- (2) $[e_0] \geq n \left[1 - \sum_{i=0}^{m-1} e_i \right]$,

where $[\cdot]$ denotes the equivalence class in $K_0(A \cap B')$.

This is just one of many ways to define an “approximate Rohlin property” for an automorphism of an AF algebra. The tracial Rohlin property of Osaka and Phillips [18] is another example; it is weaker than this approximate Rohlin property when both are defined. The requirement in Definition 2.9 that the stack be cyclic is natural insofar as our definition of the Rohlin property also specifies cyclic stacks. It seems less natural to require equality of the classes of the e_i in $K_0(A \cap B')$. Both properties play an important role in the proof in [10]. We shall see that we can still deduce the Rohlin property if we remove either requirement (but not both).

Remark 2.10. Setting $n = 1$ in Definition 2.9 changes nothing since for $n = \ell$ we can start with a $\frac{\varepsilon}{\ell}$ -approximate stack of height ℓm satisfying the definition for $n = 1$ and then the construction in Remark 2.2 yields an ε -approximate stack of height m satisfying the definition for $n = \ell$.

3. General results on the Rohlin property

Theorem 3.1. An automorphism of a unital AF algebra having the approximate Rohlin property of Definition 2.9 also has the Rohlin property.

Remark 3.2. An approximate Rohlin property was introduced in [2], in the context of the one-sided shift on the CAR algebra $\bigotimes_{\infty} M_2$. The formulation for AF algebras in Definition 2.9 comes from [10], where it is shown (Theorem 4.1) that this approximate Rohlin property implies the Rohlin property for unital simple AF algebras when K_0 is

of finite rank and has no infinitesimal elements. In this setting the Rohlin property also implies the approximate Rohlin property, but we do not know whether the converse of Theorem 3.1 holds for all AF algebras. Osaka and Phillips have recently shown in [18] that their tracial Rohklin property is implied by the Rohlin property for simple, stably finite, unital algebras, under various sets of additional hypotheses.

Proof of Theorem 3.1. Given m, ε and F we shall use the approximate Rohlin property to construct a partition of unity by projections $e_{0,0}, \dots, e_{0,m-1}, e_{1,0}, \dots, e_{1,m}$ satisfying Definition 2.1. The technique is similar to that in the proof of Lemma 4.4 in [10] in which additional assumptions (that $K_0(A)$ be of finite rank and have no infinitesimals) are used.

We shall assume that the elements of F are of norm 1 as there is no loss of generality in doing so. The construction depends on parameters $n, \ell \in \mathbb{N}$, to be specified later, which determine heights of intermediate stacks. We require that $n \equiv 1 \pmod{m+1}$ and $\ell > 4$. Set $N = n\ell$ and choose $\eta < \frac{1}{8}(Nm)^{-4}$.

Let B be a finite-dimensional C^* -subalgebra of A such that $\text{dist}(\alpha^{-k}(x), B) < (Nm)^{-3}$ for all $x \in F, k = 0, 1, \dots, Nm - 1$. Let e_0, \dots, e_{Nm-1} be an η -approximate cyclic stack in $A \cap B'$ such that $[e] \leq [e_0]$, where $[\cdot]$ denotes equivalence class in $K_0(A \cap B')$ and

$$e = 1 - \sum_{i=0}^{Nm-1} e_i.$$

By Lemma 2.8 there is a partial isometry p_0 in $A \cap B'$ such that $p_0^* p_0 = e$ and $p_0 p_0^* < e_0$. By the triangle inequality,

$$\|[e_i, x]\| < 2(Nm)^{-3}, \quad x \in F, \quad i = 0, 1, \dots, Nm - 1 \quad (1)$$

and

$$\|[p_0, \alpha^{-k}(x)]\| < 2(Nm)^{-3}, \quad x \in F, \quad k = 0, 1, \dots, Nm - 1. \quad (2)$$

Remarks. The AF property of A and the approximate Rohlin property of α play no further role; the proof, from this point forward, requires only an η -approximate cyclic stack e_0, \dots, e_{Nm-1} and a partial isometry p_0 such that $p_0^* p_0 = 1 - (e_0 + \dots + e_{Nm-1})$ and $p_0 p_0^* < e_0$, satisfying inequalities (1) and (2) above. In particular, we do not assume equivalence of the e_i in $K_0(A \cap B')$.

We have

$$\|\alpha(p_0)^* \alpha(p_0) - e\| = \|\alpha(e) - e\| < Nm\eta$$

and since

$$\alpha(p_0) \alpha(p_0)^* < \alpha(e_0),$$

$\alpha(p_0)\alpha(p_0)^*$ is within η of some subprojection of e_1 . By Lemma 2.4, there is a partial isometry $p_1 \in A$ such that $p_1^*p_1 = e$ and $p_1p_1^* < e_1$ and $\|\alpha(p_0) - p_1\| < 8Nm\eta < (Nm)^{-3}$. Continuing inductively, we can find partial isometries $p_k, k = 0, 1, \dots, Nm - 1$, such that $p_k^*p_k = e$ and $p_kp_k^* < e_k$ for each k and

$$\|\alpha(p_k) - p_{k+1}\| < (Nm)^{-3} \quad \text{for } k = 0, 1, \dots, Nm - 2.$$

If $x \in F$ and $k \in \{0, 1, \dots, Nm - 1\}$ then

$$\begin{aligned} \|[p_k, x]\| &\leq \|p_kx - \alpha^k(p_0)x\| + \|[\alpha^k(p_0), x]\| + \|x\alpha^k(p_0) - xp_k\| \\ &\leq 2\|p_k - \alpha^k(p_0)\| + \|[p_0, \alpha^{-k}(x)]\| \\ &< 2k(Nm)^{-3} + 2(Nm)^{-3} \\ &\leq 2(Nm)^{-2}. \end{aligned}$$

Loosely, p_0, \dots, p_{Nm-1} is an approximate stack of partial isometries, with domain e and orthogonal ranges, that almost commutes with F . To complete the proof as in Lemma 4.4 of Kishimoto [10] we shall need a cyclic such stack. For $k = 0, 1, \dots, nm - 1$ the partial isometry

$$q_k = \frac{1}{\sqrt{\ell}} \sum_{j=0}^{\ell-1} p_{jnm+k}$$

has domain e and its range is a subprojection of $\sum_{j=0}^{\ell-1} e_{jnm+k}$. In effect, q_k “smears” its domain, e , across the ranges of all the p_i with $i \equiv k \pmod{nm}$, so that $\alpha(q_{nm-1})$ is close to q_0 even though $\|\alpha(p_{Nm-1}) - p_0\| \leq 2$ is the best possible estimate. By the triangle inequality,

$$\|\alpha(q_k) - q_{k+1}\| < \sqrt{\ell}(Nm)^{-3} < 3/\sqrt{\ell} \quad k = 0, 1, \dots, nm - 2,$$

$$\|\alpha(q_{nm-1}) - q_0\| < \frac{1}{\sqrt{\ell}} \left[(\ell - 1)(Nm)^{-3} + 2 \right] < 3/\sqrt{\ell}$$

and

$$\|[q_k, x]\| < 2\sqrt{\ell}(Nm)^{-2}, \quad x \in F, \quad k = 0, 1, \dots, nm - 1.$$

Define $e_{0,k}$ for $k = 0, 1, \dots, m - 1$ by setting

$$e_{0,k} = \sum_{j=0}^{N-1} e_{jm+k} - \sum_{j=0}^{n-1} q_{jm+k}q_{jm+k}^*.$$

Another application of the triangle inequality, together with the bounds above, yields

$$\|\alpha(e_{0,k}) - e_{0,k+1}\| < N\eta + 6n/\sqrt{\ell} < 7n/\sqrt{\ell}, \quad k = 0, 1, \dots, m - 1,$$

where $e_{0,m} = e_{0,0}$, and

$$\|[e_{0,k}, x]\| < 2N(Nm)^{-3} + 4n\sqrt{\ell}(Nm)^{-2} < 5/N, \quad x \in F, \quad k = 0, 1, \dots, m - 1.$$

Now we produce the second approximate stack, just as in the proof of Lemma 4.4 in [10]. The C^* -subalgebra D generated by $\{q_0, q_1, \dots, q_{nm-1}\}$ is isomorphic to M_{nm+1} . Indeed, D has $nm + 1$ orthogonal projections, $e, q_0q_0^*, q_1q_1^*, \dots, q_{nm-1}q_{nm-1}^*$, and for each i , q_i is a partial isometry from e to $q_iq_i^*$. Thus, for each pair i, j , $q_iq_j^*$ is a partial isometry from $q_jq_j^*$ to $q_iq_i^*$, and we see that our generators for D are part of a full set of matrix units (which they generate) for an $(nm + 1)$ -dimensional C^* -algebra. Let u be the unitary in D defined by

$$(\text{Ad } u)(q_k) = q_{k+1}$$

for $k = 0, 1, \dots, nm - 1$, where $q_{nm} = q_0$, and

$$(\text{Ad } u)(e) = e.$$

From the triangle inequality we obtain the generous estimate

$$\|\alpha|_D - (\text{Ad } u)|_D\| < 3(nm + 1)^2/\sqrt{\ell}.$$

The eigenvalues of u are $e^{\frac{2\pi ij}{nm}}$, $j = 0, 1, \dots, nm$, and there is a unitary $v \in D$ with eigenvalues $e^{\frac{2\pi ij}{nm+1}}$, $j = 0, 1, \dots, nm$, such that $\|u - v\| < \frac{2\pi}{nm}$. To see this, note that unitaries in M_{nm+1} can be diagonalized by unitaries in M_{nm+1} and there is a path of diagonal unitaries from $\text{diag}\left(1, \frac{2\pi i}{nm}, \dots, \frac{2\pi i(nm-1)}{nm}, 1\right)$ to $\text{diag}\left(1, \frac{2\pi i}{nm+1}, \dots, \frac{2\pi i(nm-1)}{nm+1}, \frac{2\pi i(nm)}{nm+1}\right)$ having length $\frac{2\pi}{nm+1}$. There is a projection r in D such that $\sum_{j=0}^{nm} (\text{Ad } v)^j(r)$ equals the identity in D (v acts as a permutation of least period $nm + 1$ on some orthonormal basis for \mathbb{C}^{nm+1} ; we can take r to be the projection onto one of these basis vectors) and we set

$$e_{1,k} = \sum_{j=0}^{\frac{nm+1}{m+1}-1} (\text{Ad } v)^{j(m+1)}(r), \quad k = 0, 1, \dots, m,$$

which makes sense because $n \equiv 1 \pmod{m+1}$. For $k = 0, 1, \dots, m$ we have $(\text{Ad } v)(e_{1,k}) = e_{1,k+1}$ and

$$\begin{aligned} \|\alpha(e_{1,k}) - e_{1,k+1}\| &\leq \|\alpha(e_{1,k}) - (\text{Ad } u)(e_{1,k})\| + \|(\text{Ad } u)(e_{1,k}) - (\text{Ad } v)(e_{1,k})\| \\ &\leq 3(nm + 1)^2/\sqrt{\ell} + \frac{4\pi}{nm}, \end{aligned}$$

where $e_{1,m+1} = e_{1,0}$. For $x \in F$ we bound $\|[e_{1,k}, x]\|$ by $4(nm + 1)^2\sqrt{\ell}(Nm)^{-2} < 4/\sqrt{\ell}$, using the triangle inequality with our previous estimates.

Now $\{e_{i,j} : i = 0, 1, j = 0, 1, \dots, m+i-1\}$ is a partition of unity by projections. Our estimates show that these approximate Rohlin stacks have the desired properties if we take $n > \frac{8\pi}{m\varepsilon}$ and $\ell > \frac{36(nm+1)^4}{\varepsilon^2}$. \square

Inspection of the proof shows that one can deduce the Rohlin property from weaker hypotheses, by simply replacing the hypotheses involving K_0 in the approximate Rohlin property with a new requirement that a certain partial isometry exists. This implies an inequality in K_0 but is not implied by it for algebras without nice properties like cancellation.

Theorem 3.3. *Let α be an automorphism of a unital C^* -algebra A with the property that for any finite subset F of A and for any $m \in \mathbb{N}$, $\varepsilon > 0$, there are projections e_0, e_1, \dots, e_{m-1} in A and a partial isometry p in A such that*

- (1) $\|e_i e_j\| < \varepsilon$ for $i, j = 0, 1, \dots, m-1, i \neq j$,
- (2) $\|\alpha(e_i) - e_{i+1}\| < \varepsilon$ for $i = 0, 1, \dots, m-1$, where $e_m = e_0$,
- (3) $\left\| p^* p - \left(1 - \sum_{i=0}^{m-1} e_i \right) \right\| < \varepsilon$ and $\|pp^*(1 - e_0)\| < \varepsilon$,
- (4) $\|[x, y]\| < \varepsilon$ for any $x \in F, y \in \{e_0, p\}$.

Then α has the Rohlin property.

Proof. Given m, ε and F , let n, ℓ, N and η be as in the proof of Theorem 3.1. Assume as before that the elements of F have norm 1. In the proof of Theorem 3.1, the approximate Rohlin property for an automorphism of an AF algebra is used only in the construction of an η -approximate cyclic stack e_0, \dots, e_{Nm-1} and a partial isometry p_0 such that

- $p_0^* p_0 = 1 - \sum_{i=0}^{Nm-1} e_i$ and $p_0 p_0^* < e_0$,
- $\|[e_i, x]\| < 2(Nm)^{-3}$ for $x \in F$ and $i = 0, \dots, Nm-1$, and
- $\|[p_0, x]\| < 2(Nm)^{-3}$ for $x \in \cup_{j=0}^{Nm-1} \alpha^{-j}(F)$.

We can construct these, first approximately using the hypotheses here, then exactly using the basic approximation lemmas. The rest of the proof of Theorem 3.1 can be copied verbatim. \square

We end the section with a condition which is sometimes easier to verify, to be used in the proof of Theorem 6.1. The hypothesis that the top of the stack be taken near the bottom is removed at the expense of a requiring a partial isometry between the first two levels of the stack. This implies equality of the $[e_i]$ always, but is not implied by it in algebras without cancellation.

Theorem 3.4. *Let α be an automorphism of a unital C^* -algebra A with the property that for any finite subset F of A and for any $m \in \mathbb{N}$, $\varepsilon > 0$, there are an ε -approximate stack e_0, e_1, \dots, e_{m-1} for α and partial isometries p, q in A such that*

- (1) $p^*p = 1 - \sum_{i=0}^{m-1} e_i$ and $pp^* < e_0$,
- (2) $q^*q = e_0$ and $qq^* = e_1$,
- (3) $\|[x, y]\| < \varepsilon$ for any $x \in F, y \in \{p, q\}$.

Then α has the Rohlin property.

Remark 3.5. We could allow e_i to be almost orthogonal instead of orthogonal and relax the equalities involving p and q to approximations as in Theorem 3.3 and the conclusion still holds.

Proof. We show that α satisfies the hypothesis of Theorem 3.3. Our proof uses many of the same ideas as in Lemmas 2.1 and 4.3 in [10]. The construction depends on parameters $\ell \in \mathbb{N}, \eta > 0$ to be specified later. Let F be a finite set, $m \in \mathbb{N}$, and $\varepsilon > 0$. Let $e_0, e_1, \dots, e_{(\ell-1)(m+2)m-1}$ be an η -approximate stack for α and let p, q be partial isometries in A such that

- $p^*p = 1 - \sum_{i=0}^{(\ell-1)(m+2)m-1} e_i$ and $pp^* < e_0$,
- $q^*q = e_0$ and $qq^* = e_1$,
- $\|[x, y]\| < \eta$ for all $x \in \bigcup_{i=0}^{(\ell-1)(m+2)} \alpha^{-i}(F), y \in \{p, q\}$.

By Lemma 2.5 there is a unitary $u \in A$ such that

$$\|1 - u\| < 8(\ell - 1)(m + 2)m\eta$$

and

$$u\alpha(e_i)u^* = e_{i+1}, \quad i \in \{0, 1, \dots, (\ell - 1)(m + 2)m - 2\}.$$

Write β for $(\text{Ad } u) \circ \alpha$. Define a system of matrix units $\{q_{i,j} : 0 \leq i, j \leq (\ell - 1)(m + 2)m - 1\}$ by setting

$$q_{i,i} = e_i$$

and for $i < j$

$$q_{i,j} = \beta^i(q)\beta^{i+1}(q) \cdots \beta^{j-1}(q), \quad q_{j,i} = q_{i,j}^*.$$

One readily checks that

$$q_{i,j}q_{i',j'} = \delta_{j,i'}q_{i,j'}$$

and

$$\beta(q_{i,j}) = q_{i+1,j+1}, \quad 0 \leq i, j \leq (\ell - 1)(m + 2)m - 2.$$

For $j = 0, 1, \dots, m - 1$ put

$$\begin{aligned} f_j = & \sum_{i=0}^{\ell-2} \left[\frac{i+1}{\ell} q_{im+j,im+j} + \frac{\ell-1-i}{\ell} q_{((m+1)(\ell-1)+i)m+j,((m+1)(\ell-1)+i)m+j} \right. \\ & + \frac{\sqrt{(i+1)(\ell-1-i)}}{\ell} q_{im+j,((m+1)(\ell-1)+i)m+j} \\ & \left. + \frac{\sqrt{(i+1)(\ell-1-i)}}{\ell} q_{((m+1)(\ell-1)+i)m+j,im+j} \right] \\ & + \sum_{i=\ell-1}^{(m+1)(\ell-1)-1} q_{im+j,im+j}. \end{aligned}$$

Then f_0, f_1, \dots, f_{m-1} is a stack of height m for β . We have

$$\begin{aligned} \beta(f_{m-1}) - f_0 = & \frac{1}{\ell} \sum_{i=0}^{\ell-1} [(e_{((m+1)(\ell-1)+i)m} - e_{im}) \\ & + (\sqrt{i(\ell-i)} - \sqrt{(i+1)(\ell-1-i)}) (q_{im,((m+1)(\ell-1)+i)m} \\ & + q_{((m+1)(\ell-1)+i)m,im})] \end{aligned}$$

and since, for $i = 0, 1, \dots, \ell - 1$,

$$\begin{aligned} \left| \sqrt{i(\ell-i)} - \sqrt{(i+1)(\ell-1-i)} \right| & \leq \sqrt{|i(\ell-i) - (i+1)(\ell-1-i)|} \\ & = \sqrt{|1+2i-\ell|} \\ & < \sqrt{\ell}, \end{aligned}$$

we see from orthogonality that

$$\|\beta(f_{m-1}) - f_0\| < \frac{1}{\ell} + \frac{1}{\sqrt{\ell}}.$$

This shows that f_0, f_1, \dots, f_{m-1} is a $\frac{2}{\sqrt{\ell}}$ -approximate cyclic stack for β , and thus by choosing ℓ large enough we can ensure that f_0, f_1, \dots, f_{m-1} is a ε -approximate cyclic stack for α . We have constructed an approximate cyclic stack from an approximate noncyclic stack, and the sum of this new stack is smaller than the sum of the old one by $m(\ell - 1)$ projections equivalent to e_0 , specifically,

$$1 - \sum_{j=0}^{m-1} f_j = p^*p + \sum_{j=0}^{m-1} \sum_{i=0}^{\ell-2} \left\{ \frac{\ell - 1 - i}{\ell} q_{im+j, im+j} + \frac{i + 1}{\ell} q_{((m+1)(\ell-1)+i)m+j, ((m+1)(\ell-1)+i)m+j} + \frac{\sqrt{(i + 1)(\ell - 1 - i)}}{\ell} q_{im+j, ((m+1)(\ell-1)+i)m+j} + \frac{\sqrt{(i + 1)(\ell - 1 - i)}}{\ell} q_{((m+1)(\ell-1)+i)m+j, im+j} \right\}.$$

The base of our new stack, f_0 , has a subprojection equivalent to $1 - \sum_{j=0}^{m-1} f_j$, indeed

$$r = \sum_{j=0}^{m-1} \sum_{i=0}^{\ell-2} \left(\sqrt{\frac{\ell - 1 - i}{\ell}} q_{(\ell+im+j-1)m, im+j} + \sqrt{\frac{i + 1}{\ell}} q_{(\ell+im+j-1)m, ((m+1)(\ell-1)+i)m+j} \right) + \left(\sqrt{\frac{1}{\ell}} e_0 + \sqrt{\frac{\ell - 1}{\ell}} q_{(m+1)(\ell-1)m, 0} \right) p$$

is a partial isometry satisfying

$$rr^* = 1 - \sum_{j=0}^{m-1} f_j \quad \text{and} \quad r^*r < f_0.$$

The $q_{i,j}$ almost commute with F if η is chosen small relative to $1/\ell$, and since f_0, \dots, f_{m-1} and r are linear combinations of the $q_{i,j}$ these too almost commute with F . By Theorem 3.3, α has the Rohlin property. \square

Remark 3.6. In the proof of Theorem 3.4 we constructed an approximate cyclic stack f_0, f_1, \dots, f_{m-1} from an approximate (noncyclic) stack $e_0, e_1, \dots, e_{Nm-1}$ using only the partial isometry q from e_0 to e_1 . To construct the partial isometry r from $1 - \sum f_j$ to a subprojection of f_0 we needed the partial isometry p from $1 - \sum e_i$ to a subprojection of e_0 . The f_j and r lie in $C^*(p, q, \alpha(q), \dots, \alpha^{Nm-1}(q))$ and the proof shows that these can be made to almost commute with any finite set provided the e_i and q and p can be made to almost commute with any finite set. This justifies the assertion following

Definition 2.9 that we can drop the “cyclic” requirement, since once we make the f_j and r almost commute with a system of matrix units for B , a small inner perturbation takes the f_j and r into the commutant of B .

4. Shifts of finite type and the bilateral algebras

We refer the reader to the excellent book of Lind and Marcus [17] for background on shifts of finite type. Let T be a nonnegative integral $r \times r$ matrix. Let G_T be the directed graph with r vertices labeled $1, 2, \dots, r$ and having $T(i, j)$ edges from vertex i to vertex j . Write \mathcal{E}_T for the edge set of G_T , and for $e \in \mathcal{E}_T$ let $i(e)$ and $t(e)$ be the initial and terminal vertices of e , respectively, following Definition 2.9

Definition 4.1. A path on G_T is a finite or infinite sequence x in \mathcal{E}_T such that $i(x_{i+1}) = t(x_i)$ for all relevant indices i .

Notation. For a finite path $x = x_0x_1 \dots x_{\ell-1}$ we let $i(x) = i(x_0)$ and $t(x) = t(x_{\ell-1})$ and we write $|x| = \ell$ for the length of x . By convention, each vertex i is regarded as a path of length 0 having i as its initial and terminal vertices. If a and b are integers with $a \leq b$ and x is a path then we put

$$x_{[a,b]} = x_a x_{a+1} \dots x_b$$

when the right-hand side makes sense.

We use the following fact often.

Lemma 4.2. For $i, j \in \{1, 2, \dots, r\}$, the number of paths of length ℓ in G_T starting at i and ending at j is $T^\ell(i, j)$.

Definition 4.3. The underlying space X_T of the shift of finite type determined by T is the set of all bi-infinite paths on G_T .

Definition 4.4. Basic cylinders are sets of the form

$$\text{cyl}(x, k) = \{y \in X_T : y_{i+k} = x_i \text{ for } 0 \leq i < |x|\},$$

where $x = x_0x_1 \dots x_{|x|-1}$ is a finite path in G_T and $k \in \mathbb{Z}$.

Basic cylinders form a basis of clopen sets for a topology on X_T in which X_T is a Cantor set. Though we shall not specifically use a metric, it is helpful to think of two points being close if their sequences agree on a large interval of indices about 0.

The paradigm of symbolic dynamics is that all the complexity of a system is encoded in sequence space.

Definition 4.5. The shift homeomorphism $\sigma_T : X_T \rightarrow X_T$ is defined by

$$(\sigma_T((x_i)_{i \in \mathbb{Z}}))_i = x_{i+1}.$$

Example 4.6. The full n -shift is the shift of finite type associated with the 1×1 matrix (n) , the underlying space of which is $\{1, 2, \dots, n\}^{\mathbb{Z}}$.

We shall assume our matrix T is primitive, i.e., some positive power of T has strictly positive entries, and that $T \neq (1)$. Then X_T is nonempty and σ_T is topologically mixing (see, for example, [17, Proposition 4.5.10]). By the Perron–Frobenius theorem, T has a unique (up to scalar multiplication) strictly positive (right) eigenvector; the corresponding eigenvalue λ_T is algebraically simple, greater than 1, and strictly larger in absolute value than any other eigenvalue of T . Let v, w^* be positive Perron–Frobenius eigenvectors for T and T^* , respectively, normalized so that $wv = 1$.

The *measure of maximal entropy* on X_T is the Markov measure μ_T defined from transition probabilities

$$p(e) = \lambda_T^{-1} v_{\tau(e)} / v_{\dot{\imath}(e)}, \quad e \in \mathcal{E}_T$$

and stationary distribution

$$\pi_i = w_i v_i, \quad 1 \leq i \leq r$$

by

$$\mu_T(\text{cyl}(x, k)) = \pi_{\dot{\imath}(x)} p(x_0) p(x_1) \dots p(x_{|x|-1}) \quad (3)$$

$$= \lambda_T^{-|x|} w_{\dot{\imath}(x)} v_{\tau(x)}. \quad (4)$$

We have the following standard result.

Lemma 4.7. μ_T is an ergodic measure on (X_T, σ_T) which gives every cylinder positive measure.

We are interested in the symmetric, or bilateral, C^* algebras associated to shifts of finite type. The earliest references of which we are aware for these C^* -algebras are from 1988, a construction by Wagoner in [20] specifically for shifts of finite type, and Ruelle's more general construction in [19]. These algebras are implicit in Krieger's earlier K -theoretic invariants for shifts of finite type, introduced in [15] (1982).

We write M_n for the C^* -algebra of complex $n \times n$ matrices. For integers $a \leq b$ let $H_T^{[a,b]}$ be the C^* -algebra generated by matrix units $E_{x,y,a,b}$, where x and y are paths in G_T of length $b - a + 1$ with $\dot{\imath}(x) = \dot{\imath}(y)$ and $\tau(x) = \tau(y)$. Matrix units means

$$E_{x,y,a,b} E_{x',y',a,b} = \delta_{y,x'} E(x, y', a, b)$$

and

$$E_{x,y,a,b}^* = E_{y,x,a,b}.$$

Lemma 4.8. For each interval $I = [a, b]$ we have

$$H_T^I \cong \bigoplus_{i,j} M_{T^{|I|+1}(i,j)},$$

one summand for each pair of vertices, the size of the ij th being the number of paths of length $|I| + 1$ in G_T beginning at vertex i and ending at vertex j .

Proof. For each pair of vertices i, j , enumerate the $T^{|I|+1}(i, j)$ paths of length $|I| + 1$ in G_T with initial vertex i and terminal vertex j . If x, y are the k th and ℓ th such paths from i to j then we identify $E_{x,y,a,b}$ with the $T^{|I|+1}(i, j) \times T^{|I|+1}(i, j)$ matrix having a 1 in the k, ℓ th position and zeros everywhere else, and we embed this in $\bigoplus_{i,j} M_{T^{|I|+1}(i,j)}$

by taking the direct sum of it with matrices of zeros for the other summands. It is easy to see that this induces an isomorphism. \square

Let $I = [a, b]$ and $J = [c, d]$ be intervals with $I \subset J$. Let $\psi_{T,I,J} : H_T^I \rightarrow H_T^J$ be the embedding defined for matrix units by

$$\psi_{T,I,J}(E_{x,y,a,b}) = \sum \{E_{z,u,c,d} : z_{[a-c,a+b-c]} = x \text{ and } u_{[a-c,a+b-c]} = y\}. \tag{5}$$

Lemma 4.9. The matrix of partial multiplicities for $\psi_{T,I,J}$ is the transpose of $(T^*)^{(a-c)} \otimes T^{(d-b)}$.

Proof. Simply count the number of paths z and u which extend x and y , respectively, using Lemma 4.2. \square

Lemma 4.10. If $I \subset J \subset K$ then

$$\psi_{T,I,K} = \psi_{T,J,K} \circ \psi_{T,I,J}.$$

Proof. This follows from the fact that if z is a path extending u and u is a path extending x then z is a path extending x . \square

Definition 4.11. A_T is the closure of $\varinjlim(\varphi_{T,I,J} : H_T^I \rightarrow H_T^J)$.

The proof of the following lemma is straightforward and we leave it to the reader.

Lemma 4.12. A_T is simple and has a unique trace. The trace τ_T is given on each H_T^I by

$$\tau_T \left(\sum_{x,y} c_{x,y} E_{x,y,a,b} \right) = \sum_x c_{x,x} \mu_T(\text{cyl}(x, 0)),$$

where the sums are over all paths of length $|I| + 1$. The trace determines the order on $K_0(A_T)$ up to infinitesimals.

The shift σ_T on X_T promotes to an automorphism α_T of A defined by

$$\alpha_T(E_{u,v,a,b}) = E_{u,v,a-1,b-1}. \tag{6}$$

Uniqueness of the trace implies that it is invariant under α_T , i.e.,

$$\tau_T \circ \alpha_T = \tau_T.$$

The C^* -algebra $C(X_T)$ of continuous complex-valued functions on X_T is a subalgebra of A_T in a natural way. Specifically, $\varphi_T : C(X_T) \rightarrow A_T$ is defined for characteristic functions of basic cylinders by

$$\varphi_T(\chi_{\text{cyl}(u,k)}) = E_{u,u,k,k+|u|-1}$$

and extended to all of $C(X_T)$ in the usual way. The next two lemmas follow from the definition of α_T in Eq. (6).

Lemma 4.13. The C^* -embedding φ_T takes the induced map $f \mapsto f \circ \sigma_T^{-1}$ on $C(X_T)$ to α_T , i.e.,

$$\varphi_T(f \circ \sigma_T^{-1}) = \alpha_T(\varphi_T(f)).$$

Lemma 4.14. The restriction $\alpha_T|_{H_T^I}$ is an isomorphism from H_T^I to H_T^{I-1} .

The following is a very useful fact about these algebras.

Lemma 4.15. If I and J are disjoint intervals then elements of H_T^I commute with elements of H_T^J in A_T .

Proof. We may assume that $I = [a, b]$ and $J = [c, d]$ with $b < c$. Let $K = [a, d]$. It follows from the definitions that

$$\psi_{T,I,K}(E_{x,y,a,b})\psi_{T,J,K}(E_{x',y',c,d}) = \sum_t E_{xtx',yty',a,d}$$

where the sum is over all paths t from $\mathfrak{t}(x)$ to $\mathfrak{i}(x')$ of length $c - b - 1$ and where xtx' means the concatenation of the paths x, t, x' , and similarly for yty' . Nothing here depends on the order of multiplication and $\psi_{T,I,K}(E_{x',y',c,d})\psi_{T,J,K}(E_{x,y,a,b})$ is easily seen to be the very same sum. \square

5. Shift equivalence

Shift equivalence and strong shift equivalence for shifts of finite type were introduced by Williams in [21].

Definition 5.1. Nonnegative integral square matrices U, V are said to be shift equivalent if there exist nonnegative integral matrices R, S and a positive integer ℓ such that

$$RS = U^\ell, \quad SR = V^\ell,$$

$$SU = VS, \quad UR = RV.$$

Shift equivalence is an equivalence relation. The integer ℓ is called the lag of the shift equivalence.

Definition 5.2. An elementary strong shift equivalence between nonnegative integral square matrices U and V is a pair of nonnegative integral matrices R, S such that $RS = U$ and $SR = V$. Strong shift equivalence is the equivalence relation generated by elementary strong shift equivalence.

Shifts of finite type (X_U, σ_U) and (X_V, σ_V) are *conjugate* if there is a homeomorphism $\rho : X_U \rightarrow X_V$ such that $\rho \circ \sigma_U = \sigma_V \circ \rho$ and *eventually conjugate* if (X_U, σ_U^ℓ) and (X_V, σ_V^ℓ) are conjugate for all sufficiently large ℓ . Williams showed [21] that U and V are strong shift equivalent if and only if (X_U, σ_U) and (X_V, σ_V) are conjugate, and that U and V are shift equivalent if and only if (X_U, σ_U) and (X_V, σ_V) are eventually conjugate.

The Shift Equivalence Conjecture is the proposition that shift equivalence is the same as strong shift equivalence. The first counterexample was a pair of reducible shifts of finite type found in 1992 by Kim and Roush [8]. The question for the irreducible case remained open until 1997, when Kim and Roush exhibited in [9] a pair of primitive 7×7 matrices which are shift equivalent but not strong shift equivalent. The invariant used to prove the matrices are not shift equivalent was the sign-gyration-compatibility-condition class sgc_2 , which turns out to be the same as Wagoner's K_2 -invariant, Φ_2 .

There are still many open questions. For one, it is not known whether the conjecture holds for matrices shift equivalent to full shifts. We do not even know if the automorphisms α_U and α_V associated with shift equivalent matrices U and V are conjugate, though they are of course eventually conjugate. We do have the following.

Proposition 5.3. *If U and V are shift equivalent primitive matrices then there is an isomorphism $\Psi : A_U \rightarrow A_V$ such that $(\alpha_U)_* = (\Psi^{-1} \circ \alpha_V \circ \Psi)_*$.*

Sketch of Proof. This can be proved using the standard bipartite graph construction (see [17], §7.2) to obtain a conjugacy between (X_U, σ_U^ℓ) and (X_V, σ_V^ℓ) for some ℓ . A conjugacy produced in this way induces a isomorphism of C^* -algebras with the desired property. \square

We now show that our automorphisms are usually not approximately inner. One consequence of this is that we cannot deduce the Rohlin property for α_T from the result of Kishimoto [10].

Notation. For an $s \times s$ integral matrix S we write \mathcal{R}_S for the eventual range of S ,

$$\mathcal{R}_S = \bigcap_{n \geq 0} (\mathbb{Q}^s)^* S^n.$$

By rank considerations, we have $\mathcal{R}_S = (\mathbb{Q}^s)^* S^s$. The eventual rank of S is the dimension of the rational vector space $\mathcal{R}_S \otimes \mathbb{Q}$.

Proposition 5.4. *The following are equivalent:*

- (1) $K_0(A_T)$ has no infinitesimals.
- (2) T has eventual rank 1.
- (3) T is shift equivalent to the 1×1 matrix (n) for some $n \geq 2$.
- (4) α_T is approximately inner.

Proof. $1 \Leftrightarrow 2$. As ordered groups, $K_0(A_T) \cong \varinjlim (\mathbb{Z}^r \otimes \mathbb{Z}^r, T^* \otimes T)$, thus

$$\text{rank } K_0(A_T) = \text{rank } \mathcal{R}_{T^* \otimes T} = (\text{rank } \mathcal{R}_T)^2.$$

We shall use the fact that $\varinjlim (\mathbb{Z}^r \otimes \mathbb{Z}^r, T^* \otimes T)$ is isomorphic to the (additive) subgroup of $M_r(\mathbb{Q})$ consisting of those matrices U in $T^r M_r(\mathbb{Q}) T^r$ such that $T^n U T^n \in M_r(\mathbb{Z})$ for some $n \in \mathbb{N}$. The positive cone is the set of U such that $T^n U T^n \in M_r(\mathbb{Z}_+)$ for some $n \in \mathbb{N}$. Let us identify $(K_0(A_T), \leq)$ with this ordered group. By Perron–Frobenius theory, $U \in K_0(A_T)_+$ if and only if $w U v > 0$, thus the infinitesimals are precisely those U for which $w U v = 0$. It is possible to replace v with a multiple so that its entries lie in and span $\mathbb{Q}[\lambda_T]$. The same is true of w . It follows that we can find $r^2 - \text{deg } \lambda_T$ linearly independent matrices $U_i \in M_r(\mathbb{Z})$ such that $w U_i v = 0$ for each i . The infinitesimal subgroup of $K_0(A_T)$ is generated by the matrices $T^r U_i T^r$, and its rank is easily seen to be $(\text{rank } \mathcal{R}_T)^2 - \text{deg } \lambda_T$. This last quantity is zero if and only if $\text{rank } \mathcal{R}_T = 1$.

$2 \Leftrightarrow 3$ is well known.

1 \Rightarrow 4. If $K_0(A_T)$ has no infinitesimals then α_* must be the identity on $K_0(A_T)$, since $\alpha_*(g)$ can only differ from g by an infinitesimal.

4 \Rightarrow 2. The existence of infinitesimals implies that T has an eigenvalue λ with $0 < |\lambda| < \lambda_T$. We can find in the above representation of $K_0(A_T)$ a matrix U with large entries such that $UT \approx \lambda U$ and $TU \approx \lambda_T U$, and then $\alpha_*(U) \approx \lambda^{-1} \lambda_T(U) \not\approx U$. \square

6. The Rohlin property for α_T

Theorem 6.1. *If the matrix T is primitive then α_T has the Rohlin property.*

Fix a primitive matrix T . To simplify notation we drop the subscript T from the various objects defined in the previous sections. Before proving the theorem we need some preliminary results.

Lemma 6.2. *For any $x, y \in A$*

$$\|[\alpha^n(x), y]\| \rightarrow 0 \text{ as } |n| \rightarrow \infty.$$

Proof. Given x and y we can find an interval I and elements $x', y' \in H_T^I$ such that $\|x - x'\|$ and $\|y - y'\|$ are as small as desired. If $n > |I|$ then $I \cap (I + n) = \emptyset$ and since $\alpha^n(x') \in H_T^{I+n}$ we see from Lemma 4.15 that $\alpha^n(x')$ commutes with y' . Thus

$$\begin{aligned} \|\alpha^n(x)y - y\alpha^n(x)\| &\leq \|\alpha^n(x)y - \alpha^n(x')y\| + \|\alpha^n(x')y - \alpha^n(x')y'\| \\ &\quad + \|\alpha^n(x')y' - y'\alpha^n(x')\| + \|y'\alpha^n(x') - y\alpha^n(x')\| \\ &\quad + \|y\alpha^n(x') - y\alpha^n(x)\| \\ &\leq \|x - x'\|\|y\| + \|x'\|\|y - y'\| + 0 + \|y' - y\|\|x'\| + \|y\|\|x' - x\| \\ &\leq 2\|y\|\|x - x'\| + 2(\|x\| + \|x - x'\|)\|y - y'\| \end{aligned}$$

for all $n > |I|$. By making $\|x - x'\|$ and $\|y - y'\|$ sufficiently small we can make this last quantity as small as desired for all sufficiently large n . \square

Remark 6.3. Lemma 6.2 provides an immediate simplification of our situation. In proving the Rohlin property for α we may ignore the almost commuting requirements, since for any finite collection of projections and partial isometries satisfying all the other properties their images under α^n , for sufficiently large n , have the additional property of almost commuting with the finite set F .

Our next lemma says that α_* has arbitrarily small noninfinitesimal fixed points.

Lemma 6.4. *For every $n > 0$ there exists $g \in K_0(A)_+$ such that $\alpha_*(g) = g$ and $ng < [1]$.*

Proof. For any interval $I = [a, b]$, $K_0(H^I)$ is isomorphic to a subspace of $M_r(\mathbb{Z})$ via the map that takes the class of a minimal projection $E_{u,u,a,b} \in H^I$ to the matrix having $(i(u), t(u))$ th entry equal to one and all other entries equal to zero. With this identification the class of the unit in $K_0(H^I)$ is

$$[1]_I = T^{|I|+1}.$$

Choose m such that $nI < T^m$. Let $g \in K_0(A)$ be the element represented by the identity matrix I in $K_0(H^{[1,m]})$. It is clear that g is positive and $ng < 1$. We claim that this g is fixed by α_* . Indeed, g is represented by TI in $K_0(H^{[0,m]})$, while $\alpha_*(g)$ is represented by I in $K_0(H^{[0,m-1]})$ and thus by IT in $K_0(H^{[0,m]})$. \square

Next we show that the fixed points of α_* are order-dense.

Lemma 6.5. *For any $m \in \mathbb{N}$ there exists $g \in K_0(A)$ such that $\alpha_*(g) = g$ and $mg < [1] < (m + 1)g$.*

Proof. By Lemma 6.4, we can find $h \in K_0(A)_+$ fixed by σ_* such that $m(m+1)h < [1]$. Let N be the greatest positive integer for which $Nh < [1]$. While $(N + 1)h$ may differ from $[1]$ by an infinitesimal we certainly have $(N + 2)h > [1]$. Set $g = \lfloor N/m \rfloor h$. We obviously have $m \lfloor N/m \rfloor \leq N$. Write $N = \ell m + k$ with $0 \leq k \leq m - 1$. Since $N \geq m(m+1)$ we must have $\ell \geq m + 1$. Now

$$(m + 1)\lfloor N/m \rfloor = (m + 1)\ell = \ell m + (m + 1) \geq \ell m + k + 2 = N + 2$$

and thus g has the desired properties. \square

Lemma 6.6. *For any $m \in \mathbb{N}$ there exists a clopen subset C of X such that*

- (1) $C, \sigma(C), \dots, \sigma^{m-1}(C)$ are pairwise disjoint,
- (2) $1 - m[\varphi(\chi_C)] < [\varphi(\chi_C)]$, and
- (3) $\alpha_*([\varphi(\chi_C)]) = [\varphi(\chi_C)]$.

Proof. Fix m and let g be as in Lemma 6.5. By Lemma 4.12, we can find $\varepsilon > 0$ such that if U is any clopen set in X with $\mu U < \varepsilon$ then $[\varphi(\chi_U)] < [1] - mg$. Let x be a finite path in G long enough so that

$$\mu \left(\bigcup_{k=1}^{m-1} \text{cyl}(x, -k) \right) < \varepsilon/2.$$

We can find $N \in \mathbb{N}$ large enough so that

$$\mu \left(\bigcup_{k=0}^{Nm-1} \text{cyl}(x, k) \right) > 1 - \varepsilon/2.$$

Let C' be the set of points $y \in \bigcup_{k=0}^{Nm-1} \text{cyl}(x, k)$ for which the least $k \geq 0$ such that $y \in \text{cyl}(x, k)$ satisfies $k \equiv m - 1 \pmod m$. The sets $\sigma^k(C'), 0 \leq k \leq m - 1$ are pairwise disjoint and

$$\left(\bigcup_{k=0}^{Nm-1} \text{cyl}(x, k) \right) \setminus \left(\bigcup_{k=1}^{m-1} \text{cyl}(x, -k) \right) \subset \bigcup_{k=0}^{m-1} \sigma^k(C')$$

whence

$$\mu \left(X \setminus \left(\bigcup_{k=0}^{m-1} \sigma^k(C') \right) \right) < \varepsilon.$$

This implies that $[\varphi(\chi_{C'})] > g$.

By Lemma 2.6, $[\varphi(\chi_{C'})] - g$ is the class of some projection $e \in A_T$ and by Lemma 2.8, e is equivalent to some projection $f < \varphi(\chi_{C'})$. Every projection in A is equivalent to one of the form $\varphi(\chi_E)$ for some clopen set E (this is just the statement that every projection in a finite direct sum of full matrix algebras is equivalent to a diagonal matrix), so we can find a set E such that $[\varphi(\chi_E)] = [f]$, and because $f < \varphi(\chi_{C'})$ we can take E to be a subset of C' . Set $C = C' \setminus E$. Then $[\varphi(\chi_C)] = g$ and properties (1)–(3) are satisfied. \square

Proof of Theorem 6.1. Given $m \in \mathbb{N}, \varepsilon > 0$, and a finite subset F of A we let C be as in Lemma 6.6 and set $e_i = \varphi(\chi_C)$. By Remark 6.3, we need not consider F at all. Our construction guarantees the $[e_i]$ are all the same, and that $[1 - \sum_i e_i] < [e_0]$, so the partial isometries needed to apply Theorem 3.4 exist. \square

Proposition 6.7. *If T_1 and T_2 are shift equivalent primitive integral matrices then $A_{T_1} \times_{\alpha_{T_2}} \mathbb{Z} \cong A_{T_2} \times_{\alpha_{T_2}} \mathbb{Z}$.*

Proof. Let us write α_i for α_{T_i} and A_i for $A_{T_i}, i = 1, 2$. Shift equivalence implies that there is an isomorphism $\rho : A_1 \cong A_2$ such that

$$\rho_* \circ (\alpha_1)_* = (\alpha_2)_* \circ \rho_*$$

and

$$\rho_*(K_0(A_1)_+) = K_0(A_2)_+.$$

By Theorem 4.1 of Evans and Kishimoto [6], there is an automorphism β of A_2 and a unitary $u \in A_2$ such that

$$\alpha_1 = \rho^{-1} \circ \text{Ad } u \circ \beta \circ \alpha_2 \circ \beta^{-1} \circ \rho.$$

For $i = 1, 2$, the crossed product $A_i \times_{\alpha_i} \mathbb{Z}$ is generated by A_i and a unitary v_i such that

$$v_i a = \alpha_i(a) v_i.$$

Define $\Phi : A_1 \times_{\alpha_1} \mathbb{Z} \rightarrow A_2 \times_{\alpha_2} \mathbb{Z}$ by setting

$$\Phi(v_1) = \beta^{-1}(u)v_2$$

and

$$\Phi(a) = \beta^{-1}(\rho(a)), \quad a \in A_1.$$

The map Φ is a well-defined $*$ -homomorphism because

$$\begin{aligned} \Phi(v_1 a) &= \beta^{-1}(u)v_2\beta^{-1}(\rho(a)) \\ &= \beta^{-1}(u)(\alpha_2 \circ \beta^{-1} \circ \rho)(a)v_2 \\ &= \beta^{-1}(u(\beta \circ \alpha_2 \circ \beta^{-1} \circ \rho)(a)u^*u)v_2 \\ &= (\beta^{-1} \circ \rho \circ \alpha_1)(a)\beta^{-1}(u)v_2 \\ &= \Phi(\alpha_1(a)v_1). \end{aligned}$$

To see that it is invertible observe that $\beta^{-1} \circ \rho$ is an isomorphism and

$$\Phi(\rho^{-1}(u^*)v_1) = \beta^{-1}(\rho(\rho^{-1}(u^*)))\beta^{-1}(u)v_2 = \beta^{-1}(u^*u)v_2 = v_2. \quad \square$$

Remark 6.8. Strong shift equivalence of T_1 and T_2 certainly implies that the C^* -dynamical systems (A_{T_1}, α_{T_1}) and (A_{T_2}, α_{T_2}) are isomorphic. What about the converse? We do not know whether an isomorphism of the C^* -dynamical systems $(A_{T_1}, \alpha_{T_1}) \cong (A_{T_2}, \alpha_{T_2})$ implies strong shift equivalence, or even shift equivalence, for T_1 and T_2 .

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