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SEQUENTIAL COLORING VERSUS WELSH–POWELL BOUND

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We comment in this note on the relations between sequential coloring and the Welsh–Powell upper bound for the chromatic number of a graph.

In 1967, Welsh and Powell introduced in [3] an upper bound to the chromatic number $\chi(G)$ of a graph G that is *usually* formulated in the following form (in what follows, n denotes the number of vertices of a graph):

If the vertices of a graph G are arranged in non-increasing order of their degrees, i.e.

$$d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n) \tag{1}$$

then

$$\chi(G) \leq u_S(G; v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} \max_{1 \leq i \leq n} \min\{d(v_i) + 1, i\}. \tag{2}$$

We have checked all available standard texts on the theory and applications of graphs and papers on graph coloring published after appearing [3] (including [3]), and in all of these publications the bound (2) is preceded by the assumption (1).

On the other hand, the bound (2) is quite often presented in the context of the *sequential coloring algorithm* (algorithm S, for short) which can be described as follows:

Given an ordering $u_1, u_2, \ldots, u_n$ of the vertices of a graph G. Assign color 1 to vertex u_1, and for $i = 2, 3, \ldots, n$, assign to u_i the smallest color not used for any neighbor u_j of u_i such that $j < i$.

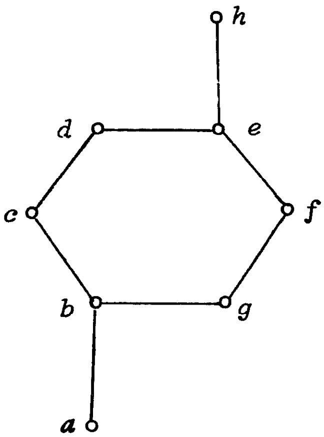
Let $\chi_S(G; u_1, u_2, \ldots, u_n)$ denote the number of colors used by algorithm S. It is clear that each vertex u_i can be colored by the smaller of two colors, i and $d(v_i) + 1$. Therefore,

for *any* vertex ordering $u_1, u_2, \ldots, u_n$ of a graph G we have

$$\chi(G) \leq \chi_S(G; u_1, u_2, \ldots, u_n) \leq u_S(G; u_1, u_2, \ldots, u_n). \tag{3}$$

We encourage the reader to prove, for instance by applying an exchange argument, that

$u_S(G; u_1, u_2, \ldots, u_n)$ is minimized by $u_1, u_2, \ldots, u_n$ arranged in non-increasing order of their degrees. (4)



vertex ordering	χ_S	u_S
$b\ e\ c\ d\ g\ f\ a\ h$	3	3
$a\ b\ c\ d\ e\ f\ g\ h$	2	4

Fig. 1. A bad example for the Welsh–Powell sequential coloring.

The fact (4) is probably an implicit motivation behind appearing (1) as the assumption for (2). There are however two issues that should be emphasized here and which support our claim that (1)–(2) should be rather presented in the form of two separate statements, as (3) and (4).

First, it is well-known that the number of colors used by the algorithm S, $\chi_S(G; u_1, u_2, \ldots, u_n)$ is *almost always* much smaller than $u_S(G; u_1, u_2, \ldots, u_m)$ the upper bound to χ_S (and also to χ) resulting from the algorithm, and there is no relation between these two numbers.

Second, the smaller bound $u_S(G; u_1, u_2, \ldots, u_n)$ does not necessarily guarantee the better sequential coloring. In particular, for certain graphs *no* ordering of vertices satisfying (1) may produce a chromatic coloring. Fig. 1 shows a bipartite (i.e. 2-chromatic) graph for which both vertices of degree 3 cannot be colored with the same color in any optimal coloring. However they will get the

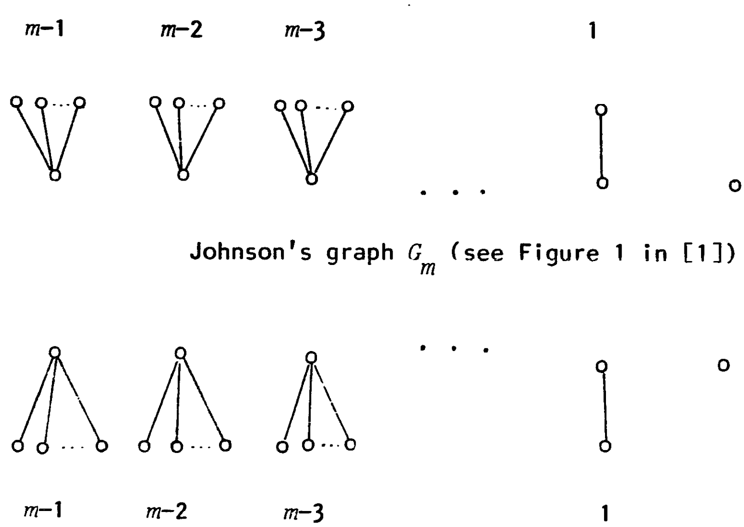


Fig. 2. Another bad example.

same color when vertices are ordered according to (1). Note that in this case we have $\chi_S(a\ldots h) < \chi_S(b\ldots h)$ although $u_S(a\ldots h) > u_S(b\ldots h)$.

In general, if two vertices of a bipartite graph located at odd distance from each other are forced by the algorithm S to receive the same color, then the coloring produced is not optimal. Fig. 2 shows a member of an infinite family of bipartite graphs, for which the number of colors used by the algorithm S applied to *any* vertex ordering satisfying (1) is of order $\sqrt{n}$.

To summarize, the upper bound (2) to the chromatic number of a graph is valid for *any* vertex ordering and is minimized for ordering (1). However, it may happen for some graphs, that for no vertex ordering satisfying (1), the algorithm S produces an optimal coloring. Hence some other vertex orderings may be of interest while searching sequentially for a chromatic coloring. This motivates our claim that (1)–(2) should be rather presented as two separate statements, (3) and (4).

We refer the reader to Section 4.1 in [2] devoted entirely to coloring algorithms, where the results of some computational experiments with sequential coloring algorithms are also presented. An extensive annotated bibliography is also included there.

References

[1] D.S. Johnson, Worst case behavior of graph coloring algorithms, Proc. 5th S-E Conf. on Combinatorics, Graph Theory and Comput., Utilitas Mathematica (Winnipeg, 1979) 513–527.

[2] M.M. Sysło, N. Deo, and J.S. Kowalik, Discrete Optimization Algorithms with Pascal Programs (Prentice Hall, Englewood Cliffs, NJ., 1983).

[3] D.J.A. Welsh and M.B. Powell, An upper bound for the chromatic number of a graph and its application to timetabling problems, Comput. J. 10 (1967) 85–86.