Abelian 1-Factorizations of the Complete Graph

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Extending a result by Hartman and Rosa (1985, Europ. J. Combinatorics 6, 45–48), we prove that for any Abelian group \( G \) of even order, except for \( G \approx \mathbb{Z}_{2^n} \) with \( n > 2 \), there exists a one-factorization of the complete graph admitting \( G \) as a sharply-vertex-transitive automorphism group.

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1. INTRODUCTION

A 1-factorization of a graph \( \Gamma = (V, E) \) is a partition \( \mathcal{F} \) of \( E \) into classes (1-factors) each of which is, in its turn, a partition of \( V \). An automorphism group of \( \mathcal{F} \) is a group of bijections on \( V \) leaving \( \mathcal{F} \) invariant.

For general background on 1-factorizations of the complete graph we refer to [1, 9, 10]. Here we are interested in 1-factorizations of the complete graph admitting a sharply-vertex-transitive automorphism group. Some papers on this subject are [2, 7, 8].

The following question naturally arises.

PROBLEM 1.1. For which groups \( G \) does there exist a 1-factorization of the complete graph admitting \( G \) as a sharply-vertex-transitive automorphism group?

The complete solution is still unknown but the problem has been settled in the cyclic and dihedral cases, as presented in the following theorems.

THEOREM 1.2 ([7]). There exists a 1-factorization of \( K_{2v} \) admitting a cyclic sharply-vertex-transitive automorphism group for all \( v \)'s but \( v \neq 2^n \), \( n \geq 2 \).

THEOREM 1.3 ([2]). There exists a 1-factorization of \( K_{2v} \) admitting a dihedral sharply-vertex-transitive automorphism group for all values of \( v \).

Bonisoli and Labbate [2] also give some partial results in the Abelian case. Here this case is completely settled; we will prove, in fact, that any Abelian group of even order that is not a cyclic 2-group is a solution of Problem 1.1.

2. STARTERS AND 1-FACTORIZATIONS

A 1-factorization of the complete graph admitting \( G \) as an automorphism group fixing one vertex and acting sharply transitively on the others is said to be 1-rotational under \( G \). It has been known for a long time (see [5]) that for any group \( G \) of odd order there exists a 1-rotational 1-factorization of the complete graph under \( G \).

We recall that a starter (see [6]) in a group \( G \) of odd order \( 2n + 1 \) is a set \( \{[x_i, y_i] | i = 1, \ldots, n \} \) of \( 2 \)-subsets of \( G \) with the property that

\[
\bigcup_{i=1}^{n} [x_i - y_i, y_i - x_i] = \bigcup_{i=1}^{n} [x_i, y_i] = G - \{0\}.
\]

The existence of a starter in \( G \) is equivalent to the existence of a 1-rotational 1-factorization of the complete graph under \( G \). The equivalence may be also viewed as a consequence of a general result on 1-rotational resolvable Steiner 2-designs (see [4], Theorem 1).
Our aim is to define a new concept of a starter in a group of *even* order able to describe all 1-factorizations admitting a sharply-vertex-transitive automorphism group.

Given a group $G$ written in additive notation, let $K_G = (G, \binom{G}{2})$ be the complete graph on $G$. Speaking of the action of $G$ on itself, namely on the vertices of $K_G$, we always mean the regular right representation of $G$ ($g(x) = x + g$ for $x \in G$). Of course, this action also induces actions of $G$ on edges and 1-factors of $K_G$.

An edge $e = [x, y]$ of $K_G$ is said to be short or long according to whether $-x + y$ is or is not an involution. In the first case $e$ is fixed by $-x + y$ and its $G$-orbit has length $|G|/2$ while, in the second case, $e$ has trivial $G$-stabilizer so that its $G$-orbit has length $|G|$.

We set $\partial[x, y] = \{x - y, y - x\}$ when $[x, y]$ is long and $\partial[x, y] = \{x - y\}$ when $[x, y]$ is short. Also, we set $\phi[x, y] = \{x, y\}$ when $[x, y]$ is long and $\phi[x, y] = \{x\}$ or (at pleasure) $\{y\}$ when $[x, y]$ is short.

More generally, given $S = \{e_1, \ldots, e_n\} \subset \binom{G}{2}$, we set $\partial S = \partial e_1 \cup \cdots \cup \partial e_n$ and $\phi(S) = \phi(e_1) \cup \cdots \cup \phi(e_n)$.

In the above equalities and throughout the paper the union is understood to be strong (elements have to be counted with their respective multiplicities) so that $\partial S$ and $\phi(S)$ are, in general, multisets of elements of $G$.

We are now ready to define the new concept of a starter in a group of even order.

**Definition 2.1.** A starter in a group $G$ of even order is a set $\Sigma = \{S_1, \ldots, S_n\}$ of subsets of $\binom{G}{2}$ satisfying the following conditions:

$$\partial S_1 \cup \cdots \cup \partial S_n = G - \{0\}.$$  

$\phi(S_i)$ is a left transversal of $G/H_i$ for a suitable $H_i \leq G$ containing all the involutions fixing the short edges of $S_i$, $i = 1, \ldots, n$.

The following theorem shows that the concept defined above is equivalent to that of a 1-factorization of the complete graph admitting $G$ as a sharply-vertex-transitive automorphism group. As a matter of fact the theorem is a very special case of a much more general theory able to describe, in particular, any regular resolution of a Steiner 2-design (see [3]). For this reason we give a quick proof without entering into the smallest details.

**Theorem 2.2.** The existence of a starter in a group $G$ of even order is equivalent to the existence of a 1-factorization of the complete graph admitting $G$ as a sharply-vertex-transitive automorphism group.

**Proof.** Let $\Sigma = \{S_1, \ldots, S_n\}$ be a starter in $G$, i.e., a set of subsets of $\binom{G}{2}$ satisfying the conditions of Definition 2.1. For $i = 1, \ldots, n$, let $F_i = \bigcup_{e \in S_i} \text{Orb}_G(e)$ be the union of the $H_i$-edge-orbits represented by the edges of $S_i$.

Given $i \in \{1, \ldots, n\}$ and $g \in G$, since $\phi(S_i)$ is a left transversal of $G/H_i$, there exists exactly one pair $(x, h) \in \phi(S_i) \times H_i$ such that $g = x + h$. One may check that the only edge of $F_i$ containing $g$ is $e + h$ where $e$ is the edge of $S_i$ through $x$. This means that each $F_i$ is a 1-factor of $K_G$.

Now, consider the set $\mathcal{F} = \text{Orb}_G(F_1) \cup \cdots \cup \text{Orb}_G(F_n)$ of 1-factors of $K_G$.

Given any $[g, h] \in \binom{G}{2}$, let $e = [x, y] \in S_i$ be the only edge of $\bigcup_{j=1}^n S_j$ such that $g - h \in \partial e$, say $g - h = x - y$. Then $F_i + (-x + g)$ is the only factor of $\mathcal{F}$ containing $[g, h]$. Thus, any $[g, h] \in \binom{G}{2}$ belongs to exactly one factor of $\mathcal{F}$, i.e., $\mathcal{F}$ is a 1-factorization of $K_G$. Obviously, $\mathcal{F}$ admits $G$ as a sharply-vertex-transitive automorphism group.

Now, let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a 1-factorization of the complete graph admitting $G$ as a sharply-vertex-transitive automorphism group. For $i = 1, \ldots, n$, let $H_i$ be the $G$-stabilizer of
Given an Abelian group \( G \), we denote by \( I(G) \) and by \( 2G \) the kernel and, respectively, the image of the group endomorphism \( \alpha : g \in G \rightarrow 2g \in G \). Thus, the nonzero elements of \( I(G) \) are the involutions of \( G \).

We shall need the following elementary lemmas.

**Lemma 3.1.** Let \( G \) be an Abelian group. We have:

(i) A subset \( X \) of \( G \) is a transversal of \( G/I(G) \) if and only if the map \( \beta : x \in X \rightarrow 2x \in 2G \) is bijective.

(ii) For any \( g \in G − 2G \) there exists at least one subgroup of index two in \( G \) not containing \( g \).

**Proof.**

(i) It follows from the proof of the Homomorphism theorem.

(ii) From the fundamental theorem on the structure of finite Abelian groups we may assume that \( G = \bigoplus_{i=1}^{n} C_{i} \) where each \( C_{i} \) is a cyclic group of even order while \( K \) is an Abelian group of odd order. Note that \( 2G = \bigoplus_{i=1}^{n} 2C_{i} \) and hence, if \( g = (x_{1}, x_{2}, \ldots, x_{i}, k) \in G − 2G \), we have that \( x_{j} \in C_{j} \). Then \( \bigoplus_{i=1}^{n} C_{i} \) is a subgroup of \( G \) of index 2 not containing \( g \).

**Lemma 3.2.** Let \( G \) be an Abelian group of even order and let \( \Sigma = \{ S_{1}, \ldots, S_{m} \} \) be a set of subsets of \( \binom{G}{2} \) satisfying the following conditions:

(1) \( \bigcup_{i=1}^{m} \partial S_{i} \) has no repeated elements and contains \( 2G − I(G) \);

(2) \( \phi(S_{i}) \) is a transversal of \( G/H_{i} \) for a suitable \( H_{i} \leq G \) containing all the involutions fixing the short edges of \( S_{i} \), \( i = 1, \ldots, m \).

Then \( \Sigma \) can be extended to a starter in \( G \).

**Proof.** Let \( A \) be the set of nonzero elements of \( G \) that are not covered by \( \bigcup_{i=1}^{m} \partial S_{i} \). We may express \( A \) in the form \( A = A' \cup -A' \cup A'' \) where \( A'' \) is the set of involutions in \( A \). For each \( a \in A' \cup A'' \) set \( T_{a} = \{0, a\} \). We have \( \bigcup_{a \in A' \cup A''} \partial T_{a} = A \). Also, for each \( a \in A' \) we have \( \phi(T_{a}) = \{0, a\} \) so that, by (1) and Lemma 3.1(ii), \( \phi(T_{a}) \) is a transversal of \( G/L_{a} \) for a suitable subgroup \( L_{a} \) of index 2 in \( G \). Finally, for each \( a \in A'' \) we have that \( \phi(T_{a}) = \{0\} \), which, obviously, is a transversal of \( G/G \).

It follows that \( \Sigma \cup \{ T_{a} \mid a \in A' \cup A'' \} \) is a starter in \( G \).

**Theorem 3.3.** For any Abelian group \( G \) of even order, but \( G \not\cong \mathbb{Z}_{2^{n}} \) with \( n > 2 \), there exists a 1-factorization of the complete graph admitting \( G \) as a sharply-vertex-transitive automorphism group.

**Proof.** For our purpose it is sufficient to construct a set \( \Sigma \) of subsets of \( \binom{G}{2} \) satisfying the conditions of Lemma 3.2. We realize the construction distinguishing four cases according to the form of the Sylow 2-subgroup, say \( H \), of \( G \).

1st case: \( H \) is elementary Abelian, i.e., \( H = I(G) \).

In this case \( G \) is the direct sum \( H + K \) where \( K \) is an Abelian group of odd order. Express \( K \) as union of two disjoint sets \( K' \) and \( -K' \), choose a nonzero element \( h \in H \) and consider the following subset of \( \binom{G}{2} \):

\[
S = \{[k, -k] \mid k \in K'\} \cup \{[0, h]\}.
\]
We have $\partial S = (K - \{0\}) \cup \{h\}$ and $\phi(S) = K$ so that $\phi(S)$ is a transversal of $G/H$.

Thus, since $2G = K$, $\Sigma = \{S\}$ satisfies the conditions of Lemma 3.2

2nd case: $H$ is neither elementary Abelian nor cyclic.

In this case $G$ may be expressed as direct sum of two groups $C$ and $K$ where $C = \langle y \rangle$ is cyclic of order $2^n \geq 4$ and $K$ is an Abelian group of even order.

Take a subset $X$ of $G$ in such a way that the map $\beta : x \in X \longrightarrow 2x \in 2G$ is bijective. It is easy to see that we may choose $X$ of the form $X' \cup -X' \cup X''$ where $\beta(X'') = 1(G) \cap 2G$. We may also assume that $X' \cap C = \{iy | 1 \leq i < 2^{n-2}\}$ and that $X'' \cap C = \{0, 2^{n-2}y\}$.

Consider the following subset of $\binom{\frac{n}{2}}{2}$:

$$S = \{[x, -x] | x \in (X' \cup X'') \setminus C \} \cup \{[-iy, (i + 1)y] | 0 \leq i < 2^{n-2}\}.$$ 

One may check that $\partial S = (2G - 2C) \cup (C - 2C)$ and that $\phi(S) = X$.

Then, by Lemma 3.1(i), $\phi(S)$ is a transversal of $G/1(G)$.

Since $K$ has even order, $K - 2K$ is not empty. Take an element $k \in K - 2K$ and consider the following subset of $\binom{\frac{n}{2}}{2}$:

$$T = \{[i, y, (2^{n-1} - i)y] | 1 \leq i < 2^{n-2}\} \cup \{[0, 2^{n-2}y + k]\}.$$ 

One may check that $\partial T = (2C - [0, 2^{n-1}y]) \cup [2^{n-2}y + k, -2^{n-2}y - k]$ and that $\phi(T)$ is a transversal of $G/L$ where $L = [0, 2^{n-1}y] + K$.

Note that $\partial S \cup \partial T$ has no repeated elements and covers all the nonzero elements of $2G$ with the only exception of the involution $2^{n-1}y$. Thus $\Sigma = \{S, T\}$ satisfies the conditions of Lemma 3.2.

3rd case: $H = [0, h, 2h, 3h]$ is cyclic of order 4.

Here $G$ is the direct sum of $H$ with an Abelian group $K$ of odd order. Express $K - \{0\}$ as union of two disjoint sets $K'$ and $-K'$. Consider the following subset of $\binom{\frac{n}{2}}{2}$:

$$S = \{[k, -k] | k \in K' \} \cup \{[h - k, 3h + k] | k \in K' \} \cup \{[0, h]\}.$$ 

We have $\partial S = (2G - [0, 2h]) \cup [h, 3h]$. Also, $\phi(S)$ is a transversal of $G/\{0, 2h\}$. Thus, $\Sigma = \{S\}$ satisfies the conditions of Lemma 3.2.

4th case: $H$ is cyclic of order greater than 4.

In this case $G$ is the direct sum of $H$ with an Abelian group $K$ of odd order >1. Take a subset $X \subseteq G$ in such a way that the map $\beta : x \in X \longrightarrow 2x \in 2G$ is bijective with $X$ of the form $X' \cup -X' \cup \{0, j\}$ where $2j$ is the only involution of $G$.

Choose an element $y = h + z \in X'$ with $(h, z) \in (H - 2H) \times (K - \{0\})$ and consider the following subset of $\binom{\frac{n}{2}}{2}$:

$$S = \{[x + y, -x + y] | x \in X' \setminus \{y\} \} \cup \{[y, 2y], [0, j + y]\}.$$ 

One checks that

$$\partial S = (2G - [0, 2y, -2y, 2j]) \cup \{y, -y, j + y, -j - y\}$$

and that $\phi(S) = X + y$. Thus, since $X$ is a transversal of $G/\{0, 2j\}$ by Lemma 3.1(i), we have that $\phi(S)$ is also a transversal of $G/\{0, 2j\}$.

Now, express $K - \{0, z, -z\}$ as union of two disjoint subsets $K'$ and $-K'$ and consider the set

$$T = \{[k, h - k] | k \in K' \} \cup \{[0, 2j], [h + z, -h - z]\}.$$
We have \( \partial T = (\bigcup_{k \in K'} \{ h - 2k, -h + 2k \}) \cup \{ 2y, -2y, 2f \} \) so that \( \partial S \cup \partial T \) has no repeated elements and covers all the elements of \( 2G - \{ 0 \} \).

Also, the projection of \( \phi(T) \) on \( K \) gives all of \( K \) exactly once so that \( \phi(T) \) is a transversal of \( G/H \).

Hence, also here, \( \Sigma = \{ S, T \} \) satisfies the conditions of Lemma 3.2. The assertion follows.

Remark 3.4. The number of \( G \)-invariant factors in the factorization produced by the above theorem is \( |I(G)| - 2 \) in the 1st case, \( |I(G)| - 2G| + 1 \) in the 2nd case, one in the 3rd case. In the 4th case we have no \( G \)-invariant factors.

References

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