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Note

Spheres Tangent to All the Faces of a Simplex

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There are at most 2^n spheres tangent to all $n+1$ faces of an n-simplex. It has been shown that the minimum number of such spheres is $2^n - c(n, \frac{1}{2}(n+1))$ if n is odd and $2^n - c(n, \frac{1}{2}(n+1))$ if n is even. The object of this note is to show that this result is a consequence of a theorem in graph theory.

It is known that there are at most 2^n spheres tangent to all $n+1$ faces of an n-simplex.

H. E. Vaughan and Hyman Gabai [1] conjectured that the minimum number of such spheres is

$$2^n - C(n, \tfrac{1}{2}(n+1))$$

or

$$2^n - C(n, \tfrac{1}{2}(n+2))$$

according as n is odd or even. By an ingenious argument, Leon Gerber [2] proved their conjecture. The object of this note is to show that Gerber's result is a consequence of a very general theorem of Paul Erdös.

We begin with a lemma from [1, p. 387] but in the notation of [2].

LEMMA 1. *Let* $\mathbf{e} = (e_0, e_1, ..., e_n)$ *with* $e_i \in \{-1, 1\}$. *For* $i \in \{0, 1, ..., n\}$, *let* A_i, $\mathbf{a}_i$, F_i, f_i *denote respectively a vertex of an n-simplex, its position vector, the hyperplane of the opposite face, and its* $(n-1)$*-dimensional measure. Let P be the center of a sphere which is tangent to* F_i *for all* $i \in \{0, 1, ..., n\}$, *and let* e_i *be* 1 *or* -1 *according as P and* A_i *are on the same or opposite sides of* F_i. *Then* $S(\mathbf{e}) = \sum_{i=0}^{n} e_i f_i > 0$ *and the position vector of P is given by*

$$\mathbf{P} = \sum_{i=0}^{n} e_i f_i \mathbf{a}_i / S(\mathbf{e}). \tag{1}$$

Conversely, for any choice of $\mathbf{e}$ *such that* $S(\mathbf{e}) > 0$, *the point whose position vector is given by* (1) *is equidistant from the* F_i.

If N is the number of solutions $\mathbf{e}$ to $S(\mathbf{e}) = 0$, then the number of $\mathbf{e}$'s such that $S(\mathbf{e}) > 0$ is $(1/2)(2^{n+1} - N)$, which is the number of spheres tangent to all the faces of the simplex.

To prove the Vaughan–Gabai conjecture, it is therefore enough to show that $N \leqslant 2C(n, (1/2)(n+1))$ when n is odd and $N \leqslant 2C(n, (1/2)(n+2))$ when n is even. To do this we need another lemma.

LEMMA 2. *Let* $f_0, ..., f_n$ *be positive and not all equal, and let* N *be the number of solutions* $\mathbf{e}$ *of*

$$\sum_{i=0}^{n} e_i f_i = 0. \tag{1}$$

Then $N \leqslant 2C(n, (n+2)/2)$ *if* n *is even and* $N \leqslant C(n+1, (n+3)/2)$ *if* n *is odd.*

Proof. Since $f_0, f_1, ..., f_n$ are not all equal, we may suppose without loss of generality that $f_0 > f_1$. Then

$$-f_0 - f_1 < -f_0 + f_1 < f_0 - f_1 < f_0 + f_1 .$$

Let the N solutions of (1) be $\mathbf{e}^{(i)}$, $1 \leqslant i \leqslant N$. Then the $(n-1)$-tuples

$$\tilde{\mathbf{e}}^{(i)} = (e_2^{(i)}, ..., e_n^{(i)}), \qquad 1 \leqslant i \leqslant N,$$

must be distinct, for if $e^{(i)} = e^{(j)}$ for some indices i and j, then we see from (1) that

$$\sum_{k=0}^{n} e_k^{(i)} f_k = \sum_{k=0}^{n} e_k^{(j)} f_k ,$$

$$e_0^{(i)} f_0 + e_1^{(i)} f_1 = e_0^{(j)} f_0 + e_1^{(j)} f_1 ,$$

$$(e_0^{(i)}, e_1^{(i)}) = (e_0^{(j)}, e_1^{(j)}),$$

$\mathbf{e}^{(i)} = \mathbf{e}^{(j)}$, and $i = j$.

For each i, let $\tilde{S}_i$ be the set of those j, $2 \leqslant j \leqslant n$, for which $e_j^{(i)} = 1$. Then there are no inclusion chains of length five among the $\tilde{S}_i$.

To see this, suppose that

$$\tilde{S}_i \subset \tilde{S}_j \subset \tilde{S}_k \subset \tilde{S}_l \subset \tilde{S}_m$$

for some distinct i, j, k, l, and m.

Putting $\sigma_s = \sum_{r=2}^{n} e_r^{(s)} f_r$, we have $\sigma_i < \sigma_j < \sigma_k < \sigma_l < \sigma_m$. But there are only four possible values for $\sigma_s = \pm f_0 \pm f_1$.

Hence there are indeed no inclusion chains of length five among the $\tilde{S}_i$, and by a theorem of Paul Erdös [3; p. 900, Theorem 5], the number N of the $\tilde{S}_i$ is at most equal to the sum of the 4 largest binomial coefficients $C(n-1, j)$, $0 \leqslant j \leqslant n-1$. Calculation shows this sum to be $2C(n, (1/2)(n+2))$ when n is even and $C(n+1, (1/2)(n+3))$ when n is odd.

We now prove the Vaughan–Gabai conjecture. We consider separately the cases n odd and n even.

Suppose that n is even; then the number of terms in the sum of (1) is odd. If the f_i are all equal, then $N = 0$; otherwise, by Lemma 2, $N \leqslant 2C(n, (n+2)/2)$.

Suppose that n is odd; if the f_i are all equal, then the solutions of (1) are those $\mathbf{e}$ having exactly $((n+1)/2)$ e_i such that $e_i = 1$, and $N = C(n+1, (n+1)/2)$. If the f_i are not all equal, then by Lemma 2

$$\begin{aligned} N &\leqslant C(n+1, (n+3)/2) < C(n+1, (n+1)/2) \\ &= 2C(n, (n+1)/2), \text{ as required.} \end{aligned}$$

In addition to proving the conjecture, this also shows that the maximum of N is isolated. That is, when n is odd, there are no values of N between $C(n+1, (n+3)/2)$ and $2C(n, (n+1)/2)$. Consequently, in odd dimensional space, there are no simplices having T tangent spheres where

$$2^n - C(n, (1/2)(n+1)) < T < 2^n - (1/2)\, C(n+1, (1/2)(n+3)).$$

The existence of gaps in the value of T is not suggested by the example of three-dimensional space. When $n = 3$, the above states merely that there are no values of T strictly between 5 and 6, and indeed when $n = 3$ all values between the maximum and the minimum are achieved. But when $n = 5$, we see that there are no values of T between 22 and $24\frac{1}{2}$; that is, $T \neq 23, 24$. As n gets larger the gap widens.

If n is even, then the f_i cannot be all equal and the existence of gaps is still undecided.

References

1. H. E. Vaughan and H. Gabai, Hyperspheres associated with an n-simplex, *Amer. Math. Monthly* **74** (1967), 384–392.
2. L. Gerber, Spheres tangent to all the faces of a simplex, *J. Combinatorial Theory* (*A*) **12** (1972), 453–456.
3. P. Erdös, On a lemma of Littlewood and Offord, *Bull. Amer. Math. Soc.* **51** (1945), 898–902.