ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT•

Advances in Mathematics 185 (2004) 207–214

ADVANCES IN Mathematics

http://www.elsevier.com/locate/aim

Degree bounds for syzygies of invariants

Harm Derksen[1]

Department of Mathematics, University of Michigan, 2074 East Hall, Ann Arbor, MI 48109-1109, USA

Received 29 November 2002; accepted 20 May 2003

Communicated by David J. Benson

Abstract

Suppose that G is a linearly reductive group. Good degree bounds for generators of invariant rings were given in (Proc. Amer. Math. Soc. 129 (4) (2001) 955). Here we study minimal free resolutions of invariant rings. For finite linearly reductive groups G it was recently shown in (Adv. Math. 156 (1) (2000) 23, Electron Res. Announc. Amer. Math. Soc. 7 (2001) 5, Adv. Math. 172 (2002) 151) that rings of invariants are generated in degree at most the group order $|G|$. In characteristic 0 this degree bound is a classical result by Emmy Noether (see Math. Ann. 77 (1916) 89). Given an invariant ring of a finite linearly reductive group G, we prove that the ideal of relations of a minimal set of generators is generated in degree at most $\leqslant 2|G|$.

MSC: 13A50 (Primary); 13D02

Keywords: Noether's bound; Free resolutions; Invariant theory

1. Introduction

Let us fix a linearly reductive group G over a field K. If V is a representation of G, then G acts on the coordinate ring $K[V]$. The ring of invariant functions is denoted by $K[V]^G$. Let $\beta_G(V)$ be the smallest positive integer d such that all invariants of degree at most d generate the invariant ring.

E-mail address: hderksen@umich.edu.
[1] Supported by NSF grant DMS 0102193.

doi:10.1016/S0001-8708(03)00167-1

Noether proved that $\beta_G(V) \leqslant |G|$ for a finite group G in characteristic 0. A group G over a field K is linearly reductive if and only if the characteristic of the base field does not divide $|G|$. This situation is often referred to as the *non-modular case*. Noether's bound was recently extended by Fleischmann (see [6]) to the general non-modular case. Fogarty found another proof of this independently in [7]. A third proof follows from the subspaces conjecture in [2], which was solved by Sidman and the author in [4]. For connected linearly reductive groups, upper bounds for $\beta_G(V)$ which depend polynomially on the weights appearing in the representation were given in [3].

In this paper, we discuss good degree bounds for syzygies of invariant rings. Suppose that G is a linearly reductive group and that V is an n-dimensional representation. Let $f_1, f_2, \ldots, f_r$ be minimal homogeneous generators of the invariant ring $R = K[V]^G$ whose degrees are $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r$. Define the graded polynomial ring $S = K[x_1, x_2, \ldots, x_r]$ by $\deg(x_i) = d_i$ for all i. Now R is an S-module via the surjective ring homomorphism $\varphi : S \twoheadrightarrow R$ defined by $\varphi(x_i) = f_i$ for all i. Let

$$0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to R \to 0$$

be the minimal free graded resolution of R as an S-module. We define $\beta_G^i(V)$ as the smallest integer such that F_i is generated as an S-module in degree at most d. Note that $F_0 = S$ and the image of F_1 in F_0 is the syzygy ideal J of S defined by

$$J = \{h \in k[x_1, \ldots, x_r] | h(f_1, f_2, \ldots, f_r) = 0\}.$$

The invariant ring $K[V]^G$ is Cohen–Macaulay (see [8]). By standard methods, one can estimate the degrees in the minimal resolution of $K[V]^G$ by cutting the invariant ring down by hypersurfaces. Using an estimate of Knop for the a-invariant (see [1, Definition 4.3.6, p. 169] for a discussion of the a-invariant) of the invariant ring, we will prove the following bound:

Theorem 1. *We have*

$$\beta_G^i(V) \leqslant d_1 + d_2 + \cdots + d_{s+i} - s \leqslant (s+i)\beta_G(V) - s$$

where s is the Krull dimension of $K[V]^G$.

If G is connected and linearly reductive, then Theorem 1 and the polynomial bounds of $\beta_G(V)$ in [3] give polynomial bounds for $\beta_G^i(V)$. From the case $i = 1$ follows that the syzygy ideal J is generated in degree at most $\beta_G^1(V) \leqslant (s+1)\beta_G(V) - s$.

Let G be a finite group in the non-modular case. Theorem 1 and the inequality $\beta_G(V) \leqslant |G|$ imply that

$$\beta_G^i(V) \leqslant (n+i)|G| - n$$

and that J is generated in degree at most $\beta^1_G(V) \leqslant (n+1)|G| - n$. This last inequality will be improved in Theorem 2.

Suppose that M is a graded module over the ring $T = K[V]$ with minimal free resolution

$$0 \to H_l \to H_{l-1} \to \cdots \to H_0 \to M \to 0.$$

Recall that the Castelnuovo–Mumford regularity $\mathrm{reg}(M)$ is the smallest integer d such that H_i is generated in degree at most $d+i$ for all i. Let I be the ideal generated by all homogeneous invariants of positive degree. Define $\tau_G(V)$ as the smallest integer d such that every homogeneous polynomial of degree d lies in I. It is well-known that the Castelnuovo–Mumford regularity of a finite length graded T-module M is exactly the maximum degree appearing in M (see [5, Exercise 20.15] or Theorem 6). Application to the module T/I gives us $\mathrm{reg}(T/I) = \tau_G(V) - 1$. From [5, Corollary 20.19] and the exact sequence $0 \to I \to T \to T/I \to 0$ follows that

$$\mathrm{reg}(I) = \mathrm{reg}(T/I) + 1 = \tau_G(V).$$

Fogarty's proof of the Noether bound in the non-modular case (see [7]), shows that $\tau_G(V) \leqslant |G|$. The following theorem gives a similar bound for the degrees of the generators of the syzygy ideal.

Theorem 2. *Suppose that G is a finite group in the non-modular case. Suppose that $\{f_1, f_2, \dots, f_r\}$ is a minimal set of homogeneous generators of the invariant ring $K[V]^G$ and let $J \subseteq K[x_1, x_2, \dots, x_r]$ be the syzygy ideal. Then J is generated in degree at most*

$$2\tau_G(V) \leqslant 2|G|.$$

The proof of this theorem does not seem to extend to higher syzygies. We conjecture though that similar bounds will hold for higher syzygies:

Conjecture 3. *If G is a finite group in the non-modular case, then*

$$\beta^i_G(V) \leqslant (i+1)\tau_G(V) \leqslant (i+1)|G|.$$

2. Examples

Example 4. Let S_n be the symmetric group acting on $V = K^n$ by permutation of the coordinates and let $A_n \subset S_n$ be the alternating group. The coordinate ring $K[V]$ can be identified with the polynomial ring $K[y_1, \dots, y_n]$. The invariant ring of S_n is

$$K[y_1, y_2, \dots, y_n]^{S_n} = K[e_1, e_2, \dots, e_n],$$

where e_j is the elementary symmetric polynomial of degree j defined by

$$e_j = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n} y_{i_1} y_{i_2} \cdots y_{i_j}$$

for all j.

It is also well known that the ring of A_n-invariants is

$$K[y_1, \dots, y_n]^{A_n} = K[e_1, e_2, \dots, e_n, \Delta],$$

where Δ is the A_n-invariant of degree $n(n-1)/2$ defined by

$$\Delta = \prod_{1 \leqslant i < j \leqslant n} (y_i - y_j).$$

Let us again define a surjective ring homomorphism

$$K[x_1, x_2, \dots, x_{n+1}] \twoheadrightarrow K[y_1, \dots, y_n]^{A_n},$$

where x_i maps to e_i for $i \leqslant n$ and x_{n+1} maps to Δ. The kernel of the homomorphism is the syzygy ideal J.

Theorem 1 says that J is generated in degree at most

$$n(n-1)/2 + n + (n-1) + \cdots + 1 - n = n(n-1).$$

The polynomials $e_1, e_2, \dots, e_n$ form a regular sequence in $K[y_1, \dots, y_n]$ and the Hilbert series of $K[y_1, \dots, y_n]/(e_1, e_2, \dots, e_n)$ is

$$(1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = 1 + (n-1)t + \cdots + t^{n(n-1)/2}.$$

From this we see that the highest degree for which $K[y_1, \dots, y_n]/(e_1, e_2, \dots, e_n)$ is non-zero is $n(n-1)/2$. The degree $n(n-1)/2$ part of $K[y_1, \dots, y_n]/(e_1, \dots, e_n)$ is one dimensional and it is spanned by the invariant Δ. This shows that every polynomial of degree at least $n(n-1)/2$ lies in the ideal $(e_1, e_2, \dots, e_n, \Delta)$, so $\tau_{A_n}(V) = n(n-1)/2$. By Theorem 2 we get that J is generated in degree at most

$$2 \cdot n(n-1)/2 = n(n-1).$$

Both theorems are sharp in this example. We have that Δ^2 is S_n-invariant and therefore Δ^2 is a polynomial in $e_1, e_2, \dots, e_n$. This gives a relation of degree $n(n-1)$ and it is known that this relation generates the ideal J.

Example 5. Let G be the cyclic group of order m, generated by σ. Let σ act on $V = K^n$ by scalar multiplication with a primitive mth root of unity ζ. This defines a group action of G on V. We identify again $K[V] = K[y_1, \dots, y_n]$ where y_i is the ith coordinate function. The invariant ring $K[V]^G$ is generated by the set $\mathscr{M}$ of all monomials in $y_1, y_2, \dots, y_n$ of degree m. To every monomial $M \in \mathscr{M}$ we attach a formal variable x_M. We consider the surjective ring homomorphism

$$K[\{x_M\}_{M\in\mathscr{M}}] \twoheadrightarrow K[V]^G$$

which maps x_M to M for every monomial $M\in\mathscr{M}$. The kernel of this homomorphism is again the syzygy ideal J.

By Theorem 1, J is generated in degree at most $(n+1)m-n$. Since every monomial of degree m lies the ideal J generated by all homogeneous invariants of positive degree, we have $\tau_G(V)=m$. By Theorem 2, J is generated in degree at most $2m$ (which means that J is generated by polynomials which are quadratic in the variables $\{x_M\}_{M\in\mathscr{M}}$). Indeed, J is generated by relations of the form

$$x_{y_iM}x_{y_jN}-x_{y_jM}x_{y_iN},$$

where M and N are monomials of degree $m-1$. Now Theorem 2 is sharp, but Theorem 1 is not.

3. A general degree bound for Syzygies

Suppose that $S=K[x_1,x_2,\dots,x_r]$ is the graded polynomial ring where $\deg(x_i)=d_i$ is a positive integer for all i. We will assume that $d_1\geqslant d_2\geqslant\cdots\geqslant d_r$. Let M be a finitely generated graded Cohen–Macaulay S-module. The minimal resolution of M is

$$0\to F_k\to F_{k-1}\to\cdots\to F_1\to F_0\to M\to 0$$

where

$$F_i\cong \mathrm{Tor}_i^S(M,K)\otimes_K S.$$

Here $\mathrm{Tor}_i^S(M,K)$ is a finite-dimensional graded vector space, and this makes $\mathrm{Tor}_i^S(M,K)\otimes_K S$ into a graded module.

If M is a finite-dimensional graded vector space, then $\deg(M)$ is the maximal degree appearing in M if M is non-zero, and $\deg(M)=-\infty$ if M is zero. For a finitely generated module M, $a(M)$ is the degree of the Hilbert series $H(M,t)$, seen as a rational function (the so-called a-invariant of M).

Theorem 6. *We have the inequality*

$$\deg(\mathrm{Tor}_i^S(M,K))\leqslant d_1+d_2+\cdots+d_{s+i}+a(M),$$

where s is the dimension of M.

Proof. We prove the theorem by induction on $s=\dim M$. Suppose that M has dimension 0. In this case we prove the inequality by induction of the length $\dim_K M$

of M. If M has length 0, then M is the trivial module and the inequality is obvious. Suppose that M is non-zero. Note that $a := a(M)$ is the maximum degree appearing in M. Let M_a be the part of M of degree a. Then M_a is a submodule of M. We have an exact sequence of S-modules

$$0 \to M_a \to M \to M/M_a \to 0.$$

Since $\dim_K M/M_a < \dim_K M$ and $a(M/M_a) < a$ we get by induction that

$$\deg(\operatorname{Tor}_i^S(M/M_a, K)) \leqslant d_1 + \cdots + d_i + a - 1.$$

The submodule M_a is isomorphic to the module $K^m[-a]$ which is the module K^m whose degree is shifted by a. Since

$$\deg(\operatorname{Tor}_i^S(K, K)) \leqslant d_1 + d_2 + \cdots + d_i$$

by the Koszul resolution, we have that

$$\deg(\operatorname{Tor}_i^S(M_a, K)) \leqslant d_1 + d_2 + \cdots + d_i + a.$$

From the long exact sequence

$$\cdots \to \operatorname{Tor}_i^S(M_a, K) \to \operatorname{Tor}_i^S(M, K) \to \operatorname{Tor}_i^S(M/M_a, K) \to \cdots$$

follows that

$$\deg(\operatorname{Tor}_i^S(M, K)) \leqslant d_1 + d_2 + \cdots + d_i + a.$$

Now suppose that $s > 0$. Since M is Cohen–Macaulay we can find a homogeneous non-zero divisor p of degree $e > 0$ and M/pM is again Cohen–Macaulay. First, note that $H(M/pM, t) = (1 - t^e)H(M, t)$, so $a(M/pM) = a(M) + e$. From the short exact sequence

$$0 \to M[-e] \to M \to M/pM \to 0$$

we obtain a long exact sequence

$$\cdots \to \operatorname{Tor}_{i+1}^S(M/pM, K)) \to \operatorname{Tor}_i^S(M, K)[-e] \to \operatorname{Tor}_i^S(M, K) \to \cdots .$$

Any element of $\operatorname{Tor}_i^S(M, K)[-e]$ of maximal degree must map to 0 in $\operatorname{Tor}_i^S(M, K)$, and therefore it must come from $\operatorname{Tor}_{i+1}^S(M/pM, K)$. This shows that

$$\begin{aligned} e + \deg(\operatorname{Tor}_i^S(M, K)) &= \deg(\operatorname{Tor}_i^S(M, K)[-e]) \leqslant \deg(\operatorname{Tor}_{i+1}^S(M/pM, K)) \\ &\leqslant d_1 + d_2 + \cdots + d_{(s-1)+(i+1)} + a(M/pM) = d_1 + d_2 + \cdots + d_{s+i} + a(M) + e, \end{aligned}$$

so finally

$$\deg(\operatorname{Tor}_i^S(M, K)) \leqslant d_1 + d_2 + \cdots + d_{s+i} + a(M). \qquad \square$$

Proof of Theorem 1. Let us choose $M = R$ in the previous theorem. Then

$$\beta_G^i(V) = \deg(\mathrm{Tor}_i^S(M, K)) \leqslant d_1 + d_2 + \cdots + d_{s+i} + a(R).$$

Knop proved that $a(R) \leqslant -s$ (see [9,10, Satz 4]) and Theorem 1 follows.

4. Bounds for the syzygy ideal for finite groups

Proposition 7. *Suppose that $R = \bigoplus_{d \geqslant 0} R_d$ is a graded ring with $R_0 = K$ and that $\{f_1, f_2, \dots, f_r\}$ is a minimal set of homogeneous generators of R. Let $S = K[x_1, \dots, x_r]$ be the graded polynomial ring and let $\varphi : S \twoheadrightarrow R$ be the surjective ring homomorphism defined by $x_i \mapsto f_i$ for all i. We have an exact sequence of graded vector spaces*

$$\mathrm{Tor}_2^S(K, K) \to \mathrm{Tor}_2^R(K, K) \to \mathrm{Tor}_1^S(R, K) \to 0.$$

Proof. From Exercise A3.47 (with the role of R and S interchanged) in [5], we get a five-term exact sequence

$$\mathrm{Tor}_2^S(K, K) \to \mathrm{Tor}_2^R(K, K) \to \mathrm{Tor}_1^S(R, K) \to \mathrm{Tor}_1^S(K, K) \to \mathrm{Tor}_1^R(K, K) \to 0.$$

Let $\mathfrak{n} = (x_1, \dots, x_r)$ be the maximal homogeneous ideal of S and let $\mathfrak{m} = (f_1, \dots, f_r)$ be the maximal homogeneous ideal of R. Now $\mathrm{Tor}_1^S(K, K)$ and $\mathrm{Tor}_1^R(K, K)$ can be identified with $\mathfrak{n}/\mathfrak{n}^2$ and $\mathfrak{m}/\mathfrak{m}^2$ respectively. In particular, both $\mathrm{Tor}_1^S(K, K)$ and $\mathrm{Tor}_1^R(K, K)$ are r-dimensional. The proposition follows.

Proof of Theorem 2. Let us write $T = K[V]$. We consider the T-module U, defined by

$$U = \{(w_1, w_2, \dots, w_r) \in T[-d_1] \oplus \cdots \oplus T[-d_r] \mid \sum_{i=1}^{r} w_i f_i = 0\}.$$

Since $I = (f_1, f_2, \dots, f_r)$ is $\tau_G(V)$-regular (in the sense of Mumford and Castelnuovo), we get that U is generated in degree $\leqslant \tau_G(V) + 1$. The module

$$M = \{(w_1, w_2, \dots, w_r) \in R[-d_1] \oplus \cdots \oplus R[-d_r] \mid \sum_{i=1}^{r} w_i f_i = 0\},$$

gives an exact sequence

$$0 \to M \to R[-d_1] \oplus \cdots \oplus R[-d_r] \to R \to K \to 0.$$

We can identify $M/\mathfrak{m}M$ with $\mathrm{Tor}_2^R(K, K)$. The module M is equal to U^G. We have that $((f_1, f_2, \dots, f_r)U)^G = (f_1, f_2, \dots, f_r)U^G = \mathfrak{m}M$ since $f_1, \dots, f_r$ are invariant and G is linearly reductive. We can view $M/\mathfrak{m}M$ as a submodule of $U/(f_1, \dots, f_r)U$. It is easy to see that every element of U of degree $\geqslant 2\tau_G(V)+1$, must lie in $(f_1,\dots,f_r)U$ since U is generated in degree $\leqslant \tau_G(V)+1$ and every polynomial of degree $\geqslant \tau_G(V)$ lies in $(f_1,\dots,f_r)$. This shows that

$$\deg(\mathrm{Tor}_2^R(K,K)) = \deg(M/\mathfrak{m}M) \leqslant \deg(U/(f_1,\dots,f_r)U) \leqslant 2\tau_G(V).$$

By the previous proposition, we also get $\deg(\mathrm{Tor}_1^S(R,K)) \leqslant 2\tau_G(V)$. $\square$

References

[1] W. Bruns, J. Herzog, Cohen-Macaulay Rings, Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge University Press, Cambridge, 1993.
[2] H. Derksen, Computation of invariants for reductive groups, Adv. Math. 141 (1999) 366–384.
[3] H. Derksen, Polynomial bounds for rings of invariants, Proc. Amer. Math. Soc. 129 (4) (2001) 955–963.
[4] H. Derksen, J. Sidman, A sharp bound for the Castelnuovo–Mumford regularity of subspace arrangements, Adv. Math. 172 (2002) 151–157.
[5] D. Eisenbud, Commutative Algebra with a View toward Algebraic Geometry, Graduate Texts in Mathematics, Springer, Berlin.
[6] P. Fleischmann, The Noether bound in invariant theory of finite groups, Adv. Math. 156 (1) (2000) 23–32.
[7] J. Fogarty, On Noether's bound for polynomial invariants of a finite group, Electron. Res. Announc. Amer. Math. Soc. 7 (2001) 5–7.
[8] M. Hochster, J.L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay, Adv. in Math. 13 (1974) 115–175.
[9] F. Knop, Der kanonische Modul eines Invariantenrings, J. Algebra 127 (1) (1989) 40–54.
[10] E. Noether, Der Endlichkeitssatz der Invarianten endlicher Gruppen, Math. Ann. 77 (1916) 89–92.