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On Projections of Ruled and Veronese Surfaces

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Let X be a closed subscheme in $\mathbb{P}^n$. We say that X is of maximal rank if for every $k \geqslant 1$ the natural map of restriction $r_{X,n}(k)$: $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(\mathcal{O}_X(k))$ is injective or surjective. In [12, 4.3.4], R. Hartshorne raised the following projection conjecture: "Let Z be a projectively normal curve in $\mathbb{P}^N$. Take n with $3 \leqslant n \leqslant N$ and let $X \subset \mathbb{P}^n$ be a general projection of Z. Is C of maximal rank?" Examples are known where the answer is negative [11, 14, 2, 6, Sects. 9, 10] while in certain ranges this is true [3, 4, 6, Sects. 9, 10]. Of particular interest (after [11, 14]) was the case of canonical curves. For curves with general moduli, the problem has an affirmative answer (Chang [7] in $\mathbb{P}^3$, [6] in $\mathbb{P}^n, n > 3$) except for the exceptions found in [11]. But it remained open the corresponding result for canonical curves with non-general moduli. By the counterexamples in [6] (general projection in $\mathbb{P}^4$ of any trigonal canonical curve of genus 6) it seemed useful to study Hartshorne's projection conjecture for the canonical embeddings of trigonal curves. On this topic we prove in this paper the following result.

THEOREM 1. *Fix integers g, k, with $g \geqslant k+2$, $k \geqslant 5$, $(k+3)(k+2)(k+1)/6 \geqslant 6g-8$. Fix a linearly normal trigonal canonical curve C in $\mathbb{P}^{g-1}$, C of genus g. Then the general projection of C into $\mathbb{P}^k$ has maximal rank.*

We mention here that we work over $\mathbb{C}$.

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Theorem 1 is a trivial corollary (see the beginning of Section 2) of the following result.

THEOREM 2. *Fix integers* d, k *with* $d > k \geqslant 5$, $(k+3)(k+2)(k+1)/6 \geqslant 6d+4$. *Let* S *be any smooth ruled surface in* $\mathbb{P}^{d-1}$, $\deg(S) = d$. *Then the general projection of* S *into* $\mathbb{P}^k$ *has maximal rank.*

It seems very natural to consider the "projection problem" for certain higher dimensional subvarieties of a projective space. For instance theorem 2 has (trivially) application to the projection problem for a few curves, for instance, hyperelliptic curves (see Corollary 2.1). A paper related to the projection problem is [17].

In Section 1 we prove the following result.

THEOREM 3. *Let* $V(d) \subset \mathbb{P}^{N(d)}$, $N(d) := (d^2+3d)/2$, $d \geqslant 3$, *be a* d*-ple Veronese embedding of* $\mathbb{P}^2$. *Then for every* $k \geqslant 2d$ *a general projection of* $V(d)$ *into* $\mathbb{P}^k$ *has maximal rank.*

We prove Theorems 2 and 3 by degeneration techniques, essentially degenerating a ruled or a Veronese surface to a suitable union of planes (as in [18]). The reader wil recognize in some proofs a mild application of an inductive procedure, the so-called "méthod d'Horace" used in [13] and in several related papers.

The projection of Veronese varieties was considered in several papers [10, 16, 21, 22], but their authors were interested only in projections defined by a family of monomials.

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For $S \subset M \subset \mathbb{P}^k$, let $r_{S,M}(t)$: $H^0(M, \mathcal{O}_M(t)) \to H^0(S, \mathcal{O}_S(t))$ be the restriction map. If $M = \mathbb{P}^k$, we write often $r_{S,k}(t)$ instead of $r_{S,M}(t)$.

Let T be a closed subscheme of the scheme M and $\mathcal{I}_{T,M}$ its ideal sheaf. If M is a projective space and $\dim(M) = q$, we will write often $\mathcal{O}_q$, $\mathcal{I}_{T,q}$, instead of $\mathcal{O}_M$, $\mathcal{I}_{T,M}$. Set $N(d) := (d^2+3d)/2$ and $\mathbb{P}(d) := \mathbb{P}^{N(d)}$. Let $V(d) \subset \mathbb{P}(d)$ be the d-ple Veronese embedding of $\mathbb{P}^2$. We define a reduced, connected scheme $W(d) \subset \mathbb{P}(d)$, $W(d)$ spanning $\mathbb{P}(d)$, $W(d)$ union of d^2 planes L_{ij}, A_{hk}, $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant i$, $2 \leqslant h \leqslant d$, $1 \leqslant k \leqslant h-1$, with the following incidence relations (see [18, Fig. 3, p. 316]); $L_{ij} \cap A_{ij}$ is a line, D_{ij}, $L_{ij} \cap A_{i+1j}$ is a line B_{ij}, $A_{ij} \cap L_{ij+1}$ is a line C_{ij}; in the remaining cases $L_{ij} \cap A_{hk}$, $L_{ij} \cap L_{uv}$, $A_{uv} \cap A_{ce}$ is as small as possible (either empty or a point); $L_{ij} \cap L_{hk} = \varnothing$, $i \leqslant h$, unless $h = i$, $k = j-1, j$ or $j+1$ or $h = i+1$, $k = j$; $A_{ij} \cap A_{hk} = \varnothing$, $i \leqslant h$, unless $i = h$, $k = j-1, j, j+1$ or $h = i+1$, $k = j$; $L_{ij} \cap A_{hk} = \varnothing$ unless $i = h$, $j = k$ (line D_{ij}) or $i = h$, $j = k-1 > 0$ (the line

C_{ij-1}) or $i=h+1$, $j=k+1$ or $i=h-1$, $j=k-1, k, k+1$. Note that up to a projective transformation, there is a unique $W(d)$ spanning $\mathbb{P}(d)$: use induction on the number of planes in $W(d)$. Using (d^2-1) Mayer–Vietoris exact sequences, we obtain that $W(d)$ and $V(d)$ have the same Hilbert polynomial. Hence it is reasonable to ask if $W(d)$ is in the closure in $\mathrm{Hilb}(\mathbb{P}(d))$ of the set of d-ple Veronese embeddings of $\mathbb{P}^2$ into $\mathbb{P}(d)$, i.e., of the set of projective transformations of $V(d)$. We will show that this is true. By [18] there is N very big and a family of embeddings i_t of $V(d)$ in $\mathbb{P}^N$ (with $i_t^*(\mathcal{O}_N(1))=\mathcal{O}_{V(d)}(1)$), $i_t(V(d))$ spanning a linear space of dimension $N(d)$, with a limit Z with $Z_{\mathrm{red}}=W(d)$ (recall that the Hilbert scheme is complete, hence the limit exists). By a general projection from $\mathbb{P}^N$ into $\mathbb{P}(d)$, we assume $N=N(d)$. Let F be the sheaf of nilpotents on Z. By degree reason, $\dim(\mathrm{Supp}(F))\leqslant 1$. Considering the Hilbert polynomial of a general hyperplane section, we find $\dim(\mathrm{Supp}(F))\leqslant 0$. Since Z and Z_{red} have the same Hilbert polynomial, Z must be reduced.

By (d^2-1) Mayer–Vietoris exact sequences, we find $h^0(W(d), \mathcal{O}_{W(d)}(1))=N(d)+1$. We claim that we may find such a Z spanning $\mathbb{P}(d)$, i.e., that $W(d)$ is in the closure in $\mathrm{Hilb}(\mathbb{P}(d))$ of the projective transformations of $V(d)$. Let $p\colon V\to T$ be a flat family, $0\in T$, T a disc, with $p^{-1}(0)=Z$, $p^{-1}(t)\cong V(d)$ for general t, $V\subset\mathbb{P}(d)\times T$ and p induced by the projection on T. Reembed V in $\mathbb{P}(d)\times T$ by the sections of $p^*(\mathcal{O}_{V/T}(1))$. Now the image $V_t\cong V(d)$ for general t, while $V_0\cong Z$, V_0 spans $\mathbb{P}(d)$, i.e., $V_0=W(d)$ (up to a projective transformation).

Note that for a general projection E of $W(d)$ in $\mathbb{P}^k$, $k\geqslant 6$, $d\geqslant 3$, $E\cong W(d)$ because $W(d)$ has embedding dimension 6 and only at finitely many points. Furthermore $W(2)$ can be projected isomorphically into $\mathbb{P}^4$. Note that $h^i(\mathcal{O}_{V(d)}(2-i)=0$, $i=1,2$. Hence for any smooth projection Y of $V(d)$ in $\mathbb{P}^k$, $k\geqslant 2d$, $h^i(\mathcal{J}_{V(d),k}(3-i))=0$, $i=2,3$. Thus by Castelnuovo–Mumford's lemma [19, p. 99] and a dimensional count, Y has maximal rank if and only if the restriction map $r_{Y,k}(2)$ is surjective; in particular it is sufficient to consider the case $k=2d$. By semicontinuity, it is sufficient to prove the following lemma.

LEMMA 1.1 *Let $T\subset\mathbb{P}^{2d}$, $d\geqslant 2$, be the general projection into $\mathbb{P}^{2d}$ of $W(d)$. Then $r_{T,2d}(2)$ is injective (hence bijective).*

Proof. By induction on d. First assume $d\geqslant 3$ and the result true for $d-1$. By semicontinuity it is sufficient to find just X in $\mathbb{P}^{2d}$, X projection of $W(d)$, with $r_{X,2d}(2)$ injective. By the uniqueness (up to a projective transformation of $W(d)$ any union of planes X in $\mathbb{P}^{2d}$ with $X\cong W(d)$ as abstract scheme a projection of $W(d)$: use that $h^0(\mathcal{O}_X(1))=N(d)+1$. Fix a hyperplane H of $\mathbb{P}^{2d}$ and a hyperplane M of H. Let $W\subset M$ be a general projection of $W(d-1)$, hence with $r_{W,2d-2}(2)$ injective, W union of planes

J_{ij}, A_{hk}, $1 \leqslant i \leqslant d-1$, $1 \leqslant j \leqslant i$, $2 \leqslant h \leqslant d-1$, $1 \leqslant k \leqslant h-1$. Let X be the union of W and the union U of $2d-1$ suitable planes L_{dj}, $1 \leqslant j \leqslant d$, A_{dh}, $1 \leqslant h \leqslant d-1$, with the incidence relations of the planes in $W(d)$. We assume that U is as general as possible, with the only constraint of the incidence relations. Assume that $r_{W \cup (U \cap H), H}(2)$ is injective. Then we claim that $r_{X,2d}(2)$ is injective. Take $f \in H^0(\mathbb{P}^{2d}, \mathscr{I}_{X,2d}(2))$. By the assumption $f|H$ vanishes. Hence f is divided by the equation z of H. Note that for general U, U spans $\mathbb{P}^{2d}$. Since f/z vanishes on U, $f=0$. We claim that the same proof gives the injectivity of $r_{W \cup (U \cap H), H}(2)$. Indeed by induction $r_{W,M}(2)$ is injective, while for general U, $U \cap (H \backslash H)$ contains d lines not in W and $U \cap (H \backslash M)$ spans H.

To conclude the proof, we have to check the starting case of the induction, $d=2$. The same proof applies, since a general union $W \cup (U \cap H)$ of a plane W and 2 lines intersecting W, both contained in H, $\dim(H)=3$, is contained in no quadric surface. ∎

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In the first part of this section we show that Theorem 2 implies Theorem 1 and we give another application (Corollary 2.1) of Theorem 2. In the last part of this section we show how to degenerate a smooth ruled surface to a suitable configuration of planes (a chain of planes).

Let $S(d,e) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \subset \mathbb{P}^{d+1}$, $\deg(S(d,e)) = d$, be a smooth ruled surface. $\mathrm{Pic}(S(d,e))$ has two generators h, f with the relations $h^2 = -e$, $h \cdot f = 1$, $f^2 = 0$, $e \geqslant 0$; furthermore $0 \leqslant e \leqslant d-2$, $d \equiv e \bmod(2)$. Any smooth ruled surface of degree d in $\mathbb{P}^{d+1}$ is of the form $S(d,e)$ for some $e \equiv d \bmod(2)$, $0 \leqslant e \leqslant d-2$. The hyperplane section of $S(d,e)$ is given by $|h + xf|$, $x = (d+e)/2$.

Proof of Theorem 1. Fix a linearly normal curve canonical trigonal curve $C \subset \mathbb{P}^{g-1}$. Each group of points of the g_3^1 on C spans a line by Riemann–Roch and the union of these lines is a surface S of minimal degree [9]; by [20, 4.10], S is smooth, i.e., it is of the form $S(g-2, e)$ for some e. Since C is projectively normal, the restriction map $r_{C,S}(t)$: $H^0(S, \mathcal{O}_S(t)) \to H^0(C, \mathcal{O}_C(t))$ is surjective for every $t \geqslant 2$. Furthermore $r_{C,S}(2)$ is bijective because the lines in S are trisecant to C. By Castelnuovo–Mumford's lemma and Theorem 2, if $(k+2)(k+1)/2 \geqslant 3(g-2)+3$, the restriction maps $r_{S,k}(t)$ are surjective for every $t \geqslant 2$, while if $3(g-2)+3 > (k+2)(k+1)/2$ and $(k+3)(k+2)(k+1)/6 \geqslant 6g-8$, $r_{S,k}(2)$ is injective and $r_{S,k}(t)$ surjective for every $t \geqslant 3$. Hence Theorem 1 follows from Theorem 2. ∎

Theorem 2 has some corollaries for the postulation of a general projection of any curve contained in a smooth ruled surface. We prove here only the case of hyperelliptic curves.

COROLLARY 2.1. *Let $X \subset \mathbb{P}^m$ be a projectively normal hyperelliptic curve. Assume that neither $M := \mathcal{O}_X(1)$ is of the form $(p+1)\, g_2^1$, p genus of X, nor pg_2^1 of the form $M(-P)$ for some P in X. Take an integer k with $5 \leqslant k < m$, $3m \leqslant (k+2)(k+1)/2$. Then a general projection of X into $\mathbb{P}^K$ has maximal rank.*

Proof. Set $d := \deg(X) = m + p$. Since X is projectively normal, $m \geqslant p$ [15, Corollary 3.4]. By [9] X is contained in a surface S of minimal degree, S union of the lines spanned by the divisors in the g_2^1 on X. By the assumption on M, S cannot be a cone over a rational normal curve of degree $m-1$ (project from the vertex). Hence $S = S(m-1, e)$ for some e; we conclude as in the proof of Theorem 1; however, here we use only the weaker version of Theorem 2 proved in Section 3. ∎

If $\mathcal{O}_X(1)$ is in one of the excluded cases, X is in a cone S over a rational curve of degree $m-1$ (see [9, 8] if $\deg(X) = 2p+2$) and the thesis of 2.1 is trivially false by a dimensional count.

A bichain of type (i, j), $i \geqslant 0$, $j \geqslant 0$, in $\mathbb{P}^k$ is a reduced subscheme X in $\mathbb{P}^k$ with $i+j$ irreducible components $A_1, ..., A_i$, $L_1, ..., L_j$, A_u planes, L_v lines, with the following incidence relations: $A_u \cap L_v = \varnothing$ unless $u = i$, $v = 1$; $A_u \cap A_{u+2}$ is a point for $u = 1, ..., i-2$; $A_u \cap A_{u+1}$ is a line, D_u; $A_u \cap A_v = \varnothing$ if $|u-v| \geqslant 3$; $A_i \cap L_1$ is a point; $L_u \cap L_v = \varnothing$ if $|u-v| \geqslant 2$; $L_u \cap L_{u+1}$ is a point for $u = 1, ..., j-1$. A chain of i planes is a bichain of type $(i, 0)$. A chain of j lines (or a bamboo in the terminology of [3]) is a bichain of type $(0, j)$. Let $X(n) \subset \mathbb{P}^{n+1}$ be a chain of n planes spanning $\mathbb{P}^{n+1}$. By induction on n, we see that $X(n)$ is unique, up to a projective transformation. Using $n-1$ Mayer–Vietoris exact sequences, we find that $h^0(X(n), \mathcal{O}_{X(n)}(1)) = n+2$ and that $X(n)$ has the same Hilbert polynomial of $S(n, e)$, any e. Hence it is natural to ask if, for a fixed e, $X(n)$ is in the closure in $\mathrm{Hilb}(\mathbb{P}^{n+1})$ of the set of projective translates $gS(n, e)$, $g \in \mathrm{Aut}(\mathbb{P}^{n+1})$. This is true for every e (2.2).

LEMMA 2.2. *For every $n \geqslant 2$, $0 \leqslant e \leqslant n-2$ with $n \equiv e$ $\mathrm{mod}(2)$, $X(n)$ is in the closure in $\mathrm{Hilb}(\mathbb{P}^{n+1})$ of the set of projective transformations of $S(n, e)$.*

Proof. Think of $\mathbb{P}^{n+1}$ has a hyperplane H of $\mathbb{P}^{n+2}$ and let M be a general hyperplane of $\mathbb{P}^{n+2}$. For a point P in $\mathbb{P}^{n+2} \backslash M$, let t_P: $\mathbb{P}^{n+2} \backslash \{P\} \to M$ be the projection from P. Fix a line L of the ruling of $S(n, e) \subset H$. For a general point P in L, $t_P(S(n, e))$ is a smooth surface isomorphic to $S(n-1, |e-1|)$. Fix a general $P \in L$ and a general line R in $\mathbb{P}^{n+2}$ with $P \in R$. For a general point $Q \in R$, $Q \neq P$, $t_Q(S(n, e))$ is isomorphic to $S(n, e)$. We obtain a flat family $\{t_Q(S(n, e))\}$, $Q \in R \setminus \{P\}$, of subschemes of M. Since Hilb(M) is complete, this family has a limit Z for Q going to P. By the picture in [1], Z_{red} contains $t_P(S(n, e))$ and a plane V with V intersecting $t_P(S(n, e))$ at a line of the ruling and at most another finite set A. Since $t_P(S(n, e)) \cup V$ spans M for general P, $A \neq \emptyset$; indeed as in the picture in [1] we find that V is the intersection with M of the span of R and the tangent plane to $S(n, e)$ at P. Set $J := t_P(S(n, e)) \cup V$. Let N be the sheaf of nilpotents of Z. By degree reasons, $\dim(\mathrm{Supp}(N)) \leqslant 1$. By a Mayer–Vietoris exact sequence J and Z have the same Hilbert polynomial, as well as their general hyperplane sections; as in Section 1, $N = 0$. Then we continue considering $t_P(S(n, e))$ instead of $S(n, e)$. Choosing carefully the line L_i from the general point of which we project at each step, after $n-1$ similar steps we degenerate $S(n, e)$ to a chain of n planes spanning $\mathbb{P}^{n+1}$, i.e., to $X(n)$. For instance in the second step we have to project from a general point of the line $V \cap (t_P(S(n, e)))$. ∎

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By semicontinuity and 2.2 to prove Theorem 2, it is sufficient to prove that, with the given restrictions on d, k, a general projection T of $X(n)$ into $\mathbb{P}^k$ has maximal rank. In this section this will be proved under the stronger restriction that $(k+2)(k+1)/2 \geqslant 3d+3$ (3.1). The remaining cases will be proved in the next section.

Let $T \subset \mathbb{P}^k$, $k \geqslant 5$, be a general projection of $X(d)$ or $S(d, e)$. We have $h^i(T, \mathcal{O}_T(2+a-i)) = 0$ for 1, 2 and for every $a \geqslant 0$; if T is a projection of $X(d)$, this follows from $d-1$ Mayer–Vietoris exact sequences. Hence $h^{i+1}(\mathbb{P}^k, \mathcal{I}_{T,k}(2+a-i)) = 0$ for every $i > 0$, $a \geqslant 0$. Hence if $r_{T,k}(t)$ is surjective for some $t \geqslant 2$, then $r_{T,k}(j)$ is surjective for every $j \geqslant t$ [19, p. 99].

LEMMA 3.1. *Fix integers d, k with $d \geqslant k \geqslant 5$, $(k+2)(k+1)/2 \geqslant 3d+3$. Then a general projection T of $X(d)$ into $\mathbb{P}^k$ has $r_{T,k}(2)$ surjective, hence maximal rank.*

Proof. Since $X(d)$ is unique up to a projective transformation and for every chain Z of d planes $\mathrm{H}^0(Z, \mathcal{O}_Z(1)) = d+1$ (Mayer–Vietoris), it is sufficient to find a chain Z of d planes with $r_{Z,k}(2)$ surjective.

We use induction on k, the cases with $k \leqslant 6$ being considered at the end of the proof. Take a hyperplane H of $\mathbb{P}^k$. Take as T a chain of d planes as general as possible with the restriction that $T = U \cup V$ with U chain of $k-1$ planes, V chain of $d-k+1$ planes, and $V \subset H$. In particular we assume that U spans $\mathbb{P}^k$.

Step 1. The lemma is true if the restriction map $r_{T \cap H, H}(2)$ is surjective.

Set $x := h^0(\mathbb{P}^k, \mathcal{O}_k(2)) - h^0(T, \mathcal{O}_T(2))$. Note that $x = h^0(H, \mathcal{O}_H(2)) - h^0(T \cap H, \mathcal{O}_{T \cap H}(2))$. Assume that $r_{T \cap H, H}(2)$ is surjective. Take $S \subset H$, $\text{card}(S) = x$, S general. By the assumption $r_{(T \cap H) \cup S, H}(2)$ is bijective. As in the proof of 1.1, we obtain that $r_{T \cup S, k}(2)$ is injective, hence surjective.

Step 2. proof of the surjectivity of $r_{T \cap H, H}(2)$. Note that $T \cap H$ is a bichain in H of type $(d-k+1, k-2)$ and that any general bichain of that type occurs as $T \cap H$, for some T as above. Hence it is sufficient to find a bichain Y of type $(d-k+1, k-2)$ with $r_{Y,H}(2)$ surjective. Fix a hyperplane M in H (if $k \geqslant 7$). Take as Y the following bichain. $Y = A \cup B \cup R \cup E$; $A \cup B$ is a chain of $d-k+1$ planes, A is a chain of $d-k$ planes, $A \cup B \cup R$ is a bichain of type $(d-k+1, k-3)$, $B \cap R \neq \emptyset$, $A \cup E \subset M$, $B \cup R$ intersects transversally M and spans H; furthermore we assume that Y is general (with these constraints). As in Step 1, it is sufficient to prove that $r_{Y \cap M, M}(2)$ is surjective. $Y \cap M$ is the disjoint union of A, E, and $R \cap (M \backslash (E \cup B))$. Since $R \cap (M \backslash (E \cup B))$ can be formed by $k-4$ general points of M, it is sufficient to prove that $r_{A \cup E, M}(2)$ is surjective. Note that $A \cup E$ is contained in a chain of $d-k+3$ plains: fix points $w \in A \cap B$, $c \in E$, z in the $\mathbb{P}^4$ spanned by $A \cup E$ and add to A the planes spanned respectively by $z \cup E$, $z \cup (A \cap B)$, $w \cup z \cup c$. Hence if $k-4 \geqslant 6$, we may take such a chain F general and with $r_{F,M}(2)$ is surjective; thus $r_{A \cup E, M}(2)$ is surjective. If $5 \leqslant k \leqslant 9$, we have to handle directly $A \cup E$ as in the first part of the proof of this step. For instance if $k = 5$, we have $d \leqslant 6$ and if, say, $d = 6$, in H we have a bichain of type (2, 3). We may take in $M = \mathbb{P}^3$ a chain I of 2 planes, and link I with a general chain of 3 lines. For another method, see the proof of 3.2. ∎

LEMMA 3.2. *Fix integers $k \geqslant 5$, $i > 0$, $j \geqslant 0$. If $3i + 3 + 2j \leqslant (k+2)(k+1)/2$, there is a bichain Z in $\mathbb{P}^k$, Z of type (i, j), with $r_{Z,k}(2)$ surjective. If $3i + 3 + 2j > (k+2)(k+1)/2$, there is a bichain Y in $\mathbb{P}^k$, Y of type (i, j), with $r_{Y,k}(2)$ injective. If $k \leqslant 4$, the same statements holds if $i \leqslant k-1$.*

Proof. The last part is easier; note that a chain of i planes exists in $\mathbb{P}^k$, $k \leqslant 4$, if and only if $i \leqslant k$.

If $j = 0$ the first part is exactly 2.1. The same proof gives also the injective part for $j = 0$. The same proof (plus initial cases) could be used for $j > 0$, but we prefer to use the method of [5, 2.1], hence double induction on k and j. Suppose there is a bichain T of type $(i, j-1)$ in $\mathbb{P}^k$ with $r_{T,k}(2)$ surjective. Let P be a point in the irreducible component B of T to which we want to link another line; B is a line if $j \geqslant 2$. Take a general point Q in $\mathbb{P}^k$; the line PQ imposes at least a condition to $W := H^0(\mathbb{P}^k, \mathcal{I}_{T,k}(2))$ if $W \neq 0$. Suppose $w := \dim(W) \geqslant 2$ and that every such line imposes only one condition to W. Fix a general $F \in \mathbb{P}(W)$ and a general $Q \in F$, Q not in the base locus U of W. Then F must contain the line PQ. Since any point $Q' \in F, Q'$

near to Q, is not in the base locus U, F must contain PQ' for general $Q' \in F$. Thus every quadric in W is a cone with vertex P, for general P in B, hence of vertex B. Let $X \subset \mathbb{P}^{k-1}$ be the projection from P of $T \setminus B$; for general T, P, X is a general bichain of type $(i, j-2)$ if $j \geqslant 2$, $(i-1, 0)$ if $j=1$. By the inductive assumption X cannot be contained in w quadrics, contradiction. For this induction step, it is useful to use also the injective part of the lemma for $k-1$. The injective part is very similar, but even easier. ∎

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To conclude the proof of Theorem 2, it is sufficient to prove the following lemma.

LEMMA 4.1. *Fix integers k, d with $d > k \geqslant 5$, $(k+2)(k+1)/2 < 3d+3$, $6d+4 \leqslant (k+3)(k+2)(k+1)/6$. Then the general projection, T, of $X(d)$ into $\mathbb{P}^k$ has $r_{T,k}(2)$ injective and $r_{T,k}(3)$ surjective, hence maximal rank.*

Proof. The injectivity part is contained in 3.2.

For every integer k, define intergers $a(k)$, $b(k)$, $c(k)$, $d(k)$ using the following relations:

$$3a(k)+3+b(k)=(k+2)(k+1)/2, \qquad 0 \leqslant b(k) \leqslant 2 \tag{1}$$

$$6c(k)+4+d(k)=(k+3)(k+2)(k+1)/6, \qquad 0 \leqslant d(k) \leqslant 5. \tag{2}$$

Note that $b(k)=0$ if $k \not\equiv 0 \bmod(3)$, while $b(k)=1$ if $k \equiv 0 \bmod(3)$. We consider first the cases in which $b(k) \leqslant d(k)$, i.e., we exclude until Step 6 the case $k \equiv 9, 18, 27 \bmod(36)$. We will prove the lemma by induction on k, using also the excluded values $k \equiv 9, 18, 27 \bmod(36)$. We will see in Step 2 why the induction works; indeed if $k \equiv 9, 18, 27 \bmod(36)$, the proof in $\mathbb{P}^{k+1}$ uses only a weaker statement in $\mathbb{P}^k$.

Step 1 (numerical). We have $c(k)-a(k) \leqslant c(k-2)-e$ with $e=3$ if $k \geqslant 26$, $e=2$ if $k \geqslant 20$, $e=1$ if $k \geqslant 14$, $e=0$ if $k \geqslant 7$.

Proof. By (1) and (2) we obtain

$$6(c(k)-a(k))-2+d(k)-2b(k)=(k-3)(k+2)(k+1)/6. \tag{3}$$

Comparing (2) for $k'=k-2$ and (3) we find

$$6(c(k)-a(k)-c(k-2))=6-d(k)+2b(k)+d(k-2)-(k+1). \tag{4}$$

Step 1 follows from (4) if $k>8$. For low k we use the explicit values of $c(k)$, $a(k)$, $c(k-2)$.

Step 2. Take a hyperplane H in $\mathbb{P}^k$ and a chain $T=U\cup V$ of $c(k)$ planes, $V\subset H$, U chain of $a(k)$ planes, and T general with these properties. In particular by 3.1 we may assume that $r_{U,k}(2)$ is surjective. Since $b(k)\leqslant d(k)$ by the assumption on k, the proof in 3.1, Step 1, shows that it is sufficient to prove that $r_{T\cap H,H}(3)$ is surjective. If $k\equiv 9$, 18, or 27 mod(3) the same reasoning works if we are trying to prove only the existence in $\mathbb{P}^k$ of a chain E of $c(k)-1$ planes with $r_{E,k}(3)$ surjective.

Step 3. We want to find a chain $T=U\cup V$ as in step 2 with $r_{T\cap H,H}(3)$ surjective. $T\cap H$ is a bichain of type $(c(k)-a(k), a(k)-1)$ and any such bichain is of the form $T\cap H$ for suitable T. For $k\geqslant 6$ define integers i, j with the following restrictions: $i>0, j\geqslant 0$, $a(k)-3\leqslant j\leqslant a(k)-1$, $3i+3+2j=k(k+1)/2$. This is possible for $k\geqslant 9$ because $2a(k)+9\leqslant k(k+1)/2$ for $k\geqslant 9$ by (1); for $k=8$ take $j=12$, $i=3$; for $k=7$, take $j=8$, $i=3$; for $k=6$, take $i=3, j=6$. If $k=5$ set $i=2, j=3$. By 3.2 we may find a bichain I of type (i, j) in H with maximal rank. Hence we may assume $c(k)-a(k)>i$. Take a hyperplane M of H. Take as $T\cap H$ a bichain $N\cup K$ with N bichain of type (i, j), $K=K'\cup K''$, with $K'\cap K''=\varnothing$, K'' chain of $a(k)-j+1$ lines, $N\cup K''$ bichain of type $(i,a(k)-1)$, K' chain of $c(k)-a(k)-i$ planes, $N\cup K'$ bichain of type $(c(k)-a(k), j)$. As in 3.1, proof of Step 1, it is sufficient to check that the restriction map $r_{K\cup(N\cap M),M}(3)$ is surjective. Note that if $k-2=\dim(M)\leqslant 4$, $c(k)-a(k)-i\leqslant k-3$, hence there are chains of $c(k)-a(k)-i$ planes in M. The surjectivity of $r_{K\cup(N\cap M),M}(3)$ is sufficient (and not impossible by dimensional counts) even if $k=5$, because $d(5)-c(5)=4>1$. $K\cup(N\cap M)$ is the union of a bichain Z of type $(c(k)-a(k)-i, i-1)$, of K'' and of $j-2$ general points of M. Since $h^0(M, \mathcal{O}_M(3))\geqslant h^0(K\cup(N\cap M), \mathcal{O}_{K\cup(N\cap M)}(3))$, it is sufficient to prove the surjectivity of $r_{Z\cup K'',M}(3)$. Note that $0\leqslant a(k)-1-j\leqslant 2$. Hence in any case $Z\cup K''$ is contained in a chain of $c(k)-a(k)+2$ planes; $c((k)-a(k)+2\leqslant c(k-2)$ if $k\geqslant 20$. By induction (using the result for $k'=k-2$) we may assume the surjectivity of $r_{Z\cup K'',M}(3)$; if $k-2\equiv 9, 18, 27$ mod(36) we use Step 1 for $k\geqslant 26$, i.e., that $c(k)-a(k)+2\leqslant c(k-2)-1$. For $k\leqslant 19$ we have to use the explicit values of i, j; for instance, if $j=a(k)-1$, then $K''=\varnothing$ and Z is contained in a chain of $c(k)-a(k)-1$ planes. In the next two steps we will show the initial cases $k=5, 6$.

Step 4: $k=5$. By the previous steps, it is sufficient to find a bichain E in $H:=\mathbb{P}^4$ with $r_{E,H}(3)$ surjective. Take a hyperplane M of H. In H take a bichain $E=U\cup V$ with V reducible quadric in M, U spanning H and intersecting $M\setminus V$ at 4 points spanning M. Since $r_{E\cap M,M}(3)$ is bijective, it is sufficient to prove that $r_{U,H}(2)$ is surjective. This follows either from the proof of 3.2 or using another hyperplane J of M, J containing two lines of J.

Step 5: $k=6$. It is sufficient to prove that a general bichain E in $H := \mathbb{P}^5$, E of type (5, 7), has $r_{E,H}(3)$ surjective; take a hyperplane M of H and take $E = U \cup V$ with $V \subset M$ and U bichain of type (5, 1). It is sufficient to check that in M the general disjoint union of a chain I of 5 lines and a chain J of 6 lines has $r_{I \cup J,M}(3)$ surjective. Since for any union R of two skew lines in $\mathbb{P}^3$, $r_{R,3}(2)$ is surjective, the general union F in $\mathbb{P}^4$ of a chain of 4 lines and 2 skew lines has $r_{F,4}(2)$ bijective. Hence it is sufficient to check that the general union D in $\mathbb{P}^3$ of a line and a chain of 4 lines has $r_{D,3}(3)$ surjective: use a plane containing two of the lines of D.

Step 6: $k \equiv 9, 18, 27 \bmod(36)$. Set $E := \mathbb{P}^5$ and let J be a hyperplane in E. Let $\{A_t\}$, $\{B_t\}$, $t \in U$, $0 \in U$, be 2 families of planes in E with $A_t \cap B_t = \emptyset$ if $t \neq 0$, $A_0 \cup B_0 \subset J$, $A_0 \cap B_0$ a point, P. Let $\chi_E(P)$ be the first infinitesimal neighbourhood of P in E (it has $\mathscr{I}^2_{P,E}$ as ideal sheaf). The family $\{A_t \cup B_t\}$, $t \in U$, $t \neq 0$, is flat. Since $\mathrm{Hilb}(E)$ is complete, it has a limit at $0 \in U$. It is easy to check that this limit is $A_0 \cup B_0 \cup \chi_E(P)$ (see [13] for the corresponding case of lines in $\mathbb{P}^3$ instead of planes in $\mathbb{P}^5$). Now take 2 planes C, D in J, with $C \cap D$ a line, $D \cap A_0$ a line, $B_0 \cap C$ a line, no line contained in 3 of the planes A_0, B_0, C, D or point contained in all of them. Then $T := A_0 \cup D \cup C \cup B_0 \cup \chi_E(P)$ is a degeneration in E of a flat family of chains of 4 planes in E. For any hypersurface M (with equation z) of any E, the residual scheme $\mathrm{Res}_M(X)$ of any X in E with respect to M has as ideal sheaf the germs f with zf vanishing on X. Then $\mathrm{Res}_M(T)$ is P with its reduced structure. Fix a chain $U \cup A_0 \cup C \cup D$ of $m-1$ planes in $\mathbb{P}^k$, $k \geqslant 5$, $m \geqslant 4$, and a point $P \in A_0 \setminus U$; let E be a $\mathbb{P}^5$ containing A_0, C, D and take a plane B_0 in E with $A_0 \cap B_0 = P$, $B_0 \cap U = \emptyset$. We say that $Q := U \cup A_0 \cup C \cup D \cup B_0$ is a chain with m planes with a tail. $Q \cup \chi_E(P)$ is the limit of a flat family of chains if m planes. Take a hyperplane H in $\mathbb{P}^k$. Note that $b(k) = 1$, $d(k) = 0$. Take the following degeneration W of a chain of $c(k)$ planes. In H there is the reduced scheme associated to a chain Y of $c(k) - a(k)$ planes with a tail; for a suitable 5-dimensional E (not in H) $\chi_E(P)$ is the nilpotent on the critical point of Y; since $E \not\subset H$, $\mathrm{Res}_H(\chi_E(P)) = P$. The consider a general chain of $a(k)$ planes such that $I \cup J$ is a chain of $c(k)$ planes with tail. Set $L = I \cup J \cup \chi_E(P)$. Note that $\mathrm{Res}_H(L) = I \cup \{P\}$; since P is general enough, by 3.1 we may obtain that $r_{I \cup \{P\},k}(2)$ is bijective. It is sufficient to prove that $r_{L \cap H,H}(3)$ is bijective. $L \cap H$ is reduced. We may prove the bijectivity we need in two steps, using a hyperplane M of H and a hyperplane S of M; use twice the usual inductive procedure and in S the inductive assumption in $\mathbb{P}^{k-3}$. Since $k \geqslant 9$, this can be left safely to the reader. The proof of Theorem 2 is over. ∎

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