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A Representation Theorem for Distribution Semigroups*

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1. INTRODUCTION

Let E be a complex Banach space, $S(t)$, $t \geqslant 0$ a strongly continuous semigroup in E, A its infinitesimal generator. It is well-known that if A is bounded, then

$$S(t) = \exp(tA), \tag{1.1}$$

where the right side can be computed by means of the power series of the exponential function or by contour integrals ([6], p. 287). If A is unbounded this kind of interpretation fails but we can still attach a meaning to (1.1). For instance, we have

$$S(t)u = \lim_{\lambda \to \infty} \exp(t\lambda AR(\lambda; A))u, \tag{1.2}$$

where $R(\lambda; A) = (\lambda I - A)^{-1}$ and the limit is uniform on compacts of $t \geqslant 0$ ([6], Chapter XII). Since

$$\lambda AR(\lambda; A)u \to Au \tag{1.3}$$

as $\lambda \to \infty$ for all $u \in D(A)$ we can think of the right side of (1.3) as defining $\exp(tA)$, and thus of (1.2) as a generalization of (1.1). (See also [6], p. 365 for a similar formula for a different class of semigroups.) Formula (1.2) is one of the "exponential formulas" in Hille–Phillips, p. 354 and is called by Yosida a "representation theorem" for the semigroup S ([11], p. 248, formula (7)).

We show in this paper that the formula (1.2) is valid in the more general context of distribution semigroups (Lions, [7]) provided the

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limit is taken in the sense of distributions. The result is a simple consequence of the characterization of the infinitesimal generators of distribution semigroups due to Chazarain ([1,2]). We also establish a sort of converse to this result, which can be used as a "generation theorem" for an operator A. Finally, results of the same sort are obtained for operator-valued distributions related to the second-order equation $u'' = Au$ in the same way distribution semigroups are related to the equation $u' = Au$. An application to the approximation of distribution solutions of differential equations is indicated.

2. We use the definitions and notations of [8–10] for scalar and vector-valued distributions and those of [7] for distribution semigroups. In particular, $\mathscr{D}$ (resp. $\mathscr{D}_-$) is the space of all infinitely differentiable complex-valued functions $\varphi, \psi, \ldots$ defined in $-\infty < t < \infty$ and with compact support (resp. with support bounded above), $\mathscr{S}$ is the space of all infinitely differentiable functions decreasing at infinity (together with its derivatives) faster than any power of $|t|$. The spaces $\mathscr{D}$, $\mathscr{D}_-$, $\mathscr{S}$ will be endowed with their usual L. Schwartz topologies ([7], pp. 64, 172, 234). If X is any complex Banach space we denote by $\mathscr{D}'(X)$ the space of X-valued distributions; by definition $\mathscr{D}'(X) = \mathscr{L}(\mathscr{D}, X)$, the space of all linear continuous maps of $\mathscr{D}$ into X endowed with the topology of uniform convergence on bounded subsets of $\mathscr{D}$ ([8], §2, p. 49). We shall also use the spaces $\mathscr{D}_+'(X) = \mathscr{L}(\mathscr{D}_-, X)$, $\mathscr{S}(X) = \mathscr{L}(\mathscr{S}, X)$ endowed with similar topologies. As is well-known both $\mathscr{D}_+'(X)$, $\mathscr{S}'(X)$ can be identified with subspaces of $\mathscr{D}'(X)$, $\mathscr{D}_+'(X)$ with the subspace consisting of distributions with support bounded below. Finally, we shall say that an element of $\mathscr{D}'(X)$ belongs to $\mathscr{D}'_{[0,\infty)}(X)$ if its support is contained in $[0, \infty)$ and we shall assign to $\mathscr{D}'_{[0,\infty)}(X)$ the topology that it inherits from $\mathscr{D}'(X)$.

If A is a closed operator with domain $D(A)$ in X and range in X we will usually think of $D(A)$ as normed with $|u|_{D(A)} = |u|_X + |Au|_X$ (the "graph" norm); $D(A)$ is a Banach space relative to this norm.

Recall ([7], Théorème 4.1, p. 147), that if $S \in \mathscr{D}'(\mathscr{L}(E, E))$ is a distribution semigroup, A its infinitesimal generator then A is a closed operator with domain $D(A)$ dense in E, $S \in \mathscr{D}'(\mathscr{L}(E, D(A))$, has support in $t \geqslant 0$ and satisfies

$$S' - AS = \delta \otimes I \tag{2.1}$$

($S' = (d/dt)\, S$, I the identity operator in E). Moreover, A commutes with S, that is $S(\varphi)\, A = AS(\varphi)$ for any $\varphi \in \mathscr{D}$. Conversely, if A is a closed operator with domain $D(A)$ dense in E, $S \in \mathscr{D}_{[0,\infty)}(\mathscr{L}(E, D(A))$, commutes with A and satisfies (2.1) then S is a distribution semigroup, A its infinitesimal generator (see [2], p. 2 and [7], Théorème 5.1, p. 149). We shall make use in the sequel of the following result of Chazarain ([1], [2]).

2.1. THEOREM. *Let A be a closed operator in E with dense domain $D(A)$. Then A generates a distribution semigroup S if, and only if, $R(\mu; A)$ exists in a logarithmic region Λ and satisfies there the estimate*

$$| R(\mu; A)| \leqslant \text{pol}(| \mu |) \tag{2.2}$$

pol$(\cdot)$ *a (nonnegative) polynomial.*

Recall that a logarithmic region Λ in the complex plane is defined by means of an inequality

$$\operatorname{Re} \mu \geqslant \max(\alpha \log | \operatorname{Im} \mu | + \beta, \omega), \tag{2.3}$$

where $0 \leqslant \alpha, \beta, \omega < \infty$. Application of the distribution S to an element $\varphi \in \mathscr{D}$ can be written

$$S(\varphi) = \frac{1}{2\pi i} \int_\Gamma R(\mu; A)\, \Phi(\mu)\, d\mu \tag{2.4}$$

where Γ is, say, the boundary of Λ oriented clockwise with respect to Λ and

$$\Phi(\mu) = \int e^{\mu t} \varphi(t)\, dt. \tag{2.5}$$

The convergence of (2.4) is assured by the fact (immediately verifiable from (2.5)) that, for each $n \geqslant 0$ there exists a constant C_n such that

$$| \Phi(\mu)| \leqslant \frac{C_n e^{b \operatorname{Re} \mu}}{(1 + | \mu |)^n}, \tag{2.6}$$

$b = \sup(\operatorname{supp} \varphi) < \infty$. (Observe incidentally that (2.4) itself can be thought of as an "exponential formula" analogous to formulas E_5, E_6, and E_7 of [6], p. 354).

In all that follows we will use the notation $A_\lambda = \lambda A R(\lambda; A)$, $S_\lambda(t) = h(t) \exp(t A_\lambda)$, where h is the Heaviside function ($h(t) = 1$ if $t \geqslant 0$, $h(t) = 0$ if $t < 0$).

2.2. THEOREM. *Assume A generates a distribution semigroup S. Then*

$$S = \lim_{\lambda\to\infty} S_\lambda$$

the limit understood in the topology of $\mathscr{D}'(\mathscr{L}(E, E))$.

Proof. A simple computation shows that if λ, μ are two complex numbers, $\lambda + \mu \neq 0$, $\lambda\mu(\lambda + \mu)^{-1} \in \rho(A)$, the resolvent set of A, then $\mu \in \rho(A_\lambda)$ and

$$R(\mu; A_\lambda) = \frac{1}{\lambda + \mu}\left[I + \frac{\lambda^2}{\lambda + \mu} R\left(\frac{\lambda\mu}{\lambda + \mu}; A\right)\right]. \tag{2.7}$$

On the other hand, it is not difficult to verify that if $\omega' > \omega$, $\lambda\omega(\lambda - \omega)^{-1} \leqslant \omega'$ then the map $\mu \to \lambda\mu(\lambda + \mu)^{-1}$ maps the logarithmic region Λ',

$$\operatorname{Re}\lambda \geqslant \max(\alpha \log |\operatorname{Im}\lambda| + \beta, \omega') \tag{2.8}$$

into the region Λ defined by (2.3); this shows in particular that $\sigma(A_\lambda)$ is contained in the complement of Λ' for these values of λ.

Let now Γ' be the boundary of Λ' and write formula (2.4) for S_λ and any $\varphi \in \mathscr{D}$. We obtain, making use of (2.7)

$$S_\lambda(\varphi) = \int_0^\infty S_\lambda(t)\,\varphi(t)\,dt = \frac{1}{2\pi i}\int_{\Gamma'} R(\mu; A_\lambda)\,\Phi(\mu)\,d\mu. \tag{2.9}$$

Using now formula (2.7) we see that the integrand in (2.9) tends to

$$R(\mu; A)\,\Phi(\mu)$$

in $\mathscr{L}(E, E)$ as $\lambda \to \infty$. On the other hand, if $\mu \in \Lambda'$, $\lambda\omega(\lambda - \omega)^{-1} < \omega'$ we get, again from (2.7) and from the estimate (2.2) that

$$\begin{aligned} |R(\mu; A_\lambda)| &\leqslant \frac{1}{|\lambda + \mu|}\left[1 + \frac{\lambda^2}{|\lambda + \mu|}\operatorname{pol}\left(\left|\frac{\lambda\mu}{\lambda + \mu}\right|\right)\right] \\ &\leqslant K \operatorname{pol}\left(\left|\frac{\lambda\mu}{\lambda + \mu}\right|\right) + L \end{aligned} \tag{2.10}$$

for convenient constants K, L. Observing now that

$$\left|\frac{\lambda\mu}{\lambda + \mu}\right| \leqslant \frac{\lambda\,|\mu|}{\lambda + \omega'}$$

combining (2.10) with (2.2) and making use of the estimate (2.6) for

Φ—for n, say, equal to deg(pol) + 2—we see that the integrand in the right side of (2.9) is bounded—uniformly in λ—by a constant times $(1 + |\mu|)^{-2}$. This and the bounded convergence theorem of Lebesgue show that (2.9) converges to (2.4) as $\lambda \to \infty$—the change of Γ by Γ' in (2.4) is easily justifiable. Finally we note that by definition of bounded sets in $\mathscr{D}$ ([8], Chapter III, p. 68) the estimates (2.6) are uniform for φ in a bounded set of $\mathscr{D}$, which shows that (2.9) converges to $S(\varphi)$ uniformly on such sets. This ends the proof of Theorem 2.2.

2.3. *Remark.* Following Lions ([7], p. 154) we say that S has, *exponential growth* if, and only if, there exists a $\omega < \infty$ such that

$$\exp(-\omega t)S \in \mathscr{S}(\mathscr{L}(E, E)). \tag{2.11}$$

Infinitesimal generators of these semigroups can be characterized by the fact that the logarithmic region of Theorem 2.1 reduces to a half-plane

$$\operatorname{Re} \lambda \geqslant \omega \tag{2.12}$$

(see [7], Théorème 6.1, p. 157). If $\omega' \geqslant \omega$ it is easy to see by means of a modification of the formula (2.4) that, for $\varphi \in \mathscr{S}$ we have

$$S_{\omega'}(\varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\omega' + i\eta; A)\, \hat{\varphi}(\eta)\, d\eta, \tag{2.13}$$

where we have set $S_{\omega'} = \exp(-\omega' t)S$ and

$$\hat{\varphi}(\eta) = \int e^{i\eta t} \varphi(t)\, dt$$

is the Fourier transform of φ, convergence of (2.13) being assured by the fact that $\hat{\varphi}$ itself belongs to $\mathscr{S}$ ([8], p. 248) and thus satisfies the sequence of inequalities,

$$|\hat{\varphi}(\eta)| \leqslant \frac{C_n}{(1 + |\eta|)^n}, \tag{2.14}$$

$n = 1, 2,\ldots$ for convenient constants $C_1, C_2, \ldots$. Let us observe, moreover, that by virtue of the characterization of bounded sets given in [8], p. 234 the estimates (2.14) are *uniform* for φ in a bounded set in $\mathscr{S}$. Proceeding now exactly as in the proof of Theorem 2.2 and using the fact that the map $\mu \to \lambda\mu(\lambda + \mu)^{-1}$ takes the half-plane $\operatorname{Re} \lambda \geqslant \omega'$ into $\operatorname{Re} \lambda \geqslant \omega$ if $\lambda\omega(\lambda - \omega)^{-1} \leqslant \omega'$ we easily obtain that

$$(S_\lambda)_{\omega'} \to S_{\omega'},$$

as $\lambda \to \infty$ in the topology of $\mathscr{S}(\mathscr{L}(E, E))$.

We end this section by showing that A_λ approximates A—although not as well as in the semigroup case (see (1.3)).

2.4. Proposition. *Let m be the degree of the polynomial in* (2.2). *Then*

$$A_\lambda u \to Au \tag{2.15}$$

as $\lambda \to \infty$ for $u \in D(A^{m+2})$.

Proof. The family of operators

$$\{\lambda^{-m}R(\lambda; A); \lambda > \omega\}$$

is uniformly bounded. If $u \in D(A)$,

$$\lambda^{-m}R(\lambda; A)u = \lambda^{-m-1}u + \lambda^{-m-1}R(\lambda; A)Au \to 0$$

as $\lambda \to \infty$; since $D(A)$ is dense in E, this means that $\lambda^{-m}R(\lambda; A)u \to 0$ as $\lambda \to \infty$ for all $u \in E$. This and the formula

$$\lambda R(\lambda; A) = \sum_{k=0}^{m} \lambda^{-k}A^k u + \lambda^{-m}R(\lambda; A)\, A^{m+1}u$$

([6], p. 348) that can be obtained by induction from the first resolvent equation show that $\lambda R(\lambda; A)u \to u$ as $\lambda \to \infty$ for $u \in D(A^{m+1})$ which implies the desired conclusion. Note, finally that λ in (2.15) can be allowed to assume complex values, the condition "$\lambda \to \infty$" replaced by "$\lambda \in \Lambda, |\lambda| \to \infty$."

3. Throughout this section A_λ, S_λ are defined as in Section 2.

3.1. Theorem. *Let A be a closed operator with dense domain in E. Assume that* ($R(\lambda; A)$ *exists for $\lambda > \omega > 0$ and that the set*

$$\{S_\lambda(\cdot); \lambda > \omega\} \tag{3.1}$$

is bounded in $\mathscr{D}(\mathscr{L}(E, E))$. Then A generates a distribution semigroup S and we have

$$S = \lim_{\lambda\to\infty} S_\lambda$$

in the sense of distributions.

Proof. We assume first that $A^{-1} \in \mathscr{L}(E, E)$ (this restriction will be removed later). Then $A_\lambda^{-1} = A^{-1} - \lambda^{-1}I$ exists and it is bounded for all $\lambda > \omega$. Using the fact that $S_\lambda'(t) = A_\lambda S_\lambda(t)$ for $t > 0$ and that $S_\lambda(0) = I$, we get

$$\begin{aligned} S_\lambda(t) - S_\mu(t) &= S_\lambda(s)\, S_\mu(t-s)\big|_0^t \\ &= \int_0^t \frac{d}{ds}\,[S_\lambda(s)\, S_\mu(t-s)]\, ds \qquad (3.2) \\ &= (A_\lambda - A_\mu) \int_0^t S_\lambda(s)\, S_\mu(t-s)\, ds \end{aligned}$$

for all λ, $\mu > \omega$. Combining this equality with

$$\begin{aligned} A_\lambda^{-2}S_\lambda(t) - A_\mu^{-2}S_\mu(t) &= A_\lambda^{-1}A_\mu^{-1}(S_\lambda(t) - S_\mu(t)) \\ &\quad + (A_\lambda^{-1} - A_\mu^{-1})(A_\lambda^{-1}S_\lambda(t) + A_\mu^{-1}S\ (t)) \end{aligned}$$

and

$$(A_\lambda - A_\mu)\, A_\lambda^{-1}A_\mu^{-1} = A_\mu^{-1} - A_\lambda^{-1},$$

we finally obtain

$$\begin{aligned} A_\mu^{-2}S_\lambda(t) - A_\mu^{-2}S_\mu(t) &= (A_\mu^{-1} - A_\lambda^{-1}) \int_0^t S_\lambda(s)\, S_\mu(t-s)\, ds \\ &\quad + (A_\lambda^{-1} - A_\mu^{-1})(A_\lambda^{-1}S_\lambda(t) + A_\mu^{-1}S_\mu(t)) \qquad (3.3) \end{aligned}$$

for λ, $\mu > \omega$. By virtue of [10]. Proposition 39, p. 167 the convolution product is a bilinear continuous mapping from

$$\mathscr{D}'_{[0,\infty)}(\mathscr{L}(E, E)) \times \mathscr{D}'_{[0,\infty)}(\mathscr{L}(E, E))$$

into $\mathscr{D}'_{(0,\infty]}(\mathscr{L}(E, E))$, thus it maps (Cartesian products of) bounded sets into bounded sets. But then the family of functions

$$\left\{\int_0^t S_\lambda(s)\, S_\mu(t-s)\, ds,\ \lambda, \mu > \omega\right\}$$

is bounded in $\mathscr{D}'(\mathscr{L}(E, E))$ and so is

$$\{A_\lambda^{-1}S_\lambda(t) + A_\mu^{-1}S_\mu(t),\ \lambda, \mu > \omega\}$$

($A_\lambda^{-1}S_\lambda$ is the convolution of S_λ with hI, h the Heaviside function). Since $A_\lambda^{-1} \to A^{-1}$, $A_\mu^{-1} \to A^{-1}$ in $\mathscr{L}(E, E)$, $A_\lambda^{-1} - A_\mu^{-1} \to 0$; this, the preceding considerations and formula (3.3) show that the generalized sequence $\{A_\lambda^{-2}S_\lambda\,;\,\lambda > \omega\}$ is Cauchy in $\mathscr{D}'_{[0,\infty)}(\mathscr{L}(E, E))$, hence convergent in that space. But

$$(A_\lambda^{-2}S_\lambda)'' = S_\lambda + \delta \otimes A_\lambda^{-1} + \delta' \otimes A_\lambda^{-2};$$

thus, the same is true of S_λ itself. Multiplying now the equality $S_\lambda' - A_\lambda S_\lambda = \delta \otimes I$ by A_λ^{-1} we get

$$A_\lambda^{-1}S_\lambda' - S_\lambda = \delta \otimes A_\lambda^{-1}.$$

Letting $\lambda \to \infty$,

$$A^{-1}S' - S = \delta \otimes A^{-1},$$

which implies that $S \in \mathscr{D}'(\mathscr{L}(E, D(A))$ and that (2.1) is satisfied. This shows that S is a distribution semigroup, A its infinitesimal generator. We now remove the restriction $A^{-1} \in \mathscr{L}(E, E)$. Firstly, observe that Theorem 3.1—in the particular case proved—remains true if the family $\{A_\lambda\}$ is replaced by any family $\{\tilde{A}_\lambda\}$ of invertible, mutually commuting operators such that $A_\lambda^{-1} \to A^{-1}$ as $\lambda \to \infty$. Let now A satisfy the conditions of Theorem 3.1, and let $\omega' > \omega$, $B = A - \omega' I$, $\tilde{B}_\lambda = A_\lambda - \omega' I$. We use now formula (2.8); according to it we have

$$(A_\lambda - \omega' I)^{-1} = -R(\omega'; A_\lambda) = -\frac{1}{\lambda + \omega'}\left[I + \frac{\lambda^2}{\lambda + \omega'} R\left(\frac{\lambda\omega'}{\lambda + \omega'}\,;\, A\right)\right];$$

thus, $\tilde{B}_\lambda^{-1} \to -R(\omega'; A) = B^{-1}$ as $\lambda \to \infty$. If we now define $\tilde{T}_\lambda(t) = h(t)\exp(t\tilde{B}_\lambda)$ we have $\tilde{T}_\lambda(t) = \exp(-\omega' t)\, S_\lambda(t)$, thus all the conditions of Theorem 3.1 are satisfied for B, $\tilde{B}_\lambda$, $\tilde{S}_\lambda$. Applying the particular case just proved and the comments following it we see that $\tilde{T}_\lambda$ converges in $\mathscr{D}_+'(\mathscr{L}(E, E))$ to a distribution semigroup T generated by B. But then S_λ converges to $(\exp \omega' t)T$, which is a semigroup distribution generated by $B + \omega' I = A$. This ends the proof of Theorem 3.1.

3.2. *Remark.* Theorem 3.1 remains valid if the following modifications are made in the hypotheses:

(a) $R(\mu; A)$ is assumed to exist for some $\mu_0 \in C$.

(b) $\{A_\lambda\}$ is an arbitrary family of bounded, mutually commuting operators such that $R(\mu_0; A_\lambda)$ exists for all λ and $R(\mu_0; A_\lambda) \to R(\mu_0; A)$ in $\mathscr{L}(E, E)$.

(c) The family $\{S_\lambda\}$ is bounded in $\mathscr{D}'(\mathscr{L}(E, E))$, where $S_\lambda(t) = h(t)\exp(tA_\lambda)$. The proof follows exactly the same lines and will thus be omitted.

3.3. *Remark.* Suppose the boundedness assumption in Theorem 3.1 is strenghtened as follows: there exists $\omega' < \infty$ such that

$$\{\exp(-\omega' t) S_\lambda\,;\, \lambda > \omega\}$$

is bounded in $\mathscr{S}(\mathscr{L}(E, E))$. Then it is possible to show that $\exp(-\omega' t)\, S_\lambda \to \exp(-\omega' t) S$ in $\mathscr{S}(\mathscr{L}(E, E))$; thus, S has exponential growth. To adapt the proof of Theorem 3.1 to this case we only need to observe that if we set $X = \mathscr{D}'_{[0,\infty)}(\mathscr{L}(E, E)) \cap \mathscr{S}(\mathscr{L}(E, E))$, the convolution operator from $X \times X$ into X is continuous. This result also admits of a generalization along the lines of Remark 3.2.

4. As in the previous sections A is here a closed operator with dense domain in E. We study here distributions in $\mathscr{D}'_{[0,\infty)}(\mathscr{L}(E, D(A))$ commuting with A and satisfying

$$S'' - AS = \delta \otimes I \tag{4.1}$$

($S'' = (d/dt)^2 S$). Such distributions could be defined in the way distribution semigroups are, that is by means of a functional equation (similar to the one satisfied by the function $(1/a) \sin ax$) and by regularity conditions at the origin; however we shall not attempt here to do so, since the study of distributions satisfying (4.1) may be carried out by means of semigroup distribution theory. In this way, Chazarain has obtained the following:

4.1. Theorem ([2], Théorème 5). *There exists*

$$S \in \mathscr{D}'_{[0,\infty)}(\mathscr{L}(E, D(A))$$

satisfying (4.1) *and commuting with A if and only if $R(\mu^2; A)$ exists for μ in a logarithmic region Λ and satisfies there*

$$| R(\mu^2; A)| \leqslant \operatorname{pol}(| \mu |) \tag{4.2}$$

for some nonegative polynomial pol($\cdot$).

Recall ([2]) that if $\varphi \in \mathscr{D}$,

$$S(\varphi) = \frac{1}{2\pi i} \int_\Gamma R(\mu^2; A)\, \Phi(\mu)\, d\mu, \tag{4.3}$$

where Γ is the boundary of Λ, Φ defined by (2.5).

As in past sections, $A_\lambda = \lambda AR(\lambda; A)$, but we now define

$$S_\lambda(t) = h(t) \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A_\lambda{}^k$$

(We can write symbolically $S_\lambda(t) = hA_\lambda^{-1/2} \sin h(tA_\lambda^{1/2})$.) S_λ satisfies

$$S_\lambda'' - A_\lambda S_\lambda = \delta \otimes I \tag{4.3}$$

as can be verified by direct computation.

4.2. THEOREM. *Let* $S \in \mathscr{D}_{[0,\infty)}(\mathscr{L}(E, D(A))$ *satisfy* (4.1) *and commute with* A. *Then*

$$S = \lim_{\lambda \to \infty} S_\lambda$$

in $\mathscr{D}'(\mathscr{L}(E, E))$.

The proof follows closely that of Theorem 2.2 and will thus only be sketched. Applying formula (2.7) we obtain

$$R(\mu^2; A_\lambda) = \frac{1}{\lambda + \mu^2} \left[I + \frac{\lambda^2}{\lambda + \mu^2} R\left(\frac{\lambda \mu^2}{\lambda + \mu^2}; A \right) \right]$$

if $\lambda + \mu^2 \neq 0$, $\lambda\mu^2(\lambda + \mu^2)^{-1} \in \rho(A)$. Let the region Λ be given by (2.3) and let Λ' be the logarithmic region

$$\operatorname{Re} \mu \geqslant \max(\alpha \log | \operatorname{Im} \mu | + \beta', \omega')$$

with $\beta' > \beta$, $\omega' > \omega$. A (somewhat lengthy) computation shows that there exists a $\lambda_0 < \infty$ such that if $\lambda > \lambda_0$ then the map

$$\mu \to \left(\frac{\lambda \mu^2}{\lambda + \mu^2} \right)^{1/2} \tag{4.5}$$

transforms the region Λ' into Λ. We call now Γ' the boundary of Λ' and apply formula (4.3) to S_λ ; we have

$$S_\lambda(\varphi) = \int_0^\infty S_\lambda(t)\, \varphi(t)\, dt = \frac{1}{2\pi i} \int_{\Gamma'} R(\mu^2; A_\lambda)\, \Phi(\mu)\, d\mu. \tag{4.6}$$

Proceeding now as in the proof of Theorem 2.2 we can show that $R(\mu^2; A_\lambda) \to R(\mu^2; A)$ in $\mathscr{L}(E, E)$ and that $| R(\mu^2; A_\lambda)|$ is bounded—uniformly in λ—by a polynomial in $| \mu |$. This and the Lebesgue bounded convergence theorem yield the desired result.

4.3. *Remark.* An analogous of Remark 2.3 holds in our case; if, for some $\omega < \infty$

$$\exp(-\omega t)S \in \mathscr{S}(\mathscr{L}(E, E))$$

then

$$\exp(-\omega' t)S_\lambda \to \exp(-\omega' t)S$$

in $\mathscr{S}'(\mathscr{L}(E, E))$ if $\omega' > \omega$. The proof imitates that of Remark (2.3). In fact, it can be proved as in the case of first-order equations that if $\exp(-\omega t)\, S \in \mathscr{S}'(\mathscr{L}(E, E))$ then the logarithmic region Λ of Theorem 4.1 reduces to a half-plane ($\operatorname{Re}\lambda \geqslant \sqrt{\omega}$) and then we only have to observe that the map (4.5) transforms the half-plane $\operatorname{Re}\lambda \geqslant \sqrt{\omega}'$ into $\operatorname{Re}\lambda \geqslant \sqrt{\omega}$ if $\omega' > \omega$ for $\lambda \geqslant a$ convenient λ_0.

The analogous of Theorem 3.1 for our case is

4.4. THEOREM. *Assume $R(\lambda; A)$ exists for* $\operatorname{Re}\lambda > \omega$ *and that*

$$\{S_\lambda(\cdot);\, \lambda > \omega\} \tag{4.7}$$

is bounded in $\mathscr{D}'(\mathscr{L}(E, E))$. Then

$$S = \lim_{\lambda\to\infty} S_\lambda$$

exists in the topology of $\mathscr{D}(\mathscr{L}(E, E))$ and satisfies (4.1).

Proof. It will be carried out by making use of Theorem 3.1—or, rather, of the extension of it furnished by Remark 3.2. Let $\mathfrak{E}$ be the Cartesian product $E \times E$ endowed with the product topology. We define an operator in $\mathfrak{E}$ by

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

(the matrix notation is self-explanatory). $D(\mathfrak{A}) = D(A) \times E$. It is easy to see that

$$R(\mu; \mathfrak{A}) = \begin{pmatrix} \mu & I \\ A & \mu \end{pmatrix} R(\mu^2; A) \tag{4.8}$$

if $\mu^2 \in \rho(A)$. We now define

$$\mathfrak{A}_\lambda = \begin{pmatrix} 0 & I \\ A_\lambda & 0 \end{pmatrix}.$$

Applying (4.8) to $\mathfrak{U}_\lambda$ we get

$$R(\mu; \mathfrak{A}_\lambda) = \begin{pmatrix} \mu & I \\ A_\lambda & \mu \end{pmatrix} R(\mu^2; A_\lambda)$$

and then it follows from (4.4) and (4.7) that for μ large enough $R(\mu; \mathfrak{U}_\lambda) \to R(\mu; \mathfrak{U})$ in $\mathscr{L}(\mathfrak{E}, \mathfrak{E})$. On the other hand, a simple computation shows that

$$\mathfrak{A}_\lambda^{2n} = \begin{pmatrix} A_\lambda{}^n & 0 \\ 0 & A_\lambda{}^n \end{pmatrix},$$

$$\mathfrak{A}_\lambda^{2n+1} = \begin{pmatrix} 0 & A_\lambda{}^n \\ A_\lambda^{n+1} & 0 \end{pmatrix}$$

thus

$$\mathfrak{S}_\lambda(t) = h(t)\exp(t\mathfrak{A}_\lambda) = \begin{pmatrix} S_\lambda{}'(t) & S_\lambda(t) \\ A_\lambda S_\lambda(t) & S_\lambda{}'(t) \end{pmatrix}. \tag{4.9}$$

Since the family (4.7) is bounded in $\mathscr{D}'(\mathscr{L}(E, E))$ so is

$$\{\mathfrak{S}_\lambda\,;\,\lambda > \omega\}$$

in $\mathscr{D}'(\mathscr{L}(\mathfrak{E}, \mathfrak{E}))$. We have verified at this stage all the hypotheses in Remark 3.2, and we can thus conclude that $\mathfrak{S}_\lambda$ will converge in $\mathscr{D}'(\mathscr{L}(\mathfrak{E}, \mathfrak{E}))$ to a distribution $\mathfrak{S} \in \mathscr{D}'_{(0,\infty]}(\mathscr{L}(\mathfrak{E}, \mathfrak{E}))$ satisfying

$$\mathfrak{S}' - \mathfrak{A}\mathfrak{S} = \delta \otimes \mathfrak{I}$$

and commuting with $\mathfrak{U}$. By virtue of the representation (4.9) for $\mathfrak{S}_\lambda$ this distribution will be given by

$$\mathfrak{S} = \begin{pmatrix} S' & S \\ AS & S' \end{pmatrix},$$

where S is some distribution in $\mathscr{D}_{[0,\infty)}(\mathscr{L}(E, E))$. But it is not difficult to deduce from this and from the form of $\mathfrak{U}$ that S will commute with A and satisfy (4.1); on the other hand, it is clear that all the entries in the matrix of $\mathfrak{S}_\lambda$ will converge to the corresponding entries of the matrix of $\mathfrak{S}$; in particular, $S_\lambda \to S$ in $\mathscr{D}'(\mathscr{L}(E, E))$ as claimed.

The case in which the distribution S in (4.1) coincides with a differentiable $\mathscr{L}(E, E)$ valued function with strongly continuous derivative has been examined in [3], therein approximations to

S', S are found in the strong topology. Finally, observe that there is no point in looking for results of the type considered in this paper for solutions of the equation $S^{(n)} - AS = \delta \otimes I$, $n \geqslant 3$; for there can only be solutions of this equation if A is bounded and everywhere defined and S coincides in $t \geqslant 0$ with the entire function,

$$S(t) = h(t) \sum_{k=0}^{\infty} \frac{t^{nk+n-1}}{(nk+n-1)!} A^k$$

(for a proof of this result see [2] or [5]).

5. We present here a simple application of the preceding results. Recall that the Cauchy problem for the equation

$$U' = AU + T \tag{5.1}$$

is *well set* if, and only if, (a) for every $T \in \mathscr{D}_+'(\mathscr{L}(E, E))$ there exists a unique $U \in \mathscr{D}_+'(\mathscr{L}(E, D(A))$ satisfying (5.1), and (b) the linear map $T \to U$ from $\mathscr{D}_+'(\mathscr{L}(E, E))$ into $\mathscr{D}_+'(\mathscr{L}(E, D(A)))$ defined by (5.1) is continuous (see [2], [5] for further details). It is known (see [2], p. 2) that the Cauchy problem for (5.1) is well set if, and only if, there exists $S \in \mathscr{D}_{[0,\infty)}(\mathscr{L}(E, D(A)))$ satisfying $S' - AS = \delta \otimes I$; the map $T \to U$ given by Eq. 5.1 can be written

$$U = S * T. \tag{5.3}$$

Applying the results of Section 2 to the first-order Eq. 5.1, we see that the solutions of the Cauchy problem for (5.1) can be approximated arbitrarily well by the solutions of the Cauchy problem for the "smooth" equation,

$$U' = A_\lambda U + T,$$

whose solutions can in turn be computed by convolution of T by the power series $S_\lambda(t) = h(t) \exp(tA_\lambda)$.

References

1. J. Chazarain, Problèmes de Cauchy au sens des distributions vectorielles et applications, *C. R. Acad. Sci. Paris Ser. A* **266** (1968), 10–13.
2. J. Chazarain, Problèmes de Cauchy abstraits et applications, Seminaire Lions-Schwartz, exposé du 26 Janvier 1968.

3. G. DA PRATO AND E. GIUSTI, Una caratterizzazione dei generatori di funzioni coseno astratte, *Boll. Un. Mat. Ital.* **22** (1967), 357–362.
4. H. O. FATTORINI, Ordinary differential equations in linear topological spaces, II, *J. Differential Equations* **6** (1969), 50–70.
5. H. O. FATTORINI, On a class of differential equations for vector-valued distributions, *Pacific Jour. Math.* **32** (1970), 79–104.
6. E. HILLE AND R. S. PHILLIPS, Functional analysis and semi-groups, *Amer. Math. Soc. Colloquium Publications XXXI*, Providence, R. I. 1957.
7. J. L. LIONS, Les semi groupes distributions, *Portugal. Math.* **19** (1960), 141–164.
8. L. SCHWARTZ, "Théorie des Distributions," nouvelle ed., Hermann, Paris, 1966.
9. L. SCHWARTZ, Théorie des distributions a valeurs vectorielles, I, *Ann. Inst. Fourier* **7** (1957), 1–141.
10. L. SCHWARTZ, Théorie des distributions a valeurs vectorielles, II, *Ann. Inst. Fourier* **8** (1958), 1–209.
11. K. YOSIDA, "Functional Analysis," Springer-Verlag, 1965.