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Note

The chromaticity of wheels with a missing spoke II

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Abstract

In the previous paper, it was shown that the graph U_{n+1} obtained from the wheel W_{n+1} by deleting a spoke is uniquely determined by its chromatic polynomial if $n \geqslant 3$ is odd. In this paper, we show that the result is also true for even $n \geqslant 4$ except when $n = 6$ in which case, the graph W given in the paper is the only graph having the same chromatic polynomial as that of U_7. The relevant tool is the notion of nearly uniquely colorable graph.

As in [2], only finite undirected graphs without loops or multiple edges will be considered. A graph G is *chromatically unique* if it is uniquely determined by its chromatic polynomial $P(G;\lambda)$. The *wheel* W_{n+1} is obtained by taking the join of a single vertex and the cycle C_n on n vertices. The graph U_{n+1} is obtained from W_{n+1} by deleting a *spoke* which is an edge joining the single vertex to a vertex on C_n. In [2], it is shown that U_{n+1} is chromatically unique if $n \geqslant 3$ is odd. In this paper we shall show that the result extends to all $n \geqslant 3$ except for $n = 6$ in which case, the graph W of Fig. 1 is the only graph having the same chromatic polynomial as that of U_7. The relevant tool is the notion of nearly uniquely colorable graph. A graph is *nearly uniquely s-colorable* if it has chromatic number s and there are precisely two ways of partitioning its vertex set into s independent subsets, up to permutation of these independent subsets. In terms of chromatic polynomial, if a graph G is nearly uniquely s-colorable, then $P(G;s) = 2(s!)$. Notice that if $n \geqslant 4$ is even, then U_{n+1} is nearly uniquely 3-colorable. Another example of a nearly uniquely 3-colorable graph is the graph W of Fig. 1.

Suppose G is nearly uniquely s-colorable. Let $V_0, \ldots, V_{s-1}$ be the color classes of an s-coloring of G. Then for any $0 \leqslant i \leqslant s-1$, we may write the color class V_i as $V_i = Z_i \cup X_i$ where Z_i is the set of those vertices always sharing the same color in any s-coloring of G; X_i has the similar property except that in a different s-coloring, all vertices in X_i may get possibly a color different from those in Z_i.

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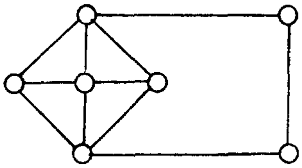

Fig. 1. The graph W.

Let $T_0, \ldots, T_{s-1}$ be the color classes of another different s-coloring of G. Then $T_i = Z_i \cup X_{\sigma(i)}$ where σ is a permutation on the set $0, 1, \ldots, s-1$. In other words, if φ_1 and φ_2 are two different s-colorings of G, then Z_i is the set of all vertices x such that $\varphi_1(x) = i = \varphi_2(x)$ and X_i is the set of all vertices y such that $\varphi_1(y) \neq \varphi_2(y)$ if $\sigma(i) \neq i$. Since G is nearly uniquely s-colorable, σ is a non-identity map, and it is unique.

Note that, for any i in which $\sigma(i) = i$, Z_i and X_i always share the same color in any s-coloring of G. However, for any i in which $\sigma(i) \neq i$, either Z_i and X_i or else Z_i and $X_{\sigma(i)}$ share the same color.

If H is a subset of the vertex set of G, we let $\langle H \rangle$ denote the subgraph of G induced by H. If $V_0, \ldots, V_{s-1}$ are the color classes of an s-coloring of G, then the subgraph $\langle V_{i_1} \cup \cdots \cup V_{i_r} \rangle$, $i_1 < \cdots < i_r$, is called an *r-color subgraph* of G. As in [2], we let $N(x)$ denote the neighborhood of x. In certain cases, we also use it to denote the subgraph induced by $N(x)$.

Theorem 1. *Suppose G is nearly uniquely s-colorable. Then all the 2-color subgraphs of G, except possibly one, are connected; in the event that one of them is disconnected, it has exactly two connected components. Further, at most one of the components is an isolated vertex.*

Proof. Since G is nearly uniquely s-colorable, there are exactly two different partitions of the vertex set of G into s color classes. Let $V_0, \ldots, V_{s-1}$ be the color classes of an s-coloring of G. Suppose for some $i \neq j$, $\langle V_i \cup V_j \rangle$ has c connected components $S_1, \ldots, S_c$. Then, for any $1 \leqslant k \leqslant c$, at least one of $S_k \cap V_i$ and $S_k \cap V_j$ is non-empty. By interchanging the color of $S_k \cap V_i$ with those in $S_k \cap V_j$, we get a different partition of the vertex set of G into s color classes. But if $c \geqslant 3$, there are at least four different ways of doing this. So $c \leqslant 2$.

If there are two or more subgraphs which are disconnected, we can also argue in a like manner to get a contradiction that there are more than two ways of partitioning the vertices of G into s color classes.

If the disconnected 2-color subgraph consists of two isolated vertices, then these two vertices may be colored with the same color giving an $(s-1)$-coloring of G, which is a contradiction. □

Proposition 2. *Let G be a nearly uniquely s-colorable graph on n vertices. Suppose* $s \geqslant 3$ *and assume that each 2-color subgraph is a tree. Then G has at most* $\binom{s-1}{2}n - 2\binom{s}{3} - 2$ *triangles.*

Proof. For $0 \leqslant i \leqslant s-1$, let V_i, Z_i, X_i and $\sigma(i)$ be as defined before Theorem 1. If σ is written as a product of disjoint cycles, then σ contains no cycles of length 2 because any 2-color subgraph is connected. Let σ_0 be a cycle of length t ($\geqslant 3$) in σ. By relabeling the color classes, if necessary, we may write $\sigma_0 = (01 \cdots t-1)$.

Consider the subgraph $J = \langle V_0 \cup \cdots \cup V_{t-1} \rangle$. Note that no vertex in Z_i is adjacent to any vertex in $X_i \cup X_{i+1}$. The subscripts are reduced modulo t. Hence, in the subgraph $\langle V_i \cup V_{i+1} \rangle$, vertices in X_{i+1} are adjacent only to vertices in X_i. Also, some vertices in X_i must be adjacent to some vertices in Z_{i+1}.

Let $Q = \langle V_i \cup V_j \cup V_k \rangle$, $i<j<k$, be a 3-color subgraph of G and consider the number of triangles in Q. Note that $\langle V_j \cup V_k \rangle$ is a tree. For each $x \in V_i$, let $\Delta(x)$ denote the number of triangles in Q containing x. Then, by the lemma given in [2], $\Delta(x) = |N(x)| - \omega(x)$ where $\omega(x)$ denotes the number of components in the subgraph $N(x) \cap \langle V_j \cup V_k \rangle$. Summing up $\Delta(x)$ for all $x \in V_i$, we see that the number of triangles in Q is

$$\sum_{x \in V_i} (|N(x)| - \omega(x)),$$

which is equal to $|Q| + |V_i| - 2 - \sum_{x \in V_i} \omega(x)$ since $\sum_{x \in V_i} |N(x)|$ is the number of edges from V_i to $\langle V_j \cup V_k \rangle$. So the maximum number of triangles in Q is $|Q|-2$ and this occurs if and only if $\omega(x) = 1$ for all $x \in V_i$. If each 3-color subgraph Q of G attains this maximum number of triangles, then the number of triangles in G is $\binom{s-1}{2}n - 2\binom{s}{3}$. To establish the lemma, it suffices to show that there is a 2-color subgraph J_0 of J and two vertices x in $J - J_0$ such that $N(x) \cap J_0$ has two components, or there exist two 2-color subgraphs J_1 and J_2 and some vertex x_i in $J - J_i$ such that $N(x_i) \cap J_i$ has two components, $i = 1, 2$.

If $s = 3$, then $t = 3$. Evidently, there is a vertex $w \in X_0$ that is adjacent to a vertex $w_1 \in X_1$ and to a vertex $z_1 \in Z_1$. Now, in the subgraph $\langle V_1 \cup V_2 \rangle$, there is a path from w_1 to z_1 via a vertex in Z_2. This implies that $N(w) \cap \langle V_1 \cup V_2 \rangle$ has two components (since no vertex in X_0 is adjacent to a vertex in Z_2). Similar argument shows that there is a vertex z in Z_0 such that $N(z) \cap \langle V_1 \cup V_2 \rangle$ has two components.

Suppose $s \geqslant 4$. If $t = 3$, applying similar argument as before (to the subgraph $\langle V_0 \cup V_1 \cup V_2 \rangle$) will lead to similar conclusion.

Assume that $t \geqslant 4$. For any i such that $0 \leqslant i \leqslant t-1$, consider the subgraph $Q_i = \langle V_i \cup V_{i+1} \cup V_{i+2} \rangle$. By applying the action of σ, we see that there is an edge joining a vertex z in Z_i and a vertex w in X_{i+2}. Now, by the notes given in the second paragraph, we see that w must be adjacent to some vertex in X_{i+1}. Evidently, $N(w) \cap \langle V_i \cup V_{i+1} \rangle$ has at least two components. Since there are t choices of Q_i, we can have t such vertices.

This completes the proof. □

Lemma 3. *Let G be a nearly uniquely 3-colorable graph without cut vertices. Suppose G has n (>3) vertices, $2n-3$ edges and $n-3$ triangles. Then in any 3-coloring of G, there is a 2-color subgraph X such that X is disconnected with exactly two components.*

Further, there are exactly two vertices x in $G - X$ such that $N(x)$ has exactly two components, one in each of the connected component of X; for all other vertices, $N(x)$ is connected.

Proof. Since G has $n-3$ triangles, it follows from Proposition 2 that not every 2-color subgraph is a tree. Since G has $2n-3$ edges, there must be a 2-color subgraph X of G that is disconnected. By Theorem 1, X has exactly two components A_1 and A_2. Also, any other 2-color subgraph is connected. Consequently, there exists a vertex x_1 in $G - X$ such that x_1 is adjacent to vertices in A_1 and A_2. Since x_1 is not a cut vertex, there is also another vertex $x_2 \in G - X$ with similar property as x_1.

The number of edges in X is $|X| - k$ for some $k \in \{1, 2\}$. So the number of edges from $G - X$ to X is $2n + k - |X| - 3$ which is the sum of all $|N(x)|$ with x ranging over $G - X$. Let $\Delta(x)$ denote the number of triangles containing x, and let $\omega(x)$ denote the number of components in $N(x)$.

If X is a forest, then $k = 2$ and $\Delta(x) = |N(x)| - \omega(x)$. If X is unicyclic, then $k = 1$ and the cycle C in X is of even length. In this case, $\Delta(x)$ is either $|N(x)| - \omega(x)$ or else $|N(x)| - \omega(x) + 1$. The latter case is possible only if x is adjacent to all the vertices of C (and there is only one such x).

To finish the proof, we need only sum up $\Delta(x)$ for all x in $G - X$ to find out that there are precisely two vertices x for which $\omega(x) = 2$, and that $\omega(x) = 1$ for all other vertices. □

Before going into the proof of the main theorem, we shall recall some necessary conditions for two graphs to have the same chromatic polynomial. Let C_n^* denote a chordless C_n, $n \geqslant 3$. Let A_G, B_G and D_G denote, respectively, the number of C_n^*, $n = 3, 4$ and 5 in the graph G. Also, let C_G, L_G, M_G, P_G and R_G denote, respectively, the number of $K_4, K_5, K_{2,3}, U_5$ and W_5 in G. It is well-known that if two graphs G and Y have the same chromatic polynomial, then they have the same number of vertices and the same number of edges. Further, $A_G = A_Y$ and $B_G - 2C_G = B_Y - 2C_Y$ by Theorem 1 of [4]. If in addition, G has chromatic number 3, then $C_G = 0 = C_Y$, and so $B_G = B_Y$ in this case.

In the event that $G = U_{n+1}$, we have $A_G = n - 2$, $B_G = 1$, $C_G = 0$, $D_G = 0$, $L_G = 0$, $M_G = 0$ and $R_G = 0$. Also, if $n \geqslant 6$, then $P_G = 0$. So if there is a graph Y such that $P(Y; \lambda) = P(U_{n+1}; \lambda)$, $n \geqslant 6$, then by an application of Theorem 2 of [4], it follows that $D_Y = 3R_Y$.

Theorem 4. *For any $n \geqslant 3$, U_{n+1} is chromatically unique except when $n = 6$, in which case, W is the only graph having the same chromatic polynomial as that of U_7.*

Proof. The case odd $n \geqslant 3$ has been treated in [2]. We may assume that $n \geqslant 6$ is even since U_5 is known to be chromatically unique.

Suppose Y and U_{n+1} have the same chromatic polynomial. Then Y is nearly uniquely 3-colorable. Let V_0, V_1 and V_2 be three color classes of Y. Further, let $G_i = \langle V_{i+1} \cup V_{i+2} \rangle$ where $i = 0, 1, 2$, and the subscripts are reduced modulo 3.

Since Y satisfies the conditions of Lemma 3, we may assume that G_0 is disconnected with components $G_{0,1}$ and $G_{0,2}$. The proof takes advantage of the fact that only one of the G_i is unicyclic. For this reason, we may sometimes interchange the colors of $G_{0,j}$ in order to get a cycle. For each $i = 1, 2$, let $x_i \in V_0$ be such that $N(x_i)$ has two components, one in each $G_{0,j}$.

Let $N = N(x_1) \cap N(x_2)$. Then $|N| \leqslant 3$ since $B_Y = 1$. Let $N_j = N \cap G_{0,j}$. If $|N_j| > 0$, we treat N_j as a single vertex. Let u_j be an end-vertex of $G_{0,j}$ furthest away from N_j and such that its neighbor w (in $G_{0,j}$) is of least possible degree. If $|N_j| = 0$, we treat $N(x_i) \cap G_{0,j}$ as a single vertex (regardless of the choice of i), and let u_j be an end-vertex of $G_{0,j}$ with similar property as before.

Since $(\lambda - 2)^2$ does not divide $P(Y; \lambda)$, for each end-vertex v of $G_{0,j}$ with $v \in N(w)$, v is adjacent to at least two vertices of V_0; and for each z in V_0 that is adjacent to v, z is also adjacent to w and to at least one vertex y in $N(w) \cap G_{0,j} - \{v\}$ if z is neither x_1 nor x_2. Let F be the set of all vertices z in V_0 such that z is adjacent to some vertices of $N(w) \cap G_{0,j}$. Let $H = (N(w) \cap G_{0,j}) \cup F$.

Because of the choice of u_j, $N(w) \cap G_{0,j}$ has at most two non-end-vertices of $G_{0,j}$. Suppose v_1 and v_2 are two vertices of $N(w) \cap G_{0,j}$ that are not end-vertices of $G_{0,j}$. Then G_0 is unicyclic and v_1, w and v_2 are three consecutive vertices on the cycle C of G_0. In this case, note that there is a vertex u in F that is adjacent to all the vertices of C (for otherwise the number of triangles in Y is less than $n - 2$). Except possibly either v_1 or v_2, each vertex in $\langle H \rangle$ has degree at least 2. This means that $\langle H \rangle$ contains a cycle. If $N(w) \cap G_{0,j}$ has only one vertex that is not an end-vertex of $G_{0,j}$, then again $\langle H \rangle$ contains a cycle. If all the vertices in $N(w) \cap G_{0,j}$ are end-vertices, then $|N_j| = 1$ and $G_{0,j}$ is a star $K_{1,r}$ where $r = |G_{0,j}| - 1$.

Assume that G_0 is unicyclic. Then the argument in the preceding paragraph implies that $G_{0,1} = C$ and $G_{0,2}$ is a path. If $|G_{0,2}| = 1$, then by changing the color of $G_{0,2}$ if necessary, we see that there is a cycle in G_i for some $i > 0$ and this is clearly a contradiction. So $|G_{0,2}| \geqslant 2$. If $|G_{0,1}| \geqslant 6$, then we will obtain a contradiction because it is readily seen that either $B_Y = 0$, or else $B_Y \geqslant 2$. If $|G_{0,1}| = 4$, then $R_Y \geqslant 1$ and this implies that $D_Y \geqslant 3$; but there is no way that we could arrive at $D_Y \geqslant 3$.

Assume now that G_0 is a forest. Suppose $|N| = 3$. This is possible only if $G_{0,1} = K_{1,2}$, for otherwise $B_Y \geqslant 2$. Since $R_Y = 1$, it follows that $D_Y = 3$. Now this is possible only if $G_{0,2} = K_2$, in which case Y is isomorphic to W.

Suppose $|N| = 2$. Assume that $|N_j| = 1$ for each j. Then $\{x_1, x_2, N_1, N_2\}$ is a cycle in G_i (by interchanging the colors of $G_{0,j}$ if necessary). This means that u_j is of distance at most one from N_j. Also, $G_{0,1} = K_{1,r}$ where $r = |G_{0,1}| - 1$ and $G_{0,2} = K_1$. This will then result in Y isomorphic to U_{n+1}.

Assume that $|N_1| = 2$ or that $|N_1| \leqslant 1$. Since $B_Y = 1$, we see that C_4^* must be contained in some W_5, but then it is impossible to have $D_Y = 3$.

This completes the proof. □

In [3], it is shown that $K_m + U_{n+1}$, the join of K_m and U_{n+1}, is chromatically unique for all $m \geqslant 1$ if $n \geqslant 3$ is odd. I wonder if the result is still true for all $m \geqslant 1$ and all even $n \geqslant 4$ except $n = 6$. Also, it is probably the case that $K_m + W$ is the only graph having the same chromatic polynomial as that of $K_m + U_7$.

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