Representations of Cartan type Lie algebras in characteristic $p$

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Abstract

In this paper, we discuss the representations of Cartan type Lie algebras in characteristic $p > 2$, from the viewpoint of reducing rank. When the character is regular semisimple for generalized Witt algebras, we can essentially reduce higher-rank representations to lower-rank representations.

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Introduction

The purpose of this paper is to describe representations of Cartan type Lie algebras of prime characteristic, from the viewpoint of rank reduction.

This work is motivated by the representation theory of classical Lie algebras with characters of standard Levi forms, developed by Friedlander–Parshall and Jantzen [1,2]. For a classical Lie algebra $\mathfrak{g}$ of characteristic $p$ under some mild conditions, one can reduce a reduced representation of an arbitrary character to the case of a nilpotent character. And, if a nilpotent character has a standard Levi

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form corresponding to a subset $I$ of the set of all simple roots, the representations $\mathfrak{g}$ are much related to the gradations, Weyl groups, and subalgebras corresponding to the root basis $I$. This reduction idea can be applied to our cases although the situation for Cartan type Lie algebras are quite different. Unlike the situation of classical Lie algebras, there are no universal tools like Weyl groups and the highest weight theory, exploited in the study of their representations because of some non-symmetry of the structures. Up to now, except for very few examples of such Lie algebras with low ranks, we are far away from the determination of their simple modules. Generally speaking, the lower-rank case is easier than the higher-rank case in the study of their representations. For rank-one Cartan type Lie algebras of type $W$, there is a complete determination in $[3,4]$ for the restricted case, and in $[5]$ for the nonrestricted case. So, it is also a natural idea to reduce representations of higher-rank cases to those of lower-rank cases. Here, we give an explanation of this idea in the case of generalized Witt algebras $L = W(m : 1)$. For an index subset $I \subset \{1, 2, \ldots, m\}$ and its supplement $\hat{I} \subset \{1, 2, \ldots, m\} \setminus I$, we have the divided power algebra $U(\hat{I})$ and the two generalized Witt algebras $W(I)$ and $W(\hat{I})$ (see Section 1.1). For any Lie algebra $\mathfrak{g}$, we can define a loop algebra associated with a divided algebra $U$: $U \otimes \mathfrak{g}$ (see Section 4.1). We can identify $W(m : 1)$ with $W(I) \oplus U(\hat{I}) \otimes W(I)$. And $W(\hat{I})$ normalizes each loop subalgebra of $U(\hat{I}) \otimes W(I)$ (see Section 4.3). This makes it possible to consider some induced representations related to $W(I)$ and $W(\hat{I})$. From this viewpoint we mainly discuss in this paper representations of generalized Witt algebras with regular semisimple characters. We can reduce their representations to the lower-rank representations, in a large extent.

In our discussion, the loop algebras resulting from classical Lie algebras will play an important role. Here we will be only concerned with the case when the character is regular semisimple. Their generic representations will be a topic of further discussion. There is a close relation between this work and Shen’s theory of graded modules and filtered modules $[6–9]$. In Section 3, an idea similar to that in $[6]$ gives rise to the simplicity of the relevant modules. It should be mentioned that, when $I$ coincides with $\{1, 2, \ldots, m\}$, then the corresponding grade-zero component $L_I$ is just $\mathfrak{h}$, the canonical torus of $L$ (see Section 1.2). The representations concerned have been included in the consideration of $[8,10–12]$.

This paper is organized as follows. In Section 1, the $\Lambda_I$-gradation, associated with an index subset $I$, is introduced for a Cartan type Lie algebra $L$. For an arbitrary $\chi \in L^*$, there exists the smallest index subset $I$ such that each simple module of character $\chi$ is $\Lambda_I$-graded (Section 1.4). From Section 2 on, the discussion is limited to the case $L = W(m : 1)$. In Section 2, for the grade-zero subalgebra $L_I$, the descriptions of simple modules are given (Sections 2.5 and 2.9). In Section 3, a result about the simplicity of induced modules of $L$ is given (Section 3.2). In Section 4, a class of loop algebras are discussed, associated with classical Lie algebras and divided power algebras.
character is regular semisimple, the result obtained (Section 4.4) is analogous to the Friedlander–Parshall’s one in the classical Lie algebra case [1]. In the last section, the main theorem shows that when the character is regular semisimple, the simple representations of $L$ are in one-to-one correspondence with those of $L_I$. The latter can be described through the simple representations of $W(\hat{I})$ and the weights of the abelian loop algebra $\mathfrak{L}(\hat{I}) \otimes \mathfrak{h}(I)$ associated with the canonical torus $\mathfrak{h}(I)$ of $W(I)$. When $I$ is the whole index set $\{1, 2, \ldots, m\}$, it has been known that simple $L$-modules of character $\chi$ is just parameterized by the weights of the torus $\mathfrak{h}$ of $L$ aforementioned [8].

1. Gradations of Lie algebras associated with an index set $I$

In this paper, we always assume that $K$ is an algebraically closed field of characteristic $p > 2$. Let $L = X(m : 1)^{(2)}$, $X \in \{W, S, H, K\}$ be a simple restricted Lie algebra of Cartan type over $K$. By definition, each $X(m : 1)^{(2)}$ is a Lie (sub-)algebra of special derivations of the divided power algebra $\mathfrak{L}(m : 1)$. In this paper, we simply denote $\mathfrak{L}(m : 1)$ by $\mathfrak{L}(m)$, even sometimes we denote by $\mathfrak{L}(I)$ the divided power subalgebra of $\mathfrak{L}(m : 1)$, corresponding to a subset $I \subset \{1, 2, \ldots, m\}$. For the notation concerning Cartan type Lie algebras we refer the readers to [13,14].

1.1. Notations

For a function $\alpha$ on the set $S$ over a number ring $R$, $\alpha: S \rightarrow R$ and for a subset $I$ of $S$, we often use a notation $\alpha(I) := \alpha|_I$. In order to emphasize the index subset $I$ of $S$ we sometimes write $\alpha_I$ for the elements of $R^I := \{f | f: I \rightarrow R\}$.

Set $P := \{0, 1, 2, \ldots, p-1\}$. For the set $S := \{1, 2, \ldots, m\}$ and for a function $\alpha$ on $S$ over $P$, we can write $(\alpha(1), \alpha(2), \ldots, \alpha(m))$ for $\alpha$. Denote $A(m) := P^S = \{\alpha | \alpha: S \rightarrow P\}$. Suppose $I = \{i_1 < i_2 < \cdots < i_l\} \subset S$. Then $\alpha(I)$ just means $(\alpha(i_1), \ldots, \alpha(i_l))$. We denote $A(I) := P^I$, which is actually equal to $\{\alpha(I) | \alpha \in A(m)\}$. Remember the meaning of $\alpha_I$ introduced in the foregoing paragraph. So naturally, for $I \subset S = \{1, 2, \ldots, m\}$ and for all $\alpha_I \in A(I)$, we have $\mathfrak{L}(I) = K$-span\{$x^{\alpha_I} | \alpha_I \in A(I)$\}, a divided power subalgebra of $\mathfrak{L}(m)$ with product

$$x^{\alpha_I} \cdot x^{\beta_I} = \begin{cases} (\alpha_I + \beta_I)_{\alpha_I} & \text{if } \alpha_I + \beta_I \subset A(I), \\ 0 & \text{otherwise}, \end{cases}$$

where

$$(\alpha_I + \beta_I)_{\alpha_I} = \prod_{j=1}^{l} \left(\alpha_I(i_j) + \alpha_I(i_j)\right).$$
Hence, we can define the Cartan type Lie subalgebra $X(I)$ of $X(m : 1)^{(2)}$, corresponding to $I$ for $X \in \{W, S, H, K\}$. In addition, we always denote $|\alpha| = \sum_i \alpha(i)$ for $\alpha = (\ldots, \alpha(i), \ldots)$.

1.2. To define the gradation associated to a subset $I$ of $\{1, 2, \ldots, m\}$, we introduce a degree function $\deg$ for the standard basis $E$ of $L$; it is applied in a more general context than just for the type $W$ because we will give some general facts about all four series of Cartan type Lie algebras, as a start point of our basic idea. However, in this paper, we need it just for the type $W$.

First, for $X = H$ or $K$, put $\theta_i = ^{\varepsilon_i} + ^{\varepsilon_i}$, $1 \leq i \leq r$, with $\varepsilon_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})$ where $m = 2r$ for $X = H$, or $m = 2r + 1$ for $X = K$ and $\tilde{i} = i + r$. Additionally, $\theta_m := \varepsilon_m$ for $X = K$. Denote by $\Theta$ the set of elements $\sum a_k \theta_k$ satisfying $\sum a_k = 0$. Then $\Theta$ is a subgroup of the additive group $\mathbb{Z}^m$. Denote $\langle \mathbb{Z}^m \rangle$ the quotient group of $\mathbb{Z}^m$ by $\Theta$, whose elements denoted by $\langle a \rangle = \langle a_1, \ldots, a_m \rangle$ correspond to $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$.

For $\alpha = (\alpha_1, \ldots, \alpha_m) \in A(m)$, define $\deg : E \to \mathbb{Z}^m$ by

\[
\deg x^\alpha D_j = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_m) \quad \text{for } X = W,
\]

\[
\deg D_{ij}(x^\alpha) = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_i - 1, \ldots, \alpha_m) \quad \text{for } X = S,
\]

where $\alpha_i - 1$ and $\alpha_j - 1$ lie in the $i$th and $j$th entries, respectively, and

\[
\deg D_X(x^\alpha) = \langle \alpha - \theta_1 \rangle \quad \text{for } X = H \text{ or } K.
\]

1.3. For a given index subset $I = \{i_1 < i_2 < \cdots < i_l\}$ satisfying

(1.3.1) $I \subset \{1, 2, \ldots, m\}$ for $X = W$ or $S$,

(1.3.2) $I \subset \{1, 2, \ldots, r\}$ where $m = 2r$ for $X = H$ and $m = 2r + 1$ for $X = K$,

set

\[
L_I := \mathcal{K}\text{-span}\{x^\alpha D_j \mid (\deg x^\alpha D_j)(I) = 0\} \quad \text{for } X = W,
\]

\[
L_I := \mathcal{K}\text{-span}\{D_{ij}(x^\alpha) \mid (\deg D_{ij}(x^\alpha))(I) = 0\} \quad \text{for } X = S,
\]

and

\[
L_I := \mathcal{K}\text{-span}\{D_X(x^\alpha) \mid \deg D_X(x^\alpha) = \sum_{i=1}^m a_i \varepsilon_i \text{ satisfying } a_i = a_i \}
\]

for $I \in I$ for $X = H$ or $K$.

Write $\Lambda(L) := \{\deg(D) \mid D \in \mathcal{E}\}$. If the context is clear, denote it directly by $\Lambda$.

Set

\[
\Lambda_I = \begin{cases} 
\{\alpha(I) \mid \alpha \in \Lambda\} & \text{for } X = W \text{ or } S, \\
\{a(I - \tilde{i}) \mid a = (a_1, a_2, \ldots, a_m) \in \Lambda\} & \text{for } X = H \text{ or } K,
\end{cases}
\]
where \( \tilde{I} := \{ \tilde{i} \mid i \in I \} \) and
\[
\mathbf{a}(I - \tilde{I}) := (a_{i_1} - a_{\tilde{i}_1}, \ldots, a_{i_l} - a_{\tilde{i}_l}) \quad \text{for} \quad I = \{ i_1 < i_2 < \cdots < i_l \}.
\]

Here \( \mathbf{a}(I - \tilde{I}) \) is really well defined because \( \langle a_1, a_2, \ldots, a_m \rangle = \langle b_1, b_2, \ldots, b_m \rangle \) implies \( a_i - a_{\tilde{i}} = b_i - b_{\tilde{i}} \) for \( i = 1, 2, \ldots, r \) by the definition [11, 3.2]. The following lemma is obvious.

**Lemma.** Suppose both \( I \) and \( J \) are as in (1.3.1) or (1.3.2).

1. If \( I \subset J \), then \( L_I \supset L_J \).
2. For \( X = W \) or \( S \), if \( I = \{ 1, 2, \ldots, m \} \) then \( L_I = \mathfrak{h} \), the canonical torus in \( L \).
   
   If \( I = \emptyset \), then \( L_I = L \).

1.4. The index subset \( I \) gives rise to a \( \Lambda_I \)-gradation of \( L \):
\[
L = \bigoplus_{\alpha \in \Lambda_I} L_{\alpha},
\]
\[
L_{\alpha} = \begin{cases} 
\mathcal{K}\text{-span}\{ D \in \mathcal{E} \mid \deg D(I) = \alpha \} & \text{for } X = W \text{ or } S, \\
\mathcal{K}\text{-span}\{ D \in \mathcal{E} \mid \deg D(I - \tilde{I}) = \alpha \} & \text{for } X = H \text{ or } K.
\end{cases}
\]

For \( \Lambda_I \), we can define an order \( >_I \) similarly to [11, Section 3.1]: for \( X = W \) or \( S \), \( \alpha(I) >_I \beta(I) \) if and only if \( |\alpha(I)| > |\beta(I)| \) or \( |\alpha(I)| = |\beta(I)| \) and \( \alpha(I) > \beta(I) \) (note: \( \alpha(I) > \beta(I) \) means lexicographically \( a_{i_1} = b_{\tilde{i}_1}, \ldots, a_{i_l} = b_{\tilde{i}_l} \) but \( a_{i_{l+1}} > b_{\tilde{i}_{l+1}} \) for \( 1 \leq t < l \), \( \alpha(I) = (a_{i_1}, \ldots, a_{i_l}) \), and \( \beta(I) = (b_{\tilde{i}_1}, \ldots, b_{\tilde{i}_l}) \)). For \( X = H \) or \( K \), \( \mathbf{a}(I - \tilde{I}) >_I \mathbf{b}(I - \tilde{I}) \) is just similarly defined as above.

Set
\[
L^+(I) = \bigoplus_{\alpha_{I} >_I 0} L_{\alpha I}, \quad L^-(I) = \bigoplus_{\alpha_{I} >_I 0} L_{\alpha I}, \quad L^0(I) := L_I.
\]

Then we have a triangular decomposition for \( L \): \( L = L^-(I) \oplus L^0(I) \oplus L^+(I) \).

All three subalgebras are restricted Lie algebras. We will see that \( L_I \) plays an important role in representations of \( L \) the following proposition and the general theory of representations for Lie algebras of characteristic \( p \) (see the next section).

**Proposition.** For any \( \chi \in L^* \), there exists the smallest index set \( I \) such that \( \chi(L^\pm(I)) = 0 \).

**Proof.** If there exist \( I_1, i = 1, 2 \), such that \( \chi(L^\pm(I_i)) = 0 \), then for \( I := I_1 \cap I_2 \), \( \chi(L^\pm(I)) = 0 \). This is because \( L_I \subset L_{I_1} + L_{I_2} \) by Lemma 1.3(1). And
\[
L^+(I) + L^-(I) \subset (L^+(I_1) + L^-(I_1)) \cap (L^+(I_2) + L^-(I_2)).
\]

This implies the existence of the desired index set. \( \square \)
Remark. This result implies that any simple \( \chi \)-reduced module is \( \Lambda I \)-graded for the corresponding smallest index subset \( I \) (see Section 2.2).

1.5. In particular, in the case \( X = W \) we obtain by direct verification the following lemma.

**Lemma.** Set \( \hat{I} := \{1, 2, \ldots, m\} \setminus I \). Then \( L_I \) is a semidirect sum of an abelian ideal and a simple Lie algebra of type \( W \) with less rank than that of \( L \) itself: Concretely:

1. \( L_I = L^c_I \oplus L^0_I \), where \( L^c_I = K\text{-Span}\{x^{\alpha_i+\epsilon_i} D_i \mid \alpha_i \in A(\hat{I}), i \in I\} \) is an abelian subalgebra, and \( L^0_I = K\text{-span}\{x^{\alpha_i} D_k \mid \alpha_i \in A(\hat{I}) \text{ and } k \in \hat{I}\} \) can be regarded as a simple generalized Witt algebra \( W(\hat{I}) \) corresponding to \( \hat{I} \).

2. \( [L^c_I, L^0_I] \subset L^c_I \).

**Remark.** (1) In Section 4, we will introduce so-called loop algebras associated with the divided power algebra \( \mathfrak{U}(\hat{I}) \). Then we will be able to identify \( L^c_I \) with \( \mathfrak{U}(\hat{I}) \otimes h(I) \), where \( h(I) = K\text{-span}\{x^{\epsilon_i} D_i \mid i \in I\} \) is the canonical torus of \( W(I) \).

(2) The second statement about \( L^c_I \) can be extended in Lemma 4.3 to \( \mathfrak{U}(\hat{I}) \otimes g' \) for any subalgebra \( g' \) of \( W(I) \).

(3) As a convention, we define \( L^c_I \) to be 0 when \( I = \emptyset \) and \( L^0_I \) to be 0 when \( I = \{1, 2, \ldots, m\} \) (then \( L_I \) is just \( L \) itself or \( h \), respectively).

2. Representations of \( L_I \)

In the sequel, we consider just the case \( X = W \). This is to say \( L = W(m : 1) \). For the rank-one case, it is just the Witt algebra, which representations have been well known, due to [3,4]. So there should be expected an inductive way to describe the representations of higher-rank Lie algebras through that of lower-rank Lie algebras, via the relations of representations resulting from \( L_I \).

2.1. Recall that any simple \( g \)-module \( V \) for a restricted Lie algebra \( (g, [p]) \) is dependent on a linear function \( \chi \) on \( g \), called a character of \( g \), which satisfies the following equations for the corresponding representation \( \rho \) on \( V \), by Schur’s Lemma [15]:

\[
\rho(x)^p - \rho(x^{[p]}) = \chi(x)^p \text{ Id}_V \quad \forall x \in g. \tag{2.1.1}
\]

Conversely, each representation \( (\rho, V) \) satisfying (2.1.1) is called a \( \chi \)-reduced representation of \( g \). All \( \chi \)-reduced representations of \( g \) constitute a full subcategory \( \mathcal{M}_\chi \) of the \( g \)-module category. This category is equivalent to the representation category of the corresponding finite-dimensional algebra \( u_\chi(g) \), which is a
quotient of the universal enveloping algebra $U(g)$ of $g$ by the ideal generated by all $x^p - x^{[p]} - \chi(x)^p$, $x \in g$. If $\chi$ is zero, then $\mathcal{M}_0$ is just the restricted module category of $g$. And in this case $u_0(g)$ is just the restricted enveloping algebra of $g$, simply denoted by $u(g)$.

2.2. In this paper, we always suppose $\chi(L^\pm(I)) = 0$. Let $(\rho, V)$ be a $\chi$-reduced representation. Notice that for any standard basis element $D \in L^\pm(I)$, $D^{[p]} = 0$. Hence $\rho(D)^p = 0$, due to (2.1.1). By the weakly-closed set theorem of Jacobson [16], any elements of $L^+(I)$ (respectively $L^-(I)$) act on $V$ nilpotently. Hence every simple $\chi$-reduced module is $\Lambda_f$-graded [6, 1.5], which is uniquely determined by simple modules of $L^0(I) (= L_I)$ [17, 3.1].

2.3. Note that $L^c_I$ is commutative. Any simple $\chi$-reduced module of $L^c_I$ is one-dimensional, which is determined by a function $f \in (L^c_I)^*$ satisfying

$$f(G)^p - f(G^{[p]}) = \chi(G)^p$$

for $G \in L^c_I$. (2.3.1) We call such an $f$ a weight of $L^c_I$, associated with $\chi$. Denote the set of all weights satisfying (2.3.1) by $X(\chi)$. When $\chi(L^c_I) = 0$, we call $f$ a restricted weight of $L^c_I$. Set

$$h_0 := h(I) = \mathcal{K}\text{-span}\{x^{\alpha_i}D_i \mid i \in I\}$$

and

$$h_1 := \mathcal{K}\text{-span}\{x^{\alpha_i^+ + \alpha_i}D_i \mid \alpha^+_i (\neq 0) \in A(\hat{I}), \ i \in I\}.$$ 

Then $L^c_I = h_0 \oplus h_1$, as vector spaces. For any standard basis element $G_1 = x^{\alpha_i^+ + \alpha_i}D_i \in h_1$, $G_1^{[p]} = 0$. Hence (2.3.1) implies $f(G_1) = \chi(G_1)$. Thus $f|_{h_1} = \chi|_{h_1}$. For a standard basis element of $h_0$, $G_0 = x^{\alpha_i}D_i$, $G_0^{[p]} = G_0$. Hence we have $f(G_0)^p - f(G_0) = \chi(G_0)^p$. For any other $g \in X(\chi)$, we have the same equality $g(G_0)^p - g(G_0) = \chi(G_0)^p$. Hence $(f - g)(G_0)^p = (f - g)(G_0)$. This is to say: $(f - g)(G_0) \in \mathbb{F}_p$. Thus $(f - g)|_{h_0}$ is a restricted weight of $h_0$. Combining with $f|_{h_1} = \chi|_{h_1} = g|_{h_1}$, we have the following lemma.

**Lemma.** For any $f, g \in X(\chi)$, $f - g$ is a restricted weight of $L^c_I$. $\#X(\chi) = p^{#I}$.

2.4. Let me first recall some facts about representations of nilpotent Lie algebras. Suppose $g$ is a nilpotent Lie algebra over an algebraically closed field with a finite-dimensional representation $\rho : g \rightarrow gl(V)$. Then $V$ can be decomposed into direct sum of vector spaces:

$$V = \bigoplus_{\tau \in \mathfrak{g}^*} V_\tau, \text{ where } V_\tau = \{v \in V \mid \exists n \in \mathbb{N}, (\rho(g) - \tau(g) \text{ Id}_V)^n.v = 0\}.$$ 

If $V_\tau \neq 0$, then $V_\tau$ is called the $\tau$-weight space of $V$ and $\tau$ is called a weight of $V$. Denote by $P(V)$ the set of all weights of $V$. 
Lemma. Let \( g \) be a nilpotent subalgebra of Lie algebra \( g' \) over an algebraically closed field.

1. Suppose \( g' \) and the \( g' \)-module \( V \) have the weight space decompositions with respect to \( g \):

\[
  g' = \bigoplus_{\alpha \in P(g')} g'_\alpha \quad \text{and} \quad V = \bigoplus_{\tau \in P(V)} V_\tau.
\]

Then \( g'_\alpha V_\tau \subset V_{\alpha + \tau} \).

2. Furthermore, if \( \varphi \) is a \( g' \)-module homomorphism from \( V = \bigoplus_{\tau \in P(V)} V_\tau \) to \( W = \bigoplus_{\mu \in P(W)} W_\mu \), then \( \varphi(V_\tau) \subset W_\tau \).

3. Suppose \( (g, [p]) \) is a restricted Lie algebra. Denote for a given \( \chi \in g^* \),

\[
  X(\chi) = \{ f \in g^* \mid f(x)p - f(x^[p]) = \chi(x)p \quad \text{for all} \quad x \in g \}.
\]

Then for any \( \chi \)-reduced representation \( (\rho, V) \), \( P(V) \subset X(\chi) \).

Proof. (1) It may be directly referred to [16, III.5]. (2) It follows from the direct computation, according to the definition.

(3) For any \( \tau \in P(V) \) and \( v(\neq 0) \in V_\tau \), there exists \( n \in \mathbb{N} \) such that \( (\rho(x) - \tau(x) \text{Id}_V)P^n v = 0 \). Hence \( \rho(x)^{p^q} v = \tau(x)^{p^q} v \) for any integer \( q \geq n \). Notice that

\[
  \tau(x)^{p^{n+1}} v = \rho(x^{[p]})^{p^n} v + \chi(x)^{p^{n+1}} \text{Id}_V
\]

Thus, \( (\tau(x)^P - \tau(x^{[p]}) - \chi(x)^P)^{p^n} v = 0 \). From this it follows that \( \tau \in X(\chi) \).

2.5. Now we turn back to the case \( g' = L_I \) and \( g = L^c_I \) (note that \( L^c_I \) is abelian). Suppose \( V \) is an \( L_I \)-module with character \( \chi \). As an \( L^c_I \)-module, \( V \) can be decomposed into direct sum of weight spaces, \( V = \bigoplus_{\tau \in P(V)} V_\tau \).

By Lemma 2.4(3), \( P(V) \subset X(\chi) \). On the other hand, \( L^c_I \) is an abelian ideal of \( L_I \). Hence, under adjoint action \( L_I = \bigoplus_{a \in P(L_I)} (L_I)_a \) with \( P(L_I) = \{ 0 \} \). By Lemma 2.4(1), \( L_I V_\tau = (L_I)_0 V_\tau \subset V_f \). This is to say, each weight space \( V_\tau \) of \( V \) is an \( L_I \)-submodule of \( V \). Now we suppose that \( V \) is a simple \( L_I \)-module; then \( \# P(V) = 1 \). Suppose further \( P(V) = \{ f \} \). Then any simple \( L^c_I \)-module in \( V \) is a stabilized line \( K_vf \) with \( Gf = f(G)v_f \) for all \( G \in L_I \). And \( V = u_\chi(L_I)v_f \). Denote by \( K_f \) the one-dimensional module of \( L^c_I \) corresponding to the weight \( f \).

Set \( Z_{\chi, f}(f) = u_\chi(L_I) \otimes_{u_\chi(L^c_I)} K_f \). From the universality of the tensor product in \( Z_{\chi, f} \), it follows that \( V \) is a quotient of \( Z_{\chi, f} \). So we have the following proposition.

Proposition. Each \( L^c_I \)-weight space of any \( L_I \)-module of character \( \chi \) is still an \( L_I \)-module. In particular, we have:
(1) Any simple $L_I$-module of character $\chi$ admits a unique weight in $X(\chi)$. Especially, a simple module corresponding to $f \in X(\chi)$ is called an $f$-weighted simple module and is a quotient of $Z_{X,I}(f)$.

(2) As $L_I$-modules, all composition factors of $Z_{X,I}(f)$ are $f$-weighted simple modules.

**Proof.** We only need to prove statement (2). Since $Z_{X,I}(f)$ is generated by an $f$-weight vector $\omega_f := 1 \otimes 1$, we have $\omega_f \in (Z_{X,I}(f))_f$. By the same argument as above, we have $P(Z_{X,I}(f)) = \{f\}$. By the statement (1), $\bar{M}_q$ must be a $g$-weighted simple module for a unique $g \in X(\chi)$. Denote by $(\rho, Z_{X,I}(f))$ and by $(\bar{\rho}_q, M_q)$ respectively the corresponding representations of $L_{c,I}$, then $\bar{\rho}_q$ is a quotient representation of $\rho_q := \rho|_{M_q}$. Fix a nonzero vector $v \in M_q \setminus M_q - 1$. On one side, for any $G \in L_{c,I}$ we have $(\rho_q(G) - g(G) \text{Id}_{M_q})^n v = 0$ (mod $M_q - 1$) for a certain $n_1 \in \mathbb{N}$. And on the other side, $(\rho(G) - f(G) \text{Id}_{Z_{X,I}(f)})^n v = 0$ for a certain $n_2 \in \mathbb{N}$, thereby $(\rho_q(G) - f(G) \text{Id}_{M_q})^n v = 0$ (mod $M_q - 1$). Choose $n \in \mathbb{N}$ such that $p^n > \max\{n_1, n_2\}$. Then we have $(f(G) - g(G))^n v = 0$ (mod $M_q - 1$). By the choice of $v$, it must hold that $g(G) = f(G)$ for any $G \in L_{c,I}$. Hence $g = f$. $\Box$

Hence, we have the following corollary.

**Corollary.** Any simple $u_{X}(L_I)$-module is an $f$-weighted simple module corresponding to a unique $f \in X(\chi)$. All non-isomorphic composition factors of $Z_{X,I}(f)$ constitute a set of representatives of isomorphism classes of $f$-weighted simple $L_I$-modules.

2.6. Next, we will give further discussions about $f$-weighted simple modules of $L_I$. Denote $C = L_{c,I}^c$, $A = L_{0,I}^0$. Both of them are restricted subalgebras of $L_I$.

For a $C$-module $V$, call $v \in V$ a regular $f$-weight vector if $Gv = f(G)v$ for all $G \in C$. All regular $f$-weight vectors constitute a $C$-submodule, denoted by $V_f^{\text{reg}}$.

With respect to $C$, $u_{X}(L_I)$ has the weight space decomposition

$$u_{X}(L_I) = \bigoplus_{f \in X(\chi)} u_{X}(L_I)_f.$$  

It is easily seen that $u_{X}(L_I)_f \neq 0$ for all $f \in X(\chi)$. This is because the canonical representation $\Gamma_f : u_{X}(L_I) \rightarrow Z_{X,I}(f)$ with $\Gamma_f(a) = a(1 \otimes 1)$ can be regarded an $L_I$-module homomorphism. By Lemma 2.4(2), $\Gamma_f(u_{X}(L_I)_g) \subset (Z_{X,I}(f))_g = \delta_{f,g} Z_{X,I}(f)$. Hence $\Gamma_f(u_{X}(L_I)_f) = \Gamma_f(u_{X}(L_I)) \neq 0$. Furthermore $u_{X}(L_I)_f = u_{X}(C)_f u_{X}(A)$ because $u_{X}(L_I)$ is $u_{X}(C)$-free. By Lemma 2.4(1) again, $u_{X}(L_I)_f$
is an ideal of $u_\chi(L_1)$. Hence for a $u_\chi(A)$-module $W$, the $L_I$-module induced from it has the weight space decomposition which is simultaneously the direct-sum decomposition of $C$-submodules:

$$u_\chi(L_1) \otimes_{u_\chi(A)} W = \bigoplus_{f \in X(\chi)} u_\chi(L_1)_f \otimes_{u_\chi(A)} W.$$

Denote by $\tilde{W}^f$ the direct summand corresponding to $f$ which is by Proposition 2.5 an $L_I$-module, equal to $u_\chi(C)_f \otimes W$ as a vector space.

**Lemma.** (1) For a given $f \in X(\chi)$, the $C$-module $u_\chi(C)$ admits a unique regular $f$-weight vector $r_f(\in u_\chi(C)_f)$, up to scalars.

(2) Suppose $W$ is a $u_\chi(A)$-module. Then as a $u_\chi(C)$-module, $\tilde{W}^f$ has the socle $r_f \otimes W$.

**Proof.** (1) Let $\{G_i\}_{i=1}^h$ be the standard basis of $C$, $h = \dim C$. Then by the PBW theorem,

$$u_\chi(C) = \bigoplus_{a \in p^h} \mathbb{K}G^a := \bigoplus_{a \in p^h} \mathbb{K}G_1^a G_2^a \cdots G_h^a.$$

Suppose $c_f$ is a nonzero regular $f$-weight vector in $u_\chi(C)$. We can write $c_f = \sum_{q=0}^{p-1} K_q G_1^q$ with $K_q \in u_\chi(C_1)$, where $C_1 := \bigoplus_{i=2}^h \mathbb{K}G_i$. Note that $C$ is abelian and $u_\chi(C)$ is a free module over the commutative algebra $u_\chi(C_1)$. Hence from $G_1 c_f = f(G_1)c_f$ we can obtain $c_f = K_{p-1} r_G$ where

$$r_G := \begin{cases} 
\sum_{q=0}^{p-1} f(G_i)^q G_i^{p-1-q} & \text{if } G_i \in \mathfrak{h}_1, \\
\sum_{q=0}^{p-1} f(G_i)^q G_i^{p-1-q} - 1 & \text{if } G_i \in \mathfrak{h}_0.
\end{cases}$$

Here the meaning of $\mathfrak{h}_0$ and $\mathfrak{h}_1$ are the same as in Section 2.3. By induction, we have $c_f = a \prod_{i=1}^h r_{G_i}$ for $a \in \mathbb{K}$. Set $r_f := \prod_{i=1}^h r_{G_i}$. It can be verified by straightforward computation that $r_f$ is a regular $f$-weight vector in $u_\chi(C)$. Hence $u_\chi(C)^{\text{reg}} = K r_f$.

(2) Observe that $(u_\chi(L_1) \otimes_{u_\chi(A)} W)^{\text{reg}} = (\tilde{W}^f)^{\text{reg}}$. If $v_f = \sum c_f^{(i)} \otimes w_i$ is a regular $f$-weight vector in $\tilde{W}^f$, then all $c_f^{(i)}$’s must be regular $f$-weight vectors in $u_\chi(C)$ because $u_\chi(L_1) \otimes_{u_\chi(A)} W$ is a free $u_\chi(C)$-module. Due to (1), $v_f \in r_f \otimes W$. On the other hand, $r_f \otimes W \subset (\tilde{W}^f)^{\text{reg}}$. Hence the $C$-module socle $\text{Soc}_C(\tilde{W}^f) = (\tilde{W}^f)^{\text{reg}} = r_f \otimes W$. □

Thus we have an $L_I$-submodule of $\tilde{W}^f$ generated by $r_f \otimes W$, denoted by $R_f(W)$, which is equal to $u_\chi(A)r_f \otimes W$, as a vector space.
2.7. Before giving Proposition 2.8, we have to make further preparation. Let \( W \) be a \( u\chi(A) \)-module. Consider the adjoint representation \( \text{ad}_{L_1/A} : A \rightarrow \mathfrak{gl}(L_1/A) \). Since the mapping \( A \rightarrow K, y \mapsto \text{tr}(\text{ad}_{L_1/A} y) \), sends \( y^{[p]} \) to \( \text{tr}(\text{ad}_{L_1/A} y)^p \), there is a unique \( K \)-algebra automorphism \( \sigma \) of \( u\chi(A) \) satisfying \( \sigma(y) \mapsto y - \text{tr}(\text{ad}_{L_1/A} y) \cdot 1 \) for all \( y \in A \). Hence we can define the \( A \)-module \( W_{\sigma} \) with underlying vector space \( W \) and the action

\[ y \cdot w = \sigma(y)w, \quad \forall w \in W, \ y \in A, \]

which satisfies \( \sigma(y)p - \sigma(y^{[p]}) = \chi(y)p \cdot \text{Id}_W \). This is to say, \( W_{\sigma} \) is still a \( u\chi(A) \)-module.

**Lemma.** (1) As \( A \)-modules, \( W_{\sigma} \) coincides with \( W \).

(2) If \( V \) is a \( u\chi(L_1) \)-module, then \( \text{Hom}_{u\chi(L_1)}(V, u\chi(L_1) \otimes_{u\chi(A)} W) \cong \text{Hom}_{u\chi(A)}(V, W) \).

**Proof.** (1) It suffices to prove that \( \text{tr}(\text{ad}_{L_1/A} y) = 0 \) for any \( y \in A \) For this, we only verify it for all standard basis elements \( y = x^{\beta_i} D_k, k \in \hat{I} \). Notice that \( L_1/A \) is isomorphic to \( C \) as a vector space, which has the standard basis: \( x^{\alpha_i} D_i, \ i \in I, \alpha_i \in A(\hat{I}) \). Recall that for \( G = x^{\alpha_i} D_i \in C \),

\[ [y, G] = \left( \beta_i + \alpha_i - \varepsilon_k \right) x^{\alpha_i + \beta_i - \varepsilon_k} D_i \begin{cases} \notin KG & \text{ if } \beta_i \neq \varepsilon_k, \\ = \alpha_i D_i(k) G & \text{ if } \beta_i = \varepsilon_k. \end{cases} \]

Hence \( \text{tr}(\text{ad}_{L_1/A} y) = 0 \) if \( \beta_i \neq \varepsilon_k \) and \( \text{tr}(\text{ad}_{L_1/A} y) = \frac{1}{2} p(p - 1) p^{#I - 1} = 0 \) in \( K \).

Hence \( \text{tr}(\text{ad}_{L_1/A} y) = 0 \) for all \( y \in A \).

(2) From [18, 1.4] and (1), it follows that

\[ u\chi(L_1) \otimes_{u\chi(A)} W \cong \text{Hom}_{u\chi(A)}(u\chi(L_1), W_{\sigma}) = \text{Hom}_{u\chi(A)}(u\chi(L_1), W). \]

Hence we have (with referring to [19, p. 118] for the second isomorphism below)

\[ \text{Hom}_{u\chi(A)}(V, u\chi(L_1) \otimes_{u\chi(A)} W) \cong \text{Hom}_{u\chi(A)}(V, \text{Hom}_{u\chi(A)}(u\chi(L_1), W)). \]

2.8. **Proposition.** Any \( f \)-weighted simple module \( M(f) \) of \( L_I \) is contained in the \( L_I \)-socle of \( R_f(W) \) up to an isomorphism, where \( W \) is a simple \( A \)-quotient module of \( M(f) \), and \( R_f(W) \) is defined in Section 2.6. Conversely, for any simple \( A \)-module \( W \) with character \( \chi \), each simple \( L_I \)-module in \( R_f(W) \) has an \( A \)-quotient isomorphic to \( W \).

**Proof.** Let \( W \) be a simple quotient of \( M(f) \), both taken as \( A \)-modules. Then \( \text{Hom}_{u\chi(A)}(M(f), W) \neq 0 \). Note that \( f \) is the unique weight of \( M(f) \) (Proposition 2.5). By Lemmas 2.7 and 2.4(2) we have
\[ \text{Hom}_{\chi(A)}(M(f), W) \cong \text{Hom}_{\chi(L)}(M(f), \chi(L) \otimes W) = \text{Hom}_{\chi(L)}(M(f), \tilde{W}f). \]

Hence \( \text{Hom}_{\chi}(M(f), \tilde{W}f) \neq 0 \). Thus \( M(f) \) can be regarded a simple \( L_I \)-submodule in \( \tilde{W}f \). Furthermore, any simple \( L_I \)-submodule of \( \tilde{W}f \) admits a non-zero regular \( f \)-weight vector which is contained in \( \tilde{W}f_{\text{reg}} = r_f \otimes W \) (Lemma 2.6). Hence \( \text{Soc}_{L_I} \tilde{W}f = \text{Soc}_{L_I}(R_f(W)) \). This is to say \( M(f) \subset \text{Soc}_{L_I}(R_f(W)) \).

The proof for the first conclusion is completed. By the same reason, one can understand that the second conclusion is true, too. \( \square \)

3. Induced representations of \( L = W(m : 1) \)

All notations introduced in the first sections will be kept here. Fix a \( \chi \in L^* \). Then there is the smallest index set \( I \) such that \( \chi(L^\pm(I)) = 0 \).

3.1. Let

\[ L_{[0],I} = L_I + \sum_{\alpha_i \in A(I)} \sum_{(\neq)j \in I} \mathcal{K} \chi_{\alpha_i} \varepsilon_j D_j, \]

\[ L_{[q],I} = \sum_{\alpha_j \in A(I)} \sum_{\alpha_i \in A(I)} \sum_{i \in I} \mathcal{K} \chi_{\alpha_i} \varepsilon j D_i \]

for \( q = -1 \) and \( q > 0 \). We have a \( \mathbb{Z} \)-gradation of \( L: L = \bigoplus_{q \geq -1} L_{[q],I} \). Set \( L_{q,I} = \sum_{k \geq q} L_{[k],I} \) for \( q \geq -1 \). Then \( \{L_{q,I}\} \) is a filtration of \( L \). When \( I = \{1, 2, \ldots, m\} \), they coincide with the usual situation.

Thus we have \( L = L_{[-1],I} \oplus L_{[0],I} \oplus L_{1,I} \). If \( V \) is a simple \( L_{[0],I} \)-module with character \( \chi \), then \( V \) can be extended to a simple \( L_{0,I} \)-module via the trivial \( L_{1,I} \)-action. This is because the standard basis elements of \( L_{1,I} \) under the \([p]\)-mapping are carried to zero and thereby they operate nilpotently on any \( \chi \)-reduced modules. Thus the weakly closed set theorem of Jacobson ensures that \( L_{1,I} \) nilpotently operates on any \( \chi \)-reduced modules. Hence, we have the induction functor \( \text{Ind}^L_{L_{0,I}} \) from the \( \chi(L_{[0],I}) \)-module category \( \chi(L_{[0],I})-\text{Mod} \) to the \( \chi(L) \)-module category \( \chi(L)-\text{Mod} \): \( \text{Ind}^L_{L_{0,I}}(V) = \chi(L) \otimes u_{\chi(L_{0,I})} V \).

On the other hand, for a \( \chi(L) \)-module \( W \), set \( W^{L_{1,I}} := \{w \in W \mid L_{1,I}w = 0\} \). Since \( L_{[0],I} \) normalizes \( L_{1,I} \), \( W^{L_{1,I}} \) is an \( L_{[0],I} \)-module. Thus we have the fixed-point functor \( R := (-)^{L_{1,I}} \) from \( \chi(L)-\text{Mod} \) to \( \chi(L_{[0],I})-\text{Mod} \).

3.2. \textbf{Proposition.} Suppose \( \chi(G_i) \neq 0 \) for a certain \( i \in I \) and \( G_i := x^{\pi_i + \varepsilon_i} D_i \), where \( \pi_i := \sum_{j \in I} (p - 1) \varepsilon_j \). Then the functor \( \text{Ind}^L_{L_{0,I}} \) is one-to-one correspondence between the isomorphism classes of simple \( L_{[0],I} \)-modules and simple \( L \)-modules with the same character \( \chi \).
Proof. Set \( \mathcal{J} = \text{Ind}_{L_{0,1}}^L \). It is easily verified that \( \mathcal{J} \) is left adjoint to \( \mathcal{R} \). Next we have to prove that both of functors \( \mathcal{J} \) and \( \mathcal{R} \) carry the simple objects to the simple objects. Suppose \( V \) is a simple object of \( u_x(L_{\{0,1\}})\text{-}\text{Mod} \). Let \( W := \mathcal{J}(V) \). Then \( W = u_x(L_{\{-1,1\}}) \otimes V \) as vector spaces. For any nonzero vector \( w \in W \) we can express it as

\[
w = \sum_{a \in T} F^a \otimes v_a, \tag{3.2.1}\]

where \( F^a = F_{1}^{a(1)} F_{2}^{a(2)} \cdots F_{n}^{a(n)} \), the \( F_q \)'s are standard basis elements like \( F_q = x^{a_i} D_i \), \( i \in I \), \( T \) is a certain subset of \( P^n \) for \( n = \dim L_{\{-1,1\}} \). If \( w \notin V \), then there is an \( a \neq 0 \) and \( v_a \neq 0 \) in the sum expression of \( w \) (3.2.1). Without loss of generality we may suppose \( F_1 = x^{a_i} D_i \) with \( a(1) \neq 0 \) for a certain \( a \in T \) such that \( |a_j| \) is the smallest one among all the values \( |\alpha| \) as long as \( F_q = x^{a_i} D_i \), \( b(q) \neq 0 \) for a certain \( b \in T \) appearing in the sum expression of \( w \) (3.2.1), and simultaneously such that \( i_1 = i \) if \( x^{a_i} D_i \) appears there, as a factor of a certain summand. Thus we can rewrite \( w \) as follows:

\[
w = F_{1}^{t_1} \tilde{F}_{t_1} + F_{t_1-1}^{t_1} \tilde{F}_{1(t_1-1)} + \cdots + F_{1} \tilde{F}_{11} + \tilde{F}_{10}, \tag{3.2.2}\]

where

\[
\tilde{F}_{1q} = \sum_{a \in T} F_{a_1^{(q)}} \otimes v_a, \quad q = 0, 1, 2, \ldots, t_1 (< p),
\]

\[
a_1^{(q)} := (a(2), \ldots, a(n)).
\]

According to the assumption, \( \tilde{F}_{1t_1} \neq 0 \). Set \( \tilde{F}_1 := x^{\pi_j - \alpha_j + \varepsilon_j + \varepsilon_1} D_i \in L_{1,1}^+ \). Then for all \( x^{\alpha} D_{i_q} \) appearing in the sum expression (3.2.1),

\[
[\tilde{F}_1, x^{\alpha} D_{i_q}] = \begin{cases} -\left(\frac{\pi_i}{\alpha_i}\right) G_i & \text{if } \alpha = \alpha_i, i_q = i_1, \\ 0 & \text{otherwise}. \end{cases}
\]

Hence \( [\tilde{F}_1, F_q] = -\delta_{1q}\left(\frac{\pi_i}{\alpha_i}\right) G_i \), which implies the trivial action of \( F_{1} \) on \( \tilde{F}_{1q} \) for \( q = 1, \ldots, t_1 \). Furthermore,

\[
\tilde{F}_1 \cdot f_{1}^{k} = \sum_{q=0}^{k} (-1)^q \left(\begin{array}{c} k \\ q \end{array}\right) F_{1}^{k-q} (\text{ad } F_{1})^q \tilde{F}_1
\]

\[
= F_{1}^{k} \tilde{F}_1 - k\left(\frac{\pi_i}{\alpha_i}\right) F_{1}^{k-1} G_i + \delta_{F_1,D_I} \left(\begin{array}{c} k \\ 2 \end{array}\right) \left(\frac{\pi_i}{\alpha_i}\right) F_{1}^{k-2} x^{\pi_i} D_i.
\]

Due to the choice of \( F_{1} \), \( x^{\pi_i} D_i \) does not appear in \( \{F_2, \ldots, F_q\} \). Hence \( [G_i, F_q] = 0 \) for \( q \geq 2 \). In addition, \( x^{\pi_i} D_i \) is in the abelian restricted subalgebra \( \bigoplus_{j=2}^{n} \mathbb{K} F_j \) when \( F_{1} = D_i \). From the above, we first have:
\[ \tilde{F}_1 \cdot F_{t_1}^{i_1} \tilde{F}_{1t_1} = -t_1 \left( \frac{\pi_i}{\alpha_i} \right) F_1^{t_1-1} G_i \tilde{F}_{1t_1} + \delta_{F_{t_1} D_i} \frac{t_1}{2} \left( \frac{\pi_i}{\alpha_i} \right) F_1^{t_1-2} x^{\pi_i} D_i \tilde{F}_{1t_1} \neq 0. \]

This is due to the fact \( G_i^p \tilde{F}_{1t_1} = \chi(G_i)^p \tilde{F}_{1t_1} \neq 0 \), along with the fact that both \( G_i \tilde{F}_{1t_1} \) and \( x^{\pi_i} D_i \tilde{F}_{1t_1} \) are in \( u_x (\bigoplus_{j=2}^n \mathcal{K} F_j) \otimes V \). Then we have
\[
\tilde{F}_1 \cdot w = F_1^{t_1-1} \tilde{F}_{1(t_1-1)} + F_1^{t_1-2} \tilde{F}_{1(t_1-2)} + \cdots + F_1 \tilde{F}_{11} + \tilde{F}_0,
\]
such that \( \tilde{F}_{iq} q = 1, 2, \ldots, t_1 - 1 \), are all in \( u_x (\sum_{j=2}^n \mathcal{K} F_j) \otimes V \) and \( \tilde{F}_{1(t_1-1)} \neq 0 \). Hence \( \tilde{F}_1 \cdot w \neq 0 \). The above argument can be repeated. Thus, we have \( \tilde{F}_1^{t_1} \cdot w \neq 0 \) and there is no factor \( F_1 \) appearing in any nonzero component of the sum expression of \( \tilde{F}_1^{t_1} w \), similar to (3.2.1). Iterating the above process, we will have
\[
0 \neq \tilde{F}_h^{t_h} \cdots \tilde{F}_1^{t_1} \cdot w \in 1 \otimes V,
\]
where the meaning of \( \tilde{F}_q (q = 2, \ldots, h) \) is similar to that of \( \tilde{F} \). This implies the simplicity of \( \text{Ind}_{L_{0,1}}^L (V) \).

Conversely, suppose \( W \) is an arbitrary simple object of \( u_x (L) - \text{Mod} \). If \( \mathcal{R}(W) \) contains a simple submodule \( W' \), by the above argument \( \mathcal{J}(W') \) is a simple \( u_x (L) - \text{module} \). Hence \( \mathcal{J}(W') \cong W' \) and \( \mathcal{R} \circ \mathcal{J}(W') \cong \mathcal{R}(W') \). Furthermore, the above argument shows that for any \( w \in \mathcal{J}(W') \setminus W' \), there is an \( \tilde{F}_1 \in L_{1,1} \) such that \( \tilde{F}_1 w \neq 0 \). Hence \( \mathcal{R} \circ \mathcal{J}(W') = W' \). This implies that \( W' \) coincides with \( \mathcal{R}(W) \). Hence \( \mathcal{R}(W) \) is \( L_{0,1} - \text{simple} \).

Thus, the adjunction morphisms \( \mathcal{J} \circ \mathcal{R} \mapsto \text{Id}_{u_x (L) - \text{Mod}} \) and \( \mathcal{R} \circ \mathcal{J} \mapsto \text{Id}_{u_x (L_{0,1}) - \text{Mod}} \) are both isomorphisms for simple objects. The proposition follows. \( \square \)

**Remark.** When \( L = W(m : 1) \), the corresponding result can be seen in [8,10].

### 4. The representations corresponding to loop algebras

We still set \( L = W(m : 1) \) in this section, and keep the notations used in the preceding sections.

#### 4.1. For the divided power algebra \( \mathfrak{U} = \mathfrak{U}(s : 1) \) and a restricted Lie algebra (g, [p]), we can define a loop algebra \( \mathcal{L}(g) = \mathfrak{U} \otimes g \) with Lie product:
\[
[x^\alpha \otimes X, x^\beta \otimes Y] = \left( \frac{\alpha + \beta}{\alpha} \right) x^{\alpha + \beta} \otimes [X, Y].
\]

It is not hard to verify that \( \mathcal{L}(g) \) is still a restricted Lie algebra with
\[
(x^\alpha \otimes X)^{[p]} = \begin{cases} 0 & \text{if } \alpha \neq 0, \\ 1 \otimes X^{[p]} & \text{if } \alpha = 0. \end{cases} \quad (4.1.1)
\]
When \( g \) is centerless, there is a unique \([p]\)-mapping for \( \mathcal{L}(g) \). In fact, in this case \( \mathcal{L}(g) \) is also centerless because \( \sum_{\alpha=1}^{n} \alpha \otimes X_{\alpha} \overline{\otimes} \mathcal{L} \neq 0 \) implies \( x_{\alpha} \otimes \mathcal{L} \neq 0 \) for the smallest \( |\alpha| \) among the values \( |\alpha| \) with \( x_{\alpha} \otimes X_{\alpha} \neq 0 \). Here \( \pi \) denotes \( \sum_{i=1}^{n} (p - 1) \varepsilon_{i} \). Thus the \([p]\)-mapping is uniquely determined by (4.1.1) for a basis \( \{x_{\alpha} \otimes X_{\alpha}\} \) of \( \mathcal{L}(g) \), where \( \alpha \) runs over \( A(s) \) and \( X_{1}, \ldots, X_{s} \) form a set of basis of \( g \) [13, II.2.2].

4.2. In the rest of this paper, we only deal with \( gl(I) \) for \( I \) as in the preceding sections. Here \( gl(I) \) is the classical Lie subalgebra of \( gl(m) \) corresponding to the index subset \( I = \{i_{1}, i_{2}, \ldots, i_{l}\} \subset (1, 2, \ldots, m) \), which is isomorphic to \( gl(\# I) \).

Denote by \( U(\widehat{I}) \) the divided power subalgebra of \( U(m) \), with generators \( x_{\alpha} \otimes E_{ij} \) for all \( \alpha \otimes E_{ij} \) \( \in \mathcal{A}(\widehat{I}) \). Thus the \([p]\)-mapping is uniquely determined by (4.1.1) for a basis \( \{x_{\alpha} \otimes X_{\alpha}\} \) of \( \mathcal{L}(g) \), where \( \alpha \) runs over \( A(s) \) and \( X_{1}, \ldots, X_{s} \) form a set of basis of \( g \) [13, II.2.2].

\[ g_{j} := \sum_{\alpha \in A(i) \cap j} \sum_{i,j \in I} \mathcal{K}x_{\alpha}^{i+\varepsilon_{i}} D_{j}. \]

**Lemma.** With the notations as above, \( g_{j} \) is Lie-isomorphic to the loop algebra \( \widehat{gl}(I) \).

**Proof.** Set \( \phi : x_{\alpha}^{i+\varepsilon_{i}} D_{j} \mapsto x_{\alpha}^{i} \otimes E_{ij} \), where \( E_{ij} \) denotes the \( m \times m \) matrix with \( (i,j) \)-entry being one and with the others being zero. This is a desired isomorphism. \( \square \)

4.3. Hence, we can identify \( g_{j} \) with \( \widehat{gl}(I) \) in the following discussion. In particular, \( g_{j} \) is a \( p \)-subalgebra of \( (L, [p]) \) with \([p]\)-mapping satisfying

\[ (x_{\alpha}^{i+\varepsilon_{i}} D_{j})^{[p]} = \begin{cases} x_{\alpha}^{i} D_{i} & \text{if } \alpha_{j} = 0 \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases} \]

This is admissible to the \([p]\)-mapping as defined in (4.1.1) for \( \widehat{gl}(I) \). Notice that for \( k \in \widehat{I} \),

\[ [x_{\beta}^{i} D_{k}, x_{\alpha}^{i} \otimes E_{ij}] = \left( \begin{array}{c} \beta_{j} + \alpha_{j} - \varepsilon_{k} \\ \beta_{j} \end{array} \right) x_{\alpha}^{i+\varepsilon_{i}-\varepsilon_{k}} \otimes E_{ij}. \]  

(4.3.1)

The following lemma is obvious.

**Lemma.** If \( g' \) is a subalgebra of \( W(I) \), then the loop algebra \( \mathcal{U}^{0} \otimes g' \) is normalized by \( L^{0}_{j} \).

4.4. Denote \( b_{0}^{0} := \mathcal{L}(b) \oplus L^{0}_{j} \) and \( n_{0}^{\pm} = \mathcal{L}(n^{\pm}) \) (notice the difference between the notations \( n_{0}^{\pm} \) and \( b_{0}^{0} \), as well as \( R_{i}, b_{i} \) and some others to appear after
Section 4.5. Basically, such notations involving Borel or parabolic algebras mean the sums of the corresponding loop algebras and $L_I^0$). By the foregoing arguments, $(x^a \otimes e_\alpha)^{|p|} = 0$ for all $a, \alpha$ and $(x^a \otimes h_\alpha)^{|p|} = 0$ for $a \neq 0$ and all $\alpha$. Thus all elements of $n_0^\pm$ act nilpotently on any $\chi$-reduced modules of $n_0^\pm$. In particular, for any simple module $V$ of $L_I$, $V$ can act as a $b_0$-module with the trivial $n_0^\pm$-action (note: $b_0 = L_I \oplus n_0^\pm$). Hence we have an induced module:

$$\text{Ind } V := u\chi(L_{-0}) \otimes u\chi(b_0) V.$$ 

Similarly to the definition of a character being regular semisimple in the case of classical Lie algebras [1], we call $\chi \in L^*$ regular semisimple if $\chi(G_\alpha) \neq 0$ for all $\alpha \in \Phi^+$ and $\chi(L_I^\pm(I)) = 0$, where $G_\alpha := x^{\pi i} \otimes h_\alpha$.

**Proposition.** Suppose $\chi \in L^*$ is regular semisimple. Then $u\chi(L_I)$ and $u\chi(L_{-0})$ are Morita equivalent. In particular, the modules $\text{Ind } V$ constitute the corresponding set for $u\chi(L_{-0})$ when $V$ runs over a set of representatives of the isomorphism class of simple $u\chi(L_I)$-modules.

In the next subsections, we will prove this result, applying the same argument as in [1] to our case.

### 4.5

Let $\Phi$ be the root system of $g := gl(I)$. For $\alpha \in \Phi$, let $e_\alpha \in g$ be a nonzero root vector. We can assume that those root vectors are normalized so that if $h_\alpha = [e_\alpha, e_{-\alpha}]$ then the system $\{e_{\pm \alpha}, h_\alpha\}$ satisfies $[h_\alpha, e_{\pm \alpha}] = \pm 2e_{\pm \alpha}$. We have the decomposition of root spaces, $g = \bigoplus_{\alpha \in \Phi} g_\alpha$. Canonically, denote $n^\pm = \bigoplus_{\alpha \in \Phi^\pm} g_\alpha$. Let $b$ be the Borel subalgebra of $g$ associated with $\Phi^+$. Suppose $\mathfrak{N}$ is a parabolic subalgebra of $g$ containing the Borel subalgebra $b$, and $\mathfrak{N}$ has a Levi decomposition $\mathfrak{Z} \oplus \mathfrak{N}$, where $\mathfrak{Z}$ is the Levi factor and $\mathfrak{N}$ is the nil-radical. Let $\mathfrak{R}_{-0} = \mathfrak{L}(\mathfrak{N}) \oplus L^0_I$, which contains $b_0$. Due to Lemma 4.3 (as well as (4.3.1)), $\mathfrak{L}(\mathfrak{N})$ is still the nil-radical of $\mathfrak{R}_{-0}$. Let $R$ (respectively $Z$, $N$, $B$) denote the subalgebra of $u\chi(L_{-0})$ generated by $\mathfrak{R}_{-0}$ (respectively $\mathfrak{L}(\mathfrak{Z})$, $\mathfrak{L}(\mathfrak{N})$ and $b_0$). Let $I(-) = R \otimes_B -$, which is the induction functor from $B$-module to $R$-module. Then $I$ is an exact functor since $R$ is a free $B$-module. Set $(\cdot)^N$ (or $(-)^{\mathfrak{L}(\mathfrak{N})}$) be the fixed-point functor with the image being the subspace consisting of all vectors annihilated by $\mathfrak{L}(\mathfrak{N})$. By Lemma 4.3 $\mathfrak{L}(\mathfrak{N})$ is an ideal of $\mathfrak{R}_{-0}$, $I(M)^N$ and $I(M^N)$ are still $\mathfrak{R}_{-0}$-modules for a $B$-module $M$. So by the same argument as in [1, Section 8.1], we have the following lemma.

**Lemma.** Suppose $N$ is isomorphic to the restricted enveloping algebra $u(\mathfrak{L}(\mathfrak{N}))$ of $\mathfrak{L}(\mathfrak{N})$. Then for any finite-dimensional $B$-module $M$, we have a natural isomorphism of $R$-modules $I(M)^N \simeq I(M^N)$.
4.6. The notations are as in the above subsections.

**Lemma.** Let $\mathcal{R}$ be a parabolic subalgebra of $\mathfrak{gl}(I)$ containing the Borel subalgebra $\mathfrak{b}$, and $\mathfrak{z}$ be its Levi factor. Suppose the derived subalgebra $\mathfrak{z}' := [\mathfrak{z}, \mathfrak{z}]$ of the Levi factor is isomorphic to $\mathfrak{sl}(2)$, spanned by $\{e_\alpha, h_\alpha, f_\alpha\}$ with $\chi(x^{2i} \otimes h_\alpha) \neq 0$. If $V$ is a finite-dimensional $B$-module with the trivial $E_\alpha$-action, then $I(V)^{E_\alpha} = V$. Here $E_\alpha := \mathfrak{L}(\mathcal{K} e_\alpha)$.

**Proof.** Obviously, $V \subset I(V)^{E_\alpha}$. It is sufficient to prove that $\dim(I(V)^{E_\alpha}) = \dim V$. If $\dim(I(V)^{E_\alpha}) > \dim V$, then there is at least one vector $w$ such that for a certain $a \in A(I)$ and $F_a := x^a \otimes f_\alpha \in F_\alpha := \mathfrak{L}(\mathcal{K} f_\alpha)$,

$$w = \sum_{q=0}^t \hat{F}_q \in I(V)^{E_\alpha},$$

where $\hat{F}_q \in u_{\chi}(\sum_{b \neq a} (b) \otimes v_b, v_b \in V,

\hat{F}_q \neq 0$ (note: $F_\alpha$ is an abelian restricted algebra). Then

$$[\hat{E}_a, F_b] = \begin{cases} G_a & \text{if } a = b, \\ 0 & \text{for } b \neq a \text{ with } |b| \geq |a|. \end{cases}$$

Note that $E_\alpha V = 0$ and $[G_a, F_b] = 0$ for $b \neq a$ with $|b| \geq |a|$. In addition, $[G_a, F_a] = -2\delta_a,0 x^{2i} \otimes f_\alpha \in F_\alpha$. By the same reason as in the proof of Proposition 3.2, we have

$$\hat{E}_a w = -t F_a^{t-1} \hat{F}_{t-1} \oplus \sum_{q < t-1} F_a^{q} \hat{F}_q,$$

where all $\hat{F}_q$ are still in $u_{\chi}(\sum_{b \neq a, |b| \geq |a|} F_b) \otimes v_b'$ with $\hat{F}_{t-1} = G_a \hat{F}_{t} \neq 0$ because of the condition $\chi(G_a) \neq 0$. As $G_\alpha^{\{p\}} = 0$, $G_\alpha^{\{p\}}$ acts on $I(V)$ as the scalar $\chi(G_a)^p \neq 0$. This implies $\hat{E}_a w \neq 0$, which contradicts with $w \in I(V)^{E_\alpha}$. This implies $\dim I(V)^{E_\alpha} = \dim V$, thereby $I(V)^{E_\alpha} = V$. □

4.7. The following lemma is a special case of [1, 8.4] when $\Phi_s = \emptyset$, which is necessary for the further discussion.

**Lemma.** The notations are as above. Let $\Phi^+$ be a positive root system (associated with $\mathfrak{b}$). Then we can enumerate $\Phi^+$ as $\delta_1, \ldots, \delta_r$, $\alpha = |\Phi^+|$, such that for each $q$, $\Phi_q^+ := \{-\delta_1, \ldots, \delta_{q-1}, \delta_q\}$ is a system of positive roots for $\Phi$ in which $\delta_q$ is a simple root. For each $q$, $\Psi_q := \{-\delta_1, \ldots, -\delta_q\}$ is a closed subsystem of $\Phi$.

Thus associated with $\Phi^+_i$ and $\Phi^+_i \cup \Phi^+_{i+1}$ we have for $\mathfrak{gl}(I)$ respectively the Borel subalgebra $\mathfrak{b}_i$ and the parabolic subalgebra $\mathfrak{R}_i$, whose nil-radical is denoted by $\mathfrak{N}_i$. In particular, the derived subalgebra of the Levi factor of $\mathfrak{N}_i$ is isomorphic
to \(sl(2)\) spanned by \(\{e_{\delta_i}, h_{\delta_i}, e_{-\delta_i}\}\). Set \(b_i := \mathcal{L}(b_i) \oplus L_0^0\) and \(\mathfrak{N}_i := \mathcal{L}(\mathfrak{N}_i) \oplus L_0^0\). Then \(\mathcal{L}(\mathfrak{N}_i)\) is the nil-radical of \(\mathfrak{N}_i\), as argued in Section 4.5. Similarly, \(R_i, B_i,\) and \(N_i\) will stand for the subalgebras of \(u_x(L_0)\) generated respectively by \(\mathfrak{N}_i, b_i,\) and \(\mathcal{L}(\mathfrak{N}_i)\). In particular, \(N_i\) is isomorphic to the restricted enveloping algebra \(u(\mathcal{L}(\mathfrak{N}_i))\) of \(\mathcal{L}(\mathfrak{N}_i)\) when \(\chi \in L^*\) is regular semisimple.

4.8. With Lemmas 4.5–4.7, we can prove the following key lemma by applying the argument in [1, 8.5] to our case. For the convenience, we provide the proof in the details. Below we will keep the notations appearing above in this section.

**Lemma.** Suppose \(\chi \in L^*\) is regular semisimple. Then for any simple \(u_x(L_0)\)-module \(M\), \(M^N\) is a simple \(B\)-submodule and the natural mapping \(\text{Ind} M^N \to M\) is a \(u_x(L_0)\)-module isomorphism. Any simple \(u_x(L_0)\)-module is \(N\)-projective. Here the functor \(\text{Ind}\) is defined as in Section 4.4, \(B = u_x(b_0)\) and \(N = u(\mathcal{L}(n^+))\) (note: \(\chi(\mathcal{L}(n^+)) = 0\)).

**Proof.** Let \(W_1\) be a nonzero simple \(B\)-module in \(M^N\). We can define an increasing filtration \(\{W_q\}\) of \(\text{Ind} W_1\) inductively: \(W_{q+1} = R_q \otimes B_q W_q\) for \(q \geq 1\). Then \(W_{r+1} = \text{Ind} W_1\). And each \(W_q\) is a free \(J_{q-1}\)-module, where \(J_{q-1}\) is the subalgebra of \(u_x(L_0)\) generated by \(\mathcal{L}(g_\alpha)\) for \(\alpha \in \Psi_{q-1}\). And \(J_{q-1}\) is actually the restricted enveloping algebra of the restricted Lie algebra \(K\)-span\(\{\mathcal{L}(g_\alpha) \mid \alpha \in \Psi_{q-1}\}\) because \(\chi|_{n^+_\mathcal{L}} = 0\) and the \(\Psi_{q-1}\)'s are closed. Moreover, \(J_{q-1} \subset N_q\).

If \(\text{Ind} W_1\) is not simple, then there is a proper simple submodule \(W'\) in \(\text{Ind} W_1\). This is to say, \(W' \cap W_1 = 0\) and \(W' \cap W_{r+1} = W'\). Hence we can suppose \(W' \cap W_{i+1} \neq 0\) and \(W' \cap W_i = 0\) for a certain \(i \geq 0\). However, we find that there are some nonzero vectors in \(W_i^{N_i} \cap W'\), and thereby show that the above assumption impossibly occurs. For this we first consider \(W_i^{N_i}\). Since \(J_{i-1} \subset N_i, W_i^{N_i} \subset W_i^{J_i-1}\). Note that \(J_{i-1}\) is a Hopf algebra, in which there exists a nonzero left integral \(\Omega\); i.e., \(u\Omega = \varepsilon(u)\Omega\) for all \(u \in J_{i-1}\), where \(\varepsilon: J_{i-1} \to K\) is the counit of \(J_{i-1}\). Hence,

\[
W_i^{N_i} \subset W_i^{J_i-1} = \Omega W_i = \Omega \otimes W_1.
\]

The third equality is because \(W_i\) is \(J_{i-1}\)-free. Furthermore, \(W_i^{N_i} \subset \Omega \otimes W_i^{N_i}\) since \(\mathcal{L}(n^+) W_1 = 0\).

Next, we will see that \(W_i^{N_i}\) is annihilated by \(E_{\delta_j}\). In fact, for \(j < i, [x^a \otimes e_{\delta_i}, x^b \otimes e_{-\delta_j}]\) is either zero or in \(\mathcal{L}(g_{\delta_i-\delta_j})\) if \(\delta_i - \delta_j\) is a root. In the latter case, \(\delta_i - \delta_j \in \Phi_i^+\). Hence \([E_{\delta_i}, \Omega] \otimes W_i^{N_i} \subset \Omega \otimes W_i^{N_i} = 0\). In addition, \(\Omega E_{\delta_i} \otimes W_i^{N_i} = 0\). So \(\Omega \otimes W_i^{N_i}\) annihilated by \(E_{\delta_i}\); i.e., \((W_i^{N_i})^{E_{\delta_i}} = W_i^{N_i}\).

Note that \(W' \cap W_{i+1} \neq 0\) and that \(N_i\) nilpotently acts on it. Therefore \((W' \cap W_{i+1})^{N_i} \neq 0\). By Lemma 4.5, \((W' \cap W_{i+1})^{N_i} \subset R_i \otimes B_{i} W_i^{N_i}\). Now from
\[ W_{i}^{N_{i}} = (W_{i}^{N_{i}})^{E_{i}} \]

we have, by applying Lemma 4.6 to the case of \( R_{i} \),

\[ 0 \neq (W' \cap W_{i+1})^{N_{i}} \subseteq (R_{i} \otimes_{B_{i}} W_{i}^{N_{i}})^{E_{i}} = W_{i}^{N_{i}}. \]

The above results are in contradiction to \( W' \cap W_{i} \neq 0 \). This implies that \( \text{Ind} W_{1} \) is simple. Hence any nonzero homomorphism of \( \text{Ind}(W_{1}) \) to \( M \) is an isomorphism. And \( M \cong \text{Ind} W_{1} \) is a free \( N^{-} \)-module for \( N^{-} = u(\Sigma(n^{-})) \).

Of course \( \dim N^{-} = \dim N \). Symmetrically, \( M \) is a free \( N^{-} \)-module. Hence \( \dim MN^{-} = \dim M/\dim N = \dim W_{1} \). This implies that \( W_{1} = M^{N} \). The proof is completed. \( \square \)

4.9. We are in a position to prove Proposition 4.4.

In the proof of Lemma 4.8, we have known that the \( L_{0} \)-module \( \text{Ind} V \) is simple, and that for any simple \( u_{\chi}(L_{0}) \)-module \( W \), \( W \) is isomorphic to \( \text{Ind}(WN) \) and \( WN \) is just a simple \( u_{\chi}(LI) \)-module. Thus the same argument as in [1, 3.2] results in the second statement.

5. The main result

Let \( L = W(m : 1) \) and \( \chi \in L^{*} \). The index set \( I \) is the smallest one satisfying \( \chi(L_{+}(I)) = \chi(L_{-}(I)) = 0 \) (see Proposition 1.4). Associated with \( I \), we have the subalgebra \( L_{I} \) of \( L: L_{I} = \hat{W}(I) + L_{I}^{c} \), where \( L_{I}^{c} \cong \hat{L}(I) \otimes \mathfrak{h}(I) \) and \( \mathfrak{h}(I) \) is the canonical torus of \( W(I) \). We still keep the notations used in the above sections. Due to Proposition 2.5, we know that any simple \( L_{I} \)-module \( V \) of character \( \chi \) admits a unique weight \( f \) in \( X(\chi) \), and \( V \) is a quotient of \( Z_{\chi,I}(f) = u_{\chi}(L_{I}) \otimes_{u_{\chi}(L_{I}^{c})} K_{f} \). Set \( B(I) = L_{I} \oplus L_{+}(I) \). For any given \( f \)-weighted simple module \( M(f) \) of \( L_{I} \), \( M(f) \) can be regarded as a simple \( u_{\chi}(B(I)) \)-module with the trivial \( L_{+}(I) \)-action. Then we have the induced module \( \text{Ind}^{L}_{B(I)} M(f) := u_{\chi}(L) \otimes_{u_{\chi}(B(I))} M(f) \).

**Theorem.** Suppose \( \chi \in L^{*} \) is regular semisimple. Then:

1. The correspondence \( \text{Ind}^{L}_{B(I)} \) is one-to-one between the isomorphism classes of simple \( u_{\chi}(L_{I}) \)-modules and simple \( u_{\chi}(L) \)-modules.

2. Any simple \( L \)-module of character \( \chi \) corresponds to a unique weight of \( L_{I}^{c} \). The set \( X(\chi) \) of weights of all \( \chi \)-reduced simple modules of \( L \) can be divided into \( p^{\#I} \) classes. Two simple modules of different classes are non-isomorphic.

3. All \( L \)-modules induced from non-isomorphic composition factors of \( Z_{\chi,I}(f) \) for \( L_{I} \) constitute a set of representatives of the isomorphism classes of simple \( L \)-modules corresponding to \( f \in X(\chi) \).

4. Any simple \( u_{\chi}(L_{I}) \)-module is contained in \( \text{Soc}_{L_{I}}(R_{f}(V)) \) for a unique \( f \in X(\chi) \) and for a simple \( W(\hat{I}) \)-module \( V \) with character \( \chi \), where \( R_{f}(V) = u_{\chi}(W(\hat{I}))r_{f} \otimes V \), and \( r_{f} \) is a unique regular \( f \)-weight vector of \( u_{\chi}(L_{I}^{c}) \), up to scalars (Section 2.6).
Proof. Note that $B(I) = b_\bar{0} \oplus L_{1,I} \subset L_{0,I} = L_\bar{0} \oplus L_{1,I}$, 
\[
\text{Ind}_{B(I)}^{L_{1,I}} M(f) = u_\chi(L) \otimes_{u_\chi(L_{0,I})} \left( u_\chi(L_{0,I}) \otimes_{u_\chi(B(I))} M(f) \right) 
\]
\[
= u_\chi(L) \otimes_{u_\chi(L_{0,I})} \left( u_\chi(L_\bar{0}) \otimes_{u_\chi(b_\bar{0})} M(f) \right) 
\]
\[
= \text{Ind}_{L_{0,I}}^{L_{1,I}} \left( \text{Ind}_{B(I)}^{L_{1,I}} M(f) \right).
\]
The second equality holds because $u_\chi(L_{0,I}) \otimes_{u_\chi(B(I))} M(f)$ is isomorphic to $\text{Ind}_{B(I)}^{L_{1,I}} M(f)$ as $u_\chi(L_\bar{0})$-modules, with the trivial $L_{1,I}$-action on both. The statement (1) directly follows from Propositions 3.2 and 4.4. The other statements are due to (1), Corollary 2.5, and Proposition 2.8.  

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