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Nontraceable detour graphs

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Abstract

The detour order (of a vertex v) of a graph G is the order of a longest path (beginning at v). The detour sequence of G is a sequence consisting of the detour orders of its vertices. A graph is called a detour graph if its detour sequence is constant. The detour deficiency of a graph G is the difference between the order of G and its detour order. Homogeneously traceable graphs are therefore detour graphs with detour deficiency zero. In this paper, we give a number of constructions for detour graphs of all orders greater than 17 and all detour deficiencies greater than zero. These constructions are used to give examples of nontraceable detour graphs with chromatic number k , $k \geq 2$, and girths up to 7. Moreover we show that, for all positive integers $l \geq 1$ and $k \geq 3$, there are nontraceable detour graphs with chromatic number k and detour deficiency l .

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1. Introduction

Unless otherwise stated we consider only finite, connected graphs without loops and multiple edges. In general, we use standard notation and terminology. For the sake of brevity, we say that “a graph H is a subgraph of a graph G ” instead of “a graph H is isomorphic to a subgraph of a graph G ”. If H is a subgraph of G , then we write $H \subseteq G$.

Let $G = (V, E)$ be a graph. If u, v are two vertices of G , then $\tau_G(u, v)$ denotes the order of a longest path in G with endvertices u and v . The *detour order* of a vertex $v \in V$ is the order of a longest path $P \subseteq G$ having v as an initial vertex. The detour order of v is denoted by $\tau_G(v)$. If there is no danger of confusion, then we simply write $\tau(v)$ and $\tau(u, v)$ instead of $\tau_G(v)$ and $\tau_G(u, v)$, respectively.

The concept of the detour order of a vertex appeared probably for the first time (with a different notation) in Problem 46 (formulated by L. Lovász) of [18, p. 366]. Also, a longest path in a graph was called a detour path by Kapoor et al. [12], and the length of such a path the detour number of the graph. In this paper, the detour order of a graph G , which we will denote by $\tau(G)$, is the maximum of the detour orders of its vertices. The difference $|V(G)| - \tau(G)$ will be

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called the *detour deficiency* of G . If we denote the vertices of G by v_1, v_2, \dots, v_n , then $(\tau_G(v_1), \tau_G(v_2), \dots, \tau_G(v_n))$ is called a *detour sequence* of G .

A longest path in a graph G is called a *detour* of G , and G is called a *detour graph* if each vertex $v \in V(G)$ is the initial vertex of a detour of G . In such a case the detour sequence of G is constant.

The invariant $\tau(G)$ can be used to define an interesting additive hereditary property of graphs (for details we refer the reader to [1]). In [3] the relationship between $\tau(G)$ and the detour chromatic number of G is described. Moreover, it is shown that the detour chromatic number is strongly related to the Path Partition Conjecture. This conjecture is discussed in [8].

In [2] Borowiecki and Mihók posed the problem of characterising detour sequences, that is, to find necessary and sufficient conditions for a given sequence of positive integers to be the detour sequence of some graph. This is analogous to the problem of characterizing degree sequences of graphs, studied, for example, in [9–11]. The detour sequences of trees have been characterised by Lesniak [13], and Dobrynin and Mel'nikov [7] have characterised the detour sequences of several families of cubic graphs, and have listed the detour sequences of all connected cubic graphs with at most 20 vertices.

A path in a graph G is called a *hamiltonian path* if it contains all the vertices of G . A graph which contains a hamiltonian path is said to be *traceable*. Similarly, a cycle in G is called a *hamiltonian cycle* if it contains all the vertices of G . In such a case G is called a *hamiltonian graph*. A graph G is called *hamiltonian connected* if each pair of distinct vertices u, v are endvertices of a hamiltonian path in G . A graph G is said to be *homogeneously traceable* if every vertex is an initial vertex of a hamiltonian path in G . Thus, a homogeneously traceable graph is a detour graph with detour deficiency zero.

The study of nonhamiltonian, homogeneously traceable graphs (NHHT graphs) was initiated by Skupień in 1975 (see [16,14]), and continued by Chartrand et al. [5]. From an existence theorem proved in [5] we know there are no NHHT graphs of order 3, 4, \dots , 8, but that NHHT graphs exist for all orders greater than 8 (see also [16]). The class of connected, nontraceable, detour graphs (CND graphs) shares many of the properties of NHHT graphs, and CND graphs can be seen as a natural generalisation of NHHT graphs.

In Section 2 we state some properties of CND graphs which are shared by NHHT graphs, and we use these properties to show that the detour order of a CND graph must be greater than 8.

In Section 3 we establish the existence of CND graphs by giving constructions for infinite families of CND graphs of all orders greater than 17, and all detour deficiencies greater than zero. These constructions also give examples of CND graphs with girths up to 7.

In Section 4 we give another construction for CND graphs. We use this construction to show that for arbitrary positive integers $k \geq 2$ and $l \geq 1$ there exist CND graphs with chromatic number k and detour deficiency at least l . In Section 5 we end by stating some open problems.

2. Properties of CND graphs

As we have already stated, many of the properties of NHHT graphs are shared by CND graphs. The proof of the following theorem is very similar to the proofs given in Skupień [16,15] and Chartrand et al. [5] for NHHT graphs. One has to only replace, where necessary, the order of the graph with the detour order, so we will not include the proofs here.

Theorem 2.1. *Let G be a CND graph of order n . Then:*

1. G is 2-connected (hence $\delta(G) \geq 2$).
2. Every vertex of G has at most one neighbour of degree 2.
3. $\Delta(G) \leq \tau(G) - 4$.
4. If T is the set of vertices of degree 2 in G , then $|V(G) \setminus T| \geq |T|$.
5. $|E(G)| \geq \lceil 5n/4 \rceil$.
6. G contains a detour P such that the endvertices of P each have degree at least 3.

We will use Theorem 2.1 to establish a lower bound on $\tau(G)$, where G is a CND graph. The following simple lemmas will be useful. We omit the obvious proof of Lemma 2.2.

Lemma 2.2. Let $P : v_1, v_2, v_3, \dots, v_{p-1}, v_p$ be a detour of a nontraceable graph G . Let H be a component of $G - V(P)$. Suppose that $v_l \in V(P)$ is adjacent to a vertex $w \in V(H)$, and a vertex $v_k \in V(P)$ is adjacent to a vertex $u \in V(H)$ ($w = u$ is allowed), where $l > k$. Then:

1. v_k and v_l are not consecutive vertices of P , i.e. $l \geq k + 2$.
2. $v_k \neq v_1$ and $v_l \neq v_p$.
3. v_1 is not adjacent to any of $v_{l-1}, v_{l+1}, v_{k+1}$, and v_p is not adjacent to any of $v_{k-1}, v_{k+1}, v_{l-1}$.
4. v_1 is not adjacent to v_p .
5. If v_1 is adjacent to v_j , then v_p is not adjacent to v_{j-1} .

Lemma 2.3. Let G be a CND graph with $\tau(G) = 8$. Let $P : v_1, v_2, \dots, v_8$ be a detour of G whose endvertices each have degree at least 3. Let H be a component of $G - V(P)$. Then H consists of a single vertex v and $\deg(v) = 2$.

Proof. Suppose that $|V(H)| > 1$. Since G is 2-connected there exist distinct vertices $v \in V(H)$ and $w \in V(H)$ such that v is adjacent to $v_l \in V(P)$ and w is adjacent to $v_k \in V(P)$, where $l > k$. We may suppose that $k \geq 3, l \leq 6$ and $l \geq k + 3 \geq 6$, otherwise we get a path longer than P . Hence $l = 6$ and $k = 3$. By Lemma 2.2, and since $\deg(v_1) \geq 3$, it follows that v_1 is adjacent to at least two (excluding v_2) vertices of P , which, again by Lemma 2.2, must be v_3 and v_6 . Similarly v_8 must also be adjacent to v_3 and v_6 . But this is not possible, since by Theorem 2.1(3) we get $\Delta(G) \leq \tau(G) - 4 = 4$. Hence $|V(H)| = 1$.

Next we show that $\deg(v) = 2$. Since G is 2-connected and $\tau(G) = 8$ we get $2 \leq \deg(v) \leq 4$. The case $\deg(v) = 4$ is ruled out by Lemma 2.2(1),(2). If $\deg(v) = 3$ there are only two cases to consider:

1. v is adjacent to v_2, v_4 and v_6 :
By Lemma 2.2(3) vertex v_1 must be adjacent to v_4 and v_6 . But for the same reason v_8 must be adjacent to either v_4 or v_6 , which contradicts $\Delta(G) \leq 4$.
2. v is adjacent to v_2, v_5 and v_7 :
Again, by Lemma 2.2(3), vertex v_1 must be adjacent to v_5 and v_7 , and v_8 must also be adjacent to v_5 , contradicting $\Delta(G) \leq 4$. \square

We are now able to give a lower bound for $\tau(G)$.

Theorem 2.4. Let G be a CND graph. Then, $\tau(G) \geq 9$.

Proof. We must have $\Delta(G) \geq 3$, otherwise G is hamiltonian, since $\delta(G) \geq 2$. Hence, $\tau(G) \geq \Delta(G) + 4 \geq 7$. If $\tau(G) = 7$, then by Theorem 2.1 we get $\Delta(G) \leq 3$. Hence, using Theorem 2.1(6), it is easy to show that $\tau(G) \neq 7$. Now suppose that $\tau(G) = 8$. Then, by Theorem 2.1, $\Delta(G) \leq 4$. Let $P : v_1, v_2, v_3, \dots, v_8$ be a detour of G with $\deg(v_1) \geq 3$ and $\deg(v_8) \geq 3$. Let $v \in V(G) \setminus V(P)$. By Lemma 2.3 v is adjacent to exactly two vertices of P . We show that the following cases, which by Lemma 2.2 exhaust all possible ways for v to be connected to P , cannot occur:

1. v adjacent to v_2 and v_4 : Consider the following subcases:
 - (a) v_1 adjacent to v_4 and v_6 :
By Lemma 2.2 vertex v_8 must be adjacent to v_6 and v_2 . Since v_2 is adjacent to v , and $\deg(v) = 2$, it follows from Theorem 2.1(2) that $\deg(v_3) \geq 3$. But v_3 cannot be adjacent to a vertex on P , other than v_2 and v_4 , without getting a path longer than P , or a vertex with degree greater than 4. Therefore v_3 is adjacent to some vertex $w \in V(G) \setminus V(P)$, and it follows from Lemma 2.3 that w is also adjacent to some vertex of P other than v_3 . But this again implies that we get a path longer than P , or a vertex with degree greater than 4.
 - (b) v_1 adjacent to v_4 and v_7 :
Then by Lemma 2.2 vertex v_8 must be adjacent to v_5 and v_2 . We then get the path $v_6, v_5, v_8, v_7, v_1, v_4, v, v_2, v_3$ which is longer than P .
 - (c) v_1 adjacent to v_6 and v_7 :
By Lemma 2.2 vertex v_8 must be adjacent to v_2 and v_4 . Then by Theorem 2.1 we get $\deg(v_5) \geq 3$. However, by using a similar argument to that in case 1(a) above, it is easy to show that v_5 cannot have degree greater than 2.

2. v adjacent to v_3 and v_5 :
 - (a) v_1 adjacent to v_3 and v_5 :
Apart from v_7 , the vertex v_8 can only be adjacent to v_6 , which contradicts $\deg(v_8) \geq 3$.
 - (b) v_1 adjacent to v_5 and v_7 :
Apart from v_7 , the vertex v_8 can only be adjacent to v_3 , contradicting $\deg(v_8) \geq 3$.
3. v adjacent to v_2 and v_5 :
Then v_1 is adjacent to v_5 and v_7 . Apart from v_7 , the vertex v_8 can only be adjacent to v_2 , contradicting $\deg(v_8) \geq 3$.
4. v adjacent to v_3 and v_6 :
Then vertex v_1 must be adjacent to v_3 and v_6 . Apart from v_7 , the vertex v_8 cannot be adjacent to any other vertices of P , again contradicting $\deg(v_8) \geq 3$.
5. v adjacent to v_2 and v_6 :
By Lemma 2.2, vertex v_1 must be adjacent to v_4 and v_6 , and v_8 must be adjacent to v_4 and v_2 (since $\Delta(G) \leq 4$). By Theorem 2.1(2) we get $\deg(v_5) \geq 3$. But once again, by using a similar argument to that in case 1(a), it is easy to show that v_5 cannot have degree greater than 2.
6. v adjacent to v_2 and v_7 :
 - (a) v_1 adjacent to v_4 and v_5 :
Then v_8 must be adjacent to v_5 and v_2 , and in that case $v_3, v_4, v_1, v_2, v, v_7, v_8, v_5, v_6$ is a path longer than P .
 - (b) v_1 adjacent to v_4 and v_7 :
 - (i) If v_8 is adjacent to v_4 and v_5 , or if v_8 is adjacent to v_2 and v_5 , we get a path $v_3, v_4, v_1, v_2, v, v_7, v_6, v_5, v_8$ which is longer than P .
 - (ii) If v_8 is adjacent to v_4 and v_2 :
By Theorem 2.1 we get $\deg(v_6) \geq 3$ and $\deg(v_3) \geq 3$. The only possibility is that v_6 is adjacent to v_3 . But then we get the path $v_8, v_7, v, v_2, v_3, v_6, v_5, v_4, v_1$ which is longer than P .

All cases have now been considered. \square

Theorem 2.4 implies immediately that there are no CND graphs with order less than 10, and in the next section we show that CND graphs of all orders greater than 17 exist. Whether or not a CND graph of order n exists, where $10 \leq n \leq 17$, remains an open problem.

3. Constructions of CND graphs

In this section we describe four types of constructions for CND graphs. The constructions in this section all follow the same pattern: we specify a certain type of multigraph L (i.e. multiple edges and loops allowed in L), inflate the vertices of L with graphs of a certain type, and also, in some cases, insert graphs of another type into the edges of L . Note that, although L is allowed to be a multigraph, our constructions are such that the CND graphs constructed will all be simple. We will denote the length of a longest trail in a multigraph L by $t(L)$, and $E(L)$ denotes the set of edges of L . We say a trail L spans G , or is a *spanning trail* of G , if every vertex of G belongs to L .

Definition 3.1. An admissible multigraph L is a multigraph with the following properties:

A-1 $t(L) < |E(L)|$.

A-2 For each vertex $v \in V(L)$, and edge $e \in E(L)$ which is incident with v , there is a trail of length $t(L)$ beginning v, e, \dots which spans L .

Lemma 3.2. Let L be an admissible multigraph. Then every vertex of L has odd degree greater than 1.

Proof. First note that it is obvious that $\deg(v) > 1$ for all $v \in V(L)$. Now suppose that a vertex v of L has even degree. Let T be a longest trail in L starting at v . Then every edge incident with v must be an edge of T (otherwise we get a longer trail), and hence the vertex v must also be the last vertex of T . Hence T is a closed trail, and since at least one edge e of L does not belong to T , we can join this edge to T to get a longer trail than T . \square

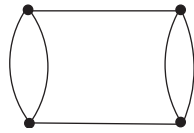


Fig. 1. $C_2 \times K_2$.

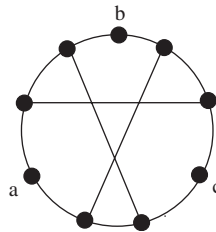


Fig. 2. PG^\odot .

It follows easily from Lemma 3.2 that the two smallest admissible multigraphs are K_4 and the multigraph $C_2 \times K_2$ shown in Fig. 1. In fact, $C_n \times K_2$, where $n \geq 2$, is an infinite family of cubic, admissible multigraphs.

3.1. First construction

We need the following type of graph:

Definition 3.3. A simple graph G is said to be an I-type graph if it has a distinguished set $D = \{a, b, c\}$ of three vertices such that

- (I-1) G has no spanning path with both endvertices in D .
- (I-2) For any vertex $v \in V(G)$ there is a path P with endvertex v and an endvertex in D , and the remaining pair of vertices in D are joined by a path Q , disjoint from P , such that P and Q together span G . (The cases where $P = a$ or $P = b$ or $P = c$ are included.)

One example of an I-type graph is the net (i.e. K_3 with a pendant leaf attached to each vertex), where we take the distinguished vertices to be the vertices of degree 1. Another example is the graph obtained by deleting a vertex from the Petersen graph, the distinguished vertices being those vertices of degree 2 which were formerly neighbours of the deleted vertex (see Fig. 2). A vertex deleted Petersen graph is denoted by PG^\odot .

Interesting properties of PG^\odot can be found in [16].

Let L be a cubic, admissible multigraph of order k . Let F be a graph obtained by inflating each vertex $v_i \in V(L)$ with an I-type graph G_i , ($i = 1, 2, \dots, k$). That is, we delete the vertex v_i , adding a copy of graph G_i in its place, and joining the three former neighbours of the deleted vertex to the three distinguished vertices of the copy of G_i by using the three edges which were incident with v_i . This process is carried out sequentially on all the vertices of L . We will say that F is obtained by inflating L with I-type graphs. For example, the inflation of the admissible multigraph shown in Fig. 1 with I-type graphs G_1, G_2, G_3 and G_4 is illustrated in Fig. 3.

The construction of CND graphs using I-type graphs is explained in Theorem 3.4.

Theorem 3.4. Let L be a cubic, admissible multigraph of order k . Let F be obtained by inflating L with I-type graphs $G_i, i = 1, 2, 3, \dots, k$. Then F is a CND graph with detour order

$$2 - k + \sum_{i=1}^k |V(G_i)|.$$

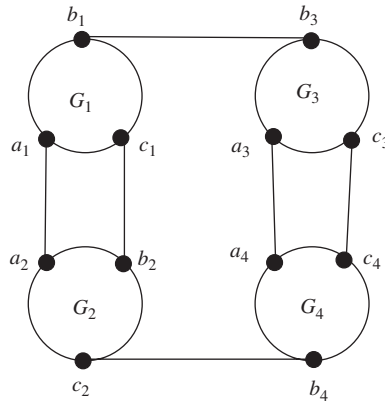


Fig. 3. Inflation of a multigraph with I-type graphs.

Proof. By Condition I-1, if a path W in F contains vertices from each copy of G_i , then at least $k - 2$ vertices of F are not in W , since at least one vertex must be omitted from every copy of G_i which does not contain an endvertex of W . Hence

$$\tau(F) \leq 2 - k + \sum_{i=1}^k |V(G_i)|$$

and F is nontraceable. Next we show that each vertex v of F is an endvertex of a path of order

$$2 - k + \sum_{i=1}^k |V(G_i)|.$$

Let $v \in V(F)$, and let G_0 be the I-type subgraph of F containing v . Then there is a path P from v to one of the three distinguished vertices of G_0 (call it a_0) such that the two remaining distinguished vertices are joined by a path Q , disjoint from P , where P and Q together span G_0 . Let e_1 be the edge incident with a_0 whose other endvertex is not in G_0 . Let v_0 be the vertex in L corresponding to G_0 . Then, by A-2, L has a longest trail T starting v_0, e_1, \dots , which spans L . Now each of the three edges of L incident with v_0 lies in T , otherwise we can add a missing edge to T to obtain a trail in L that is longer than T . Thus, v_0 occurs exactly twice in T . Let v_l be the last vertex of T . Then v_l also occurs exactly twice in T , while every vertex in $V(L) \setminus \{v_0, v_l\}$ occurs exactly once in T . We can now construct a path P in F starting at v that exits from G_0 at a_0 and then moves through the subgraphs G_i in accordance with the trail T . Such a path will visit each of G_0 and G_l (the I-type graph corresponding to v_l) twice and all the other G_i only once each. Thus, we can choose P so that it contains all the vertices of G_0 and all those of G_l , as well as all but one vertex of each of the remaining G_i . Hence $|V(P)| = 2 - k + \sum_{i=1}^k |V(G_i)|$. Since $\tau(F) \leq 2 - k + \sum_{i=1}^k |V(G_i)|$, and $v \in F$ was chosen arbitrarily, it follows that F is a CND graph. \square

Remark 3.5. We get an infinite family of cubic CND graphs (all of girth 5) by letting the cubic, admissible graphs be $C_n \times K_2, n \geq 3$, and taking all the I-type graphs to be copies of PG^\odot . If instead we use K_4 , or the multigraph $C_2 \times K_2$ shown in Fig. 1, for the cubic, admissible multigraph we get two of the four smallest presently known cubic CND graphs. We will describe the other two in Section 3.2.

3.2. Second construction

Here we need a new type of graph, which we call *homogeneously connected from two vertices*, or briefly an HCTV graph. The definition is:

Definition 3.6. A simple graph M is called homogeneously connected from two vertices if it is K_1 or it has two vertices x, y such that each vertex of M is an initial vertex of a hamiltonian path with the other endvertex either x or y . The vertices x, y are called anchors of M .

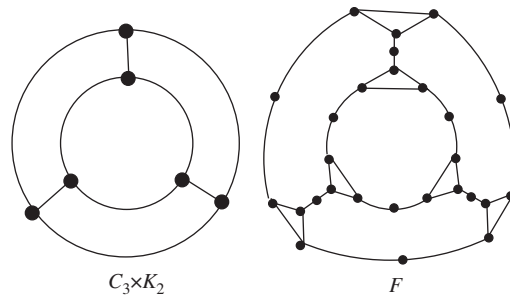


Fig. 4. An illustration of the inflation and insertion operations.

One can easily see that if a graph G is hamiltonian connected then it is an HCTV graph. On the other hand, if it is an HCTV graph, then it is homogeneously traceable. Moreover, from the definition it follows that the anchors of an HCTV graph G are endvertices of a hamiltonian path in G . Examples of HCTV graphs are the complete graphs K_n , $n \geq 1$, and the complete balanced bipartite graphs $K_{n,n}$, $n \geq 1$.

Our next construction for CND graphs is described in the following theorem. We will need a similar inflation operation on graphs or multigraphs to that described in the first construction. Starting with any graph or multigraph we can *inflate* a vertex v with a complete graph K_n , where $n \geq \deg(v)$. In other words, we delete the vertex v and add a copy of K_n in its place, joining the former neighbours of the deleted vertex to distinct vertices of K_n (which we will call the *inflation* vertices of K_n) by using the edges which were incident with v . We also need the operation of *inserting* an HCTV graph M in an edge e of any graph or multigraph. This is done by deleting e , and then joining the two vertices which were incident with e to the anchors of M by a matching if $M \neq K_1$, or to M if $M = K_1$. This is illustrated in Fig. 4 for the case where we inflate $C_3 \times K_2$ with copies of K_3 and insert K_1 in each edge of $C_3 \times K_2$ to get the graph F .

Theorem 3.7. *Let L be an admissible multigraph and $k = |E(L)|$. Let M_i , $i = 1, 2, 3, \dots, k$ be HCTV graphs, all of the same order m . Let F be obtained by first inflating each vertex v of L with a copy of K_n for some $n \geq \deg(v)$, and then inserting a copy of an HCTV graph M_i in the edge e_i of L , $i = 1, 2, \dots, k$. Then F is a CND graph.*

Proof. Suppose that the longest trail in L has length l . Then no path in F can contain vertices from more than l copies of the HCTV graphs, since otherwise we can use the path in an obvious way to construct a trail in L with length greater than l . Hence $\tau(F) \leq |V(F)| - (k - l)m$. Next we show that each vertex of F is an endvertex of a path of order $|V(F)| - (k - l)m$. Let $v \in V(F)$. We have two cases to consider:

1. Suppose that v belongs to one of the complete subgraphs, say K^* , of F obtained by inflating L . Suppose that edge e_1 in L was incident with the vertex $v_1 \in V(L)$ which was inflated to give K^* . Then L has a spanning trail T of length l starting $v_1 e_1 \dots$. We can now construct a path Q starting at v which exits K^* via e_1 , and then moves through the inflations and the insertions in accordance with the trail T . The trail T spans L , hence Q contains vertices from all the complete graphs used to inflate L . It is easy to see that we can use the properties of the complete graphs to ensure that Q passes through all the vertices of all the complete graphs used in the inflation process, and the properties of the HCTV graphs allow us to choose Q so that it contains all the vertices of the l HCTV graphs lying in the path. Hence, $|V(Q)| = |V(F)| - (k - l)m$.
2. If $v \in V(M_i)$ for some i we can similarly use a longest trail in L to construct a path in F starting at v and having order $|V(F)| - (k - l)m$.

Since $v \in V(F)$ was chosen arbitrarily, it follows that F is a CND graph. \square

Remarks 3.8. We get the following results from this construction:

1. If we take the HCTV graphs to be K_1 and inflate each vertex of L with a K_3 , and choose L to be either the graph in Fig. 1 or K_4 , then we get the two smallest presently known CND graphs, each of order 18 and size 24, with detour

deficiency one. It is proved in [4] that these two graphs are the smallest (with regard to both size and order) 2-connected, nontraceable, claw-free graphs. (A graph is claw-free if it does not contain $K_{1,3}$ as an induced subgraph.) Hence they are the smallest claw-free, CND graphs.

2. Let L be $C_n \times K_2$, $n \geq 3$, the HCTV graphs be K_2 , and we inflate L with K_3 . The resulting CND graphs have order $12n$ and size $15n$, $n \geq 3$. Hence these CND graphs all realise the lower bound on the size of CND graphs given in Theorem 2.1. If, instead of K_2 , we use K_1 for the HCTV graphs we get a family of CND graphs each having detour deficiency $n - 1$, $n \geq 3$, and, of course, girth 3.
3. Let L be K_4 or the admissible multigraph shown in Fig. 1, the HCTV graphs be K_2 , and we inflate L with K_3 . The resulting two CND graphs have order 24 and size 30, so they also realise the lower bound on the size of CND graphs. Thus, we have examples of CND graphs of order $12n$, $n \geq 2$, having the least possible number of edges.
4. In Remark 3.5 we gave two examples of cubic CND graphs of order 36. We get two more such graphs by choosing L to be either K_4 or the multigraph shown in Fig. 1, the HCTV graphs to be copies of $K_4 - e$ (i.e. the graph obtained by deleting any edge from K_4), and we inflate L with copies of K_3 .
5. It is easy to get CND graphs of all orders greater than 17. For example, let L be K_4 , the HCTV graphs be K_1 , and we inflate L with three copies of K_3 and one copy of K_{n+3} , ($n \geq 0$). Then, we get CND graphs of order $18 + n$, for $n \geq 0$.

3.3. Third construction

We need the following definition of another type of graph:

Definition 3.9. A simple graph G is said to be an R-type graph if it has a distinguished set $D = \{a, b, c\}$ of three vertices such that

- (R-1) There is a spanning path of G with both endvertices in D .
- (R-2) For any vertex $v \in G$ there is a path P with endvertex v and an endvertex in D , and the remaining pair of vertices in D are joined by a path Q , disjoint from P , such that P and Q together span G . (The cases where $P = a$ or $P = b$ or $P = c$ are included.)

Clearly complete graphs K_n , $n \geq 3$, are R-type graphs, where the distinguished vertices can be any three vertices of K_n , and in Fig. 5 we show an R-type graph with girth 4.

Using Definition 3.9 we get:

Theorem 3.10. Let L be a cubic, admissible multigraph of size k . Let M_i , $i = 1, 2, 3, \dots, k$ be HCTV graphs all of the same order m . Let F be obtained by first inflating all the vertices of L with copies of R-type graphs, and then inserting copies of the HCTV graphs M_i in each of the edges of L . Then F is a CND graph.

Proof. We omit the details of the proof since they are very similar to the proof of Theorem 3.7. We simply note that we can use a longest trail in L (let its length be l) to define a path Q , starting at an arbitrary vertex $v \in V(F)$, which contains all the vertices of the R-type graphs and omits all the vertices of $k - l$ of the HCTV graphs, and no path in F can contain more vertices than Q . □

This construction gives examples of CND graphs with girth 4, if we use the graph shown in Fig. 5 as our R-type graphs and K_1 for HCTV graphs. For L we can take any cubic, admissible multigraph, for example, K_4 .

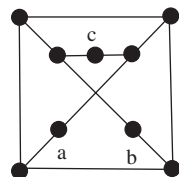


Fig. 5. An R-type graph with girth 4.

3.4. Fourth construction

Here we will need the concept of a maximal hypohamiltonian graph. A graph G is *hypohamiltonian* if G is not hamiltonian, but every vertex deleted subgraph $G - v$ of G is hamiltonian. A hypohamiltonian graph G is called *maximal hypohamiltonian* if $G + e$ is hamiltonian for each $e \in E(\overline{G})$, where \overline{G} denotes the complement of G . The Petersen graph is an example of a maximal hypohamiltonian graph.

We now describe a construction which allows us to use certain maximal hypohamiltonian graphs to construct CND graphs with girth 7 and 6.

We begin by defining the types of graphs we need for the construction.

Definition 3.11. A simple graph G is said to be a U-type graph if it contains a set $D = \{a, b, c\}$ of three distinguished vertices such that

- (U-1) For each pair of vertices in D there is a path in G having those two vertices as endvertices and containing all vertices of G except the other vertex in D .
- (U-2) There is no spanning path of G with both endvertices in D .
- (U-3) G is traceable from each vertex in D .
- (U-4) If $v \in V(G)$ and $v \notin \{a, b, c\}$ then there is a path from v to a vertex in $\{a, b, c\}$ which spans G .

The vertex deleted Petersen graph shown in Fig. 2 is a U-type graph. In fact, any maximal hypohamiltonian graph which has at least one vertex of degree 3 can be used to construct a U-type graph. This is proved in the next theorem.

Theorem 3.12. Let H be a maximal hypohamiltonian graph containing a vertex w of degree 3. Let the vertices a, b, c of H be adjacent to w . Let $G = H - w$. Then G is a U-type graph with $D = \{a, b, c\}$.

Proof.

1. $H - a$ is hamiltonian, and b, c are the only vertices of $H - a$ adjacent to w . Hence we have a hamiltonian cycle C in $H - a$ containing w and such that bwc is a path on C . Therefore the path on C from b to c which omits w contains all the vertices of $H - w$ except a . Similarly for the other vertices in D . Thus, U-1 is satisfied.
2. If there were a path spanning $H - w$ with endvertices in D then H would be hamiltonian, hence U-2 is satisfied.
3. U-3 follows since $H - w$ is hamiltonian.
4. Since v is not adjacent to w , and H is maximal nonhamiltonian, it follows that $H + (vw)$ is hamiltonian. A hamiltonian cycle C in $H + (vw)$ must contain the edge (vw) , and one of a, b, c must be adjacent to w on C . Suppose that a is adjacent to w on C . Then, the path on C from v to a which omits w contains all the vertices of $H - w$. Thus, U-4 is satisfied. \square

We also need the following type of multigraph:

Definition 3.13. A presentable multigraph S is a multigraph such that

- (S-1) S is cubic.
- (S-2) There is a longest trail in S beginning v, e, \dots from each vertex v and each edge e incident with v , and this longest trail spans S .
- (S-3) There is a hamiltonian path beginning v, e, \dots from each vertex v and edge e incident with v .

Some examples of presentable multigraphs are K_4 , $C_n \times K_2$, and the Petersen graph.

The construction of CND graphs using presentable multigraphs and U-type graphs is described in Theorem 3.14.

Theorem 3.14. Let S be a presentable multigraph of order k . Let F be obtained by inflating S with U-type graphs G_i , $i = 1, 2, 3, \dots, k$. Then F is a CND graph.

Proof. If a path P in F contains vertices from each G_i , then at least $k - 2$ vertices of F are not in P , since, by Condition U-2, at least one vertex must be omitted from every G_i which does not contain an endvertex of P . Hence

$$\tau(F) \leq \sum_{i=1}^k |V(G_i)| - k + 2,$$

and therefore F is nontraceable. Next we show that each vertex of F is an endvertex of a path of order $\sum_{i=1}^k |V(G_i)| - k + 2$. Let $v \in V(F)$. We have two cases to consider.

1. Suppose that v is a distinguished vertex of some U-type subgraph G_0 of F . Then, following the same procedure described in the proof of Theorem 3.4, we can construct a path Q in F starting at v with $|V(Q)| = \sum_{i=1}^k |V(G_i)| - k + 2$.
2. Suppose that $v \in V(G_0)$, for some U-type subgraph G_0 of F , but that v is not a distinguished vertex of G_0 . Then there is a path P in G_0 from v to some distinguished vertex of G_0 , say a_0 , which spans G_0 . Let e_1 be the edge incident with a_0 whose other endvertex is not in G_0 . Let v_0 be the vertex in S corresponding to G_0 . Then S has a hamiltonian path Q starting $v_0 e_1 \dots$. We can now construct a path P in F starting at v that exits from G_0 at a_0 and then moves through the G_i in accordance with the path Q . By properties U-3, U-4 we can choose the path P so that it contains all the vertices of the first and last U-type subgraphs in the path, and omits (by U-1 and U-2) exactly one vertex from each of the other U-type graphs. Hence,

$$|V(Q)| = \sum_{i=1}^k |V(G_i)| - k + 2.$$

Since $\tau(F) \leq \sum_{i=1}^k |V(G_i)| - k + 2$ and $v \in V(F)$ was chosen arbitrarily it follows that F is a CND graph. \square

The Coxeter graph has girth 7, is cubic, and is maximal hypohamiltonian (see Clark and Entringer [6] and Skupień [17]). Thus, we get an infinite family of CND graphs of girth 7 if we use $C_n \times K_2$, $n \geq 3$, for presentable multigraphs and copies of a vertex deleted Coxeter graph for U-type graphs. Clark and Entringer [6] showed that the Isaacs snarks J_k , $k \geq 5$, k odd, are maximal hypohamiltonian. Since these snarks, and their vertex deleted counterparts, all have girth 6 they can similarly be used to inflate the vertices of $C_n \times K_2$ to give infinite families of CND graphs of girth 6. The bipartite CND graphs constructed in Section 4 also have girth 6.

4. A construction for CND graphs with prescribed chromatic number

The constructions in the previous section did not enable us to find bipartite CND graphs. In this section we describe a construction that allows us to construct CND graphs with prescribed chromatic numbers. As before, our construction will be based on inflations and insertions, but here we will inflate the vertices of a cycle C_n with graphs which satisfy the conditions described in the next definition.

Definition 4.1. Let G be a simple graph of order $k \geq 3$ and $m \leq k - 2$ be a positive integer. We say that G is an inflator graph with drop m if it has a set of two distinguished vertices $\{a, b\}$ such that:

- (D-1) $\tau_G(a, b) = k - m$.
- (D-2) Each distinguished vertex is an initial vertex of a path in G containing all the vertices of G except the other distinguished vertex.
- (D-3) G is traceable from each distinguished vertex.
- (D-4) Let $v \in V(G) \setminus \{a, b\}$. Then there is a path P from v to a distinguished vertex, and a path Q in G with initial vertex the remaining distinguished vertex, such that $V(P) \cap V(Q) = \emptyset$ and $V(P) \cup V(Q) = V(G)$. The path Q may consist of a single distinguished vertex.

Theorem 4.2 shows that inflator graphs are closely related to NHHT graphs.

Theorem 4.2. *Let m and k be positive integers such that $m \leq k - 2$. Let G be an inflator graph of order k and drop m , with distinguished vertex set $\{a, b\}$. Let H be the graph obtained from G by adding a new vertex x and two new edges ax and bx . Then H is a NHHT graph.*

Proof. Suppose that H is hamiltonian. Then, since the vertex x is of degree 2, any hamiltonian cycle must contain the edges ax and bx . But any path connecting a and b in G has order at most

$$\tau_G(a, b) = k - m < k = |V(G)|,$$

and therefore H cannot contain a hamiltonian cycle.

Now we prove that any vertex of the graph G is an initial vertex of a hamiltonian path in H . Firstly, consider the vertex a . According to condition D-2 of Definition 4.1 there exists a path P in G starting at b and containing all the vertices of G except a . Therefore, the path $axbP$ is a hamiltonian path in H starting at a . Similarly, b is also the initial vertex of a hamiltonian path in H .

Consider now a vertex $v \in V(G) \setminus \{a, b\}$. By condition D-4 of Definition 4.1, we have a path P from v to a vertex in $\{a, b\}$, say a , and a path Q , disjoint from P , starting at b such that $V(P) \cup V(Q) = V(G)$. Then the path PxQ is a hamiltonian path in H starting at v .

Finally, consider the vertex x . According to the condition D-3 of Definition 4.1 there exists a hamiltonian path in G with initial vertex a . This path, together with the edge xa , forms a hamiltonian path in H with initial vertex x .

Thus, the proof is complete. \square

One can easily verify that the graph G^\odot shown in Fig. 6 is an inflator graph of order 8 and drop one.

Corollary 4.3. *The graph G^\odot is the smallest inflator graph (with respect to both order and size).*

Proof. Skupień in [16] proved that the smallest NHHT graph has order 9 and size 12. It can be obtained from G^\odot by adding a new vertex x and two new edges ax and bx . Therefore, applying Theorem 4.2, we obtain the desired result. \square

In the next theorem we use the operation of inflating each vertex of a cycle C_n with a copy of an inflator graph. Explicitly, we replace each vertex v of C_n with a copy of an inflator graph G by deleting v and joining the two former neighbours of v in C_n to the two distinguished vertices of G by a matching.

We will also use HCTV graphs (which we introduced in Definition 3.6) in the next theorem. Also, if $M_i, i = 1, 2, \dots, n$ are HCTV graphs and $G_i, i = 1, 2, \dots, n$ are inflator graphs, we will denote by $\mathcal{F}(G_1, G_2, \dots, G_n, M_1, M_2, \dots, M_n)$ the graph obtained by first inflating each vertex v_i of a cycle $C_n, n \geq 2$, with G_i , and then inserting M_i in each former edge e_i of $C_n, i = 1, 2, 3, \dots, n$. This construction is illustrated in Fig. 7. If $G_1 = G_2 = \dots = G_n = G$ and $M_1 = M_2 = \dots = M_n = M$ then we simply write $\mathcal{F}(n, G, M)$.

Theorem 4.4. *Let m and k_1, k_2, \dots, k_n be positive integers satisfying $m + 2 \leq \min\{k_1, k_2, \dots, k_n\}$. Let $G_i, i = 1, 2, \dots, n$, be inflator graphs of order $k_i, i = 1, 2, \dots, n$ respectively. Suppose that each G_i has the same drop m . Let $M_i, i = 1, 2, \dots, n$ be HCTV graphs, each of order at least m . Then the graph $F = \mathcal{F}(G_1, G_2, \dots, G_n, M_1, M_2, \dots, M_n)$ is a CND graph.*

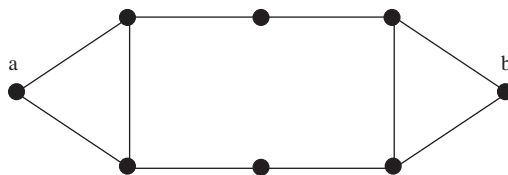


Fig. 6. The smallest inflator graph.

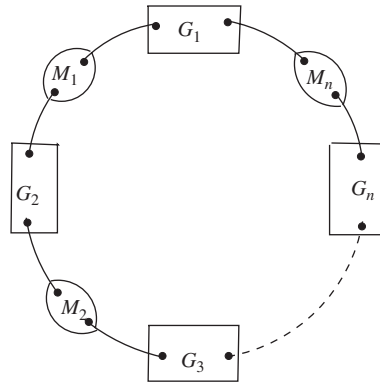


Fig. 7. The construction of $\mathcal{F}(G_1, G_2, \dots, G_n, M_1, M_2, \dots, M_n)$.

Proof. It is not difficult to see that if a path P contains at least one vertex from each M_i , then it goes through at least $n - 1$ graphs from G_1, \dots, G_n and, according to the condition D-1 of Definition 4.1, at least $(n - 1)m$ vertices of G_1, \dots, G_n are not contained in P . Hence in this case the order of P is at most

$$\sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

Similarly, if a path P contains no vertices from exactly one HCTV graph, say M_r , then P goes through at least $n - 2$ inflator graphs G_i . Therefore in this case its order is at most

$$\sum_{i=1}^n k_i + \sum_{i=1}^{r-1} |V(M_i)| + \sum_{i=r+1}^n |V(M_i)| - (n - 2)m.$$

Since $|V(M_r)| \geq m$ we get

$$\sum_{i=1}^n k_i + \sum_{i=1}^{r-1} |V(M_i)| + \sum_{i=r+1}^n |V(M_i)| - (n - 2)m \leq \sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

Therefore

$$\tau(F) \leq \sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

Next we show that each vertex of F is an endvertex of a path of order

$$\sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

We consider three cases.

- (i) Firstly, let v be a vertex of one of the HCTV subgraphs of F , say M_i . Let P be a path starting at v , then passing through all the other vertices of M_i , and exiting M_i via an anchor vertex of M_i . P then follows the underlying cycle C_n of F , passing alternately through all the inflator subgraphs of F and all the other HCTV subgraphs of F , ending in an inflator subgraph, say G_j , where $j = i$ or $j = i + 1 \pmod{n}$. By property D-3 and D-1 of Definition 4.1, and since P enters and leaves the other HCTV subgraphs via anchor vertices, it is easy to see that we can choose P to contain all the vertices of the HCTV subgraphs and all the vertices of G_j , while omitting exactly m vertices

from each of the other $n - 1$ inflator subgraphs that P passes through. Hence

$$|V(P)| = \sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

- (ii) Now suppose that v is a distinguished vertex of some inflator subgraph, say G_i , of F . Consider a path starting at v which exits G_i from v and then follows the underlying cycle C_n of F , passing through all the HCTV and other inflator subgraphs, and finally entering G_i via the (so far) unused distinguished vertex of G_i . The properties of HCTV graphs and property D-2 of Definition 4.1 imply that we can choose P such that

$$|V(P)| = \sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

- (iii) Now suppose that v is a vertex of some inflator subgraph G_i of G , but is not a distinguished vertex of G_i . By condition D-4 of Definition 4.1 there is a path P_1 from v to a distinguished vertex of G_i , say b_i , and a path Q , disjoint from P_1 , starting at the other distinguished vertex of G_i , say a_i , such that $V(P_1) \cup V(Q) = V(G_i)$. We can then construct a path P starting with the subpath P_1 , then following the underlying cycle through all the HCTV subgraphs and all the inflator subgraphs, finally ending with the subpath Q . Therefore we can choose P to contain all the vertices of G_i and all the vertices of the HCTV subgraphs while omitting exactly m vertices from each of the other $n - 1$ inflator graphs. Hence

$$|V(P)| = \sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m.$$

Since

$$\tau(F) \leq \sum_{i=1}^n k_i + \sum_{i=1}^n |V(M_i)| - (n - 1)m$$

it follows that F is a CND graph. \square

Note that $\mathcal{F}(2, G^\odot, K_1)$ gives an alternative construction for one of the two smallest, claw-free CND graphs constructed in Section 3.2.

In order to construct CND graphs with prescribed chromatic number we introduce the two graphs B^\odot and C^\odot shown in Fig. 8. It is not difficult to verify that both these graphs are inflator graphs with drop two, where the distinguished vertices of B^\odot are the vertices labelled a and b in Fig. 8, and the distinguished vertices of C^\odot are the vertices labelled c and d . Clearly both B^\odot and C^\odot are bipartite graphs.

It follows from Theorem 4.4 that $\mathcal{F}(2, B^\odot, K_2)$ is a CND graph, and, since B^\odot is bipartite, it is easy to see that $\mathcal{F}(2, B^\odot, K_2)$ is bipartite. This is the smallest bipartite CND graph that we know at present.

Using the graph C^\odot we can prove the following more general result.

Theorem 4.5. *For each positive integer $d \geq 1$ there exists a bipartite CND graph with detour deficiency at least d .*

Proof. Let $n = \lceil d/2 \rceil + 1$. Then the graph $\mathcal{F}(n, C^\odot, K_2)$ is a bipartite CND graph, and its detour deficiency is $2(n - 1) \geq d$. \square

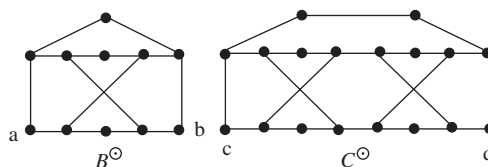


Fig. 8. Two bipartite inflator graphs.

We remark that the graph B^\odot can also be used for the construction of an infinite sequence of bipartite CND graphs. However, since the vertices a and b of B^\odot have the same colour in any 2-colouring of $V(B^\odot)$, the graph $\mathcal{F}(n, B^\odot, K_2)$ is bipartite if and only if n is even.

Using the graph G^\odot we also have

Theorem 4.6. *Let c and d be positive integers, $c \geq 3$, $d \geq 1$. Then there exists a CND graph with chromatic number c and detour deficiency d .*

Proof. Consider the graph $\mathcal{F}(d+1, G^\odot, K_c)$. Since G^\odot is 3-colourable, the new graph $\mathcal{F}(d+1, G^\odot, K_c)$ evidently has chromatic number c . By Theorem 4.4, the graph $\mathcal{F}(d+1, G^\odot, K_c)$ is a CND graph, and its detour deficiency is exactly $((d+1) - 1) = d$. \square

5. Open problems

We conclude the paper with a number of open problems. They are related to the order of CND graphs. It is proved in [4] that the order of a claw-free, CND graph is at least 18. In Section 2 we proved that the order of a CND graph is at least 10.

Problem 5.1. *What is the minimum order of a CND graph?*

An analogous problem can be formulated for bipartite graphs as well, since the smallest bipartite, CND graph provided by our construction has order 26.

Problem 5.2. *What is the minimum order of a bipartite CND graph?*

We gave four examples of cubic CND graphs of order 36, and these are the smallest presently known cubic CND graphs.

Problem 5.3. *What is the minimum order of a cubic CND graph?*

We have only given examples of graphs of order $12k$, $k \geq 2$, which realise the lower bound on size given in Theorem 2.1.

Problem 5.4. *For which other values of n does a CND graph of order n and size $\lceil 5n/4 \rceil$ exist?*

Lastly, we only have examples of CND graphs with girth up to 7. So

Problem 5.5. *Do there exist CND graphs of arbitrarily large girth?*

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