A Homological Interpretation of Skeletal Rigidity

Tiong-Seng Tay

Department of Mathematics, University of Singapore,

and

Walter Whiteley¹

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, M3J 1P3, Canada; and Centre de Recherches Mathematiques, Université de Montréal, Montreal, Canada

Email: whiteley@clid.yorku.ca.

Received January 15, 2000; accepted March 2, 2000

CONTENTS

1. Introduction.
2. Preliminaries.
3. The rigidity chain complex.
4. Homology and skeletal statics.
5. The rigidity cochain complex.
6. Cohomology and skeletal kinematics.
7. Lower homology and cohomology.
8. Coning and projection.
9. Everything on $K_n$.
10. Gluing.
13. Conjectures, open problems, and concluding Remarks.

¹ The work of this author was supported by grants from FCAR (Québec) and NSERC (Canada).
1. INTRODUCTION

In our previous papers [29, 30], we defined a series of $r$-rigidity matrices for an $n$-dimensional simplicial complex realized in $d$-space. Our goal was to develop some matrix patterns for a simplicial $(n - 1)$-complex which would capture, in the dimensions of their spaces of row (or column) dependences, the $g$-vector whose $r$-component is

$$g_r(\Delta) = \sum_{j=0}^{r} (-1)^{jr} \binom{n+1-j}{n+1-r} f_{j-1}.$$  \hspace{1cm} (1.1)

Much of the two papers was based on various ways to mimic the techniques successfully employed to study the 2-rigidity of the 1-skeleton (infinitesimal or static rigidity of the bar framework). However, when we attempted to duplicate the “gluing theorems” of 2-rigidity, we ran into a complete block. We had great difficulty unraveling the combinatorics and geometry implicit in the “trivial motions.” Unless this was successfully handled, we would not extract the desired information about the space of row dependences, i.e., the $r$-stresses.

Drawing on experience in other areas of geometry, we realized that an alternating sum such as (1.1) suggests the Euler characteristic of some chain complex. We present such a complex. This complex and its associated homology groups contain a rich array of geometry and combinatorics for the simplicial complex.

The chain complex gives us an overview of which gluing (identification of faces) between two $r$-rigid structures will preserve $r$-rigidity, or other chosen properties. Such gluing is naturally studied using Mayer–Vietoris sequences in homology. We present such an analysis, with examples of the resulting computations.

Gluing and coning have been essential tools for the study of 2-rigidity, and they play a critical role in our presentation. With these tools, we can analyze how a shelling of a complex affects the homology and the $r$-rigidity of the final complex. Section 11 presents some basic theorems.

We also analyze, in more detail, the geometry wrapped up in the lower homologies of $r$-rigidity. The earlier case of 2-rigidity continues to influence our development of this theory. For example, the general results on 2-rigidity of $d$-manifolds in $(d + 1)$-space (and lower) guarantee that all corresponding $r$-rigidity complexes have $\beta_1 = 0$. We continue to conjecture that a series of such critical cases will completely characterize the homology (and therefore the $g$-vector) of a simplicial polytope.

As we were developing and writing up this paper, we became aware of a major coincidence with previous works of Oda [24] and Ishida [14], carried out for simplicial convex polytopes, in the context of toric varieties. In retrospect, our chain complex is essentially isomorphic to Ishida’s complex. Since
we define it in a purely combinatorial and geometrical manner, our complexes are seen to work for arbitrary simplicial complexes. Convexity and an underlying spherical topology are not relevant factors. Moreover, these complexes extend immediately to non-simplicial polyhedral complexes, even ones which have no convex realizations.

For a \( d \)-sphere realized in \( d \)-or in \((d + 1)\)-space, Oda [24] analyzes his complex using a suitable spectral sequence. From this analysis, he concludes that

(i) “appropriate” lower homologies are zero;

(ii) there are interesting isomorphisms of the cohomology and dual homology for the geometric complexes.

We translate this spectral sequence analysis to arbitrary realizations in \( d \)-or in \((d + 1)\)-space of a Cohen–Macaulay simplicial \( d \)-complex. The analysis continues to show that

(i) appropriate lower homologies are zero;

(ii) interesting isomorphisms appear for the homology and cohomology of the geometric complexes.

The correspondences which result can be viewed as extensions of the classical constructive correspondence of 2-motions and 2-stresses for triangulated spheres in 3-space (Crapo and Whiteley [7]).

The recent survey [40] by one of the present authors gives a very nice overview and summary of what has been done in the area of homology extracted from discrete applied geometry. It also explains the relevance of some of the approaches taken in this paper.

We thank Henry Crapo [6] and Lou Billera [2] for demonstrating in convincing ways that the homology of an appropriate complex is the essential tool for clarifying the geometry and combinatorics of such an geometric problem. We also thank Carl Lee and Lou Billera for ongoing conversations on the combinatorics and geometry of these patterns. We thank Bernd Sturmfels for pointing out the connections with the works of Oda and Ishida. Last but not least, we thank Neil White who initiated this project and who collaborated with us for the earlier papers [29, 30]

2. PRELIMINARY NOTATION

Let \( \Delta \) denote an \((n - 1)\)-dimensional simplicial complex on the set \( X \). We write \( \langle \sigma_1, \sigma_2, \ldots, \sigma_k \rangle \), where \( \sigma_i \subseteq X \), to denote the subcomplex \{\( \tau : \tau \subseteq \sigma_i \) for some \( i \)\}. The complex which is the \((n - 1)\)-simplex is denoted by \( K_n \). Let \( \Delta^{(r)} = \{ \sigma \in \Delta : |\sigma| = r + 1 \} \) be the \( r \)-skeleton of \( \Delta \), and let
\[ f_r(\Delta) = |\Delta^{(r)}| \]. For \( 0 \leq r \leq n \), the \( r \)-th component of the \( h \)-vector of \( \Delta \) is defined by
\[
h_r(\Delta) = \sum_{j=0}^{r} (-1)^{j+r} \binom{n-j}{n-r} f_{j-1}(\Delta),
\]
and the \( r \)-th component of the \( g \)-vector is defined by
\[
g_r(\Delta) = h_r(\Delta) - h_{r-j}(\Delta) = \sum_{j=0}^{r} (-1)^{j+r} \binom{n+r-j}{n+1-r} f_{j-1}(\Delta).
\]

Usually for the \( g \)-vector, \( r \) is restricted to \( 0 \leq r \leq \lfloor (n+1)/2 \rfloor \). But we do not impose such a restriction because \( g_r \), for larger \( r \), also has interesting properties.

We will be working within the exterior algebra (Cayley algebra of joins) for sets of points in projective \( d \)-space. (See Doubilet et al. [10] for more details.) The exterior product of \( j \) copies of the vector space \( \mathbb{R}^{d+1} \) over the field of real numbers is written as \( V_{d+1}^{(j)} \) and its members are called \( j \)-tensors. Thus \( V_{d+1}^{(0)} = \mathbb{R} \), \( V_{d+1}^{(1)} = \mathbb{R}^{d+1} \), and \( \dim V_{d+1}^{(j)} = \binom{d+1}{j} \). The exterior product of \( \tilde{a}, \tilde{b} \in V_{d+1}^{(1)} \) is written as \( \tilde{a} \wedge \tilde{b} \) or simply as \( \tilde{a} \tilde{b} \). Exterior product is anti-symmetric in the sense that if \( \tilde{P} \in V_{d+1}^{(j)} \) and \( \tilde{Q} \in V_{d+1}^{(k)} \), then \( \tilde{P} \tilde{Q} = (-1)^{j+k} \tilde{Q} \tilde{P} \). A \( j \)-tensor which is the exterior product of \( j \) \( 1 \)-tensors is known as a \( j \)-extensor. A \( d+1 \)-extensor is just a real number, being the determinant of the \((d+1) \times (d+1)\) matrix whose columns are the \( d+1 \) \( 1 \)-tensors. Each \( j \)-extensor \( \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_j \) spans a \( j \) dimensional subspace of the real projective space \( \mathbb{P}^{d+1} \) which contains the points \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_j \). If \( \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_j \) span the same subspace as the \( \tilde{a}_i \)'s, then \( \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_j \) is a multiple of \( \tilde{b}_1 \tilde{b}_2 \cdots \tilde{b}_j \). In fact \( \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_j \) gives the so called Grassman Plücker coordinates of the subspace spanned by the points.

For a given \( i \)-tensor \( \tilde{\sigma} \), we define an equivalence relation \emph{modulo the kernel of} \( \tilde{\sigma} \) on \( V_{d+1}^{(i)} \): For all \( \tilde{P}, \tilde{Q} \in V_{d+1}^{(i)} 
\]
\[ \tilde{P} \tilde{\sigma} = \tilde{Q} \] if and only if \( \tilde{P} \tilde{\sigma} = \tilde{Q} \tilde{\sigma} \).

We denote this quotient space as \( V_{d+1}^{(i)}/\ker \tilde{\sigma} \). We note that \( V_{d+1}^{(0)}/\ker \tilde{\sigma} = \mathbb{R} \), and that, if \( i + j > d + 1 \), then \( \tilde{P} \tilde{\sigma} \equiv 0 \).

We also note that, if \( \tilde{P}, \tilde{P}' \in V_{d+1}^{(j)}, \tilde{Q}, \tilde{Q}' \in V_{d+1}^{(k)} \) such that \( \tilde{P} \tilde{\sigma} = \tilde{Q} \tilde{\sigma} \) and \( \tilde{P}' \tilde{\sigma} = \tilde{Q}' \tilde{\sigma} \), then \( \tilde{P} + \tilde{Q} \equiv \tilde{P}' + \tilde{Q}' \) since
\[ (\tilde{P} + \tilde{Q}) \tilde{\sigma} = \tilde{P} \tilde{\sigma} + \tilde{Q} \tilde{\sigma} = \tilde{P}' \tilde{\sigma} + \tilde{Q}' \tilde{\sigma} = (\tilde{P}' + \tilde{Q}') \tilde{\sigma} \]
and
\[ \tilde{P} \tilde{Q} \tilde{\sigma} = \tilde{P} \tilde{Q'} \tilde{\sigma} = (-1)^k \tilde{Q} \tilde{P} \tilde{\sigma} = (-1)^k \tilde{Q}' \tilde{P} \tilde{\sigma} = \tilde{P}' \tilde{Q}' \tilde{\sigma} \].
Thus the equivalence relation is well defined within the Cayley algebra.

On occasion, we will want to give specific representatives of these equivalence classes. This is done by choosing a fixed \((d + 1 - i)\)-extensor \(A_\sigma\) complementary to \(\tilde{\sigma}\) (i.e., \(A_\sigma \tilde{\sigma} \neq 0\)) and “projecting” each tensor \(\tilde{P}\) from \(\tilde{\sigma}\) onto the space spanned by \(\tilde{A}_\sigma\):

\[
\text{Proj}_\sigma(\tilde{P}) = (\tilde{P} \tilde{\sigma}) \wedge \tilde{A}_\sigma.
\]

(For \(a_1, \ldots, a_i, \tilde{b}_1, \ldots, \tilde{b}_j \in V_d^{(1)}\),

\[
a_1 \cdots a_i \tilde{b}_1 \cdots \tilde{b}_j = \sum \text{sign}[\sigma][\tilde{a}_{\sigma(1)} \cdots \tilde{a}_{\sigma(d-j)} \tilde{b}_1 \cdots \tilde{b}_j] \tilde{a}_{\sigma(d-j+1)} \cdots \tilde{a}_{\sigma(i)},
\]

where the summation is over all permutations \(\sigma\) such that \(\sigma(1) < \cdots < \sigma(d-j)\) and \(\sigma(d-j+1) < \cdots < \sigma(i)\).) This is the approach used in Tay et al. [29, Sect. 9]. This also highlights the fact that this space \(V_{d+1}^{(j)}/\text{Ker} \tilde{\sigma}\) has dimension \(\binom{d+1-i}{j}\) as a vector space over the reals.

We also need some notation for simplices and their relationships. We often regard simplices in \(\Delta\) as square-free monomials, and we often employ notation appropriate to this context, for example, \(\sigma | \rho\) for \(\sigma \subseteq \rho\), where \(\sigma\) and \(\rho\) are simplices, and \(\rho/\sigma\) for \(\rho - \sigma\) if \(\sigma \mid \rho\). If \(\sigma \rho = \rho\) for some \(x \in X\), we write \(\rho > \sigma\) (\(\rho\) covers \(\sigma\)) and \(\sigma < \rho\) (\(\sigma\) is covered by \(\rho\)). We impose an arbitrary linear order on the set \(X\) of vertices. Placing each face in this order imposes an orientation on the faces. If \(\pi\) and \(\mu\) are disjoint subsets of \(X\), we denote by \(\text{sign}[\pi, \mu]\) the sign of the permutation required to rearrange the elements of the sequence \((\pi, \mu)\) in ascending order under the given linear order, assuming that the elements of \(\pi\) and \(\mu\) are already in ascending order. Similarly, if \(\sigma \mid \rho\), we denote \(\text{sign}[\sigma, \rho] = \text{sign}[\sigma, \rho/\sigma]\).

We want to realize \(\Delta\) in \(d\)-space. So for each \(x \in X\), let \(\tilde{x}\) denote a fixed choice of homogeneous coordinates of \(x\) in \(d\)-space, i.e., \(\tilde{x} \in V_{d+1}^{(1)}\), a 1-extensor in \(\mathbb{R}^{d+1}\). The standard way to construct such homogeneous coordinates is to take the usual Euclidean coordinates and append an additional coordinate of value 1. (In fact, any realization in projective \(d\)-space is good enough, provided we fix the homogeneous coordinates for each point.) Whenever necessary, we will write \((\Delta; p)\), where \(p\) is the function that maps \(x\) to \(\tilde{x}\). We will also use \(p\) to denote restrictions to subcomplexes of \(\Delta\).

If \(\rho\) is the monomial \(x_1 x_2 \cdots x_r\), and \(x_1 \leq x_2 \leq \cdots \leq x_r\) in the linear order on the set of vertices, then \(\rho\) denotes the \(r\)-extensor \(\tilde{x}_1 \tilde{x}_2 \cdots \tilde{x}_j\); while \(\rho/\tilde{x}_j\) denotes the \((r-1)\)-extensor \(\tilde{x}_1 \cdots \tilde{x}_{j-1} \tilde{x}_{j+1} \cdots \tilde{x}_r\). (What in fact we are doing is to choose a fixed orientation for the faces and we denote a face \(\sigma\) with this orientation \([\sigma]\).) Thus \(\rho = 0\) if \(\rho\) is not square free.

We now define the spaces which appear in our chain complex. Consider an \(n\)-dimensional simplicial complex \(\Delta\) realized in projective \(d\)-space.
SKELETAL RIGIDITY

考虑向量空间

$$M^{r+1/R} = \bigoplus_{\sigma \in \Delta^{(r)}} V^{(j)}_{d+1}/\text{Ker} \bar{\sigma},$$

其中每一元素 $P$ 写成链的形式

$$P = \sum_{\sigma \in \Delta^{(r)}} \tilde{P}_{\sigma} \cdot [\sigma],$$

其中 $\tilde{P}_{\sigma} \in V^{(j)}_{d+1}/\text{Ker} \bar{\sigma}$. 两个链 $P$ 和 $Q$ 在这个空间中等价，如果我们有 $\tilde{P}_{\sigma} \bar{\sigma} = \tilde{Q}_{\sigma}$ 对所有 $\sigma \in \Delta^{(i)}$. 作为实数上的向量空间我们有

$$\text{dim} \bigoplus_{\sigma \in \Delta^{(r)}} V^{(j)}_{d+1}/\text{Ker} \bar{\sigma} = \binom{d-i}{j} f_i(\Delta).$$

我们还回忆起约定 $\Delta^{(-1)} = \emptyset$ 且注意 $V^{(j)}_{d+1}/\text{Ker} \emptyset = V^{(j)}_{d+1}$.

3. THE SKELETAL CHAIN COMPLEX

链复形的构想是一个序列的线性空间，相互连接通过“边界映射”序列使得两个相邻的映射的复合为0.

让我们 $(\Delta; p)$ 被看作维度 $n$ 写入d-空间的单纯形复形。让 $r \leq d + 1$ 是一个整数。我们假设对于每个面 $\sigma \in \Delta^{(i)}$, $i \leq r - 1$, $\bar{\sigma} \neq 0$. The r-skeletal chain complex 是

$$\mathcal{R}_r(\Delta; p) : 0 \xrightarrow{\delta_r} \bigoplus_{\rho \in \Delta^{(r-1)}} V^{(j)}_{d+1}/\text{Ker} \tilde{\rho} \xrightarrow{\delta_{r-1}} \bigoplus_{\sigma \in \Delta^{(r-2)}} V^{(j)}_{d+1}/\text{Ker} \bar{\sigma} \xrightarrow{\delta_{r-2}} \cdots \xrightarrow{\delta_2} \bigoplus_{\sigma \in \Delta^{(r)}} V^{(j)}_{d+1}/\text{Ker} \bar{\sigma} \xrightarrow{\delta_1} 0.$$ (When working with r-skeletal chain complexes and related concepts, if $\Delta$ has faces of dimension greater than $r - 1$, we can ignore them and work with $\langle \Delta^{(r-1)} \rangle$ instead. We usually assume that $n \geq r - 1$, but this is not essential. We always assume that $\bar{\sigma} \neq 0$ if $\sigma \in \Delta^{(i)}$, $i \leq r - 1$.) We also note that if $(\Delta; p)$ and $(\Delta; q)$ are different realizations of $\Delta$ in d-space, then $\mathcal{R}_r(\Delta; p)$ need not be isomorphic to $\mathcal{R}_r(\Delta; q)$.

边界映射是这样定义的：对于一般 $i$-单纯形 $\sigma$ 与系数 $\tilde{P}_{\sigma} \in V^{(j-i+1)}_{d+1}/\text{Ker} \bar{\sigma}$,

$$\partial_i \tilde{P}_{\sigma} \cdot [\sigma] = \sum_{x \mid \sigma} \tilde{P}_{\sigma} \cdot x \cdot [\sigma/x].$$
This is extended linearly to \( i \)-chains. Finally

\[
\partial_{i-1}(\tilde{P} \cdot [\emptyset]) = 0.
\]

We must check two facts. First, the maps are well defined on the equivalence classes, and, second, \( \partial_i \partial_i = 0 \). First, if \( \sigma = \pi x \) then

\[
\tilde{P} \equiv \tilde{Q} \iff \tilde{P} \tilde{\sigma} = \tilde{Q} \tilde{\sigma} \\
\iff \tilde{P} \tilde{x} \tilde{\pi} = \tilde{Q} \tilde{x} \tilde{\pi} \\
\iff \tilde{P} \tilde{x} = \tilde{Q} \tilde{x}.
\]

Second, for any \( i \)-simplex \( \sigma \),

\[
\partial_{i-1} \partial_i \tilde{P}_x \cdot [\sigma] = \partial_{i-1} \left( \sum_{x | \sigma} \tilde{P}_x \tilde{x} \cdot [\sigma/x] \right)
= \sum_{y | \sigma, y \neq x} \tilde{P}_x \tilde{x} \tilde{y} \cdot [\sigma/xy]
= \sum_{x \neq y} \tilde{P}_x (\tilde{x} \tilde{y} + \tilde{y} \tilde{x}) \cdot [\sigma/xy]
= \sum_{x \neq y} 0 \cdot [\sigma/xy] = 0.
\]

Thus we have a well-defined chain complex.

The elements of the kernel of \( \partial_i \) are the \( i \)-cycles, \( C_i(\mathcal{R}_r(\Delta; \mathbf{p})) \), and the elements of the image of \( \partial_{i+1} \) are the \( i \)-boundaries, \( B_i(\mathcal{R}_r(\Delta; \mathbf{p})) \). As usual, we have the homology spaces of the chain complex

\[ H_i(\mathcal{R}_r(\Delta; \mathbf{p})) = \text{kernel}(\partial_i) / \text{image}(\partial_{i+1}) = C_i(\mathcal{R}_r(\Delta; \mathbf{p}))/B_i(\mathcal{R}_r(\Delta; \mathbf{p})) \]

and the corresponding Betti numbers: \( \beta_i(\mathcal{R}_r(\Delta; \mathbf{p})) = \dim H_i(\mathcal{R}_r(\Delta; \mathbf{p})) \).

For a \((d-1)\)-dimensional simplicial complex, the Euler characteristic of its chain complex can be defined in two equivalent ways:

\[
\chi(\mathcal{R}_r(\Delta; \mathbf{p}_{d-1})) = \sum_{i=0}^{r-1} (-1)^i \beta_i(\mathcal{R}_r(\Delta; \mathbf{p}_{d-1}))
= \sum_{i=0}^{r-1} (-1)^i \dim \bigoplus_{\sigma \in \Delta^{(d)}} V_d^{(r-i-1)}/\ker \tilde{\sigma}
= \sum_{i=0}^{r-1} (-1)^i (d-1-i) \, f_i(\Delta) = (-1)^{r+1} h_r(\Delta) .
\]
A similar result holds when $(\Delta; p_d)$ is realized in $d$-space:

$$
\chi(\partial, (\Delta; p_d)) = \sum_{i=1}^{r-1} (-1)^i \beta_i(\partial, (\Delta; p_d))
$$

$$
= \sum_{i=1}^{r-1} (-1)^i \dim \bigoplus_{\sigma \in \Delta^{(i)}} V^{(r-i-1)}_{d+1} / \text{Ker} \partial
$$

$$
= \sum_{i=1}^{r-1} (-1)^i \left( \frac{d-i}{r-i-1} \right) f_i(\Delta) = (-1)^{r+1} g_r(\Delta). \quad (3.2)
$$

Thus our chain complex captures the combinatorics of the $h$-vector and the $g$-vector of $\Delta$. Whenever we can show that $\beta_i(\partial, (\Delta; p_d)) = 0$ for all $i \leq r-2$, we can conclude that $g_r(\Delta) = \beta_{r-1}(\partial, (\Delta; p_d)) \geq 0$. Likewise, if $\beta_i(\partial, (\Delta; p_{d-1})) = 0$ for all $i \leq r-2$, then $h_r(\Delta) = \beta_{r-1}(\partial, (\Delta; p_{d-1})) \geq 0$.

**Example 3.1.** Let $(\Delta; p)$ be a simplicial complex realized in $d$-space. Consider the $(d+1)$-skeletal chain complex $\partial_d(\Delta; p)$:

$$
0 \to \bigoplus_{\rho \in \Delta^{(0)}} V^{(0)}_{d+1} / \text{Ker} \partial \xrightarrow{\partial} \bigoplus_{\sigma \in \Delta^{(1)}} V^{(1)}_{d+1} / \text{Ker} \partial \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigoplus_{\sigma \in \Delta^{(d)}} V^{(d)}_{d+1} / \text{Ker} \partial \xrightarrow{\partial} 0.
$$

The boundary maps are

$$
\partial : \tilde{P}_{\sigma} \cdot [\sigma] = \sum_{\chi \in \sigma} \tilde{P}_{\sigma} \tilde{x} \cdot [\sigma / \chi],
$$

where $\sigma \in \Delta^{(i)}$ and $\tilde{P}_{\sigma} \in V^{(d-i)}_{d+1}$. However, since

$$
\tilde{P}_{\sigma} \tilde{x}(\tilde{\sigma} / x) = \text{sign}[\sigma / x, x] \tilde{P}_{\sigma} \tilde{\sigma}
$$

and $\tilde{P}_{\sigma} \tilde{\sigma}$ is a scalar, it is a simple exercise to see that this is the chain complex for the usual reduced homology of the simplicial complex $(\Delta^{(d)})$.

As a corollary we have the following proposition.

**Corollary 3.2.** For any simplicial $n$-complex $(\Delta; p)$ realized in $d$-space where $n \geq d$, the $(d+1)$-skeletal chain complex $\partial_d(\Delta; p)$ is (isomorphic to) the chain complex for the simplicial reduced homology of $(\Delta^{(d)})$.

**Corollary 3.3.** Let $(K_n; p)$ be the $(n-1)$-simplex realized in $d$-space. Then

$$
\beta_i(\partial, (K_n; p)) = \begin{cases} 
(n-1)_{d+1} & i = d; \\
0 & i < d.
\end{cases}
$$
Proof. Since we are considering a \((d + 1)\)-skeletal chain complex in \(d\)-space, the skeletal homology is the reduced homology of \(\langle\langle K_n^{(d)}\rangle\rangle\). Recall (Munkres [23]) that a cone has all reduced homology 0. Since \(K_{d+1}\) is such a cone (of \(K_d\)), all its reduced homology vanishes. There are three cases. The first is the case \(n = d + 1\). Here \(\langle\langle K_n^{(d)}\rangle\rangle = K_n\). Hence all skeletal homologies vanish. For the second case we have \(n \leq d\). Restricting to the \((n - 1)\)-dimensional subspace spanned by the vertices of \(K_n\), the \(n\)-skeletal homology is still the reduced homology. Hence all the homologies are 0. Nothing is changed when we move over to \(d\)-space. Thus all the homologies are still 0. For the final case, \(n \geq d + 2\). The reduced homologies of \(\langle\langle K_n^{(d)}\rangle\rangle\) all vanish except at the top. Therefore \(\beta_i(\mathcal{R}_{d+1}(K_n; p)) = 0\) if \(i \leq d\). We can use the Euler characteristic to calculate \(\beta_d(\mathcal{R}_{d+1}(K_n; p))\):

\[
(-1)^d \beta_d(\mathcal{R}_{d+1}(K_n; p)) = \sum_{i=1}^{d} (-1)^i \beta_i(\mathcal{R}_{d+1}(K_n; p)) = \chi(\mathcal{R}_{d+1}(K_n; p)) = \sum_{i=1}^{d} (-1)^i f_i(K_n) = \sum_{i=1}^{d} (-1)^i \binom{n}{i+1} = (-1)^d \binom{n-1}{d+1}.
\]

This completes the proof.

4. HOMOLOGY AND SKELETAL STATICS

Consider a simplicial complex \((\Delta; p)\) realized in \(d\)-space. We will now see that the homologies \(H_i(\mathcal{R}_r(\Delta; p))\) describe the stresses and loads of our \(r\)-rigidity matrices of \((\Delta; p)\). The equivalence of the two definitions of the Euler characteristic then points to the basic connection which we are seeking.

We recall some definitions of “skeletal statics” from Tay et al. [29, 30]. We translate the definitions (stresses, loads, etc.) from the presentation implicit in the minimal matrix and the truncated matrix.

The \(r\)-rigidity matrix \(\mathcal{R}_r(\Delta; p)\), \(r \leq d + 1\), has its rows indexed by \(\Delta^{(r-1)}\) and its columns indexed by \(\Delta^{(r-2)}\). The \((\rho, \sigma)\) entry is

\[
\tilde{x} \in V_{d+1}^{(1)}/\text{Ker} \tilde{\sigma} \quad \text{if} \ \rho = \sigma x
\]

\[
0 \quad \text{if} \ \sigma \not\parallel \rho.
\]

Working with \(\tilde{x}\), we can assume that all the entries in the row of \(\rho\) are the same 1-extensor \(\tilde{m}_\rho = \sum_{a_i \in \rho} \tilde{a}_i\) since \(\tilde{\sigma} \tilde{m}_\rho = \tilde{\sigma} \tilde{x}\) if \(\sigma x = \rho\). The 1-extensor

\[
\tilde{m}_\rho = \sum_{a_i \in \rho} \tilde{a}_i
\]

is the 1-extensor corresponding to the \(\rho\)-load vector \(\sigma \cdot \tilde{m}_\rho\), where \(\sigma \cdot \tilde{m}_\rho = \sigma \cdot \tilde{x}\). The 1-extensor

\[
\tilde{m}_\rho = \sum_{a_i \in \rho} \tilde{a}_i
\]

is the 1-extensor corresponding to the \(\rho\)-load vector \(\sigma \cdot \tilde{m}_\rho\), where \(\sigma \cdot \tilde{m}_\rho = \sigma \cdot \tilde{x}\).
\( \bar{m}_\rho \) can be considered the weighted center of mass of the face \( \rho \). Each row of this matrix can be considered a member of \( \bigoplus_{\sigma \in \Delta^{(r-2)}} V_{d+1}^{(1)}/\text{Ker} \bar{\sigma} \). The subspace generated by the rows is written \( \text{Row}_r(\bar{\Delta}; \bar{p}) \). Since

\[
\partial_{r-1}(1 \cdot [\rho]) = \sum_{x \in \rho} \bar{x} \cdot [\rho/x],
\]

and the chains \( 1 \cdot [\rho], \rho \in \Delta^{(r-1)} \), generate the space of \( (r-1) \)-chains, we see that \( \text{Row}_r(\Delta) = \text{Bd}_{r-2}(\Delta) \).

An \( r \)-stress is a row dependence of \( \mathbf{R}_r(\Delta; \mathbf{p}) \), that is, an assignment \( \lambda \) of scalars \( \lambda_\rho \) to the elements of \( \rho \in \Delta^{(r-1)} \) such that, for each \( \sigma \in \Delta^{(r-2)} \),

\[
\sum_{\rho \supset \sigma} \lambda_\rho \bar{x}_\rho \overset{\lambda}{=} \mathbf{0}.
\]

Since the \( \sigma \) component of \( \partial_{r-1}(\sum_{\rho \in \Delta^{(r-1)}} \lambda_\rho \cdot [\rho]) \) is \( \sum_{\rho \supset \sigma} \lambda_\rho \bar{x}_\rho = \mathbf{0} \), it is clear that

\[
\text{Stress}_r(\Delta; \mathbf{p}) = \text{C}_{r-1}(\mathcal{E}_r(\Delta; \mathbf{p})).
\]

Since we are at the top of our chain complex and the only \( r \)-boundary is 0, we also have

\[
\text{Stress}_r(\Delta; \mathbf{p}) = \text{H}_{r-1}(\mathcal{E}_r(\Delta; \mathbf{p})).
\]

A simplicial complex is said to be \( r \)-independent if there is only the trivial \( r \)-stress, i.e., if \( \partial_{r-1}(\mathcal{E}_r(\Delta; \mathbf{p})) = \mathbf{0} \).

An \( r \)-load is an element \( L = \sum_{\sigma \in \Delta^{(r-2)}} \bar{L}_\sigma \cdot [\sigma] \) of \( \bigoplus_{\sigma \in \Delta^{(r-2)}} V_{d+1}^{(1)}/\text{Ker} \bar{\sigma} \) such that, for each \( \pi \in \Delta^{(r-3)} \),

\[
\sum_{\sigma \supset \pi} \bar{L}_\sigma \bar{x}_\pi \overset{\pi}{=} \mathbf{0}.
\]

The entire space of \( r \)-loads is written \( \text{Load}_r(\Delta; \mathbf{p}) \). By the definition, we have \( \text{Row}_r(\Delta; \mathbf{p}) \subseteq \text{Load}_r(\Delta; \mathbf{p}) \). Also we have \( \text{Load}_r(\Delta; \mathbf{p}) = \text{C}_{r-2}(\mathcal{E}_r(\Delta; \mathbf{p})) \). This is seen as follows. For each load \( L \) and each \( \pi \in \Delta^{(r-3)} \), the \( \pi \) component of \( \partial_{r-2}(\sum_{\sigma \supset \pi} \bar{L}_\sigma \cdot [\sigma]) \) is

\[
\sum_{\sigma \supset \pi} \bar{L}_\sigma \bar{x}_\pi \overset{\pi}{=} \mathbf{0}.
\]

Thus \( L \) is an \( (r-2) \)-cycle. Conversely, for each \( (r-2) \)-cycle \( \mathbf{P} \) and each \( \pi \in \Delta^{(r-3)} \), the \( \pi \) component of \( \partial_{r-2}(\sum_{\sigma \supset \pi} \bar{P}_\sigma \cdot [\sigma]) \) is

\[
\sum_{\sigma \supset \pi} \bar{P}_\sigma \bar{x}_\pi \overset{\pi}{=} \mathbf{0}.
\]

Thus \( \mathbf{P} \) is a load.
Finally it is also immediate that Row, (Δ) ⊆ Load, (Δ). This leads to the following definition of a space which was defined implicitly in our preceding papers [29, 30] (by duality with trivial motions). The space of unresolved loads is the space UnRes, (Δ; p) = Load, (Δ; p)/Row, (Δ; p). A simplicial complex is called r-rigid if, and only if, UnRes, (Δ; p) = 0 and the row space generates the r-loads.

Given our previous translations, we have the following translation between statics and homology.

**Theorem 4.1.** For any simplicial complex (Δ; p) realized in d-space, d ≥ r − 1, we have

(i) Stress, (Δ; p) = Cy,−1 (R, (Δ; p)) = H,−1 (R, (Δ; p));
(ii) Load, (Δ; p) = Cy,−2 (R, (Δ; p));
(iii) Row, (Δ; p) = Bd,−2 (R, (Δ; p)).
(iv) UnRes, (Δ; p) = H,−2 (R, (Δ; p)).

**Corollary 4.2.** For any pure simplicial (d − 1)-complex (Δ; p) realized in d-space, d ≥ r − 1, g, (Δ) = dim Stress, (Δ; p) if, and only if,

\[ \sum_{i=1}^{r-2} (-1)^{r-i} \beta_i (R, (Δ; p)) = 0. \]

**Proof.** We know that \( \beta_{r-1}(R, (Δ; p)) = \dim \text{Stress}, (Δ; p) \). The result then follows from

\[ \dim \text{Stress}, (Δ; p) + (-1)^{r+1} \sum_{i=1}^{r-2} (-1)^i \beta_i (R, (Δ; p)) = (-1)^{r+1} \chi (R, (Δ; p)) = g, (Δ). \]

In particular, for a (d − 1)-complex (Δ; p) realized in d-space we will have the desired measure of \( g, (Δ) \) as the (non-negative) dimension of a vector space if \( \beta_i (R, (Δ; p)) = 0 \) for all \( i ≤ r − 2 \). In Section 11 we will show that any shellable (d − 1)-complex, in general position in d-space or d − 1-space, will have \( \beta_i (R, (Δ; p)) = 0 \) for all \( i ≤ r − 3 \). More generally, we will show in Section 12 that the same holds for Cohen–Macaulay \( d \)-complexes. This will yield the following corollary.

**Corollary 4.3.** Let \( \Delta \) be a Cohen–Macaulay \( d \)-complex realized in n-space, \( n = d \) or \( d + 1 \), such that the \( \sigma \neq 0 \) for every face \( \sigma ∈ \Delta \).

1. If \( n = d \) we have

\[ \dim \text{Stress}, (Δ; p) = h, (Δ). \]
2. If \( n = d + 1 \) we have
\[
\dim \text{Stress}_r(\Delta; p) - \dim \text{UnRes}_r(\Delta; p) = g_r(\Delta; p).
\]

We also have the following conjecture, which, if true, would extend the \( g \)-theorem to homology spheres.

**Conjecture 4.4.** Let \( (\Delta; p) \) be a homology \( d \)-sphere realized in \( (d + 1) \)-space, such that the \( \tilde{\sigma} \neq 0 \) for every face \( \sigma \in \Delta \). Then for \( 0 \leq r \leq [(d + 2)/2] \), we have
\[
\dim \text{Stress}_r(\Delta; p) = g_r(\Delta; p),
\]
and for \( [(d + 2)/2] < r \leq d + 2 \), we have
\[
\dim \text{UnRes}_r(\Delta; p) = -g_r(\Delta; p).
\]

**Example 4.5.** Consider the 2-skeletal chain complex of a simplicial complex \( (\Delta; p) \) realized in \( d \)-space.
\[
\mathcal{R}_2(\Delta; p) : 0 \to \bigoplus_{\pi \in \Delta^{(1)}} V_{d+1}^{(0)} / \text{Ker} \tilde{\pi} \rightarrow \bigoplus_{a_i \in \Delta^{(0)}} V_{d+1}^{(1)} / \text{Ker} \tilde{a}_i \rightarrow V_{d+1}^{(2)} \to 0.
\]
This corresponds to the standard static or infinitesimal rigidity of a bar framework on the 1-skeleton. Therefore
\[
\chi(\mathcal{R}_2(\Delta; p)) = |f_1(\Delta)| - d|f_0(\Delta)| + \binom{d + 1}{2}.
\]
If the points span at least a hyperplane of the space, we will see that \( \beta_{-1}(\mathcal{R}_2(\Delta; p)) = 0 \) (in Section 7). By Theorem 4.1, \( \text{UnRes}_2(\Delta; p) = H_0(\mathcal{R}_r(\Delta; p)) \). Thus \( \beta_0 = 0 \) if the 1-skeleton is 2-rigid. So
\[
0 \leq \beta_1(\mathcal{R}_2(\Delta; p)) = \dim \text{Stress}_1(\Delta; p) = |f_1(\Delta)| - d|f_0(\Delta)| + \binom{d + 1}{2}.
\]
In particular, standard results on the rigidity of bar frameworks \([15, 35]\) show that for a generic realization of a simplicial \( d \)-sphere (or even a simplicial minimal homology cycle) in \( (d + 1) \)-space, the 1-skeleton is 2-rigid. Therefore we have the standard lower bound for simplicial spheres (or minimal homology cycles in \( d \)-space),
\[
|f_1(\Delta)| \geq d|f_0(\Delta)| - \binom{d + 1}{2}.
\]

In our preceding paper \([29]\), we studied four other variants of the \( r \)-rigidity matrix. In the end, we showed that, for all of these, the spaces of \( r \)-stresses and what we have called here the \( r \)-unresolved loads were isomorphic. We could elaborate a distinct chain complex for each variant, but we would end up with isomorphic homologies.
5. THE RIGIDITY COCHAIN COMPLEX

Let \((\Delta; p)\) be a simplicial complex realized in (projective) \(d\)-space. The \(r\)-skeletal cochain complex, \(r \leq d + 1\), is

\[
\mathcal{R}'(\Delta; p) : 0 \leftarrow \bigoplus_{\rho \in (\Delta^{(r-1)}_d)} V_{d+1}^{(d+1-r)}/\text{Ker} \, \tilde{\rho} \bigoplus_{\sigma \in (\Delta^{(r-2)}_d)} V_{d+1}^{(d+1-r)}/\text{Ker} \, \tilde{\sigma} \mapsto 0.
\]

\[
\ldots \leftarrow \bigoplus_{a_i \in (\Delta^{(0)}_d)} V_{d+1}^{(d+1-r)}/\text{Ker} \, \tilde{a}_i \leftarrow V_{d+1}^{(d+1-r)} \leftarrow 0.
\]

For a general \(i\)-simplex \(\sigma\) with coefficient \(\tilde{P}_\sigma \in V_{d+1}^{(d+1-r)}/\text{Ker} \, \tilde{\sigma}\),

\[
\delta_i(\tilde{P}_\sigma \cdot [\sigma]) = \sum_{\rho \triangleright \sigma} \text{sign}[\sigma, \rho] \tilde{P}_\rho \cdot [\rho].
\]

This is extended linearly to \(i\)-cochains.

We must check two facts. First, the maps are well defined on the equivalence classes, and, second, \(\delta_i \delta_{i-1} = 0\). First, if \(\rho = \sigma \rho\) then

\[
\tilde{P} \tilde{\sigma} = \tilde{P} \tilde{\sigma} = \tilde{P} \tilde{\sigma} = \tilde{P} \tilde{\sigma} \Rightarrow \tilde{P} \tilde{\sigma} = \tilde{P} \tilde{\sigma}.
\]

Second, for any \(i - 1\)-simplex \(\pi \mid \rho\), where \(\rho\) is an \((i + 1)\)-simplex, there will be two \(i\) simplices in the interval \(\pi \prec \sigma', \sigma'' \prec \rho\). Then

\[
\delta_i \delta_{i-1} \tilde{P}_\pi \cdot [\pi] = \delta_i \left( \sum_{\sigma \triangleright \pi} \text{sign}[\pi, \sigma] \tilde{P}_\pi \cdot [\sigma] \right)
\]

\[
= \sum_{\rho \triangleright \pi} \text{sign}[\sigma, \rho] \left( \sum_{\sigma \triangleright \pi} \text{sign}[\pi, \sigma] \tilde{P}_\pi \cdot [\rho] \right)
\]

\[
= \sum_{\rho \triangleright \pi} \left( \text{sign}[\sigma', \rho] \text{sign}[\pi, \sigma'] \right) \tilde{P}_\rho \cdot [\rho] = 0.
\]

Thus we have a well-defined cochain complex.

The elements of the kernel of \(\delta_{i+1}\) are the \(i\)-cocycles, \(\text{CoC}y^i(\mathcal{R}'(\Delta; p))\), and the elements of the image of \(\delta_i\) are the \(i\)-coboundaries, \(\text{CoB}d^i(\mathcal{R}'(\Delta; p))\).

We have the usual cohomology spaces of the cochain complex,

\[
H^i(\mathcal{R}'(\Delta; p)) = \text{CoC}y^i(\mathcal{R}'(\Delta; p))/\text{CoB}d^i(\mathcal{R}'(\Delta; p)).
\]

and the corresponding Betti numbers, \(\beta^i(\mathcal{R}'(\Delta; p)) = \dim H^i(\mathcal{R}'(\Delta; p))\).

**Example 5.1.** Consider, again, \((d + 1)\)-skeletal cochain complex of \((\Delta; p)\) realized in \(d\)-space. Since \(V_{d+1}^{(0)}/\text{Ker} \, \tilde{\sigma} \cong \mathbb{R}\), the cochain complex is

\[
\mathcal{R}'(\Delta; p) : 0 \leftarrow \bigoplus_{\rho \in (\Delta^{(r-1)}_d)} \mathbb{R} \bigoplus_{\sigma \in (\Delta^{(r-2)}_d)} \mathbb{R} \mapsto \mathbb{R} \leftarrow \mathbb{R} \leftarrow 0,
\]

where \(r = d + 1\). This is the usual (reduced) cohomology of \(\Delta\).
Remark 5.2. Let $\Delta$ be a simplicial complex based on $X$. The Stanley-Reisner ring or the face ring of $\Delta$ over $\mathbb{R}$ is $A = \mathbb{R}[X]/I_{\Delta}$, where $I_{\Delta}$ is the ideal generated by all square free monomials which are not members of $\Delta$. We grade $A$ in a natural way by degree, $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$. The $r$-skeletal cochain complex is naturally related to the face ring as follows. Realize $\Delta$ in $d$-space. Consider the top cohomology:

$$0 \xleftarrow{\delta_0} \bigoplus_{\rho \in \Delta^{(r-1)}} V^{(d+1-r)}_{d+1}/\ker \hat{\rho} \xrightarrow{\delta_{r-1}} \bigoplus_{\sigma \in \Delta^{(r-2)}} V^{(d+1-r)}_{d+1}/\ker \tilde{\sigma} \xrightarrow{\delta_{r-2}} \cdots.$$ 

Since $\hat{\rho}$ is an $r$-extensor, we have that $P \in V^{(d+1-r)}_{d+1}$ implies $P\hat{\rho} \in V^{(d+1)}_{d+1} \cong \mathbb{R}$. Therefore,

$$\text{CoCy} \left( \mathcal{R}'(\Delta) \right) = \bigoplus_{\rho \in \Delta^{(r-1)}} V^{(d+1-r)}_{d+1}/\ker \hat{\rho} \cong A_r/\langle \Box \rangle,$$

where $A_r/\langle \Box \rangle$ is the space of polynomials generated by monomials in $\Delta^{(r-1)}$, or, equivalently, the square-free part of the graded component $A_r$ of the face ring. The isomorphism is given by

$$f : \bigoplus_{\rho \in \Delta^{(r-1)}} V^{(d+1-r)}_{d+1}/\ker \hat{\rho} \to A_r/\langle \Box \rangle,$$

where $f\left( \sum_{\rho \in \Delta^{(r-1)}} \tilde{\rho} \cdot \rho \right) = \sum_{\rho \in \Delta^{(r-1)}} (\tilde{\rho} \hat{\rho}) \rho$.

Now consider the coboundary of a single face $\sigma \in \Delta^{(r-2)}$:

$$\delta(\tilde{P}_\sigma \cdot \sigma) = \sum_{x : ax \gg \sigma} \tilde{P}_\sigma \cdot \lfloor ax \rfloor. \quad (5.1)$$

Every tensor in $V^{(d)}_{d+1}$ can be represented by a vector in $\mathbb{R}^{d+1}$ as follows. Let $\tilde{A}$ be the $d$-extensor $\tilde{A} = \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_d$. Form a $(d + 1) \times d$ matrix $B$ whose $i$th column is the column vector $\tilde{a}_i$. Then $A_i$, the $i$th component of the vector representing $\tilde{A}$, is $A_i = (-1)^{(i+1)} B_i$, where $B_i$ is the determinant of the matrix obtained from $B$ by deleting the $i$th row. This is then extended linearly to all $d$-tensors. As we remarked earlier, a $(d+1)$-extensor $\tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_{d+1}$ is the determinant of the matrix whose columns are the column vectors $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{d+1}$. Thus for each $x, \sigma$, we have

$$\tilde{P}_\sigma \cdot \tilde{x} = \sum_{i=1}^{d} (\tilde{P}_\sigma \cdot \tilde{a}_i) \tilde{x}_i,$$

where $(\tilde{P}_\sigma \cdot \tilde{a}_i)$ and $\tilde{x}_i$ are the appropriate (numerical) components of the vectors $\tilde{P}_\sigma \cdot \tilde{a}$ and $\tilde{x}$, respectively. Rewriting (5.1) in light of the preceding comments, we have

$$f \left( \sum_{x : ax \gg \sigma} \tilde{P}_\sigma \cdot \lfloor ax \rfloor \right) = \sum_{x : ax \gg \sigma} \left( \sum_{i=1}^{d} (\tilde{P}_\sigma \cdot \tilde{a}_i) \tilde{x}_i \right) ax = \sum_{i=1}^{d} (\tilde{P}_\sigma \cdot \tilde{a}_i) (\sum_{x : ax \gg \sigma} \tilde{x}_i x) \sigma.$$
Now we write

$$\theta_t = \sum_{x \in \Delta(0)} \tilde{x}_t x.$$  

The boundary is the square-free part the ideal $I_\theta = \langle \{\theta_1, \theta_2, \ldots, \theta_{d+1}\} \rangle$ in $A_r$ of the face ring.

We conclude that the cohomology

$$H^{r-1}(\mathcal{R}(\Delta; p)) = (A_r/(\square)) / I_{\theta}/(\square)) \cong A_r/I_{\theta}.$$  

Because we are working with a field for our coefficients, we have a simple isomorphism between our homology and our cohomology. Given $\sigma \in \Delta^{(r)}$, consider the quotient space $V^{(r-i-1)}/\text{Ker} \tilde{\sigma}$. We have a dual space $V^{(d+1-r)}/\text{Ker} \tilde{\sigma}$ as described below.

If $\sigma = x_1 x_2 \cdots x_{i+1}$ and $E_\sigma = \{\tilde{e}_1, \ldots, \tilde{e}_{d-i}, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{i+1}\}$ is a basis of $V$, then

$$E^{(k)}_\sigma = \{\tilde{e}_{i_1} \tilde{e}_{i_2} \cdots \tilde{e}_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq d - i\}$$

is a basis of $V^{(k)}/\text{Ker} \tilde{\sigma}$. There is a duality map or vector space isomorphism

$$\eta_\sigma: V^{(r-i-1)}/\text{Ker} \tilde{\sigma} \rightarrow V^{(d+1-r)}/\text{Ker} \tilde{\sigma}$$

which reverses step, given explicitly by

$$\tilde{e}_{i_1} \tilde{e}_{i_2} \cdots \tilde{e}_{i_{r-1}} \mapsto \text{sign}\{\ell_1 \ell_2 \cdots \ell_{r-i-1}, j_1 j_2 \cdots j_{d-r}\} \tilde{e}_{j_1} \tilde{e}_{j_2} \cdots \tilde{e}_{j_{d-r}},$$

where $\{j_1, j_2, \ldots, j_{d-r}\} = \{1, 2, \ldots, d - i\} - \{i_1, i_2, \ldots, i_{r-1}\}$ and $1 \leq j_1 < j_2 < \cdots < j_{d-r} \leq d - i$. We will simply denote $\eta_\sigma(\tilde{w})$ by $\tilde{w}^*$ for $\tilde{w} \in V^{(k)}/\text{Ker} \tilde{\sigma}$. Note that if $\tilde{P} = \tilde{Q}$ then $\tilde{P}^* = \tilde{Q}^*$, so duality is well defined with our equivalence classes. Note also that the duality map depends on $\tilde{\sigma}$. The above works because we have defined inner product (and duality) in terms of a basis of the quotient space $V^{(d-r)}/\text{Ker} \tilde{\sigma}$.

If $E_\sigma$ is chosen so that $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_{d-r} \tilde{\sigma} = 1$, then $V^{(k)}/\text{Ker} \tilde{\sigma}$ is a real inner product space under the inner product $\langle \tilde{P}, \tilde{Q} \rangle = \tilde{P}^* \tilde{Q} \tilde{\sigma}$, with orthonormal basis $E^{(k)}$. Therefore $(\tilde{P}^*)^* = \tilde{P}$. We note that the inner product depends on the choice of $E_\sigma$, which we assume is fixed for each $\sigma$ for the remainder of this paper.

The above duality can be extended naturally to a duality map

$$\eta: \bigoplus_{\sigma \in \Delta^{(d)}} V^{(r-i-1)}/\text{Ker} \tilde{\sigma} \rightarrow \bigoplus_{\sigma \in \Delta^{(d)}} V^{(d+1-r)}/\text{Ker} \tilde{\sigma}$$
and an inner product on $\bigoplus_{\sigma \in \Delta^{(i)}} V_{d+1}^{(r-i-1)}/\text{Ker} \tilde{\sigma}$. This is done as follows. For any $P, Q \in \bigoplus_{\sigma \in \Delta^{(i)}} V_{d+1}^{(r-i-1)}/\text{Ker} \tilde{\sigma}$, define $\eta(P) = P^* \in \bigoplus_{\sigma \in \Delta^{(i)}} V_{d+1}^{(d+1-r)}/\text{Ker} \tilde{\sigma}$ by

$$P^* = \sum_{\sigma \in \Delta^{(i)}} \tilde{P}_\sigma \cdot [\sigma];$$

and

$$\langle P, Q \rangle = \sum_{\sigma \in \Delta^{(i)}} \langle \tilde{P}_\sigma, \tilde{Q}_\sigma \rangle = \sum_{\sigma \in \Delta^{(i)}} \tilde{P}_\sigma \tilde{Q}_\sigma \tilde{\sigma}.$$

For any $A \in \bigoplus_{\sigma \in \Delta^{(i)}} V^{(r-i-1)}/\text{Ker} \tilde{\sigma}$ and $B \in \bigoplus_{\sigma \in \Delta^{(i)}} V^{(d+1-r)}/\text{Ker} \tilde{\sigma}$, define

$$AB = \sum_{\sigma \in \Delta^{(i)}} \widetilde{A}_\sigma \widetilde{B}_\sigma \tilde{\sigma}.$$

Then $\langle A^*, B \rangle = (A^*)^* B = AB$.

We also say that $P$ is orthogonal to $Q$ if $\langle P, Q \rangle = 0$. Using this duality $^*$ and the orthogonality as defined, we have some simple vector space isomorphisms.

**Theorem 5.3.** Let $(\Delta; p)$ be realized in $d$-space. Then for each $r$ and each $i$, we have:

1. $(\text{Cy}_i(\mathcal{R}_r(\Delta; p)))^* = (\text{CoBd}_i(\mathcal{R}_r^1(\Delta; p)))^{1^*}$
2. $(\text{Bd}_i(\mathcal{R}_r(\Delta; p)))^* = (\text{CoCy}_i(\mathcal{R}_r^1(\Delta; p)))^{1^*}$
3. $H_i(\mathcal{R}_r(\Delta; p)) \cong H_i^1(\mathcal{R}_r(\Delta; p))$.

**Proof.** For any cycle $C \in \text{Cy}_i(\Delta)$, we have

$$\partial C = \partial \sum_{\sigma} \tilde{C}_\sigma \cdot [\sigma]$$

$$= \sum_{\sigma} \sum_{\pi: \pi \equiv \sigma} \tilde{C}_\sigma \tilde{x} \cdot [\pi]$$

$$= \sum_{\pi} \sum_{\sigma: \pi \equiv \sigma} \tilde{C}_\sigma \tilde{x} \cdot [\pi]$$

$$\overset{\Delta^{i-1}}{=} 0.$$ 

Or, equivalently, for all $\pi$, $\sum_{\sigma \equiv \pi} \tilde{C}_\sigma \tilde{x} \tilde{\pi} = \sum_{\sigma \equiv \pi} \text{sign}[\pi, \sigma] \tilde{C}_\sigma \tilde{\sigma} = 0$.

For any $(i-1)$-cochain $P$, we have

$$\delta P = \sum_{\pi} \delta \tilde{P}_\pi \cdot [\pi]$$

$$= \sum_{\pi} \sum_{\sigma \equiv \pi} \text{sign}[\pi, \sigma] \tilde{P}_\pi \cdot [\sigma]$$

$$= \sum_{\sigma} \sum_{\pi \equiv \sigma} \text{sign}[\pi, \sigma] \tilde{P}_\pi \cdot [\sigma].$$
Computing the inner product, we have
\[
\langle C^*, \delta P \rangle = \sum_{\sigma} \widetilde{C}_\sigma \left( \sum_{\pi < \sigma} \text{sign}[\pi, \sigma] \tilde{P}_\pi \right) \tilde{\sigma}
= \sum_{\pi} \sum_{\sigma > \pi} \text{sign}[\pi, \sigma] \widetilde{C}_\sigma \tilde{P}_\pi \tilde{\sigma} = 0.
\]
This proves (i). Part (ii) can be proved in a similar way, and part (iii) follows from (i) and (ii).

6. COHOMOLOGY AND SKELETAL KINEMATICS

Consider the \(r\)-skeletal rigidity matrix of a simplicial complex \((\Delta; \mathbf{p})\) realized in \(d\)-space. An \(r\)-motion \(M\) is a chain in \(\bigoplus_{\sigma \in \Delta^{(r-1)}} V^{(d+1-r)}/\text{Ker} \tilde{\sigma}\) such that for every \(\rho \in \Delta^{(r-1)}\), the row corresponding to \(\rho\), \(R \in \text{Row}_{r}(\Delta)\) satisfies
\[
\langle M^*, R \rangle = MR = \sum_{\sigma \prec \rho} \text{sign}[\sigma, \rho] \tilde{M}_{\sigma} \tilde{\rho} = 0.
\]
The space of \(r\)-motions is denoted as \(\text{Motion}_{r}(\Delta)\). Clearly \((\text{Motion}_{r}(\Delta; \mathbf{p})\)\)\(^{-1}\) = \((\text{Row}_{r}(\Delta; \mathbf{p}))^*\).

**Proposition 6.1.** We have \(\text{Motion}_{r}(\Delta; \mathbf{p}) = \text{CoCyc}^{-1}(\tilde{\mathcal{H}}(\Delta; \mathbf{p}))\).

**Proof.** Consider an \((r-2)\)-cochain \(M\). We have
\[
\delta_{r-1} \sum_{\sigma} \tilde{M}_{\sigma} \cdot [\sigma] = \sum_{\sigma} \sum_{\rho \prec \sigma} \text{sign}[\sigma, \rho] \tilde{M}_{\sigma} \cdot [\rho]
= \sum_{\rho} \left( \sum_{\sigma \prec \rho} \text{sign}[\sigma, \rho] \tilde{M}_{\sigma} \right) \cdot [\rho]
\]
if and only if \(M\) is an \(r\)-motion. This completes the proof.

The space of **trivial** \(r\)-motions \(\text{Triv}_{r}(\Delta; \mathbf{p})\) is generated by the following. For every \(\pi \in \Delta^{(r-3)}\) and every tensor \(\tilde{S} \in V_{d+1}^{(d-r+1)}\), \(T_{\pi, \tilde{S}}\) is defined by
\[
T_{\pi, \tilde{S}} = \sum_{\sigma \succ \pi} \text{sign}[\pi, \sigma] \tilde{S} \cdot [\sigma].
\]
Note that the \(\sigma\) component of \(T_{\pi, \tilde{S}}\) is 0 if \(\pi \nmid \sigma\).

For the \((r-3)\)-cochain \(\tilde{S} \cdot [\pi]\),
\[
\delta_{r-2}(\tilde{S} \cdot [\pi]) = \sum_{\sigma \succ \pi} \text{sign}[\pi, \sigma] \tilde{S} \cdot [\sigma] = T_{\pi, \tilde{S}}.
\]
Thus $T_{x,S}$ is an $(r - 2)$-coboundary. It follows that $\text{Triv}_r(\Delta; p) = \text{CoBd}^{r-2}(\mathcal{R}(\Delta; p))$. The space of nontrivial motions $\text{NonTriv}_r(\Delta; p)$ is $\text{Motion}_r(\Delta; p)/\text{Triv}_r(\Delta; p)$. Therefore,

$$\text{NonTriv}_r(\Delta; p) = \text{Motion}_r(\Delta; p)/\text{Triv}_r(\Delta; p) = \text{CoCy}_{r-2}(\Delta; p)/\text{CoBd}_{r-2}(\Delta; p) = H^{r-2}(\mathcal{K},(\Delta; p)).$$

Summarizing the above discussion, we have the following theorem.

**Theorem 6.2.** For any simplicial complex $(\Delta; p)$, realized in $d$-space, we have:

(i) $\text{Triv}_r(\Delta; p) \cong \text{CoBd}^{d-r}(\mathcal{R}(\Delta; p))$;

(ii) $\text{Motion}_r(\Delta; p) \cong \text{CoCy}^{d-r}(\mathcal{R}(\Delta; p))$;

(iii) $\text{NonTriv}_r(\Delta; p) \cong H^{d-r}(\mathcal{K},(\Delta; p))$.

**Corollary 6.3.** For any pure simplicial $n$-complex $(\Delta; p)$, realized in $d$-space, $d \geq r - 1$, we have:

(i) $\dim \text{Triv}_r(\Delta; p) = \sum_{i=1}^{r-3} (-1)^{r+1-i} (d-i) (-1)^{r-i} \beta_i(\mathcal{R}(\Delta; p))$;

(ii) if $n = d - 1$, then $g_r(\Delta) = \dim \text{Stress}_r(\Delta; p) - \dim \text{NonTriv}_r(\Delta; p)$ and

$$\text{Triv}_r(\Delta; p) = \sum_{i=1}^{r-3} (-1)^{r+1-i} \left(\frac{d-i}{r-i-1}\right) \beta_i(\mathcal{R}(\Delta; p))$$

if and only if $\sum_{i=1}^{r-3} (-1)^{r-i} \beta_i(\mathcal{R}(\Delta; p)) = 0$.

**Proof.** From the identity (3.2), we have

$$\sum_{i=1}^{r-1} (-1)^i \beta_i(\mathcal{R}(\Delta; p)) = \sum_{i=1}^{r-1} (-1)^i \left(\frac{d-i}{r-i-1}\right) \beta_i(\Delta).$$

Hence the right hand side of (i) is

$$\beta_{r-1}(\mathcal{R}(\Delta; p)) - \beta_{r-2}(\mathcal{R}(\Delta; p)) - f_{r-1}(\Delta) + (d - r + 2)f_{r-2}(\Delta).$$

But $\beta_{r-1}(\mathcal{R}(\Delta; p)) = \dim \text{Stress}_r(\Delta; p)$, thus the quantity $f_{r-1}(\Delta) - f_{r-2}(\Delta)$ gives the row rank of the $r$-rigidity matrix $R_r(\Delta; p)$. The quantity $(d - r + 2)f_{r-2}(\Delta)$ gives the number of columns in $R_r(\Delta; p)$ since the entries in the columns can be considered as vectors in $\mathbb{R}^{d-r+2}$. Since $(\text{Motion}_r(\Delta; p))^\perp = (\text{Row}_r(\Delta; p))^\perp$,

$$\dim \text{Motion}_r(\Delta; p) = \beta_{r-1}(\mathcal{R}(\Delta; p)) - f_{r-1}(\Delta) + (d - r + 2)f_{r-2}(\Delta).$$

However, $\text{NonTriv}_r(\Delta; p) = \beta_{r-2}(\mathcal{R}(\Delta; p))$. So the desired identity in (i) follows. Part (ii) follows from (3.2), Theorem 6.2, and Proposition 8.3 of Tay et al. [29].
7. LOWER HOMOLOGY AND COHOMOLOGY

A major intermediate goal is to prove that $\beta_i(\partial_r(\Delta; p)) = 0$ for $i \leq r - 3$, for important classes of simplicial complexes. The next seven sections are contributions in this direction. We begin with some results for the two lowest homologies.

**Proposition 7.1.** Let $(\Delta; p)$ be a simplicial complex realized in $d$-space. If the set of vectors $\{ \bar{x} : x \in \Delta^{(0)} \}$ is of rank $m$, then

$$\beta_0(\partial_r(\Delta; p)) = f_0(\Delta) - m.$$  

**Proof.** The relevant portion of the chain complex is

$$0 \xrightarrow{\partial_r} \bigoplus_{a \in \Delta^{(0)}} V^{(0)}_{d+1}/\text{Ker} \tilde{a} \xrightarrow{\partial_r} V^{(1)}_{d+1}.$$  

The 0-chain $\sum_{a \in \Delta^{(0)}} \lambda_a \cdot [x]$, $\lambda_a \in \mathbb{R}$, bounds if and only if $\sum_{a \in \Delta^{(0)}} \lambda_a \tilde{a} = 0$. Thus the result follows.

**Proposition 7.2.** Let $(\Delta; p)$ be a simplicial complex realized in $d$-space. If the set of vectors $\{ \bar{x} : x \in \Delta^{(0)} \}$ is of rank $m$, then

$$\beta_{-1}(\partial_r(\Delta; p)) = {d + 1 - m \choose r}.$$  

In particular, $\beta_{-1}(\partial_r(\Delta; p)) = 0$ if $m \geq d + 2 - r$.

**Proof.** The relevant portion of the chain complex is

$$\bigoplus_{a \in \Delta^{(0)}} V^{(r-1)}_{d+1}/\text{Ker} \tilde{a} \xrightarrow{\partial_r} V^{(r)}_{d+1} \xrightarrow{\partial_r} 0.$$  

Each $(-1)$-boundary is of the form $\sum_{x \in \Delta^{(0)}} \tilde{P}_x \bar{x}$ and $\text{C}y_{-1}(\partial_r(\Delta; p)) = V^{(r)}_{d+1}$.

Let $\{ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m \}$ be a maximal linearly independent subset of $\{ \bar{x} : x \in \Delta^{(0)} \}$, and $\tilde{P} = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_m$. We claim that

$$H_{-1}(\partial_r(\Delta)) \cong V^{(r)}_{d+1}/\text{Ker} \tilde{P}.$$
The isomorphism is established as follows. Let \( \tilde{Q}, \tilde{Q}' \in \text{Cy}_{-1}(\mathcal{R}_r(\Delta; \mathbf{p})) = V_{d+1}^{(r)} \), \( \tilde{Q} - \tilde{Q}' \in \text{Bd}_{-1}(\mathcal{R}_r(\Delta; \mathbf{p})) \) if and only if for all \( x \in \Delta^{(0)} \), there exists \( \tilde{A}_x \in V_{d+1}^{(r-1)} \) such that

\[
\tilde{Q} - \tilde{Q}' = \sum_{x \in \Delta^{(0)}} \tilde{A}_x \tilde{x}.
\]

This is equivalent to \( \tilde{Q} \tilde{P} = \tilde{Q}' \tilde{P} \) or \( \tilde{Q} \tilde{P} = \tilde{Q}' \). Thus

\[
\beta_{-1}(\mathcal{R}_r(\Delta; \mathbf{p})) = \dim V^{(r)} / \text{Ker} \tilde{P} = \left( d + 1 - m \right)\text{.}
\]

Both \( H^{(0)}(\mathcal{R}^r(\Delta; \mathbf{p})) \) and \( H_{(0)}(\mathcal{R}_r(\Delta; \mathbf{p})) \) have interesting geometric interpretations. First we look at the case \( r = 3 \) and \( d = 3 \).

**Example 7.3.** Consider \( H_0(\mathcal{R}_3(\Delta; \mathbf{p})) \) for \( (\Delta; \mathbf{p}) \) realized in 3-space. The relevant part of the skeletal chain complex is

\[
\bigoplus_{\rho \in \Delta^{(2)}} V_{4}^{(0)} / \text{Ker} \tilde{\rho} \xrightarrow{\tilde{\rho}} \bigoplus_{\sigma \in \Delta^{(1)}} V_{4}^{(1)} / \text{Ker} \tilde{\sigma} \xrightarrow{\tilde{\sigma}} \bigoplus_{\tau \in \Delta^{(0)}} V_{4}^{(2)} / \text{Ker} \tilde{\tau} \xrightarrow{\tilde{\tau}} V_{4}^{(3)}\text{.}
\]

A direct analysis suggests we write a second matrix whose row space is isomorphic to \( \text{Bd}_{d}(\mathcal{R}_3(\Delta; \mathbf{p})) \). For each edge \( xy \), the space \( V_{4}^{(1)} \) is of dimension 2 and we choose \( \{ \tilde{S}_{ab}, \tilde{T}_{ab} \} \) to be a basis. (In fact we can choose \( \{ \tilde{S}_{ab}, \tilde{T}_{ab} \} \) to be any 1-extensors such that \( \tilde{x} \tilde{y} \tilde{S}_{ab} \tilde{T}_{ab} \neq 0 \). Then \( \tilde{S}_{ab} \tilde{T}_{ab} \) is a 2-extensor complementary to \( \sigma \). Every member \( \tilde{P} \) of \( V_{4}^{(1)} / \text{Ker} \tilde{xy} \) can be represented uniquely as a linear combination of \( \tilde{S}_{xy} \) and \( \tilde{T}_{xy} \); i.e., there exist unique scalars \( \lambda_{xy} \) and \( \mu_{yx} \) such that \( \lambda_{xy} \tilde{T}_{xy} + \mu_{yx} \tilde{S}_{xy} = \tilde{P} \). Geometrically this representation is simply the 1-extensor which is the intersection of the plane \( \tilde{P} \tilde{xy} \) and the line \( \tilde{T}_{xy} \tilde{S}_{xy} \). The chains \( T_{xy} \cdot [xy] \), \( S_{xy} \cdot [xy] \), \( xy \in \Delta^{(1)} \), form a basis for \( \bigoplus_{xy \in \Delta^{(1)}} V_{4}^{(1)} / \text{Ker} \tilde{xy} \). For any edge \( xy \in \Delta^{(1)} \), and any \( \tilde{P} \in V_{4}^{(1)} / \text{Ker} \tilde{xy} \), we have

\[
\partial_1 \tilde{P} \cdot [xy] = \tilde{P} \tilde{x} \cdot [y] + \tilde{P} \tilde{y} \cdot [x]\text{.}
\]

Now form the matrix whose columns are indexed by vertices and whose rows are indexed by edges, with two rows for every edge. The two rows corresponding to the edge \( xy \) have entries \( \tilde{T}_{xy} \tilde{y} \) and \( \tilde{S}_{xy} \tilde{x} \) in the columns corresponding to \( x \) and entries \( \tilde{T}_{xy} \tilde{x} \) and \( \tilde{S}_{xy} \tilde{y} \) in the columns corresponding to \( y \), and zeroes elsewhere. This matrix is called the \( (3, 0) \)-rigidity matrix of \( (\Delta; \mathbf{p}) \) and is denoted by \( R_{3,0}(\Delta; \mathbf{p}) \).

If \( \Delta \) is the 3-simplex with vertices \( a, b, c, d \), we have the following matrix, with two rows for every edge.
If we consider each row of the matrix as a vector in $\bigoplus_{x \in \Delta^{(0)}} V_4^{(2)}/\text{Ker } \tilde{x}$, the row space of this matrix is $Bd_0(\mathcal{R}_3(\Delta; \mathbf{p}))$.

Also for any 1-chain $\sum_{xy \in \Delta^{(1)}} \tilde{P}_{xy} \cdot [xy]$ there exist unique scalars $\lambda_{xy}, \mu_{xy}$ such that

$$\sum_{xy \in \Delta^{(1)}} \tilde{P}_{xy} \cdot [xy] = \sum_{xy \in \Delta^{(1)}} \lambda_{xy} \tilde{T}_{xy} \cdot [xy] + \mu_{xy} \tilde{S}_{xy} \cdot [xy].$$

Since

$$\partial \sum_{xy \in \Delta^{(1)}} \lambda_{xy} \tilde{T}_{xy} \cdot [xy] + \mu_{xy} \tilde{S}_{xy} \cdot [xy] = 0$$

if and only if $\{\lambda_{xy}, \mu_{xy}\}$ is in the cokernel of the matrix, there is a homomorphism from $\text{Cy}_1(\mathcal{R}_3(\Delta))$ onto the cokernel. Since $\lambda_{xy} = \mu_{xy} = 0$ for all $xy$ if and only if $\sum_{xy \in \Delta^{(1)}} \tilde{P}_{xy} \cdot [xy]$, the homomorphism is actually an isomorphism. Thus the cokernel is isomorphic with $\text{Cy}_1(\mathcal{R}_3(\Delta; \mathbf{p}))$. We call each member of the cokernel a $(3, 0)$-stress of $(\Delta; \mathbf{p})$ and the space of $(3, 0)$ stresses is denoted by $\text{Stress}_{3,0}(\Delta; \mathbf{p})$. Thus

$$\text{Stress}_{3,0}(\Delta; \mathbf{p}) \equiv \text{Cy}_1(\mathcal{R}_3(\Delta; \mathbf{p})).$$

We define the space of $(3, 0)$-loads, denoted by $\text{Load}_{3,0}(\Delta; \mathbf{p})$, to be the space consisting of vectors of the form $\sum_{x \in \Delta^{(0)}} \tilde{P}_x \cdot [x] \in \bigoplus_{x \in \Delta^{(0)}} V_4^{(2)}/\text{Ker } \tilde{x}$, such that $\sum_{x \in \Delta^{(0)}} \tilde{P}_x \tilde{x} = 0$. It is also easy to see that $\text{Cy}_0(\mathcal{R}_3(\Delta; \mathbf{p})) = \text{Load}_{3,0}(\Delta; \mathbf{p})$.

A chain $C \in \bigoplus_{x \in \Delta^{(0)}} V_4^{(1)}/\text{Ker } \tilde{x}$ is in the kernel of the matrix if and only if for every edge $xy$, $\tilde{C}_x \tilde{S}_{xy} \tilde{x} \tilde{y} + \tilde{C}_y \tilde{S}_{xy} \tilde{x} \tilde{y} = 0$ and $\tilde{C}_x \tilde{T}_{xy} \tilde{x} \tilde{y} + \tilde{C}_y \tilde{T}_{xy} \tilde{x} \tilde{y} = 0.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab</td>
<td>$\tilde{S}_{ab} \tilde{b}$</td>
<td>$\tilde{T}_{ab} \tilde{b}$</td>
<td>$\tilde{S}_{ab} \tilde{a}$</td>
<td>$\tilde{T}_{ab} \tilde{a}$</td>
</tr>
<tr>
<td>ac</td>
<td>$\tilde{S}_{ac} \tilde{c}$</td>
<td>$\tilde{T}_{ac} \tilde{c}$</td>
<td>$\tilde{S}_{ac} \tilde{a}$</td>
<td>$\tilde{T}_{ac} \tilde{a}$</td>
</tr>
<tr>
<td>ad</td>
<td>$\tilde{S}_{ad} \tilde{d}$</td>
<td>$\tilde{T}_{ad} \tilde{d}$</td>
<td>$\tilde{S}_{ad} \tilde{a}$</td>
<td>$\tilde{T}_{ad} \tilde{a}$</td>
</tr>
<tr>
<td>bc</td>
<td>$\tilde{S}_{bc} \tilde{c}$</td>
<td>$\tilde{T}_{bc} \tilde{c}$</td>
<td>$\tilde{S}_{bc} \tilde{b}$</td>
<td>$\tilde{T}_{bc} \tilde{b}$</td>
</tr>
<tr>
<td>bd</td>
<td>$\tilde{S}_{bd} \tilde{d}$</td>
<td>$\tilde{T}_{bd} \tilde{d}$</td>
<td>$\tilde{S}_{bc} \tilde{b}$</td>
<td>$\tilde{T}_{bc} \tilde{b}$</td>
</tr>
<tr>
<td>cd</td>
<td>$\tilde{S}_{cd} \tilde{d}$</td>
<td>$\tilde{T}_{cd} \tilde{d}$</td>
<td>$\tilde{S}_{cd} \tilde{c}$</td>
<td>$\tilde{T}_{cd} \tilde{c}$</td>
</tr>
</tbody>
</table>
By the choice of $S_{xy}$ and $T_{xy}$, this is equivalent to

$$\tilde{C}_x \tilde{x} \tilde{y} + \tilde{C}_y \tilde{y} \tilde{x} = 0.$$  

On the other hand, a chain $C \in \bigoplus_{x \in \Delta^{(0)}} V_4^{(1)}/\text{Ker} \, \tilde{x}$ is a 0-cocycle if and only if

$$\delta_1 C = \sum_x \sum_{r \geq x} \tilde{C}_x \cdot [r]^{(0)} = 0;$$

i.e., for every edge $xy$,

$$C_x \tilde{x} \tilde{y} + C_y \tilde{y} \tilde{x} = 0. \quad (7.1)$$

Hence $\text{CoCy}_0(\mathbb{R}^3(\Delta; p))$ is the kernel of the matrix. We call the space of 0-cocycles the space of trivial parallel redrawings, denoted by $\text{Triv}_3(\Delta; p)$.

A 0-coboundary is a chain of the form $\sum_{x \in \Delta^{(0)}} \tilde{P} \cdot [x] \in \bigoplus_{x \in \Delta^{(0)}} V_4^{(1)}/\text{Ker} \, \tilde{x}$. We call the space of 0-coboundaries the space of trivial (3, 0)-motions and denote it by $\text{Triv}_3(\Delta; p)$. The quotient space is called the space of nontrivial (3, 0)-motions, $\text{NonTriv}_3(\Delta; p) = \text{Motion}_3(\Delta; p)/\text{Triv}_3(\Delta; p)$.

There is a very interesting geometric interpretation for $\text{CoCy}_0(\mathbb{R}^3(\Delta; p))$. We assume that for all $x \in \Delta^{(0)}$, $\tilde{x} = (x_1, x_2, x_3, 1)$; i.e., all the vertices are finite points. Now let $x = (x_1, x_2, x_3)$ be the Euclidean coordinates of the point $\tilde{x}$. Let $C$ be a 0-cocycle. Let $q$ be a new realization of $\Delta$ where the homogeneous coordinates of $x \in \Delta^{(0)}$ is given by $\tilde{x}' = \tilde{x} + \tilde{C}_x - (\tilde{C}_x)_q \tilde{x}$, where $(\tilde{C}_x)_q$ is the last coordinate of $\tilde{C}_x$. Then the last coordinate of $\tilde{C}_x - (\tilde{C}_x)_q \tilde{x}$ is 0 and that of $\tilde{x}'$ is 1. Moreover, from (7.1), we have

$$((\tilde{C}_x - (\tilde{C}_x)_q \tilde{x}) - (\tilde{C}_y - (\tilde{C}_y)_q \tilde{y})) \tilde{x} \tilde{y} = 0.$$

Hence the vector $\mathbf{x}' - \mathbf{y}'$ is parallel to the vector $\mathbf{x} - \mathbf{y}$. Thus the edges of $(\Delta; p)$ are parallel to their corresponding edges in $(\Delta; q)$. Thus 0-cocycles are parallel redrawings of the 1-skeleton.

Note that a parallel redrawing of the 1-skeleton of $\Delta$ realized in $d$-space is an assignment of homogeneous coordinates $\tilde{x}'$ for every vertex $x$ such that for each edge $xy$, the lines $\tilde{x} \tilde{y}$ and $\tilde{x}' \tilde{y}'$ are parallel. The set of parallel redrawings is a linear space. Trivially every translation or dilation is a parallel redrawing. The space of trivial parallel redrawings is generated by translations and dilations. If one factors out the trivial redrawings, the resulting quotient space is the space of nontrivial parallel redrawings. The space of parallel redrawings has been studied by various authors and is connected to Minkowski sum of convex polytopes. (See [5, 8, 39].)

Consider the 0-coboundary $\sum_{x \in \Delta^{(0)}} \tilde{P} \cdot [x] \in \bigoplus_{x \in \Delta^{(0)}} V_4^{(1)}/\text{Ker} \, \tilde{x}$. If the last coordinate of $\tilde{P} = (p_1, p_2, p_3, p_4)$ is 0, then for every $x$, $\tilde{P} - p_4 \tilde{x} = \tilde{P}$. This corresponds to the translation by $p = (p_1, p_2, p_3)$. If $\tilde{P} = (0, 0, 0, p_4)$, where $p_4 \neq 0$, then $\tilde{P} - p_4 \tilde{x} = -p_4 \mathbf{x}$. Thus $\mathbf{x}' = (1 - p_4) \mathbf{x}$, and the coboundary corresponds to a dilation with ratio $(1 - p_4)$, provided $p_4 \neq 1$. Thus the 0-coboundaries correspond to the space of trivial parallel redraw-
ings. Thus $H^{(0)}(\mathcal{R}^3(\Delta; p))$ is isomorphic to the space of nontrivial parallel redrawings.

The above analysis applies whenever $d = r$. Thus we have

**Proposition 7.4.** Let $(\Delta; p)$ be a simplicial complex realized in $d$-space. Then the 0-cohomology group, $H^{(0)}(\mathcal{R}^d(\Delta; p))$, is isomorphic to the space of nontrivial parallel redrawings of the 1-skeleton of $(\Delta; p)$.

**Example 7.5.** Next we look at the case $r = 2$ and $d = 2$. We again choose the standard homogeneous coordinates, $\bar{x} = (x_1, x_2, 1)$ for each vertex $x$. The matrix for the 0-boundaries of the 3-simplex with vertices $a$, $b$, $c$, and $d$ is shown below where each row is considered to be a member of $\bigoplus_{x \in \Delta^{(0)}} V^{(1)}_3/\text{Ker} \bar{x}$.

\[
\begin{array}{cccc}
  a & b & c & d \\
  ab & \tilde{b} & \tilde{a} & \\
  ac & \tilde{c} & \tilde{a} & \\
  ad & \tilde{d} & \tilde{a} & \\
  bc & \tilde{c} & \tilde{b} & \\
  bd & \tilde{d} & \tilde{b} & \\
  cd & \tilde{d} & \tilde{c} & \\
\end{array}
\]

The row space is now $\text{Bd}_0(\mathcal{R}^2(\Delta; p))$, and the kernel is $\text{CoCy}^0(\mathcal{R}^2(\Delta; p))$. In Section 6, we showed that $\text{NonTriv}_r(\Delta; p) \cong H^{r-2}(\mathcal{R}^r(\Delta; p))$. Thus when $r = 2$, we have $\text{NonTriv}_2(\Delta; p) \cong H^0(\mathcal{R}^2(\Delta; p))$. This means the matrix that we have must be, in some sense, equivalent to the usual rigidity matrix of the 1-skeleton of $(\Delta; p)$ as given in [32], say. To see this, recall that in Section 2 we described how one can choose specific representatives of the equivalent classes in $V^{(1)}_3/\text{Ker} \bar{x}$. Choose $A = (1, 0, 0)(0, 1, 0)$; i.e., $A$ is the line at “infinity.” Then we can choose $\text{Proj}_1(\tilde{y}) = (\tilde{x} \times \tilde{y}) \wedge A$ as the representative of the equivalence class of $\tilde{y}$. This representative turns out to be the intersection of the line $\tilde{x} \tilde{y}$ with the line at infinity. Thus $\text{Proj}_1(\tilde{y}) = (x_1 - y_1, x_2, y_2, 0)$. Using this representative, the matrix is actually the usual rigidity matrix

\[
\begin{array}{cccc}
  a & b & c & d \\
  ab & a - b & b - a & \\
  ac & a - c & c - a & \\
  ad & a - d & d - a & \\
  bc & b - c & c - b & \\
  bd & b - d & d - b & \\
  cd & c - d & d - c & \\
\end{array}
\]
where \( a = (a_1, a_2) \), the Euclidean coordinates of the vertex \( a \). Thus if \((\Delta; p)\) is realized in the plane, the space \( \text{CoCy}^i(\mathbb{R}^2(\Delta; p)) \) is the space of infinitesimal motions in the plane, as well as the space of parallel redrawings of the configuration in the plane. The space \( \text{CoBd}^i(\mathbb{R}^2(\Delta; p)) \) is the space of trivial motions as well as the space of trivial parallel redrawings. These connections are well understood. (See [5, 39].)

The results in the previous example can be generalized. For \((\Delta; p)\) in \( d \)-space, we can defined an \((r, i)\)-rigidity matrix \( R_{r,i}(\Delta; p) \) for \( 0 \leq i \leq r - 2 \). The row space is then \( \text{Bd}_i(\mathbb{R}_r(\Delta; p)) \). The kernel is \( \text{CoCy}^i(\mathbb{R}^r(\Delta; p)) \), otherwise known as \( \text{Motion}_{r,i}(\Delta; p) \). The cokernel is called \( \text{Stress}_{r,i}(\Delta; p) \) and \( \text{Stress}_{r,i}(\Delta; p) \approx \text{C}_i(\mathbb{R}_r(\Delta; p)) \). The space of \((r, i)\)-loads, \( \text{Load}_{r,i}(\Delta; p) \), is \( \text{C}_i(\mathbb{R}_r(\Delta; p)) \). The space of trivial \((r, i)\)-motions, \( \text{Triv}_{r,i}(\Delta; p) \), is \( \text{CoBd}^i(\mathbb{R}^r(\Delta; p)) \). \((\Delta; p)\) is said to be \((r, i)\)-independent if \( \text{Stress}_{r,i}(\Delta; p) = 0 \) and \((r, i)\)-rigid if \( \text{Motion}_{r,i}(\Delta; p) = \text{Triv}_{r,i}(\Delta; p) \), i.e., if \( H^i(\mathbb{R}^r(\Delta; p)) = 0 \).

We omit the details here. The case \( i = r - 2 \) gives the \( r \)-skeletal rigidity described in [29, 30] as well in Sections 4 and 5. The case \( d = r \) and \( i = 0 \) is described above. We expect all these spaces have meaningful geometric interpretation.

8. PROJECTION AND CONING

In our preceding paper [29], we saw that a non-singular projective transformation induced an isomorphism of all of the relevant spaces. This invariance is implicit in our notation and is an essential prerequisite for the constructions such as coning. We complete the picture by showing that projective transformations induce an isomorphism of the chain and cochain complexes. (See also [34] for some work done in this area for \( r = 2 \).

Let \((\Delta; p)\) be realized in \( d \)-space. Let \( \psi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{m+1}, m \leq d \), be a surjective linear transformation. For any \( i \)-extensor \( \tilde{\sigma} = \tilde{a}_1 \tilde{a}_2 \cdots \tilde{a}_i \), define \( \psi(\tilde{\sigma}) = \psi(\tilde{a}_1) \psi(\tilde{a}_2) \cdots \psi(\tilde{a}_i) \). Then \( \psi \) extends to a linear transformation \( \psi : V_{d+1}^{(i)} \rightarrow V_{m+1}^{(i)} \). The real-valued weight function \( \omega : \Delta^{(0)} \rightarrow \mathbb{R} \setminus \{0\} \) also extends to \( \Delta^{(i)} \) by \( \omega_{\sigma} = \omega_{x_1} \omega_{x_2} \cdots \omega_{x_i} \), where \( \sigma = x_1 x_2 \cdots x_i \in \Delta^{(i)} \). \((\Delta; \Phi p)\) shall denote the realization of \( \Delta \) where the homogeneous coordinates for each vertex \( x \) are \( \omega_x \psi(\tilde{x}) \).

**Theorem 8.1.** Consider the chain complex \( \mathbb{R}_r(\Delta; p) \), the cochain complex \( \mathbb{R}^r(\Delta; p) \), and the functions \( \psi \) and \( \omega \) described above. These induced a chain map

\[
\Phi : \mathbb{R}_r(\Delta; p) \rightarrow \mathbb{R}_r(\Delta; \Phi p)
\]

and a cochain map

\[
\Phi : \mathbb{R}^r(\Delta; p) \rightarrow \mathbb{R}^r(\Delta; \Phi p)
\]
as follows. (Note that we use the same notation for the chain and cochain map because they are induced by the projection \( \Phi \).

For any elementary chain \( P \cdot [\sigma] \),

\[
\Phi(P \cdot [\sigma]) = \frac{1}{\omega_{\sigma}} \psi(P) \cdot [\sigma],
\]

and for any elementary cochain \( P \cdot [\sigma] \),

\[
\Phi(P \cdot [\sigma]) = \psi(P) \cdot [\sigma].
\]

If \( \psi \) is non-singular, then \( \Phi \) has an inverse \( \omega^{-1} \psi^{-1} \) and \( \Phi \) is an isomorphism of the chain complexes and the cochain complexes.

Proof. First, \( \Phi \) respects our equivalence relations. For any face \( \sigma \in \Delta \), we have

\[
P \hat{=} Q \quad \Rightarrow \quad P \hat{=} Q \tilde{\sigma}
\]

\[
\Rightarrow \quad \psi(P) \omega_\sigma \psi(\tilde{\sigma}) = \psi(Q) \omega_\sigma \psi(\tilde{\sigma})
\]

\[
\Rightarrow \quad \psi(P) \frac{\omega_\sigma \delta(\tilde{\sigma})}{\omega_\sigma} = \psi(Q).
\]

Thus \( \Phi \) induces surjective maps

\[
\Phi: \bigoplus_{\sigma \in \Delta^{(r-1)}} V^{(r)}_{d-1}/\text{Ker} \tilde{\sigma} \rightarrow \bigoplus_{\sigma \in \Delta^{(r-1)}} V^{(r)}_{d-1}/\text{Ker} \omega_\sigma \psi(\tilde{\sigma}).
\]

It remains to show that \( \Phi \) commutes with \( \partial \) and \( \delta \):

\[
\Phi(\partial(P \cdot [\sigma])) = \Phi\left(\sum_{x|\sigma} P \bar{x} \cdot [\sigma/x]\right)
\]

\[
= \sum_{x|\sigma} \frac{1}{\omega_{\sigma/x}} \psi(P) \psi(\bar{x}) \cdot [\sigma/x]
\]

\[
= \sum_{x|\sigma} \omega_x \psi(P) \psi(\bar{x}) \cdot [\sigma/x]
\]

\[
\partial(\Phi(P) \cdot [\sigma]) = \partial\left(\frac{1}{\omega_\sigma} \psi(P) \cdot [\sigma]\right)
\]

\[
= \sum_{x|\sigma} \omega_x \psi(P) \psi(\bar{x}) \cdot [\sigma/x]
\]

\[
\Phi(\delta(P \cdot [\sigma])) = \Phi\left(\sum_{\rho > \sigma} \text{sign}[\sigma, \rho] P_{\sigma} \cdot [\rho]\right)
\]

\[
= \sum_{\rho > \sigma} \text{sign}[\sigma, \rho] \psi(P) \cdot [\rho]
\]

\[
= \delta_1(\Phi(P) \cdot [\sigma]).
\]
For a non-singular $\psi$, $\omega^{-1}\psi^{-1}$ induces the inverse chain map, and we have the desired isomorphism of chain and cochain complexes.$\blacksquare$

We now consider a central projection of $(\Delta; p)$ realized in $(d + 1)$-space into $(\Delta; \Phi p)$ in $d$-space. This is an example of a singular linear transformation in the previous theorem. What is the effect on the Betti numbers of the complex? The chain map $\Phi$ carries boundaries to boundaries, and cycles to cycles. We see that the maps $\Phi$ are surjective, so that all $(i + 1)$-boundaries in $\mathcal{R}_r(\Delta; \Phi p)$ are images of $(i + 1)$-boundaries in $\mathcal{R}_r(\Delta; p)$. Therefore, we cannot guarantee a surjection from $H_i(\mathcal{R}_r(\Delta; p))$ onto $H_i(\mathcal{R}_r(\Delta; \Phi p))$. We can get the precise answer by examining an extreme case of projection—projection from a vertex of a cone.

Let $\Delta$ be a simplicial complex, and let $\Delta' = \Delta \ast a$ be the cone where $a \not\in \Delta^{(0)}$. Consider a realization $(\Delta'; p)$ in $(d + 1)$-space and centrally project it from $a$ to a $d$-dimensional subspace $H$. This projection, $\Pi_r$, gives a realization of $\Delta$ in $d$-space, $(\Delta; \Pi_r p)$.

We establish a correspondence between the $i$-cohomologies of $\Delta$ and $\Delta'$ in these realizations. First we give a concrete construction for $\Pi_r$. Choose a basis $a_1, a_2, \ldots, a_{d+1}$ for $H$. These together with $\tilde{a}$ form a basis for $\mathbb{R}^{d+2}$. Any $x \in \mathbb{R}^{d+2}$ can be written uniquely as $x = a\tilde{a} + \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{d+1} a_{d+1}$. Define $\Pi_r(x) = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{d+1} a_{d+1}$. Thus for any face $\sigma$ of $\Delta$, we have $\tilde{a}\sigma = a\Pi_r(\sigma)$. The weight function $\omega_\sigma = 1$ for all $x \in (\Delta')^{(0)}$.

We now consider the chain and cochain map

\begin{align*}
\Pi_r^*: \mathcal{R}_r(\Delta'; p) &\rightarrow \mathcal{R}_r(\Delta; \Pi_r p), \\
\Pi_r^*: \mathcal{R}'(\Delta'; p) &\rightarrow \mathcal{R}'(\Delta; \Pi_r p).
\end{align*}

First we define an important subspace of the space of cocycles. Let $\text{Fix}(\Delta') \subseteq \text{CoCy}'(\mathcal{R}'(\Delta'; p))$ be the subspace consisting of cocycles which are zero on $\pi a$, for all $\pi \in \Delta^{(i-1)}$. Then for all $M \in \text{Fix}(\Delta')$ and all elementary chains $P \cdot [\sigma]$, where $\sigma \in \Delta^{(i)}$ and $P \in V^{(r-2-r)}_{d+2}/\text{Ker} \tilde{\sigma}$, we have, writing $\rho$ for $\sigma a$,

\begin{align*}
\langle M^*, \partial(P \cdot [\rho]) \rangle &= \sum_{x \mid \rho} \text{sign}(x, \rho) M_{\rho/x} P \tilde{\rho} \\
&= M_{\rho} P \tilde{\rho} \tilde{a} = 0.
\end{align*}

Therefore, since $M_{\sigma} \in V^{(d+2-r)}_{d+2}/\text{Ker} \tilde{\sigma}$,

\begin{align*}
M_{\sigma} &= \begin{cases} 
S_{\sigma} \tilde{a} & \text{if } \sigma \in \Delta^{(i)} \\
0 & \text{otherwise},
\end{cases}
\end{align*}

where $S_{\sigma}$ is a step $d + 1 - r$ tensor in $H$. (Since every step $d + 1 - r$ tensor $S$ can be written as $S_1 + S_2 \tilde{a}$, $S_1$ in $H$ and $S\tilde{a} = S_1 \tilde{a}$, we may assume $S_{\sigma}$ is in $H$.)
Theorem 8.2 (the cone theorem). Consider the chain and cochain maps

\[ \Pi_{\Delta}; \mathcal{R}(\Delta; p) \to \mathcal{R}(\Delta; \Pi_{\Delta} p), \]
\[ \Pi_{\Delta}'; \mathcal{R}'(\Delta; p) \to \mathcal{R}'(\Delta; \Pi_{\Delta} p). \]

We have, for all \( i \leq r - 2 \),

(i) \( \text{CoCy}(\mathcal{R}'(\Delta; \Pi_{\Delta} p)) \cong \text{Fix}(\Delta; p) \),

(ii) \( H'(\mathcal{R}'(\Delta; \Pi_{\Delta} p)) \cong \text{Fix}'(\Delta; p) / \text{CoBd}(\mathcal{R}'(\Delta; p)) \cap \text{Fix}'(\Delta; p) \),

(iii) \( H'(\mathcal{R}'(\Delta; \Pi_{\Delta} p)) \cong H'(\mathcal{R}'(\Delta; p)) \), and

(iv) \( H_{r-1}(\mathcal{R}(\Delta; \Pi_{\Delta} p)) \cong H_{r-1}(\mathcal{R}(\Delta; p)) \).

Proof. (i) Let \( f: \text{CoCy}(\mathcal{R}(\Delta; \Pi_{\Delta} p)) \to \text{Fix}(\Delta; p) \) be the linear function defined by \( f(M) = M' \) where for all \( \sigma \in \Delta^{(i)} \),

\[ M'_\sigma = \begin{cases} M_\sigma \hat{a} & \text{if } \sigma \in \Delta \\ 0 & \text{otherwise.} \end{cases} \]

We first show that \( M' \in \text{CoCy}(\mathcal{R}'(\Delta; p)) \) For all \( \rho \in (\Delta')^{(i)} \), we have

\[ \delta \sum_{\sigma \in \Delta^{(i)}} M'_\sigma \cdot [\sigma] = \delta \sum_{\sigma \in \Delta^{(i)}} M_\sigma \hat{a} \cdot [\sigma] \]

\[ = \sum_{\rho} \sum_{\sigma \in \Delta^{(i)}, \sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \hat{a} \cdot [\rho] \]

\[ = \sum_{\rho} \left( \sum_{\sigma \in \Delta^{(i)}, \sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \hat{a} \right) \cdot [\rho]. \]

If \( a \mid \rho \) then \( \hat{a} \hat{\hat{a}} = 0 \). If \( a \nmid \rho \), we note that \( P \hat{a} \hat{\hat{a}} = 0 \) if and only if \( P \hat{a} \hat{\hat{a}} = 0 \) since \( \hat{a} \hat{\hat{a}} = \hat{a} \hat{\hat{a}} \). Since \( M \in \text{CoCy}(\mathcal{R}(\Delta; \Pi_{\Delta} p)) \), we have

\[ \delta \sum_{\sigma \in \Delta^{(i)}} M_\sigma \cdot [\sigma] = \delta \sum_{\sigma \in \Delta^{(i)}, \sigma \prec \rho} \sum_{\sigma \in \Delta^{(i)}, \sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \cdot [\rho] \]

\[ = \sum_{\rho} \left( \sum_{\sigma \in \Delta^{(i)}, \sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \right) \cdot [\rho]. \]

\[ \cong 0. \]

Therefore,

\[ \sum_{\sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \hat{a} \hat{\hat{a}} = 0 \]

\[ \Rightarrow \sum_{\sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \hat{a} \hat{\hat{a}} = 0 \]

\[ \Rightarrow \sum_{\sigma \prec \rho} \text{sign}[\sigma, \rho] M_\sigma \hat{a} \hat{\hat{a}} = 0. \]
Thus in all cases, $\delta M' = 0$ and $M' \in \text{CoCy}_i(\mathcal{R}'(\Delta'; p))$. Thus by the definitions, $M' \in \text{Fix}_i(\Delta'; p)$.

Next we prove that $f$ is onto. For all $M' \in \text{Fix}_i(\Delta'; p)$, we have $M'_\sigma = S_\sigma \tilde{a}$ if $\sigma \in \Delta^{(i)}$ and 0 otherwise. Define $M_\sigma = S_\sigma$ for all $\sigma \in \Delta^{(i)}$. We need to show that $M \in \text{CoCy}_i(\mathcal{R}'(\Delta; \Pi_a(p)))$. For all $\rho \in \Delta^{(i+1)}$, the $\rho$ component of $\delta M'$ is

$$\sum_{\sigma<\rho} \text{sign}[\sigma, \rho]M'_\sigma = \sum_{\sigma<\rho} \text{sign}[\sigma, \rho]S_\sigma \tilde{a} \overset{\rho}{=} 0.$$ 

Since $S_\sigma$ is a tensor in $H$, we have

$$\sum_{\sigma<\rho} \text{sign}[\sigma, \rho]S_\sigma \overset{\Pi, \rho}{=} 0.$$ 

Thus the $\rho$ component of $\delta M$ is

$$\sum_{\sigma<\rho} \text{sign}[\sigma, \rho]M_\sigma = \sum_{\sigma<\rho} \text{sign}[\sigma, \rho]S_\sigma \overset{\Pi, \rho}{=} 0.$$ 

We conclude that $M \in \text{CoCy}'(\mathcal{R}'(\Delta; \Pi_a(p)))$. It is easy to see that $f$ is one to one and the proof is complete for the first part.

(ii) We first note that when restricted to $\text{CoBd}'(\mathcal{R}'(\Delta; \Pi_a(p)))$, the map is a bijection to $\text{CoBd}'(\mathcal{R}'(\Delta'; p)) \cap \text{Fix}'(\Delta'; p)$. The desired result would then follow. To see that this is a bijection, take the coboundary of an elementary $(i-1)$-cochain in $\mathcal{R}'(\Delta; \Pi_a(p))$, $P \cdot [\pi]$, where $\pi \in \Delta^{(i-1)}$:

$$\delta(P \cdot [\pi]) = \sum_{\sigma>\pi, \sigma \in \Delta^{(i)}} \text{sign}[\pi, \sigma]P \cdot [\sigma].$$ 

The corresponding elementary $(i-1)$-cochain $P \tilde{a} \cdot [\pi]$ in $\mathcal{R}'(\Delta'; p)$ has coboundary

$$\delta(P \tilde{a} \cdot [\pi]) = \sum_{\sigma>\pi, \sigma \in \Delta^{(i)}} \text{sign}[\pi, \sigma]P \tilde{a} \cdot [\sigma]$$

$$= \sum_{\sigma>\pi, \sigma \in \Delta^{(i)}} \text{sign}[\pi, \sigma]P \tilde{a} \cdot [\sigma]$$

since $P \tilde{a} \overset{\rho}{=} 0$ if $a \mid \sigma$. Thus $f(\delta(P \cdot [\pi])) = \delta(P \tilde{a} \cdot [\pi])$. It is routine to check the restriction is onto.

(iii) We must show that any $M \in \text{CoCy}'(\mathcal{R}'(\Delta'; p))$ can be written as a linear combination of members in $\text{Fix}'(\Delta')$ and $\text{CoBd}'(\mathcal{R}'(\Delta'; p))$. For all $\sigma \in \Delta^{(i)}$ such that $a \mid \sigma$, $M_\sigma \overset{\rho}{=} S_\sigma$ for some step $(d+2-r)$-tensor $S_\sigma$ in $H$. Let

$$T = \delta(\sum_{\sigma \in \Delta^{(i)}, a \mid \sigma} S_\sigma \cdot [\sigma/a]) \in \text{CoBd}'(\mathcal{R}'(\Delta', p)).$$
The σ component of T is S_σ. Define N = M - T. For all σ ∈ Δ'(i) such that a | σ,

\[ N_σ = M_σ - S_σ = S_σ - S_σ = 0. \]

This means N ∈ Fix_σ(Δ'). We conclude that

\[ H^i(\mathcal{R}(Δ; \Pi_p) \setminus \mathcal{R}(Δ'; p)) \cong Fix(Δ'; p)/(\operatorname{CobD}^i(\mathcal{R}(Δ'; p))) \]
\[ \cap Fix^i(Δ'; p) \cong H^i(\mathcal{R}(Δ'; p)). \]

(iv) Let λ be an (r - 1)-cycle in Cy_{r-1}(\mathcal{R}(Δ'; p)). Then for all σ ∈ Δ'(r-2),

\[ \sum_{\rho = x_\sigma \sigma \in Δ'} \lambda_\rho \check{x}_\rho = \lambda_{\sigma a} \check{a} + \sum_{\rho = x_\sigma \sigma \in Δ'} \lambda_\rho \check{x}_\rho. \]

If we project with \( \Pi_a \), then

\[ \sum_{\rho = x_\sigma \sigma \in Δ'} \lambda_\rho \Pi_a(\check{x}_\rho) \overset{\Pi_a(\sigma)}{=} 0. \]

Thus λ restricted to Δ is in Cy_{r-1}(\mathcal{R}(Δ; \Pi_p)).

Next we show that every (r - 1)-cycle λ of \( \mathcal{R}(Δ; \Pi_p) \) can be extended (uniquely) to an (r - 1)-cycle of \( \mathcal{R}(Δ'; p) \). For any σ ∈ Δ'(r-2), we have

\[ \sum_{\rho = x_\sigma \sigma \in Δ'} \lambda_\rho \Pi_a(\check{x}_\rho) \overset{\Pi_a(\sigma)}{=} 0. \]

But \( \check{x}_\rho = \Pi_a(\check{a}) + \alpha_\rho \check{a} \) for some constant \( \alpha_\rho \in \mathbb{R} \). So

\[ \sum_{\rho = x_\sigma \sigma \in Δ'} \lambda_\rho \check{x} = \sum_{\rho : \sigma \in Δ} \lambda_\rho \Pi_a(\check{x}_\rho) + \left( \sum_{\rho = x_\sigma \sigma \in Δ'} \alpha_\rho \check{a} \right) \overset{\Pi_a(\sigma)}{=} \sum_{\rho = x_\sigma \sigma \in Δ'} \alpha_\rho \check{a} = \gamma_{\sigma} \check{a}. \]

Now define \( \lambda_{\sigma a} = -\gamma_{\sigma} \). This extends λ to Δ(\r-1) with the property that for all σ ∈ Δ'(r-2),

\[ \sum_{\rho : \sigma < \rho \in Δ'} \lambda_\rho \check{x}_\rho = 0, \quad \text{i.e.,} \quad \sum_{\rho : \sigma < \rho \in Δ'} \lambda_\rho \check{x}_\rho = 0. \]

To show that this extension is an (r - 1)-cycle of \( \mathcal{R}(Δ'; p) \), we need

\[ \sum_{\rho : \sigma = \pi a} \lambda_\rho \check{x}_\rho = 0, \quad \text{i.e.,} \quad \sum_{\rho : \sigma = \pi a} \lambda_\rho \check{a} \check{x}_\rho = 0, \]
Thus, we have the desired result.

Summing over all $\rho$, we have

$$\sum_{\rho \neq \sigma} \lambda_{\rho} \tilde{p}_{\rho} \tilde{d} + \sum_{\rho \neq \sigma} \sum_{\rho' \neq \rho} \lambda_{\rho} \tilde{p}_{\rho} \tilde{x}_{\rho}' = 0.$$ 

The double summation is equal to

$$\sum_{\rho', \pi | \rho' \neq \rho', \pi | \rho} \lambda_{\rho} \tilde{p}_{\rho} \tilde{x}_{\rho}'. $$

If $\rho' = \pi x y$, then $\sigma' = \pi x$, or $\pi y$. Thus

$$\sum_{\sigma' \neq \rho', \pi | \rho} \lambda_{\rho} \tilde{p}_{\rho} \tilde{x}_{\rho}' = \lambda_{\rho} \tilde{p}_{\rho} \tilde{x}_{\rho} + \tilde{p}_{\rho} \tilde{y} \tilde{x} = 0.$$ 

Thus, we have the desired result.

As a corollary to the proof, we have the desired theorem for general projections.

**Corollary 8.3.** Consider a realization $(\Delta; p)$ in $(d + 1)$-space and its $r$-skeletal chain- and cochain-complexes. A general projection of $\Delta$ is a projection $\Pi$ of the points of $\Delta$ into a hyperplane $H$ such that for each $\rho \in \Delta^{r-1}$, \{\Pi(\tilde{x}) : x \in \rho\} is linearly independent.

(i) $\Pi$ induces an injection from $H^i(\mathcal{R}'(\Delta; \Pi p))$ into $H^i(\mathcal{R}'(\Delta; p))$, $i \leq r - 2$;

(ii) $\Pi$ induces an injection from $H_{r-1}(\mathcal{R}_r(\Delta; p))$ into $H_{r-1}(\mathcal{R}_r(\Delta; \Pi p))$;

(iii) $\beta^i(\mathcal{R}'(\Delta; p)) = 0$ implies $\beta^i(\mathcal{R}'(\Delta; \Pi p)) = 0$, $i \leq r - 2$;

(iv) $\beta_{r-1}(\mathcal{R}_r(\Delta; \Pi p)) = 0$ implies $\beta_{r-1}(\mathcal{R}_r(\Delta; p)) = 0$.

**Proof.** Take a cone of $\Delta$ creating $\Delta' = \Delta \ast a$. Extend $p$ so that $p(a) = \tilde{a}$ is the center of projection. (If $\tilde{a}$ is at infinity, a projective transformation will change this to a finite point if desired.) Then $\Pi$ projects $(\Delta'; p)$ to $(\Delta; \Pi p)$. Now apply the cone theorem, Theorem 8.2, to get isomorphisms for the cone and the projection.

What happens when we pass from $(\Delta'; p)$ to $(\Delta; p)$? For $i \leq r - 2$, the elements of $\text{Fix}'(\Delta')$ restrict to $i$-cocycles of $\mathcal{R}'(\Delta; p)$, since they are 0 on all other $i$-faces. The restriction also takes $\text{CoBd}'(\mathcal{R}'(\Delta'; p)) \cap (\text{Fix}'(\Delta'))$ onto $\text{CoBd}'(\mathcal{R}'(\Delta; \Pi p))$. So we have an injection injection $H^i(\mathcal{R}'(\Delta'; p))$ into $H^i(\mathcal{R}'(\Delta; \Pi p))$, completing the desired injection.

The proof of part (iv) in the cone theorem gives an injection of $C_{r-1}(\Delta; \Pi p)$ into $C_{r-1}(\Delta; p)$. The rest follows immediately. \[\blacksquare\]
Now consider \((\Delta; p)\) realized in \((d + 1)\)-space. In Section 7 we saw that the 0-cocycles \(\text{CoCy}^0(\mathcal{R}^{d+1}(\Delta; p))\) are just “parallel redrawings” of the 1-skeleton. If we assume that the the vertices are all finite points, and project from a “general point” at infinity to a hyperplane to obtain \((\Delta; \Pi p)\) realized in \(d\)-space, then parallel redrawings project to parallel redrawings. Thus we expect some correspondence between the 0-cocycles \(\text{CoCy}^0(\mathcal{R}^{r+1}(\Delta; p))\) and \(\text{CoCy}^0(\mathcal{R}^r(\Delta; \Pi p))\). This is indeed the case and in fact we have something more general.

**Proposition 8.4.** Let \((\Delta; p)\) be a complex realized in \(d + 1\)-space such that the set \(\{\tilde{x} : x \in \Delta^{(0)}\}\) is of rank \(d + 1\), and let \(\Pi\) be the projection to a hyperplane \(H\) from any point \(\tilde{x}\) not on any line joining two vertices. (Note that \((\Delta; \Pi p)\) is a complex realized in \(d\)-space.) Then \(\Pi\) induces an injective homomorphism \(\Pi : H^0(\mathcal{R}^{r+1}(\Delta; p)) \rightarrow H^0(\mathcal{R}^r(\Delta; \Pi p))\); consequently \(\beta^0(\mathcal{R}^{r+1}(\Delta; p)) = 0\) implies \(\beta^0(\mathcal{R}^r(\Delta; \Pi p)) = 0\).

**Proof.** \(M \in \bigoplus_{x \in \Delta^{(0)}} V^{(d+1-r)}_{d+2}/\text{Ker} \tilde{x}\) is in \(\text{CoCy}^0(\mathcal{R}^{r+1}(\Delta; p))\) if for each edge \(xy\):

\[
M_x \tilde{x} \tilde{y} + M_y \tilde{y} \tilde{x} = 0.
\]

Hence

\[
\Pi(M_x)\Pi(\tilde{x})\Pi(\tilde{y}) - \Pi(M_y)\Pi(\tilde{y})\Pi(\tilde{x}) = 0.
\]

Thus \(\Pi(M) \in \text{CoCy}^0(\mathcal{R}^r(\Delta; \Pi p))\) and we have the required homomorphism of cocycles.

If \(\beta^0(\mathcal{R}^{r+1}(\Delta; p)) \neq 0\), and since \(\{\tilde{x} : x \in \Delta^{(0)}\}\) is of rank \(d + 2\), there exists a 0-cocycle \(M \in \text{CoCy}^0(\mathcal{R}^{r+1}(\Delta; p))\) which is not a 0-coboundary; i.e., for some pair of vertices \(a, d, ad \notin \Delta^{(1)}\) such that

\[
M_a \tilde{a} \tilde{d} + M_d \tilde{d} \tilde{a} \neq 0.
\]

Therefore, for almost all choices of \(\tilde{s}\),

\[
\Pi(M_a)\Pi(\tilde{a})\Pi(\tilde{b}) - \Pi(M_b)\Pi(\tilde{b})\Pi(\tilde{a}) \neq 0.
\]

Hence \(\beta^0(\mathcal{R}^r(\Delta; \Pi p)) \neq 0\) and the homomorphism is injective. Thus if \(\beta^0(\mathcal{R}^{r+1}(\Delta; p)) = 0\) then \(\beta^0(\mathcal{R}^r(\Delta; \Pi p)) = 0\).

**Remark 8.5.** The projection \(\Pi\) above induces a homomorphism from \(H^1(\mathcal{R}^{r+1}(\Delta; p))\) into \(H^1(\mathcal{R}^r(\Delta; \Pi p))\). What we cannot prove is that the homomorphism is injective. In the previous proof, under the assumption that the vertices spanned the space, the entire space of \(r\), 0-cycles is generated by the boundaries of pairs of vertices (not necessarily edges)—so a non-trivial cocycle pairs with a “missing edge” on the existing vertices. For higher faces, such as triangles, not all \(r\), 1-cycles are generated by triples for which the three edges are already present. A key step of the proof is absent.
The previous two results can be combined to give the following.

**Corollary 8.6.** Consider the complex \((\Delta; p)\) realized in \(d\)-space such that the set \(\{\tilde{x} : x \in \Delta^{(0)}\}\) is of rank \(d + 1\).

(i) If \(\beta^0(\mathcal{R}(\Delta; p)) = 0\), then \(\beta^0(\mathcal{R}^{r+1}(\Delta; p)) = 0\), for \(r \geq 2\).

(ii) If \(\beta^0(\mathcal{R}^2(\Delta; p)) = 0\), i.e., \((\Delta; p)\) is \(2\)-rigid in \(d\)-space, then \(\beta^0(\mathcal{R}(\Delta; p)) = 0\), for all \(r \geq 3\).

**Proof.** Consider a general projection \(\Pi\) into a hyperplane \(H\). For \(r \geq 2\), we have

\[
\beta^0(\mathcal{R}(\Delta; p)) = 0 \implies \beta^0(\mathcal{R}(\Delta; \Pi p)) = 0 \implies \beta^0(\mathcal{R}^{r+1}(\Delta; p)) = 0.
\]

The proof of (i) is thus complete. (ii) follows by repeated application of (i).

**Remark 8.7.** There is a large literature on \(1\)-skeletons which are \(2\)-rigid—and therefore have all \(\beta^0(\mathcal{R}(\Delta; p)) = 0\). Some of these results will be summarized in Section 13.

9. EVERYTHING ABOUT \(K_n\)

We now have a complete characterization of all Betti numbers of \(K_n\). This will be needed for the next few sections.

**Proposition 9.1.** For \(K_n\) realized in \(d\)-space we have

\[
\beta_i(\mathcal{R}_r(K_n)) = \begin{cases} 
\binom{n+r-d-2}{r} & \text{if } i = r-1; \\
0 & \text{if } 0 \leq i \leq r-2; \\
\binom{d+1-n}{r} & \text{if } i = -1.
\end{cases}
\]

In particular, \(\beta_{r-1}(\mathcal{R}_r(K_n)) = 0\) if \(n < d + 2\), \(\beta_{-1}(\mathcal{R}_r(K_n)) = 0\) if \(n > d + 1 - r\), and \(\beta_i(\mathcal{R}_r(K_n)) = 0\) if \(d+1-r < n < d+2\).

**Proof.** First we consider the case \(n \geq d + 2\). \(K_n\) realized in \(d\)-space is the \(d + 1 - r\) cone of \(K_{n+r-d-1}\) realized in \(r-1\)-space. By Corollary 3.3, we know that

\[
\beta_i(\mathcal{R}_r(K_{n+r-d-1})) = \begin{cases} 
\binom{n+r-d-2}{r} & \text{if } i = r-1 \\
0 & \text{otherwise.}
\end{cases}
\]

The coning theorem gives the desired conclusion.

Next is the case \(d + 1 - r < n \leq d + 1\). \(K_n\) realized in \(d\)-space is the \(d + 1 - r\) cone of \(K_{n+r-d-1}\) in \(r-1\)-space. So part (i) still applies, but now \(\binom{n+r-d-2}{r} = 0\). So all the Betti numbers are zero.
When $n \leq d + 1 - r$, $K_n$ is the cone of $\emptyset$ in $d - n$-space. But

$$\beta_i(\mathcal{R}_r(\emptyset)) = \begin{cases} 0 & \text{if } i \geq 0 \\ \binom{d-n+1}{r} & \text{if } i = -1. \end{cases}$$

The conclusion again follows from the coning theorem.

We will also need to know the homology of $\partial K_n$ in $d$-space, where $\partial K_n = \langle (K_n^{n-2}) \rangle$ is the complex formed by removing the unique maximal face.

**Proposition 9.2.** For a general position realization of $K_n$ in $d$-space, if $r < n$, then

$$\beta_i(\mathcal{R}_r(\partial K_n)) = \begin{cases} \binom{n+r-d-2}{r} & \text{if } i = r - 1; \\ 0 & \text{if } 0 \leq i \leq r - 2; \\ \binom{d+1-n}{r} & \text{if } i = -1. \end{cases}$$

If $r \geq n$, then

$$\beta_i(\mathcal{R}_r(\partial K_n)) = \begin{cases} \binom{d+1-n}{r} & \text{if } i = n - 2; \\ \binom{d+1-n}{r} & \text{if } i = -1; \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** We know that $K_n = \partial K_n \cup \{\rho\}$, where $\{\rho\} = K_n^{n-1}$. Thus for $r \leq n - 1$, we have

$$\mathcal{R}_r(\partial K_n) = \mathcal{R}_r(K_n),$$

whence

$$\beta_i(\mathcal{R}_r(\partial K_n)) = \begin{cases} \binom{n+r-d-2}{r} & \text{if } i = r - 1; \\ 0 & \text{if } 0 \leq i \leq r - 2; \\ \binom{d+1-n}{r} & \text{if } i = -1. \end{cases}$$

Next we consider the case $r \geq n$. Trivially $\beta_i(\mathcal{R}_r(\partial K_n)) = 0$ for $r - 1 \geq i \geq n$. For $i \leq n - 3$,

$$\beta_i(\mathcal{R}_r(\partial K_n)) = \beta_i(\mathcal{R}_r(K_n)) = \begin{cases} 0 & \text{if } 0 \leq i \leq n - 3 \\ \binom{d+1-n}{r} & \text{if } i = -1. \end{cases}$$

For $i = n - 1$, $\mathcal{R}_r(\partial K_n)$ has no $i$-chains, whence $\beta_{n-1}(\mathcal{R}_r(\partial K_n)) = 0$. However, an $(n-1)$-chain of $(\mathcal{R}_r(K_n))$ is of the form $P \cdot [\rho]$ where $\rho$ is the unique $(n-1)$-face and $P \in \mathcal{V}_d^{\langle r-n \rangle} / \ker \hat{\rho}$. If $P \neq 0$ (in $\mathcal{V}_d^{\langle r-n \rangle} / \ker \hat{\rho}$) then $P \hat{\rho} \neq 0$. Thus $\partial_{n-1} P \cdot [\rho] = \sum_{x} P \hat{x} \cdot [\rho/x] \neq 0$ in $\bigoplus_{x \in K_n^{n-2}} \mathcal{V}_d^{\langle r-n+1 \rangle} / \ker \hat{\sigma}$. This implies that the boundary map $\partial_{n-1}$ in $\mathcal{R}_r(K_n)$ is an injection. Consequently $\mathcal{Bd}_{n-2}(\mathcal{R}_r(K_n)) \cong \mathcal{V}_d^{\langle r-n \rangle} / \ker \hat{\rho}$ and

$$\dim \mathcal{Bd}_{n-2}(\mathcal{R}_r(K_n)) = \dim \mathcal{V}_d^{\langle r-n \rangle} / \ker \hat{\rho} = \binom{d+1-n}{r-n}.$$  

For $i = n - 2$, we have $\mathcal{C}_y_{n-2}(\mathcal{R}_r(\partial K_n)) = \mathcal{C}_y_{n-2}(\mathcal{R}_r(K_n))$. But $\mathcal{Bd}_{n-2}(\mathcal{R}_r(\partial K_n)) = 0$. Thus

$$\beta_{n-2}(\mathcal{R}_r(\partial K_n)) = \beta_{n-2}(\mathcal{R}_r(K_n)) + \binom{d+1-n}{r-n} = \binom{d+1-n}{r-n}. \quad \blacksquare$$
10. GLUING AND AMPLE COMPLEXES

A standard construction in 2-rigidity describes when two 2-rigid structures \( \Delta_1 \) and \( \Delta_2 \) in \( d \)-space can be “glued” together on an overlap \( A \) to create a new 2-rigid structure \( \Delta_1 \cup \Delta_2 \), provided \( A \) has at least \( d \) vertices. Although it was not obvious in that setting, the correct analogue for \( r \)-rigidity depends on a Mayer–Vietoris sequence of the homologies [23]:

**Theorem 10.1 (Mayer–Vietoris sequence).** Let \( (\Delta_1; p) \) and \( (\Delta_2; p') \) be subcomplexes of \( (\Delta; p) \) such that \( \Delta = \Delta_1 \cup \Delta_2 \) and \( \Delta' = \Delta_1 \cap \Delta_2 \). Then, abbreviating \( H_i(\Delta) \) for \( H_i(\beta_1(\Delta; p)) \), we have the following long exact sequence:

\[
\cdots \rightarrow H_i(\Delta') \rightarrow H_i(\Delta_1) \oplus H_i(\Delta_2) \rightarrow H_i(\Delta) \rightarrow H_{i-1}(\Delta') \rightarrow \cdots .
\]

Some immediate corollaries are that if \( H_i(\Delta') = 0 \) for all \( i \), then \( H_i(\Delta_1) \oplus H_i(\Delta_2) = H_i(\Delta) \) for all \( i \), and if \( H_i(\Delta_1) = H_i(\Delta_2) = 0 \) for all \( i \), then \( H_i(\Delta) = H_{i-1}(\Delta') \) for all \( i \).

**Example 10.2.** Consider the well-known example where \( d = 3 \), \( r = 2 \), \( i = 0 \). Here we have the usual infinitesimal rigidity. Take both \( \Delta_1 \) and \( \Delta_2 \) to be tetrahedra in general position. Then they are both 3-rigid, or \( \beta_i(\Delta_1) \) and \( \beta_i(\Delta_2) \) are both zero. If their intersection \( \Delta' \) has three vertices, then \( \beta_{-1}(\Delta') = 0 \); consequently, the union is also 2-rigid. If \( \Delta' \) has two vertices, then \( \beta_{-1}(\Delta') = 1 \) and \( \beta_0(\Delta) = 1 \). Thus \( \Delta \) has a nontrivial 2-motion, which is a relative rotation about the hinge formed by the intersection. If \( \Delta' \) has only one vertex, then \( \beta_{-1}(\Delta') = 3 \) and \( \beta_0(\Delta) = 3 \), and the three nontrivial 2-motions are the relative rotations about the joint formed by the intersection.

**Example 10.3.** Consider the following sequence of examples for \( r = 3 \) and \( d = 4 \). Consider two simplicial complexes \( \Delta_1 \) and \( \Delta_2 \) where \( \Delta_1 \) is one tetrahedron, and \( \Delta_2 \) is a second tetrahedron, sharing a single triangle \( \Delta' = \Delta_1 \cap \Delta_2 \). Each of these has the sequence of Betti numbers \( \beta = (\beta_2, \beta_1, \beta_0, \beta_{-1}) = (0, 0, 0, 0) \). Therefore, for each \( i \), the Mayer–Vietoris sequence for this gluing is

\[
H_i(\Delta_1) \oplus H_i(\Delta_2) \rightarrow H_i(\Delta) \rightarrow H_{i-1}(\Delta')
\]

and the union \( \Delta \) has Betti numbers \( \beta = (0, 0, 0, 0) \).

Consider also the result of gluing two triangles along a single edge. The triangles have \( \beta = (0, 0, 0, 0) \), but the single edge, being “too small,” has \( \beta = (0, 0, 1) \). Therefore, the Mayer–Vietoris sequence for this gluing is

\[
0 \oplus 0 \rightarrow H_2(\Delta) \rightarrow 0 \oplus 0 \rightarrow H_1(\Delta) \rightarrow 0 \oplus 0 \rightarrow H_0(\Delta) \rightarrow H_{-1}(\Delta').
\]

We conclude that the union has Betti numbers \( \beta = (0, 0, 1, 0) \).
Finally we glue a third tetrahedron onto the original pair of tetrahedra, along a pair of triangles sharing an edge. The Mayer–Vietoris sequence for the union is

\[ 0 \oplus 0 \rightarrow H_2(\Delta) \rightarrow 0 \rightarrow 0 \oplus 0 \rightarrow H_1(\Delta) \rightarrow H_0(\Delta') \rightarrow 0 \oplus 0 \rightarrow H_0(\Delta) \rightarrow 0 \rightarrow \cdots \]

and we conclude that the union has Betti numbers \( \beta = (0, 1, 0, 0) \).

There are some simple corollaries which follow from the Mayer–Vietoris sequence.

**Proposition 10.4.** If \( \Delta_1, \Delta_2 \) are \( r \)-rigid in \( d \)-space, with overlap \( \Delta' \), then the union \( \Delta_1 \cup \Delta_2 \) is \( r \)-rigid in \( d \)-space if and only if \( \beta_{r-3}(\Delta') = 0 \).

For \( \Delta \) realized in \( d \)-space, we say that \( \Delta \) is \( (r, k) \)-ample if \( \beta_i(\mathcal{R}_r(\Delta)) = 0 \) for all \( -1 \leq i \leq k \). \( \Delta \) is \( r \)-ample if it is \( (r, r-2) \)-ample. \( \Delta \) is \( r \)-adequate if it is \( (r, r-3) \)-ample. \( \Delta \) is \( r \)-perfect if \( \beta_i(\mathcal{R}_r(\Delta)) = 0 \) for all \( -1 \leq i \leq r-1 \).

**Proposition 10.5 (gluing lemma).** Let \( \Delta_1 \) and \( \Delta_2 \) simplicial complexes realized in \( d \)-space. Then

(i) For \( 0 \leq k \leq r-2 \), if \( \Delta_1 \) and \( \Delta_2 \) are \( (r, k) \)-independent and \( \Delta_1 \cap \Delta_2 \) is \( (r, k) \)-rigid, then \( \Delta_1 \cup \Delta_2 \) is \( (r, k) \)-independent.

(ii) \( \text{Rank}_{r-r-2} (\Delta_1) + \text{Rank}_{r-r-2} (\Delta_2) = \text{Rank}_{r-r-2} (\Delta_1 \cap \Delta_2) \) if and only if \( \Delta_1 \cap \Delta_2 \) is \( (r, r-2) \)-rigid.

(iii) If \( \Delta_1, \Delta_2 \) are \( (r, k) \)-ample and \( \Delta_1 \cap \Delta_2 \) is \( (r, k-1) \)-ample, then \( \Delta_1 \cup \Delta_2 \) is \( (r, k) \)-ample.

(iv) If \( \Delta_1, \Delta_2 \) are \( r \)-ample and \( \Delta_1 \cap \Delta_2 \) is \( r \)-adequate, then \( \Delta_1 \cup \Delta_2 \) is \( r \)-ample.

(v) If \( \Delta_1 \) and \( \Delta_2 \) are \( (r, k) \)-rigid and \( \Delta_1 \cap \Delta_2 \) is \( (r, k) \)-adequate, then \( \Delta_1 \cup \Delta_2 \) is \( (r, k) \)-rigid.

(vi) \( \text{If } H_j(\Delta_1) = 0 \text{ and } H_j(\Delta_1 \cap \Delta_2) = 0 \text{ for } j = i, i-1 \text{ then } H_i(\Delta_1 \cup \Delta_2) \cong H_i(\Delta_2) \).

(vii) If \( \Delta_1 \) and \( \Delta_1 \cap \Delta_2 \) are \( r \)-perfect, then \( \beta_i(\Delta_1 \cup \Delta_2) = \beta_i(\Delta_2) \) for all \( i \).

**Proof.** For brevity we write \( A \) for \( \Delta_1 \cap \Delta_2 \), \( B \) for \( \Delta_1 \cup \Delta_2 \), and \( H_i(\Delta_1) \) for \( H_i(\mathcal{R}_r(\Delta)) \), etc.

Case i. We have \( H_{k+1}(\Delta_1 \cap \Delta_2) = 0 \), \( H_{k+1}(\Delta_1) = 0 \), \( H_{k+1}(\Delta_2) = 0 \), and \( H_k(\Delta_1 \cap \Delta_2) = 0 \). Thus the relevant part of the Mayer–Vietoris sequence is

\[ 0 \rightarrow 0 \oplus 0 \rightarrow H_{k+1}(\Delta_1 \cup \Delta_2) \rightarrow 0. \]

Hence \( H_{k+1}(\Delta_1 \cup \Delta_2) = 0 \) and \( \Delta_1 \cup \Delta_2 \) is \( (r, k) \)-independent.
Case ii. Since \( H_r(\Delta) = 0 \) for any simplicial complex \( \Delta \), we have as in the previous case

\[
\beta_{r-1}(\Delta_1) + \beta_{r-1}(\Delta_2) = \beta_{r-1}(\Delta_1 \cap \Delta_2) + \beta_{r-1}(\Delta_1 \cap \Delta_2)
\]

if and only if \( \beta_{r-2}(\Delta_1 \cap \Delta_2) = 0 \). The rest then follows from the definition of rank and

\[
|{(\Delta_1)}^{(r-1)}| + |{(\Delta_2)}^{(r-1)}| = |{(\Delta_1)}^{(r-1)} \cap (\Delta_2)^{(r-1)}| + |{(\Delta_1)}^{(r-1)} \cup (\Delta_2)^{(r-1)}|.
\]

Case iii. For each \( i \leq k \), we have the exact sequence

\[
\cdots \rightarrow 0 \oplus 0 \rightarrow H_i(\Delta_1 \cup \Delta_2) \rightarrow 0 \rightarrow \cdots
\]

This gives \( \beta_i(\Delta \cup \Delta_2) = 0 \) as required. The same proof works for cases iv and v.

Case vi. The relevant band of the Mayer–Vietoris sequence is

\[
\cdots \rightarrow 0 \oplus 0 \rightarrow H_i(\Delta_1 \cup \Delta_2) \rightarrow H_i(\Delta_1) \rightarrow 0 \rightarrow \cdots
\]

Thus \( H_i(\Delta_2) \cong H_i(\Delta_1 \cup \Delta_2) \). Case vii follows from vi.

We now apply the gluing theorem to the family of stacked polytopes defined as follows: A \( d \)-simplex is stacked, and each simplicial \( d \)-polytope obtained from a stacked \( d \)-polytope with one fewer vertex by adding a pyramid over some facet is stacked.

**Proposition 10.6.** A stacked \( d \)-polytope \((\Delta; p)\) realized in \( d \)-space such that \( \sigma \neq 0 \) for every facet \( \sigma \) is \( r \)-perfect for every \( r \).

**Proof.** If \( \Delta \) is a \( d \)-simplex, then \((\Delta; p)\) is \( r \)-perfect. Let \((\Delta; p)\) be a stacked \( d \)-polytope obtained from the stacked \( d \)-polytope \( \Delta' \) by adding the pyramid \( \langle \sigma \rangle \ast v \) where \( \sigma \) is a facet of \( \Delta' \) and \( v \) is a new vertex. Then \((\Delta'; p), \langle \langle \sigma \rangle \rangle \ast \langle \langle \sigma \rangle \rangle \ast v; p)\), and \((\langle \langle \sigma \rangle \rangle; p)\) are all \( r \)-perfect. Thus by the gluing lemma, \((\Delta; p)\) is also \( r \)-perfect.

**11. SHELLABLE COMPLEXES**

A shellable \( d \)-complex is a complex whose facets can be arranged in a linear order: \( p_1, p_2, \ldots, p_k \), so that for each \( i = 2, \ldots, k \), there is unique minimal face \( \sigma_i \) of \( p_i \) such that \( \sigma_j \subseteq p_j \) for all \( j < i \). \( \rho_i, \sigma_i, i = 1, \ldots, k \) is called a shelling sequence of \( \Delta \). Letting \( \Delta_m \) denote \( \cup_{i=1}^m \langle \langle \rho_i \rangle \rangle \), we have

\[
\Delta_m \cap \langle \langle p_{m+1} \rangle \rangle = \partial(\langle \langle \sigma_{m+1} \rangle \rangle) \ast \langle \langle \rho_{m+1} \rangle \rangle / \sigma_{m+1}),
\]

where \( A \ast B \) denotes iterated cones of \( A \) by the vertices of \( B \). We will now see that we can say a lot.
about the homology of our skeletal complex in the case of shellable simplicial complexes, thereby giving large classes of examples which are $r$-ample. The following theorem will be generalized to Cohen–Macaulay complexes later, but the extra geometric information we get in the shellable case is worth the effort.

**Theorem 11.1.** Let $(\Delta; p)$ be a shellable $(d - j)$-complex realized in $d$-space, $j \geq 0$, with the vertices of each $(r - 1)$-dimensional face in general position. Then $H_i(\mathcal{R}_r(\Delta; p)) = 0$ for all $i, i < r - j - 1$.

**Proof.** We may assume that $r \geq j + 1$, for otherwise the conclusion is vacuous. Assume that $\rho_1, \rho_2, \ldots, \rho_k$ is a shelling order for $\Delta$. We use the same notation as in the definition of shelling and let $A = \Delta_m \cap \rho_{m+1} = \partial((\sigma_{m+1})) * (\sigma_{m+1}/\rho_{m+1})$. By Theorem 10.5 we will have $H_i(\Delta_m; p) \cong H_i(\Delta_{m+1}; p)$ if $\beta_i((\rho_{m+1}); p) = \beta_i(A; p) = 0$. Since $r \geq j + 1$, by Proposition 9.1, we have

$$
\beta_i(\langle \rho_{m+1} \rangle; p) = \begin{cases} 
\binom{r - j - 1}{i} = 0 & \text{if } i = r - 1, \\
\binom{r - j - 1}{i} = 0 & \text{if } i = -1, \\
0 & \text{otherwise}.
\end{cases}
$$

Thus the result would follow by induction using the shelling order if we can show that $\beta_{i+1}(A; p) = 0$ for $i \leq r - j - 3$.

By the coning theorem, Theorem 8.2, $\beta_i(\mathcal{R}_r(A; p)) = \beta_i(\mathcal{R}_r(\partial K_i; p))$, where $K_i = \langle \sigma_{m+1} \rangle$, and we regard $(\partial K_i; \Pi p)$ as realized in $j + \ell - 1$-space, after projection from the $d - j - \ell + 1$ vertices of $\rho_{m+1}/\sigma_{m+1}$. Applying Proposition 9.2, we have for $\ell \leq r$,

$$
\beta_i(\mathcal{R}_r(K_i; \Pi p)) = \begin{cases} 
\binom{r - \ell}{i} = 0 & \text{if } i = \ell, \\
\binom{r - \ell}{i} = 0 & \text{if } i = -1, \\
0 & \text{otherwise}.
\end{cases}
$$

If $\ell < r - j$, $\binom{j - \ell}{i} = 0$; thus $\beta_i(\mathcal{R}_r(K_i; \Pi p)) = 0$ for all $i$. If $r - j \leq \ell \leq r$, then $\ell - 2 > j - r - j - 3$; thus $\beta_i(\mathcal{R}_r(K_i; \Pi p)) = 0$ for $i \leq r - j - 3$. Applying Proposition 9.2 again, we have for $\ell > r$

$$
\beta_i(\mathcal{R}_r(K_i; \Pi p)) = \begin{cases} 
\binom{r - j - 1}{i} = 0 & \text{if } i = \ell - 2, \\
\binom{r - j - 1}{i} = 0 & \text{if } i = -1, \\
0 & \text{otherwise}.
\end{cases}
$$

Thus in all cases we have $\beta_i(\mathcal{R}_r(K_i; \Pi p)) = 0$ for $i \leq r - j - 3$ and the proof is complete.

**Corollary 11.2.** A shellable $d$-complex $(\Delta; p)$ realized in $d$-space with the vertices in every facet in general position is $r$-ample and hence $r$-rigid for all $r$. Furthermore, $\beta_{r-1}(\mathcal{R}_r(\Delta; p)) = h_r(\Delta)$.

**Proof.** This is a direct consequence of (3.1) and the previous proposition.
Theorem 11.3. A shellable \((d - 1)\)-complex \((\Delta; p)\) realized in \(d\)-space so that all the facets are in general position is \(r\)-adequate for all \(r\). Furthermore, \(\beta_{r-1}(\mathcal{R}_r(\Delta; p)) - \beta_{r-2}(\mathcal{R}_r(\Delta; p)) = \gamma_r(\Delta)\). More specifically, in adjoining \(\rho_m\), if \(\ell = r - 1\), then

\[
\beta_{r-1}(\mathcal{R}_r(\Delta_m; p)) = \beta_{r-1}(\mathcal{R}_r(\Delta_{m-1}; p)) \\
\beta_{r-2}(\mathcal{R}_r(\Delta_m; p)) = \beta_{r-2}(\mathcal{R}_r(\Delta_{m-1}; p)) + 1.
\]

If \(\ell = r\), then either

\[
\beta_{r-1}(\mathcal{R}_r(\Delta_m; p)) = \beta_{r-1}(\mathcal{R}_r(\Delta_{m-1}; p)) + 1,
\]

with all other \(\beta_i\) unchanged, or

\[
\beta_{r-2}(\mathcal{R}_r(\Delta_m; p)) = \beta_{r-2}(\mathcal{R}_r(\Delta_{m-1}; p)) - 1,
\]

with all other \(\beta_i\) unchanged.

Proof. The first part is a direct consequence of (3.2) and the previous proposition.

When \(\ell = r - 1\), from the proof of the previous proposition, \(\beta_{r-3}(\mathcal{R}_r(\partial K_i; \Pi p)) = 1\) and \(\beta_{r-2}(\mathcal{R}_r(\partial K_i; \Pi p)) = 0\). The relevant band of the Mayer–Vietoris sequence is

\[
0 \rightarrow H_i(\mathcal{R}_r(\Delta_{m-1}; p)) \rightarrow H_i(\mathcal{R}_r(\Delta_m; p)) \rightarrow H_{i-1}(\mathcal{R}_r(\partial K_i; \Pi p)) \rightarrow 0
\]

for \(i = r - 1, r - 2\). Thus we have

\[
\beta_{r-1}(\mathcal{R}_r(\Delta_m; p)) = \beta_{r-1}(\mathcal{R}_r(\Delta_{m-1}; p)) \\
\beta_{r-2}(\mathcal{R}_r(\Delta_m; p)) = \beta_{r-2}(\mathcal{R}_r(\Delta_{m-1}; p)) + 1.
\]

When \(\ell = r\), we have \(\beta_{r-2}(\mathcal{R}_r(\partial K_i; \Pi p)) = 1\). The Mayer–Vietoris sequence now looks like

\[
0 \rightarrow H_{r-1}(\mathcal{R}_r(\Delta_{m-1}; p)) \rightarrow H_{r-1}(\mathcal{R}_r(\Delta_m; p)) \rightarrow H_{r-2}(\mathcal{R}_r(\partial K_i; \Pi p)) \\
\rightarrow H_{r-2}(\mathcal{R}_r(\Delta_{m-1}; p)) \rightarrow H_{r-2}(\mathcal{R}_r(\Delta_m; p)) \rightarrow 0.
\]

We see that either 1 is added to \(H_{r-1}(\mathcal{R}_r(\Delta_m; p))\) or 1 is subtracted from \(H_{r-2}(\mathcal{R}_r(\Delta_m; p))\), as compared to \(H_{r-1}(\mathcal{R}_r(\Delta_{m-1}; p))\) or \(H_{r-2}(\mathcal{R}_r(\Delta_{m-1}; p))\), respectively.
12. CORRESPONDENCES AND SPECTRAL SEQUENCES

The previous section showed that all shellable $d$-complex are $r$-ample and hence $r$-rigid in $d$-space and $r$-adequate in $(d + 1)$-space. These results extend to arbitrary Cohen–Macaulay $d$-complexes. These are defined as follows. Consider a simplicial complex $\Delta$. For any face $\sigma$, the link of $\sigma$ is defined to be $\text{Link}_\sigma(\Delta) = \{ \tau : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta \}$. This is often abbreviated to $\text{Link}_\sigma$. $\Delta$ is a Cohen–Macaulay complex (over the real field $\mathbb{R}$) if $\tilde{H}_i(\text{Link}_\sigma) = 0$ for all $\sigma \in \Delta$ and all $i < \dim \text{Link}_\sigma$. These include shellable complexes. The Cohen–Macaulay 2-complexes are (i) topological spheres, (ii) topological discs, and (iii) topological wedges of spheres and balls.

In addition we are able to show that the $h$-vector of any simplicial $(d)$-sphere $\tilde{\Delta}$ is symmetric while the $g$-vector is anti-symmetric, i.e., $h_i(\tilde{\Delta}) = h_{d+1-r}(\tilde{\Delta})$ and $g_i(\tilde{\Delta}) = -g_{d+1-r}(\tilde{\Delta})$. When $(\Delta; \mathbf{p})$ is realized in $(d + 1)$-space, we also have the correspondence $H_{d-r}(\text{Link}_\sigma) \cong H_{d-r}(\text{Link}_{\sigma'})$. In terms of skeletal rigidity, this means that there is an isomorphism between the space of $r$-stresses and the space of nontrivial $(d + 1 - r)$-motions of $(\Delta; \mathbf{p})$. These correspondences are derived using a homological method known as spectral sequence (see [25]). What follows is a translation and extension of the results of Oda [24].

Consider an $n$-complex $(\Delta; \mathbf{p})$ realized in $d$-space. We assume that $d \geq n$ and that the vertices of each facet are in general position. The $r$-skeletal chain complex $R_r(\Delta; \mathbf{p})$ is

$$0 \to \bigoplus_{\rho \in \Delta^{(n-1)}} V_{d+1}(\mathbb{R}/\text{Ker} \rho) \to \cdots \to \bigoplus_{\rho \in \Delta^{(n-1)}} V_{d+1}(\mathbb{R}/\text{Ker} \rho) \to V_d(\mathbb{R}) \to 0.$$ 

The $(d + 1)$-skeletal cochain complex $R_{d+1}(\Delta; \mathbf{p})$ is

$$0 \leftarrow \bigoplus_{\rho \in \Delta^{(n-1)}} V_{d+1}(\mathbb{R}/\text{Ker} \rho) \leftarrow \cdots \leftarrow \bigoplus_{\rho \in \Delta^{(n-1)}} V_{d+1}(\mathbb{R}/\text{Ker} \rho) \leftarrow V_{d+1}(\mathbb{R}) \leftarrow 0.$$ 

Since $V_{d+1}(\mathbb{R}/\text{Ker} \rho) = \mathbb{R}/\text{Ker} \rho = \mathbb{R}$ for all $\rho$ if $\rho \neq 0$, the cochain complex is the usual augmented cochain complex of the abstract complex $\Delta$:

$$0 \leftarrow \bigoplus_{\rho \in \Delta^{(n-1)}} \mathbb{R} \leftarrow \cdots \leftarrow \bigoplus_{\rho \in \Delta^{(n-1)}} \mathbb{R} \leftarrow \mathbb{R} \leftarrow 0.$$ 

We now take the tensor product of these two, creating a bicomplex $R_*(\Delta; \mathbf{p})$ with entries

$$K_{p, q} = \bigoplus_{\rho \in \Delta^{(n-p)}} V_{d+1}/\text{Ker} \rho \otimes V_{d+1}/\text{Ker} \rho$$

$$= \bigoplus_{\rho \in \Delta^{(n-p)}} \mathbb{R} \otimes V_{d+1}/\text{Ker} \rho.$$
Note that $K_{p,q} = 0$ if $p < 0$ or $q < 0$ or $p > n$ or $q > r$ or $p + q > n$. The boundary operators are $d'$ and $d''$ where $d'' = (-1)^p \text{id} \otimes \partial$ and $d' = \delta \otimes \text{id}$ (see [25]). It is easy to check that $d'd' = 0$, $d''d'' = 0$. Also $d'd'' + d''d' = 0$; i.e., the following diagram anticommutes:

$$
\begin{array}{c}
\bigoplus_{\rho \in \Delta^{(n-p)}} \bigotimes_{\eta \in \Delta^{(n-1)}} \mathbb{R} \otimes V^{(r-q-1)}_{d+1}/\text{Ker} \, \tilde{\tau} & \xleftarrow{\delta \otimes \text{id}} & \bigoplus_{\rho \in \Delta^{(n-1)}} \bigotimes_{\eta \in \Delta^{(n-p)}} \mathbb{R} \otimes V^{(r-q-1)}_{d+1}/\text{Ker} \, \tilde{\tau} \\
\downarrow \text{id} \otimes \partial & & \downarrow -\text{id} \otimes \partial \\
\bigoplus_{\rho \in \Delta^{(n-p)}} \bigotimes_{\eta \in \Delta^{(n-1)}} \mathbb{R} \otimes V^{(r-q)}_{d+1}/\text{Ker} \, \tilde{\mu} & \xleftarrow{\delta \otimes \text{id}} & \bigoplus_{\rho \in \Delta^{(n-1)}} \bigotimes_{\eta \in \Delta^{(n-p)}} \mathbb{R} \otimes V^{(r-q)}_{d+1}/\text{Ker} \, \tilde{\mu}
\end{array}
$$

Thus we have the following diagram for the bicomplex $\mathcal{H}_r(\Delta; p)$:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \xleftarrow{K_0, \rho} \bigoplus_{\rho \in \Delta^{(n-p)}} \bigotimes_{\eta \in \Delta^{(n-1)}} \mathbb{R} \otimes V^{(r-q-1)}_{d+1}/\text{Ker} \, \tilde{\tau} & \xleftarrow{\delta \otimes \text{id}} & \bigoplus_{\rho \in \Delta^{(n-1)}} \bigotimes_{\eta \in \Delta^{(n-p)}} \mathbb{R} \otimes V^{(r-q-1)}_{d+1}/\text{Ker} \, \tilde{\tau} & 0 \\
\downarrow \text{id} \otimes \partial & \downarrow \text{id} \otimes \partial & \downarrow (-1)^{r-1} \text{id} \otimes \partial & \downarrow \\
\vdots & \vdots & \vdots & \vdots \\
0 & \xleftarrow{K_0, 1} \bigoplus_{\rho \in \Delta^{(n-p)}} \bigotimes_{\eta \in \Delta^{(n-1)}} \mathbb{R} \otimes V^{(r-q)}_{d+1}/\text{Ker} \, \tilde{\mu} & \xleftarrow{\delta \otimes \text{id}} & \bigoplus_{\rho \in \Delta^{(n-1)}} \bigotimes_{\eta \in \Delta^{(n-p)}} \mathbb{R} \otimes V^{(r-q)}_{d+1}/\text{Ker} \, \tilde{\mu} & 0 \\
\downarrow \text{id} \otimes \partial & \downarrow \text{id} \otimes \partial & \downarrow (-1)^{r-1} \text{id} \otimes \partial & \downarrow \\
0 & \xleftarrow{K_0, 0} \bigoplus_{\rho \in \Delta^{(n-p)}} \bigotimes_{\eta \in \Delta^{(n-1)}} \mathbb{R} \otimes V^{(r-q)}_{d+1}/\text{Ker} \, \tilde{\mu} & \xleftarrow{\delta \otimes \text{id}} & \bigoplus_{\rho \in \Delta^{(n-1)}} \bigotimes_{\eta \in \Delta^{(n-p)}} \mathbb{R} \otimes V^{(r-q)}_{d+1}/\text{Ker} \, \tilde{\mu} & 0 \\
\downarrow \text{id} \otimes \partial & \downarrow \text{id} \otimes \partial & \downarrow (-1)^{r-1} \text{id} \otimes \partial & \downarrow \\
0 & 0 & 0 & 0
\end{array}
$$

Let $H'_{p,q} = \text{kernel} (d': K_{p,q} \rightarrow K_{p-1,q})/\text{image} (d': K_{p+1,q} \rightarrow K_{p,q})$.

$H''_{p,q}$ is similarly defined. More concretely, we have, for each fixed $q$ and $\tau \in \Delta^{(q-1)}$, that the $\tau$ components of $K_{p,q}$ form the following complex:

$$
\begin{array}{c}
0 & \xleftarrow{\bigoplus_{\rho \in \Delta^{(n)}} \mathbb{R} \otimes T} & \xleftarrow{\bigoplus_{\rho \in \Delta^{(n-1)}} \mathbb{R} \otimes T} \\
\vdots & \vdots & \vdots \\
\bigoplus_{\rho \in \Delta^{(n)}} \mathbb{R} \otimes T & \xleftarrow{\bigoplus_{\rho \in \Delta^{(n-1)}} \mathbb{R} \otimes T} & \mathbb{R} \otimes T
\end{array}
$$
where $T = V_d^{(r-q)}/Ker \tilde{\tau}$. This is isomorphic to the complex

$$0 \leftarrow \left( \bigoplus_{\rho \in \text{rk}(\text{Link})^{n-q}} \mathbb{R} \right) \otimes T \xrightarrow{\partial \oplus \text{id}} \left( \bigoplus_{\sigma \in \text{rk}(\text{Link})^{n-1-q}} \mathbb{R} \right) \otimes T \xrightarrow{\partial \oplus \text{id}} \cdots$$

This is the usual augmented cochain complex of $\text{Link}$ tensor with a constant $V^{(r-q)}/Ker \tilde{\tau}$. If the links of $\Delta$ have nice homology, then we can calculate $H'$ explicitly as in the case where $\Delta$ is a homology sphere.

**Lemma 12.1.** If $(\Delta; p)$ is a homology n-sphere realized in $d$-space, then

$$H'_{p,q} = \begin{cases} \bigoplus_{\sigma \in \Delta^{(r-1)}} V^{(r-q)}/Ker \tilde{\sigma} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

Moreover, the boundary operator for the chain complex $H'_{p,q}$ coincides with the boundary operator for the r-skeletal chain complex.

**Proof.** The links of a homology sphere are also homology spheres. Thus the reduced cohomology vanishes except at the top where it is $\mathbb{R}$. The result then follows. \(\blacksquare\)

$H'$ is again a bicomplex with boundary operators induced by $d'$ and $d''$. The boundary operator $\tilde{d}'_{p,q} : H'_{p,q} \to H'_{p,q-1}$ induced by $d''$ is defined as follows:

$$\tilde{d}'_{p,q} : [z_{p,q}] \mapsto [d''_{p,q}(z_{p,q})],$$

where $[\cdot]$ denotes homology class. The boundary $\tilde{d}'$ induced by $d'$ is similarly defined.

For the case where $(\Delta; p)$ is a homology sphere, the induced boundary operators are all trivial except $\tilde{d}'_{0,q} = \tilde{\delta}_q$, the usual boundary operator for $r$-skeletal chain complex. The following shows the diagram of $H'$ for the
case in which \((\Delta; p)\) is a homology sphere:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \bigoplus_{\tau \in \Delta^{(r-1)}} V^{(0)}/\text{Ker } \hat{\tau} & \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow 0 \\
\downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \bigoplus_{v \in \Delta^{(r-1)}} V^{(r-1)}/\text{Ker } \tilde{v} & \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & V^{(r)} & \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow 0 \leftarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

**Lemma 12.2.** If \((\Delta; p)\) is a homology \(n\)-sphere realized in \(d\)-space, then

\[
H''_{p, q} H'_{p, q} = \begin{cases} 
H_{q-1}(\hat{\mathcal{O}}(\Delta; p)) & \text{if } p = 0 \\
0 & \text{if } p \neq 0.
\end{cases}
\]

**Proof.** The result is an immediate consequence of Lemma 12.1. \(\Box\)

Next we compute \(H'_{p, q}'\).

**Lemma 12.3.** For any \(n\)-complex \((\Delta; p)\) realized in \(d\)-space, \(d \geq n\), if the vertices for each face are in general position, then

\[
H'^{n}_{p, q} = \begin{cases} 
\mathbb{R} \otimes \bigoplus_{\sigma \in \Delta^{(n-p)}} V^{(r)}/\text{Ker } \tilde{\sigma} & \text{if } q = 0 \\
0 & \text{if } q \neq 0.
\end{cases}
\]

Moreover, for \(q = 0\), the chain complex \(H'^{n}_{p, 0}\) with its induced boundary oper-
ator is isomorphic with the \((d + 1 - r)\)-skeletal cochain complex of \((\Delta; p)\).

**Proof.** For each fixed \(p\) and \(\sigma \in \Delta^{(n-p)}\), we have the complex

\[
0 \rightarrow \mathbb{R} \otimes \bigoplus_{\mu \in \Delta^{(r-1)}} V^{(0)}/\text{Ker } \hat{\rho} \xrightarrow{id \otimes \phi} \mathbb{R} \otimes \bigoplus_{\mu \in \Delta^{(r-2)}} V^{(1)}/\text{Ker } \tilde{\mu} \xrightarrow{id \otimes \phi} \\
\cdots \xrightarrow{id \otimes \phi} \mathbb{R} \otimes \bigoplus_{\mu \in \Delta^{(r-1)}} V^{(r-1)}/\text{Ker } \tilde{v} \xrightarrow{id \otimes \phi} \mathbb{R} \otimes V^{(r)}_{d+1} \rightarrow 0.
\]

This is \(\mathbb{R} \otimes \hat{\mathcal{O}}_r(\{(\sigma)\}; p)\). The homology of this complex vanishes except for \(H_{-1}\) which is equal to \(V^{(r)}/\text{Ker } \tilde{\sigma}\). (Note that this vanishes also if \(n - p + 1 + r \geq d + 2\), or \(p \leq n + r - d - 1\).)
The induced boundary operators are all trivial except \( \tilde{d}_{n,0} = \delta_n \) which is the coboundary operator for the \((d + 1 - r)\)-skeletal cochain complex of \((\Delta; p)\). Thus we have the isomorphism of the following chains and cochains:

\[
0 \leftarrow H''_{n+1,0} \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} H''_{n,0} \overset{\partial}{\leftarrow} H''_{n-1,0} \leftarrow 0
\]

\[
0 \leftarrow \bigoplus_{\rho \in \Delta^{(d-r)}} V^{(r)}/\text{Ker} \tilde{\rho} \overset{\delta}{\leftarrow} \cdots \overset{\delta}{\leftarrow} \bigoplus_{v \in \Delta^{(0)}} V^{(r)}/\text{Ker} \tilde{v} \overset{\delta}{\leftarrow} V^{(r)} \leftarrow 0.
\]

As a result we have the following diagram for \( H'' \):

\[
\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
\downarrow & \downarrow & \cdots & \downarrow \\
0 & 0 & \cdots & 0 \\
\downarrow & \downarrow & \cdots & \downarrow \\
\vdots & \vdots & \cdots & \vdots \\
\downarrow & \downarrow & \cdots & \downarrow \\
0 & 0 & \cdots & 0 \\
\downarrow & \downarrow & \cdots & \downarrow \\
0 & \bigoplus_{\rho \in \Delta^{(n)}} V^{(r)}/\text{Ker} \rho \overset{\delta}{\leftarrow} \cdots \overset{\delta}{\leftarrow} \bigoplus_{v \in \Delta^{(0)}} V^{(r)}/\text{Ker} v \overset{\delta}{\leftarrow} V^{(r)} & \leftarrow 0 \\
\downarrow & \downarrow & \cdots & \downarrow \\
0 & 0 & \cdots & 0
\end{array}
\]

**Lemma 12.4.** For any \( n \)-complex \((\Delta; p)\) in \( d \)-space, \( d \geq n \), such that the vertices of each face are in general position, we have

\[
H' H''_{p, q} = \begin{cases} 
H^{n-p}(\mathcal{R}^{d+1-r}(\Delta; p)) & \text{if } q = 0 \\
0 & \text{if } q \neq 0.
\end{cases}
\]

**Proof.** This follows from Lemma 12.3. \( \square \)

For any bicomplex \( \mathcal{K}_r(\Delta; p) \) there is an associated total complex \( \text{Tot}(\mathcal{K}_r(\Delta; p)) \) whose \( n \) component is

\[
(\text{Tot}(\mathcal{K}_r(\Delta; p)))_n = \bigoplus_{p+q=n} K_{p,q}
\]

and whose boundary operator is \( d' + d'' \). This is indeed a boundary operator as

\[
(d' + d'')^2 = (d')^2 + (d'')^2 + (d'd'' + d''d') = 0.
\]

When most of \( H' \) and \( H'' \) are zero, the homology of the total complex can be computed in terms of the iterated homologies \( H' H'' \) and \( H'' H' \).
Lemma 12.5 [25, Lemma 11.20]. If $H_{p,q} = 0$ when $p \neq 0$, then $H_q(\text{Tot}(\mathcal{R}_r(\Delta; p))) = H^qH'_0$. If $H_{p,q} = 0$ when $q \neq 0$, then $H_p(\text{Tot}(\mathcal{R}_r(\Delta; p))) = H^pH'_0$.

Theorem 12.6. If $(\Delta; p)$ is a homology $n$-sphere realized in $d$-space, $d \geq n$, such that the vertices of each face are in general position, then

$$H_i(\mathcal{R}_r(\Delta; p)) \cong H^{n-1-i}(\mathcal{R}^{d+1-r}(\Delta; p)).$$

Moreover $H_i(\mathcal{R}_r(\Delta; p)) = 0$ if $i \leq n + r - d - 2$.

Proof. The first conclusion follows from Lemmas 12.2, 12.4, and 12.5. For the second conclusion we need only note that the right hand side is 0 if $n - i \geq d + 1 - r$.

Corollary 12.7. Let $(\Delta; p)$ be a homology $n$-sphere realized in $d$-space such that the vertices of each face are in general position.

(i) Suppose $n = d$. Then for all $r$, $H_i(\mathcal{R}_r(\Delta; p)) = 0$ if $i \neq r - 1$ and $H_{r-1}(\mathcal{R}_r(\Delta; p)) \cong H^{d-r}(\mathcal{R}^{d+1-r}(\Delta; p))$. In particular $(\Delta; p)$ is $r$-rigid for all $r$.

(ii) Suppose $n = d$. Then $h_r(\Delta) = \beta_{r-1}(\mathcal{R}_r(\Delta; p))$.

(iii) Suppose $n + 1 = d$. Then for all $r$, $H_i(\mathcal{R}_r(\Delta; p)) = 0$ if $i \neq r - 1$, $r - 2$, and $H_{r-1}(\mathcal{R}_r(\Delta; p)) \cong H^{d-r-1}(\mathcal{R}^{d+1-r}(\Delta; p))$. This means we have an isomorphism between the space of $r$-stresses and the space of nontrivial $d+1-r$-motions of $(\Delta; p)$.

(iv) Suppose $n + 1 = d$. Then $g_r(\Delta) = \beta_{r-1}(\mathcal{R}_r(\Delta; p)) - \beta_{r-2}(\mathcal{R}_r(\Delta; p))$.

Proof. Parts (i) and (iii) are direct consequences of Theorem 12.6. Parts (ii) and (iv) are consequences of Parts (i) and (iii) as well as identities (3.1) and (3.2).

Remark 12.8. The above result gives a correspondence between the $r$-stresses and $(d + 1 - r)$-motions of homology $d$-spheres realized in $d + 1$-space such that the vertices in every facet are in general position. This is already known for $r = 2$. In this case Maxwell’s theorem [7, 8] gives an explicit construction of 2-stresses from $(d - 1)$-motions and vice versa. An explicit construction for $r = 1$ is also known. These explicit constructions actually give a correspondence for a broader class of complexes. For $r = 1$, the correspondence works for all oriented manifolds. For $r = 2$, and homological cycles, every 2-motion generates a $d - 1$-stress. The converse, however, holds only for homology spheres. However, for $r > 2$, the question remains open.

It also gives another interpretation of the Dehn–Sommerville relations for spheres.
Corollary 12.9. Let $\Delta$ be a homology $d$-sphere. Then

(i) $h_i(\Delta) = h_{d+1-i}(\Delta)$;
(ii) $g_i(\Delta) = -g_{d+2-i}(\Delta)$.

Proof. These follow from the previous Corollary 12.7.

We can draw similar conclusions for general Cohen–Macaulay complexes (including balls). The only difference is that for a Cohen–Macaulay $n$-complex, the only non-zero cohomology which occurs at the top is no longer $\mathbb{R}$. Thus we have the following lemma.

Lemma 12.10. If $(\Delta; p)$ is a Cohen–Macaulay $n$-complex realized in $d$-space, then

$$H'_{p,q} = \begin{cases} \bigoplus_{\sigma \in \Delta^{(p-1)}} V^{(r-q)}/\ker \tilde{\sigma} \otimes H^{n-q}(\text{Link}_\sigma) & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

Moreover, $H''H'_{p,q} = 0$ if $p \neq 0$ or $q \geq r + 1$.

Proof. The first conclusion follows from the fact that the links of Cohen–Macaulay complexes are again Cohen–Macaulay and that all their cohomologies vanish except at the top dimension. That $H''H'_{p,q} = 0$ if $p \neq 0$ follows from the fact that $H'_{p,q} = 0$ if $p \neq 0$. The final conclusion follows from the fact that $V^{(r-q)}/\ker \tilde{\sigma} = 0$ if $q \geq r + 1$.

Theorem 12.11. If $(\Delta; p)$ is a Cohen–Macaulay $n$-complex realized in $d$ space such that the vertices of each facet are in general position, then $H_i(\mathring{\Delta}; \Delta; p)) = 0$ if $i \leq n + r - d - 2$.

Proof. By Lemmas 12.10, 12.5, 12.4, and 12.3 we have

$$H^{n-1}(\mathring{\Delta}^{d+1-r}(\Delta; p)) \cong H'_{i,0} \cong H''_{0,0} = 0$$

if $j \geq r + 1$. Since $r = d + 1 - (d + 1 - r)$ and $i = n - (n - i)$, we have $H_i(\mathring{\Delta}; \Delta; p)) \cong H(\mathring{\Delta}; \Delta; p)) = 0$ if $n - i \geq d + 2 - r$.

Corollary 12.12. If $\Delta$ is an $d$-dimensional Cohen–Macaulay complex, then $h_i(\Delta) \geq 0$ for all $i$.

We can say something stronger about balls because we know the cohomologies at the top dimension.

Lemma 12.13. If $(\Delta; p)$ is a homology $n$-ball realized in $d$-space and $\Delta_0$ denotes its interior, then

$$H'_{p,q} = \begin{cases} \bigoplus_{\sigma \in \Delta^{(p-1)}} V^{(r-q)}/\ker \tilde{\sigma} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$
Proof. First we note that the homologies of a ball vanish in all dimensions. Next we note that the link of a face in the boundary is again a ball while that of a face in the interior is a sphere. The result then follows.

**Theorem 12.14.** Let $\Delta$ be a homology $n$-ball, and let $\Delta_0$ denote its interior. If $(\Delta; p)$ is realized in $d$-space ($d \geq n$) such that the facets are in general position, then

$$H_1(\mathcal{R}_f(\Delta_0; p)) \cong H^{n-1-i}(\mathcal{R}^{d+1-r}(\Delta; p)).$$

Moreover $H_i(\mathcal{R}_f(\Delta_0; p)) \cong H^{n-1-i}(\mathcal{R}^{d+1-r}(\Delta; p)) = 0$ if $i \leq n + r - d - 2$. In particular, if $d = n$, we have

$$H_{r-1}(\mathcal{R}_f(\Delta_0; p)) \cong H^{d-r}(\mathcal{R}^{d+1-r}(\Delta; p))$$

and if $d = n + 1$, we have

$$H_{r-1}(\mathcal{R}_f(\Delta_0; p)) \cong H^{d-r-1}(\mathcal{R}^{d+1-r}(\Delta; p)),$$

$$H_{r-2}(\mathcal{R}_f(\Delta_0; p)) \cong H^{d-r}(\mathcal{R}^{d+1-r}(\Delta; p))$$

while all the other homologies are 0.

Remark 12.15. There is an interesting interpretation of this result in mechanics. If $n = 2$, $d = 3$, and the boundary of the ball is “pinned” then the nontrivial 2-motions of the pinned ball correspond to the 2-stresses of the unpinned ball while the 2-stresses of the pinned ball correspond to the 2-motions of the unpinned ball. Such a correspondence can be shown constructively. For general $n$ and $d \geq n + 1$, the result says there is such a correspondence between $n$-stresses and $(d + 1 - n)$-motions. However, this still cannot be shown constructively.

13. CONJECTURES, OPEN PROBLEMS, AND CONCLUDING REMARKS

In this concluding section, we discuss some areas of further work.

13.1. The $g$-Theorem

One of the recent achievements in the theory of convex polytopes is the characterization of the $f$-vectors of convex polytopes. This result is known as the $g$-theorem. The sufficiency part was established by Billera and Lee [3] while the necessity part was due to Stanley [26, 27]. The essential part of Stanley’s proof is that $g_r(\Delta) \geq 0$, $r \leq \lceil (d + 1)/2 \rceil$, for the boundary complex $\Delta$ of a convex $d$-polytope. We have shown in Corollary 12.7 that for a homology $(d - 1)$-sphere $(\Delta; p)$ realized in $d$-space, $g_r(\Delta) = \beta_{r-1}(\mathcal{R}_f(\Delta; p)) - \beta_{r-2}(\mathcal{R}_f(\Delta; p))$. Thus we can extend the $g$-theorem to the class of homology spheres if the following conjecture is true.
Conjecture 13.1. Let $\Delta$ be a homology $(d - 1)$-sphere. Then there is a realization $(\Delta; p)$ in $d$-space such that $\beta_{r-2}(R_r(\Delta; p)) = 0$ if $r \leq [(d + 1)/2]$. In other words, $(\Delta; p)$ is $r$-rigid.

A series of works [1, 4, 9, 12, 13, 15, 35, etc.] starting with that of Cauchy has shown that the conjecture is true for $r = 2$ for a large class of $\Delta$ which includes homology spheres. The strongest result is the following.

Theorem 13.2. Let $\Delta$ be a simplicial $d$-manifold (without boundary), $d \geq 2$. Then there is realization $(\Delta; p)$ in $(d + 1)$-space such that $\beta^0(R^2(\Delta; p)) = 0$. In other words $(\Delta; p)$ is $2$-rigid.

Proof. We give a sketch of the proof as it is informative. Let $\mathcal{C}$ be the class of simplicial $d$-complexes, $d \geq 2$. The $2$-complexes in $\mathcal{C}$ are just the simplicial $2$-manifolds. For any $d$-complex $\Delta \in \mathcal{C}$, $d \geq 3$, and any face $\sigma \in \Delta$ such that $\dim \text{Link}_\sigma(\Delta) \geq 2$, we have $\text{Link}_\sigma(\Delta) \in \mathcal{C}$. Then $\mathcal{C}$ includes all simplicial $d$-manifolds. The case $d = 2$ is just Fogelsanger’s theorem [12]. (In fact he has a direct proof for $d = 2$.)

Let $(\Delta; p)$ be a $d$-dimensional simplicial complex in $\mathcal{C}$ realized generically in $d + 1$-space, $d = 3$. Let $v$ be any vertex of $\Delta$. Then the projection $\Pi$ of $\text{Link}_v(\Delta)$ into a three-dimensional subspace is a two-dimensional complex of $\mathcal{C}$ realized generically in $3$-space. Hence $\beta_0(\mathcal{R}_v(\text{Link}_v(\Delta); \Pi p)) = 0$. Now $(\text{Link}_v(\Delta) * v; p)$ is a subcomplex of $(\Delta; p)$. By the coning Theorem 8.2, $\beta_0(\mathcal{R}_v(\text{Link}_v(\Delta) * v; p)) = 0$. Now $\Delta$ is the union of the cones $\text{Link}_v(\Delta) * v, v \in \Delta^0$. By Proposition 10.5 (gluing lemma), we can then prove that $\beta_0(\mathcal{R}_\sigma(\Delta; p)) = 0$ as required. The complete proof can be carried out in the same way by induction.

A number of people [11, 17, 18, 28] have tried unsuccessfully to prove the conjecture for the class of pl-spheres.

In another direction Whiteley [35] proved a version of the conjecture for nonsimplicial $d$-polytopes whose two-dimensional faces are triangulated. This result then leads immediately to a result concerning the $h_2$ component of the generalized $h$-vectors of nonsimplicial $d$-polytopes. As a step in direction for the $h_r$ component of the generalized $h$-vector, we boldly put forward the following conjecture.

Conjecture 13.3. Let $\Delta$ be the $r$-skeleton of a homology $(d - 1)$-sphere whose $r$-faces are triangulated realized. Then there is a realization $(\Delta; p)$ in $d$-space such that $\beta_{r-2}(\mathcal{R}_r(\Delta; p)) = 0$ if $r \leq [(d + 1)/2]$. In other words, $(\Delta; p)$ is $r$-rigid.

13.2. McMullen’s Polytope Algebra

McMullen [20, 21] has recently given a somewhat simpler proof of the original $g$-theorem for convex polytopes using the idea of polytope algebra
and the algebra of weights of polytopes. Lee [18] has shown that $r$-weights of a polytope are equivalent to its $r$-stresses. The weights are defined only for convex polytopes. Can one define an algebra of $r$-stresses for arbitrary simplicial complexes? This should have interesting consequences.

13.3. Lower Homologies

In Section 7, we show that for any $(\Delta; \mathbf{p})$ realized in $d$-space, $H^{(0)}(\mathcal{R}^d(\Delta))$ is isomorphic to the space of nontrivial parallel redrawings of the 1-skeleton of $\Delta$. The space of parallel redrawings of the 1-skeleton of the a convex polytope is isomorphic to its space of Minkowski summands [16, 39]. We believe that all the homologies and cohomologies have interesting geometric interpretations.

13.4. The Symmetry of the $h$-Vector

In Corollaries 12.9 and 12.7, we prove that for a homology $d$-sphere $\Delta$, $h_r(\Delta) = h_{d+1-r}(\Delta)$. This is essentially the Dehn–Sommerville relation. Let us take the case $d = 2$. Here $\Delta$ is just the boundary complex of a convex simplicial polytope. When $r = 2$, we have

$$\beta_0(\mathcal{R}_1(\Delta; \mathbf{p})) = h_1(\Delta) = h_2(\Delta) = \beta_1(\mathcal{R}_2(\Delta; \mathbf{p})).$$

where $(\Delta; \mathbf{p})$ is some realization in 2-space. From Proposition 7.1, we have $h_1(\Delta) = f_0(\Delta) - 3$ while $h_2(\Delta) = \dim \text{Stress}_2(\Delta; \mathbf{p})$. Now take a realization $(\Delta; \mathbf{q})$ in 3-space such that for a certain projection we have $\Pi \mathbf{q} = \mathbf{p}$. A polarity in 3-space will send $(\Delta; \mathbf{q})$ to its geometric dual $(\Delta'; \mathbf{q'})$. (Note that $\Delta'$ is now a simple polytope.) It then projects this pair to a pair of “reciprocal” figures in the plane $(\Delta; \Pi \mathbf{q}) = (\Delta; \mathbf{p})$ and $(\Delta'; \Pi \mathbf{q'})$. One can choose a polarity so that corresponding edges of $(\Delta; \mathbf{p})$ and $(\Delta'; \Pi \mathbf{q'})$ are perpendicular. This pair then forms what is called a pair of reciprocal figures. Geometrically, every reciprocal $(\Delta'; \Pi \mathbf{q'})$ of $(\Delta; \mathbf{p})$ gives rise to a 2-stress of $(\Delta; \mathbf{p})$. The polytope $(\Delta; \mathbf{q})$ in 3-space is called a lifting of $(\Delta; \mathbf{p})$. The lifts in which the vertices of $(\Delta; \mathbf{q})$ are coplanar are called the trivial lifts. If this space is factored out, we get what is called the space of nontrivial lifts. The dimension of this space is equal to that of the space of 2-stresses of $(\Delta; \mathbf{p})$. It is easy to see that this dimension is also $f_0(\Delta) - 3$. (Simply leave three vertices of a 2-face in the plane and lift the other vertices into 3-space.) This explains why $h_1(\Delta) = h_2(\Delta)$. (For more detail the reader can consult [7, 8, 33].) Such an explanation for the Dehn–Sommerville relations also exists for the boundary complex of an arbitrary simplicial $d$-polytope. Other aspects of duality can also be explored. (See [36, 37] for those already for the case $r = 2$.)
13.5. The anti-symmetry of the $g$-Vector

In Corollary 12.7 we proved that for a homology $(d-1)$-sphere $(\Delta; p)$ in $d$-space,

$$H_{d-1}(\mathcal{R}_r(\Delta; p)) \cong H^{d-r-1}(\mathcal{R}^{d+1-r}(\Delta; p)).$$

The left hand side is $\text{Stress}_r(\Delta; p)$ while the right hand side is $\text{NonTriv}_{d+1-r}(\Delta; p)$. For the case $r = 2$ and $d = 3$, there is a way to construct a 2-motion from a 2-stress [33]. However, such a concrete construction is unknown for other values of the parameters. From Corollary 12.9, we have $g_r(\Delta) = -g_{d+2-r}(\Delta)$. If we assume that 13.1 is true then $\beta_{d-r}(\mathcal{R}_r(\Delta; p)) = g_r(\Delta)$ and $\beta^{d-r}(\mathcal{R}^{d+2-r}(\Delta; p)) = -g_{d+2-r}(\Delta)$; $(\Delta; p)$ is some realization in $(d + 1)$-space. The construction that we seek would then give a very nice geometric explanation for the anti-symmetry of the $g$-vector.

13.6. Constructions

For the case $r = 2$, there are several methods for constructing 2-perfect frameworks starting with the simplex [31, 38]. It would nice to have analogues of these methods developed for arbitrary $r$.

REFERENCES