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On the Generation of Dual Polar Spaces of Symplectic Type over Finite Fields

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It is demonstrated that the dual polar space of type $Sp_{2n}(q)$, $q>2$, can be generated as a geometry by $\binom{2n}{n}-\binom{2n}{n-2}$ points.

1. INTRODUCTION

We assume the reader is familiar with the basic definitions relating to *undirected graphs* and *linear incidence system* or *point-line geometry* (as a standard reference see [2]). In particular: the *distance function*, *a geodesic path*, and *diameter* of a graph; the *collinearity graph* of a point-line geometry $\Gamma=(P, L)$, *a subspace* of Γ, the subspace $\langle X\rangle_\Gamma$ *generated* by a subset X of P, and *convex subspace* of Γ. We define the *generating rank*, $gr(\Gamma)$, of a point-line geometry Γ to be $\min\{|X|: X\subset P, \langle X\rangle_\Gamma=P\}$, that is, the minimal cardinality of a generating set of Γ.

Let $G=(P, L)$ be a point-line geometry. By a *projective embedding* of Γ we mean an injective mapping $e: P\to \mathbb{PG}(V)$, V a vector space over some division ring, such that (i) the space spanned by $e(P)$ is all of $\mathbb{PG}(V)$ and (ii) for $l\in L$, $e(l)$ is a full line of $\mathbb{PG}(V)$. We say that Γ is *embeddable* if some projective embedding of Γ exists. When Γ is embeddable, we define the *embedding rank*, $er(\Gamma)$, of Γ to the maximal dimension of a vector space V for which there exists an embedding into $\mathbb{PG}(V)$. Suppose now that $\Gamma=(P, L)$ is a point-line geometry and $e_i: P\to\mathbb{PG}(V_i)$, $i=1, 2$ are projective embeddings. A *morphism of embeddings* is a map $\alpha: \mathbb{PG}(V_1)\to\mathbb{PG}(V_2)$ induced by a surjective semi-linear transformation of the underlying vector spaces V_1, V_2 such that $\alpha\circ e_1=e_2$. An embedding $\hat{e}$ is said to be *universal relative to e* if there is a morphism $\hat{\alpha}: \hat{e}\to e$ such that for any other morphism $\gamma: e'\to e$, $\hat{\alpha}$ factors through γ, that is, there is a morphism $\alpha': \hat{e}\to \acute{e}$ such that $\hat{e}=\alpha'\circ\gamma$. An embedding $e: P\to\mathbb{PG}(W)$ is *relatively*

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universal if it is universal relative to itself. An immediate consequence of these definitions is the following:

1.1. *Let $\Gamma = (P, L)$ be an embeddable point-line geometry and let $e: P \to \mathbb{PG}(V)$ be an embedding. Then*

(i) *$\dim(V) \leqslant gr(\Gamma)$. Consequently, $er(\Gamma) \leqslant gr(\Gamma)$.*

(ii) *If $\dim(V) = gr(\Gamma)$ then e is relatively universal.*

In general, when we have a subset X of a point-line geometry $\Gamma = (P, L)$ and some collection of subspaces $\mathscr{A}$ then we will set $\mathscr{A}(X) = \{A \in \mathscr{A} \mid A \subset X\}$.

Polar spaces and dual polar spaces of type $Sp_{2n}(q)$. In this paper we will be interested in two related incidence systems: the polar and dual polar spaces of type $Sp_{2n}(q)$. The polar space of type $Sp_{2n}(q)$ can be described as follows: Let V be a vector space of dimension $2n$ over $\mathbb{F}_q$ and let $f: V \times V \to \mathbb{F}_q$ be a non-degenerate *alternative form*, that is, a bilinear form which satisifies: (1) $f(v, v) = 0$ for all $v \in V$ and (2) for all $v \in V$ there exists $w \in V$ with $f(v, w) \neq 0$.

We say a subspace U is *isotropic* if $f(U, U) = 0$, so note that all projective points in $\mathbb{PG}(V)$ are isotropic. The maximal dimension of an isotropic subspace is n and all such subspaces are conjugate under the action of

$$G = G(V) = \{T: V \to V \mid f(Tv, Tw) = f(v, w), \forall v, w \in V\}.$$

This is the *symplectic group*. We denote the isotropic subspaces of dimension l, $1 \leqslant l \leqslant n$, by $P(l, q)$ or $P_{2n}(l, q)$ when we need to keep track of the dimension of the ambient symplectic space. The points of the symplectic polar space are the one spaces—$P(1, q)$—and the lines are the isotropic two subspaces—$P(2, q)$. We will denote by $PSp_{2n}(q)$ the incidence geometry $(P(1, q), P(2, q))$. For convenience throughout this paper we will set $P = P(1, q)$ and $L = P(2, q)$.

We will denote by $S(l, q)$ the collection of all nondegenerate subspaces of V which have dimension $2l$, $l \leqslant n$ and identify such a subspace with the isotropic points which it contains. The *hyperbolic lines* are the subspaces in $S(1, q)$ which we will denote by H. The geometry (P, H) will be denoted by N.

The second geometry which we will be concerned with is the dual polar space of type $Sp_{2n}(q)$. This geometry has as its points the elements of $P(n, q)$. We will denote this set by $\mathscr{P}$. The lines are in one-to-one correspondence with the elements of $P(n-1, q)$ and for an isotopic subspace, A, $\dim(A) = n-1$, the line corresponding to it is $l(A) = \{M \in \mathscr{P} \mid A \subset M\}$. We will denote the set of lines by $\mathscr{L}$. We will use the notation $DSp_{2n}(q)$ for the isomorphism class of this geometry or simply D.

The main theorem of this paper is the following:

THEOREM A. *Assume* $q > 2$. *Then the generating rank of* $DSp_{2n}(q)$ *is* $\binom{2n}{n} - \binom{2n}{n-2}$.

From (Brouwer and Shpecterov [1] and Premet and Suprunenko [9]) we will be able to conclude that the geometry $DSp_{2n}(q)$ has an embedding e into a projective space $\mathbb{PG}(M)$ where $\dim(M) = \binom{2n}{n} - \binom{2n}{n-2}$. Therefore as an immediate consequence of Theorem A we will have the following:

THEOREM B. *Assume* $q > 2$. *Then embedding into* $\mathbb{PG}(M)$ *is a relatively universal embedding and the embedding rank of* $DSp_{2n}(q)$ *is* $\binom{2n}{n} - \binom{2n}{n-2}$.

In (Cooperstein and Shult [5]) a *basis* of an embeddable geometry $\Gamma = (P, L)$ is defined to be a subset X of P such that $\langle X \rangle_\Gamma = P$ and such that there exists some embedding e: $P \to \mathbb{PG}(M)$ with $e(X)$ an independent set of points. Such sets exist if and only if the embedding rank of Γ is equal to the generating rank. Therefore we have the following corollary:

THEOREM C. *For* $q > 2$, *bases exist in the symplectic dual polar space of type* $DSp_{2n}(q)$.

The layout of this paper is as follows: In section two we consider the geometry $N = (P, H)$ and prove that it is possible to find a set of points $p_1, p_2, ..., p_{2n}$ in P such that for every $j \leqslant k$ the linear space, V_j spanned by $p_1, p_2, ..., p_j$ is nondegenerate or has radical of dimension one as j is even, or odd, respectively, and such that the subspace of N generated by $p_1, p_2, ..., p_j$ is $P(V_j) \backslash rad(V_j)$. In section three we record some necessary properties of the dual polar space $DSp_{2n}(q)$. In section four we will define, for each natural number $n \geqslant 2$, a sequence of integers by a recurrence relation and get a closed expression for these numbers. Finally, in section five we prove Theorem A.

2. PROPERTIES OF THE SYMPLECTIC SPACE $Sp_{2n}(q)$

In this section we consider a nondegenerate symplectic space V of dimension $2n$ over the field $\mathbb{F}_q$. Recall that for a subspace of U of V its "perp" is defined to be $U^\perp = \{v \in V \mid f(v, U) = 0\}$ and the radical of U is $U \cap U^\perp$.

For each point $x \in P$ let r_x be the transvection group with center x and axis $x^\perp$

$$r_x = \{\tau \in GL(V) \mid [V, \tau] \leqslant x, [x^\perp, \tau] = 0\}.$$

These are subgroups of G and the actions of G on P and $\{r_x \mid x \in P\}$ are equivalent. We remark that for $x, y \in P$ that either (i) $f(x, y) = 0$ in which case $[x, y] = 1$ and $\langle x, y \rangle \cong \mathbb{F}_q \times \mathbb{F}_q$ or (ii) $f(x, y) \neq 0$ and $\langle x, y \rangle \cong SL(2, q)$. We shall require the following

LEMMA 2.1. *Let $X \subset P$. Then the group $\langle r_x \mid x \in X \rangle$ leaves $\langle X \rangle_N$ invariant.*

Proof. Set $S = \langle X \rangle_N$. It suffices to prove that for all $x \in X$ and $y \in S$ that $y^{r_x} \subset S$. If $f(x, y) = 0$ then $y^{r_x} = \{y\}$. On the other hand, if $f(x, y) \neq 0$ then $y^{r_x} \subset P(\langle x, y \rangle) = \langle x, y \rangle_N \subset S$ since $x, y \in S$ and S is a subspace. ∎

LEMMA 2.2. *There exists a basis $v_1, v_2, ..., v_{2n}$ for V such that for each $j \leqslant 2n$, $V_j = \langle v_1, v_2, ..., v_j \rangle$ is nondegenerate if j is even and has radical of dimension one if j is odd. Moreover, for $j \geqslant 2$, $V_j \cap v_{j+1}^{\perp} = V_{j-1}$.*

Proof. Let $x_1, y_1, x_2, y_2, ..., x_n, y_n$ be a hyperbolic basis of V. This means that $f(x_i, x_j) = f(y_i, y_j) = 0$ for all i, j and $f(x_i, y_j) = \delta_{ij}$. Now set $v_1 = x_1$, $v_{2i-1} = x_i + x_{i+1}$ for $2 \leqslant i \leqslant n$ and $v_{2j} = y_j$ for $1 \leqslant j \leqslant n$. Clearly $V_{2k} = \langle v_i \mid 1 \leqslant i \leqslant 2k \rangle = \langle x_j, y_j \mid 1 \leqslant j \leqslant k \rangle$ and so is non-degenerate. It follows from this that rad(V_{2k+1}) has dimension at most one. On the other hand, $x_{k+1} \in V_{2k+1}$ and is perpendicular to all the v_j, $1 \leqslant j \leqslant 2k+1$. This prove the first part. Also notice from the construction that $f(v_i, v_j) = 0$ if and only if $|i - j| > 1$ from which the second part follows. ∎

Now let $v_1, ..., v_{2n}$ be a basis for V as in (2.2) and set $p_i = \langle rv_i \rangle$. Set $R_j = rad(V_j) = V_j \cap V_j^{\perp}$ so that $R_j = 0$ for j even and has dimension one for j odd. Our final result of this section concerns the subspace of the geometry $N = (P, H)$ generated by $p_1, p_2, ..., p_j$ for $j \leqslant 2n$.

LEMMA 2.3. *Assume $q > 2$. Then for $2 \leqslant j \leqslant 2n$, $\langle p_1, p_2, ..., p_j \rangle_N = P(V_j) \backslash R_j$.*

Proof. Set $S_j = \langle p_1, p_2, ..., p_j \rangle_N$ and let $r_i = r_{p_i}$. Set $\chi_j = \langle r_1, ..., r_j \rangle$. Suppose that $j = 2l$ is even. Then from (Kantor [6] or McLaughlin [7]) $\chi_{2l} = \langle r_i \mid 1 \leqslant i \leqslant 2l \rangle$, is equal to the subgoup

$$N_G(V_{2l}) \cap C_G(V_{2l}^{\perp}) = \{g \in G \mid g(V_{2l}) = V_{2l},\ g_{|V_l^{\perp}} = 11\}.$$

It is isomorphic to $Sp(2l, q)$ and induces this group on V_{2l}. Therefore χ_{2l} is transitive on $P(V_{2l})$. Since χ_{2l} leaves S_{2l} invariant by (2.1) it follows in this case that $S_{2l} = P(V_{2l})$.

Assume now that $j = 2l+1$ is odd. Again from ([6]) we have that

$$\chi_{2l+1} = N_G(V_{2l+1}) \cap C_G(V_{2l+1}^{\perp}).$$

We describe this subgroup in a little more detail. In this instance, R_{2l+1}, the intersection of V_{2l+1} and $V_{2l+1}^{\perp}$, has dimension one. As in the previous paragraph, $\chi_{2l+2} \cong Sp_{2l+2}(q)$ and contains χ_{2l+1}. Then $\chi_{2l+1} = O^{p'}((\chi_{2l+2})_{R_{2l+1}})$, that is, the subgroup of χ_{2l+2} which fixes the point R_{2l+1} in V_{2l+2} and is generated by the unipotent elements (here p is the characteristic of $\mathbb{F}_q$). This subgroup is transitive on the points in $P(V_{2l+1}) \backslash R_{2l+1}$. By (2.1) χ_{2l+1} leaves S_{2l+1} invariant. Hence, $S_{2l+1} = P(V_{2l+1}) \backslash R_{2l+1}$. ∎

3. PROPERTIES OF THE DUAL SYMPLECTIC POLAR SPACE

We continue with the notation of the introduction and record some properties of the geometry $(\mathscr{P}, \mathscr{L})$ of type $DSp_{2n}(q)$ which we require in the sequel. We remark that for points $x, y \in \mathscr{P}$ the distance function defined by the point-collinarity graph of $DSp_{2n}(q)$ is given by $d(x, y) = \dim[x/(x \cap y)] = \dim[y/(x \cap y)]$.

For $B \in P_{2n}(t, q)$, that is a totally isotropic subspace of dimension t, denote by $U(B)$ those $p \in \mathscr{P} = P_{2n}(n, q)$ such that $B \subset p$. We then have

3.1. *For $B \in P_{2n}(t, q)$, $U(B)$ is a convex subspace of $(\mathscr{P}, \mathscr{L})$. Moreover, $U(B)$ is the convex closure of any two points in $U(B)$ whose intersection is B. The diameter of $U(B)$ is $n-t$.*

Note that for $B \in P_{2n}(t, q)$, $\bar{B} = B^{\perp}/B$ is a nondegenerate symplectic space of dimension $2(n-t)$. Moreover, the map from $U(B)$ which takes p to p/B is a bijection onto the maximal isotropic subspaces in $\bar{B}$. Consequently we have

3.2. *The geometry of $DSp_{2n}(q)$ induced on $U(B)$ for $B \in P_{2n}(t, q)$ is $DSp_{2n-2t}(q)$.*

It is well known that $(\mathscr{P}, \mathscr{L})$ is a near $2n$-gon as introduced by Shult and Yanushka ([10]). This means

3.3. *For any point line pair p, l there is a unique point on l nearest p.*

In a near $2n$-gon we say that *quads exist* if for any two points at distance two the convex closure is a *generalized quadrangle*, that is, a point-line geometry where for any non-incident point line pair x, l there is a unique point on l collinear to x. (See [8]). By (3.1), (3.2) it follows if x, y are points at distance two then the convex closure of $\{x, y\}$ is $U(x \cap y)$ and is a generalized quadrangle. Thus, quads exist.

It is also the case that any point-quad pair, p, $Q=U(C)$ is *gated*: there is a unique point x in Q nearest p and for any point $y \in Q$, $d(p, y)=d(p, x)+d(x, y)$. This implies the well known result that

3.4. $(\mathcal{P}, \mathcal{L})$ *is a classical near* $2n$-gon.

Remark 3.5. Cameron ([3]) characterized the classical near $2n$-gons and proved that they are precisely the dual polar spaces.

We complete this section with the proof of three lemmas.

LEMMA 3.6. *Assume x is an isotropic point of V and $p \in \mathcal{P}$ is a maximal isotropic subspace of V and $p \notin U(x)$. Then there is a unique point in $U(x)$ at distance one from p.*

Proof. Clearly $p' = \langle p \cap x^{\perp}, x \rangle$ is in $U(x)$ and meets p in a hyperplane and so p, p' are collinear. On the other hand, suppose $p' \in U(x)$ is collinear with p. Then $p \cap p'$ is a hyperplane in each of p and p'. Also, $p \cap x^{\perp}$ is a hyperplane in p which contains $p \cap p'$. Therefore $p \cap x^{\perp} = p \cap p'$. Since $x \notin p \cap p'$, $p' = \langle p \cap p', x \rangle = \langle p \cap x^{\perp}, x \rangle$. ∎

LEMMA 3.7. *Suppose x, y are non-orthogonal isotropic points of V. Let h be the hyperbolic line spanned by x, y. Then*

$$\langle U(x), U(y) \rangle_D = \bigcup_{z \in h} U(z).$$

Proof. We first show that $cup_{z \in h} U(z)$ is a subspace of $(\mathcal{P}, \mathcal{L})$. Suppose $p_i \in U(z_i)$, $z_i \in h$, $i=1, 2$ are collinear. Then the line on p_1, p_2 is $U(p_1 \cap p_2)$ which is identical with $\{\langle p_1 \cap p_2, z \rangle \mid z \in h\}$. This proves the claim and it follows from this that $\langle U(x), U(y) \rangle_D \subset \bigcup_{z \in h} U(z)$. On the other hand, suppose $p \in U(z)$, $z \in h$, $z \neq x, y$. By (3.6) there is a unique $p' \in U(x)$ collinear with p. By the above argument the line $U(p \cap p')$ contains a unique point $r \in U(y)$ and then $U(p \cap p') = U(r \cap p') \subset \langle U(x), U(y) \rangle_D$. ∎

LEMMA 3.8. *Assume B is a subspace of V. Then $\bigcup_{b \in B \setminus Rad(B)} U(b)$ is a subspace of $DSp_{2n}(q)$.*

Proof. Suppose $p_i \in U(b_i)$, b_i a point in B not contained in rad(B) are collinear. Suppose b_1 and b_2 are non-orthogonal and let h be the hyperbolic line which they span. Then by (3.7) the line $U(p_1 \cap p_2)$ is contained in $\bigcup_{z \in h} U(z) \subset \bigcup_{b \in B \setminus Rad(B)} U(b)$. On the other hand, if $b_1 \perp b_2$ then $p_1, p_2 \in U(\langle b_1, b_2 \rangle) = U(b_1) \cap U(b_2)$ and then clearly the line $U(p_1 \cap p_2) \subset U(\langle b_1, b_2 \rangle) \subset \bigcup_{b \in B \setminus Rad(B)} U(b)$. ∎

4. A RECURSION FORMULA

In this section we define for each natural integer n a finite sequence of natural numbers $\{f(n, j)\}_{j=0}^{n}$ by a recursion formula. We then obtain a closed expression for $f(n, j)$ as well as for $\sum_{j=0}^{n} f(n, j)$.

For $n=1$ we simply define $f(1,0)=f(1,1)=1$. Assume for some $n\geqslant 0$ we have defined $f(n, j)$, $0\leqslant j\leqslant n$. Set $\lambda(n)=\sum_{j=0}^{n} f(n, j)$. We now define $f(n+1,0)=f(n+1,1)=\lambda(n)$. For $2\leqslant k\leqslant n+2$ we define $f(n+1,k)=\sum_{j=k-1}^{n} f(n, j)$. Some of these sequences are shown in Fig. 4.1. Note that each entry in the table from the second row down is the sum of the element directly above it (with the convention that if there is no element above it then that entry is taken to be zero) and all those to the right of that element and this reflects our recursive definition.

Before proceeding to our main result we first record two lemmas on identities of binomial coefficients which we will require:

Lemma 4.1. *For natural numbers m, k*

$$\sum_{t=0}^{k} \binom{m+t}{t} = \binom{m+k+1}{k}.$$

Proof. This follows immediately by induction on k from the identity

$$\binom{l}{s-1} + \binom{l}{s} = \binom{l+1}{s}. \quad \blacksquare$$

Lemma 4.2. *For every integer $n\geqslant 2$ the following is an identity:*

$$\binom{2n-1}{n-1} + \binom{2n-2}{n-1} - \binom{2n-1}{n-3} - \binom{2n-2}{n-3} = \binom{2n}{n} - \binom{2n}{n-2}.$$

				1	1
			2	2	1
		5	5	3	1
	14	14	19	4	1
42	42	28	24	5	1

FIG. 4.1. The sequences $f(n, j)$ for $1\leqslant n\leqslant 5$.

Proof. From the identify in the proof of (4.1) we have

$$\binom{2n}{n}-\binom{2n}{n-2}=\binom{2n-1}{n}+\binom{2n-1}{n-1}-\left[\binom{2n-1}{n}-2+\binom{2n-1}{n-3}\right]$$

$$=2\binom{2n-1}{n-1}-\left[\binom{2n-1}{n}-2+\binom{2n-1}{n-3}\right]$$

$$=\binom{2n-1}{n-1}+\binom{2n-2}{n-1}+\binom{2n-2}{n-2}$$

$$-\left[\binom{2n-2}{n-2}+\binom{2n-2}{n-3}+\binom{2n-1}{n-3}\right]$$

$$=\binom{2n-1}{n-1}+\binom{2n-2}{n-1}-\binom{2n-1}{n-3}-\binom{2n-2}{n-3}. \quad \blacksquare$$

Before proceeding to the main proposition of this section a word about convention when computing binomial coefficients: we will treat as 0 any expression $\binom{m}{l}$ where $l<0$.

PROPOSITION 4.3. *For* $j=0$, $f(n, j)=f(n, 0)=\binom{2n-2}{n-1}-\binom{2n-2}{n-3}$. *For* $1\leqslant j\leqslant n$, $f(n, j)=\binom{2n-1-j}{n-j}-\binom{2n-1-j}{n-2-j}$.

Proof. We prove the result by induction on $n\geqslant =1$. By definition, $f(1, 0)=f(1, 1)=1$. On the other hand, for $n=1$, $j=0$ we get $\binom{2n-2}{n-1}-\binom{2n-2}{n-3}=\binom{0}{0}-\binom{0}{-1}=1$ and for $n=j=1$ we get $\binom{2n-1-j}{j-1}-\binom{2n-j-1}{n-2-j}=\binom{0}{0}-\binom{0}{-2}=1$. Thus, the result holds for $n=1$.

Now assume that we have demonstated the result for n, that is, for $f(n, 0)=\binom{2n-2}{n-1}-\binom{2n-2}{n-3}$ and for $1\leqslant j\leqslant n$, $f(n, j)=\binom{2n-1-j}{n-j}-\binom{2n-1-j}{n-2-j}$.

We now compute

$$\lambda(n)=\sum_{j=0}^{n} f(n, j)$$

which, by the inductive hypothesis, is equal to

$$\binom{2n-2}{n-1}-\binom{2n-2}{n-3}+\sum_{j=1}^{n}\left[\binom{2n-1-j}{n-j}-\binom{2n-1-j}{n-2-j}\right]$$

$$=\binom{2n-2}{n-1}-\binom{2n-2}{n-3}+\sum_{i=0}^{n-1}\binom{n-1+i}{i}-\sum_{i=0}^{n-3}\binom{n+1}{i}.$$

By (4.1) this is equal to

$$\binom{2n-2}{n-1}-\binom{2n-2}{n-3}+\binom{2n-1}{n-1}-\binom{2n-1}{n-3}.$$

Then, by (4.2) this is equal to

$$\binom{2n}{n}-\binom{2n}{n-2}.$$

We now prove the result for $n+1$. By definition $f(n+1,0)=f(n+1,1)=\lambda(n)=\binom{2n}{n}-\binom{2n}{n-2}$. So the result holds for $f(n+1,0)$. It also holds for $f(n+1,1)$ since

$$\binom{2(n+1)-1-1}{(n+1)-1}-\binom{2(n+1)-1-1}{(n+1)-2-1}=\binom{2n}{n}-\binom{2n}{n-2}.$$

Assume now that $2\leqslant j\leqslant n+1$. Then $f(n+1,j)=\sum_{i=j-1}^{n} f(n,i)$ which, by the inductive hypothesis, is

$$\sum_{i=j-1}^{n}\left[\binom{2n-1-i}{n-i}-\binom{2n-1-i}{n-2-i}\right]$$
$$=\sum_{t=0}^{n-j+1}\binom{n-1+t}{t}-\sum_{s=0}^{n-j-1}\binom{n+1+s}{s}.$$

By (4.1) this is equal to

$$\binom{2n+1-j}{n+1-j}-\binom{2n+1-j}{n-1-j}$$

as desired. ∎

5. PROOF OF THE MAIN THEOREM

Let $\gamma(n,q)$ denote the generating rank of the geometry $DSp_{2n}(q)$. In our first result of this section we prove that $\gamma(n,q)\geqslant\binom{2n}{n}-\binom{2n}{n-2}$. This will be an immediate consequence of

PROPOSITION 5.1. *The geometry $DSp_{2n}(q)$ has an embedding into a projective space $\mathbb{PG}(M)$ where M has dimension $\binom{2n}{n}-\binom{2n}{n-2}$ over $\mathbb{F}_q$.*

Proof. Let M be the subspace of $\bigwedge^n(V)$ spanned by all vectors of the form $\bigwedge^n(p)$, $p\in\mathscr{P}$, that is, p a totally isotropic n-dimensional subspace.

By [1] and [9] M has dimension $\binom{2n}{n}-\binom{2n}{n-2}$. Since $\bigwedge^n(V)$ affords an embedding for the the full Grassmannian geometry of all n-dimensional subspaces of V it follows that for $l \in \mathcal{L}$ the set of points $\{\bigwedge^n(p) \mid p \in \mathcal{L}\}$ is a projective line of $\mathbb{PG}(\bigwedge^n(V))$. Since all these points lie in M, M affords an embedding for $DSp_{2n}(q)=(\mathcal{P}, \mathcal{L})$. ▮

COROLLARY 5.2. *The generating rank of* $DSp_{2n}(q)$ *is at least* $\binom{2n}{n}-\binom{2n}{n-2}$.

Proof. This follows from the definition of an embedding and the generating rank of a geometry. ▮

We can now prove our main result:

THEOREM A. *Assume* $q>2$. *Then the generating rank*, $\gamma(n, q)$, *of the symplectic dual polar space*, $DSp_{2n}(q)$ *is* $\binom{2n}{n}-\binom{2n}{n-2}$.

Proof. We make use of the notation previously introduced. As in Section 2 let $p_1, p_2, ..., p_{2n}$ be isotropic points such that for $2 \leqslant j \leqslant 2n$, $\langle p_1, p_2, ..., p_j \rangle = V_j$ is nondegenerate for even j and has a one dimensional radical, R_j for j odd and for $j \geqslant 3$, $p_j^{\perp} \cap V_{j-1} = V_{j-2}$. Now set $C = V_{n+1}$. By (.32) the subspace $\langle p_1, p_2, ..., p_{n+1} \rangle_N = P(C) \backslash R_{n+1}$. From this and (3.7) it follows that

$$\langle U(p_1), U(p_2), ..., U(p_{n+1}) \rangle_D \supset \bigcup_{z \in P(C) \backslash Rad(C)} U(z).$$

However, by (3.8) the latter is a subspace of $DSp_{2n}(q)$ and therefore

$$\langle U(p_1), U(p_2), ..., U(p_{n+1}) \rangle_D = \bigcup_{z \in P(C) \backslash Rad(C)} U(z).$$

Suppose $n+1$ is odd. Set $R = Rad(C) = R_{n+1}$ and let $p \in U(R)$. Then $\dim(p \cap C) > 1$ so that

$$p \in \langle U(p_1), U(p_2), ..., U(p_{n+1}) \rangle_D = \bigcup_{z \in P(C) \backslash Rad(C)} U(z).$$

Therefore

$$\langle U(p_1), U(p_2), ..., U(p_{n+1}) \rangle_D = \bigcup_{z \in P(C)} U(z).$$

Now in the general case suppose $p \in \mathcal{P}$ so that p is an n-dimensional isotropic subspace of V. Since $\dim(V) = 2n$ and $\dim(C) = n+1$ it follows that $p \cap C \neq 0$. Therefore there is an $x \in P(C)$, $x \subset p$ and then $p \in U(x) \subset \langle U(p_1), U(p_2), ..., U(p_{n+1}) \rangle_D$. Consequently, the set of subspaces $U(p_i)$,

$1 \leqslant i \leqslant n+1$ generate $\mathscr{P}$. We will make use of the recursion of Section 4 to demonstrate that these subspaces can be generated by $\binom{2n}{n} - \binom{2n}{n-2}$ points so that the generating rank is at most $\binom{2n}{n} - \binom{2n}{n-2}$. Together with (5.2) this will imply that the generating rank of $DSp_{2n}(q)$ is exactly $\binom{2n}{n} - \binom{2n}{n-2}$.

Suppose $n=2$. Then each of $U(p_i)$, $i=1, 2, 3$ is a $DSp_2(q)$ geometry, that is, a hyperbolic line. We require two points to generate each of $U(p_i)$, $i=1, 2$. However, $U(p_3)$ contains $\langle p_1, p_3 \rangle$ which is already a point of $U(p_1)$ and so we need only one further point to generate $U(p_3)$. As a result, by the above argument, we can generate $DSp_4(q)$ with $2+2+1=5=\binom{4}{2} - \binom{4}{0}$ points.

Suppose now that $n>2$. For $j=1$ set $G(n, j) = \{A \subset U(p_1) \mid \langle A \rangle_D = U(p_1)\}$. For $1 < j \leqslant n+1$, let $B_{j-1} = \langle U(p_1), ..., U(p_{j-1}) \rangle_D$ and set $G(n, j) = \{A \subset U(p_j) \mid \langle A, B_{j-1} \cap U(p_j) \rangle_D = U(p_j)\}$.

Finally, set $g(n, j) = \min\{|A| \mid A \in G(n, j)\}$. We claim that $g(n, j) = f(n, j-1)$ for every applicable pair n, j where $f(n, j)$ are the numbers defined recursively in Section 4. We will prove this induction on n.

Suppose for some $n \geqslant 2$ that we have demonstrated that $g(n, j) = f(n, j-1)$ for $j = 1, 2, ..., n+1$. Then by the definition of the numbers $f(n, j)$ it follows that

$$\gamma(n, q) \leqslant \sum_{j=1}^{n+1} g(n, j) = \sum_{j=0}^{n} f(n, j) = f(n+1, 0) = \binom{2n}{n} - \binom{2n}{n-2}$$

from (4.3). From (5.2) $\gamma(n, q) \geqslant \binom{2n}{n} - \binom{2n}{n-2}$ and therefore we get equality. Note that the theorem will now be a consequence of the equality of $f(n, j)$ and $g(n, j-1)$ for all n, j. Notice also that as a result of this if $A_j \in G(n, j)$, $j = 1, 2, ..., n+1$ then $A = \bigcup_{j=1}^{n+1} A_j$ is a generating set for $DSp_{2n}(q)$. This implies the following:

5.3. *Assume* $j < n+1$. *Set* $E(n, j) = \{E \subset \mathscr{P} \mid \langle B_{j-1}, E \rangle_D = \mathscr{P}\}$ *and* $e(n, j) = \min\{|E| \mid E \in E(n, j)\}$. *Then*

$$e(n, j) = \sum_{i=j}^{n+1} g(n, i).$$

We now prove that $g(n, j) = f(n, j-1)$ for $j = 1, 2, ..., n+1$. Since $U(p_1)$ is isomorphic to $DSp_{2n-2}(q)$ $g(n, 1) = \gamma(n-1, q) = \binom{2n-2}{n-2}\binom{n-2}{n-2}$. Also, $g(n, 2) = \gamma(n-1, q)$ since $B_1 \cap U(p_2) = \varnothing$ as $p_2^{\perp} \cap V_1 = 0$. Assume then that $j \geqslant 3$. Now $B_{j-1} \cap U(p_j) = B_{j-2}$ and, of course, $U(p_j)$ is a subspace of $DSp_{2n}(q)$ isomorphic to $DSp_{2n-2}(q)$. From (5.3) it we have that

$$g(n, j) = c(n-1, j-1) = \sum_{i=j-1}^{n} g(n-1, i).$$

By the induction hypothesis

$$\sum_{i=j-1}^{n} g(n-1, i) = \sum_{i=j-2}^{n-1} f(n-1, i)$$

which, by (4.2) is equal to $f(n, j-1)$ so that $g(n, j) = f(n, j-1)$ as required. ▮

Remark. The result is definitely false for $q=2$. Brouwer ([1] has shown that $DSp_{2n}(2)$ has an embedding of dimension at least $(2^n+1)(2^{n-1}+1)/3$ and has conjectured that this is the embedding rank of $DSp_{2n}(2)$. For $n \leqslant 5$ it has been shown that this is the generating rank of $DSp_{2n}(2)$ (see Cooperstein [4]). By methods similar to the above it is possible to show that the generating rank of $DSp_{12}(2)$ is at most 716.

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