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FREE COLIMITS

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0. Introduction

In a category **S**, with finite limits, a morphism $X \xrightarrow{f} I$ may be viewed as an internally I-indexed family of objects of **S**. This point of view facilitates many "set-like" constructions within **S**. If **S** does not have finite limits one is led to consider completions of **S**, in particular the free finite limit completion. Here, by "free" we mean a relaxed notion that takes into account the 2-dimensional structure of **cat**. In this paper we study free colimits simply to avoid variance complications and to make easier our extensive reference to [3].

We study colimits of diagrams obtained from graphs rather than categories. This is a technical simplification which eliminates the construction of free categories on graphs in Section 2. It is natural in this context since the category structure of $\mathbf{A}^{\mathbf{I}}$ uses only graph properties of **I**, and the connected components functor $\mathbf{cat} \xrightarrow{\pi_0} \mathbf{set}$ factors over the category of graphs. In Section 1 we collect those results of [3] which we require later and observe that they are valid for graph diagrams.

The main result, which we prove in Section 2 may be stated: For a "stable" category of graphs, $\mathbf{gph}_0$, the free $\mathbf{gph}_0$ colimit completion of a category **A** is the full subcategory of $\mathbf{set}^{\mathbf{A}^{op}}$ determined by $\mathbf{gph}_0$ colimits of representables. In particular the category of finite graphs is stable and free finite colimits may be obtained in this way. In Section 3 we use the methods of the previous section to construct free categories with coequalizers.

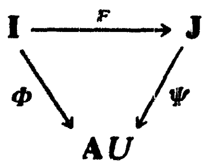
Free colimits are treated extensively in [1], however, the methods presented there are not immediately applicable to completions with respect to classes of finitary diagrams. It seems that the use of connected components and graphs can, in practice, provide simpler, more combinatoric descriptions.

I would like to thank Professor Paré for discussions on this paper.

1. Preliminaries

In what follows a *small graph* **I** is a diagram of the form $E \underset{t}{\overset{s}{\rightrightarrows}} V$ in **set**, the category of small sets. The categories of small graphs and small categories are denoted by **gph** and **cat** respectively. Extending the familiar notation for small categories, we write $|\mathbf{I}|$ for V.

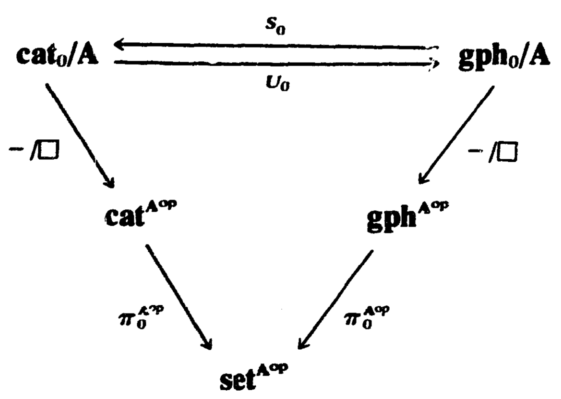
For $\mathbf{A}$ in **cat** and arbitrary full subcategories $\mathbf{gph}_0 \rightarrowtail \mathbf{gph}$, $\mathbf{cat}_0 \rightarrowtail \mathbf{cat}$, we abuse the comma category notation slightly and write $\mathbf{gph}_0/\mathbf{A}$, $\mathbf{cat}_0/\mathbf{A}$ to denote the categories whose objects are $\mathbf{gph}_0$, $\mathbf{cat}_0$ respectively, diagrams in $\mathbf{A}$. Explicitly, if $\mathbf{cat} \xrightarrow{U} \mathbf{gph}$ denotes the forgetful functor, a morphism in $\mathbf{gph}_0/\mathbf{A}$ is a commutative triangle



in **gph** with F in $\mathbf{gph}_0$.

The functor $\mathbf{cat}_0/A \xrightarrow{-/\square} \mathbf{cat}^{\mathbf{A}^{op}}$, which associates to a diagram $\mathbf{X} \xrightarrow{\Phi} \mathbf{A}$, the functor $-/\Phi$ whose value at $A \in |\mathbf{A}|$ is the comma category A/Φ, generalizes in the obvious way to a functor $\mathbf{gph}_0/\mathbf{A} \xrightarrow{-/\square} \mathbf{gph}^{\mathbf{A}^{op}}$. For any graph $\mathbf{I}: E \overset{s}{\underset{t}{\rightrightarrows}} V$ the coequalizer of the pair (s, t) defines a functor $\mathbf{gph} \xrightarrow{\pi_0} \mathbf{set}$, *connected components*, as for **cat**.

Proposition 1.1. *If there exist functors S_0 and U_0 such that both triangles*



commute up to isomorphism, then $\mathbf{A}$ *has* $\mathbf{cat}_0$ *colimits if and only if* $\mathbf{A}$ *has* $\mathbf{gph}_0$ *colimits.*

Proof. Let A have $\mathbf{cat}_0$ colimits and $\Phi \in |\mathbf{gph}_0/\mathbf{A}|$. From a slight generalization of Theorem 3.2 in [3] we have $(-/\Phi)\pi_0 \simeq (-/\Phi S_0)\pi_0$ implies $\Phi \varinjlim \cong \Phi S_0 \varinjlim$. The notation "$\cong$" indicates that the left side exists if and only if the right side exists and when either is the case they are isomorphic. Similarly, apply U_0 for the converse. ■

In practice the pair $\mathbf{cat} \overset{S}{\underset{U}{\leftrightarrows}} \mathbf{gph}$, where $\mathbf{I}S$ is the subdivision category [2], is useful. ($\mathbf{I}S$ has morphisms $\hat{I} \xrightarrow{\alpha_0} \hat{\alpha} \xleftarrow{\alpha_1} \hat{J}$, $\hat{K}$ when $\mathbf{I}$ has edges and vertices $I \xrightarrow{\alpha} J$,

K.) S and U extend to $\mathbf{cat/A} \underset{U}{\overset{S}{\rightleftarrows}} \mathbf{gph/A}$, where $(I \xrightarrow{\alpha_0} \alpha \xleftarrow{\alpha_1} J)\Phi S$ is given by $I\Phi \xrightarrow{\alpha\Phi} J\Phi \xleftarrow{J\Phi} J\Phi$. One easily verifies that the following diagram commutes up to isomorphism:

$$
(*) \qquad \begin{array}{ccccc}
\mathbf{cat/A} & & \underset{U}{\overset{S}{\rightleftarrows}} & & \mathbf{gph/A} \\
& {\scriptstyle -/\square}\searrow & & \swarrow{\scriptstyle -/\square} & \\
& \mathbf{cat}^{\mathbf{A}^{op}} & \underset{U^{\mathbf{A}^{op}}}{\overset{S^{\mathbf{A}^{op}}}{\rightleftarrows}} & \mathbf{gph}^{\mathbf{A}^{op}} & \\
& {\scriptstyle \pi_0^{\mathbf{A}^{op}}}\searrow & & \swarrow{\scriptstyle \pi_0^{\mathbf{A}^{op}}} & \\
& & \mathbf{set}^{\mathbf{A}^{op}} & &
\end{array}
$$

If $\mathbf{cat}_n$ denotes the full subcategory of $\mathbf{cat}$ determined by categories

$$\mathbf{X}: M \overset{\alpha_0}{\underset{\alpha_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\overset{i}{\longleftarrow}}}}} O$$

with card$(M) < n$ an infinite cardinal, and if $\mathbf{gph}_n$ is similarly determined by those $\mathbf{I}$ with card$(E) < n$ and card$(V) < n$; on noting that S and U restrict

$$
\begin{array}{ccc}
\mathbf{cat}_n/\mathbf{A} & \rightleftarrows & \mathbf{gph}_n/\mathbf{A} \\
\downarrow & & \downarrow \\
\mathbf{cat/A} & \underset{U}{\overset{S}{\rightleftarrows}} & \mathbf{gph/A}
\end{array}
$$

one has:

Corollary 1.2. *$\mathbf{A}$ has $\mathbf{cat}_n$ colimits if and only if $\mathbf{A}$ has $\mathbf{gph}_n$ colimits.* ∎

The functor $M: \mathbf{cat/A} \xrightarrow{-/\square} \mathbf{cat}^{\mathbf{A}^{op}} \xrightarrow{\pi_0^{\mathbf{A}^{op}}} \mathbf{set}^{\mathbf{A}^{op}}_{\Phi_n}$ has a right adjoint $\tilde{M}$, [3]. The "same formula" defines a right adjoint $\tilde{N}$ to $N: \mathbf{gph/A} \xrightarrow{-/\square} \mathbf{gph}^{\mathbf{A}^{op}} \xrightarrow{\pi_0^{\mathbf{A}^{op}}} \mathbf{set}^{\mathbf{A}^{op}}$. Explicitly, π_0 is left adjoint to the functor $\mathbf{set} \to \mathbf{gph}$ which assigns to a set X the graph $X \overset{X}{\underset{X}{\rightrightarrows}} X$, and the "Grothendieck construction", $F \mapsto (F\Sigma \xrightarrow{P} \mathbf{A})$, provides a right adjoint to $-/\square$. In the above, $F\Sigma$ is the graph whose vertices are pairs (A, a) where $A \in |\mathbf{A}|$ and $a \in |AF|$; an edge $(A, a) \xrightarrow{(f,e)} (B, b)$ consists of a morphism $A \xrightarrow{f} B$ of $\mathbf{A}$ and an edge $a \xrightarrow{e} (b)fF$ of AF. P is the projection diagram which forgets all but A, f and B.

$M = UN$; $\tilde{M}U = \tilde{N}$; and the co-unit $\tilde{N}N \xrightarrow{\varepsilon} \mathbf{set}^{\mathbf{A}^{op}}$ is an isomorphism since that for $M \dashv \tilde{M}$ is. It is convenient to note that for $F \in |\mathbf{set}^{\mathbf{A}^{op}}|$, $F\tilde{N}$ is of the form $(Y/{}^{\ulcorner}F^{\urcorner})U \xrightarrow{F\tilde{N}} \mathbf{A}$, where Y is the Yoneda embedding $\mathbf{A} \to \mathbf{set}^{\mathbf{A}^{op}}$. For the unit of $N \dashv \tilde{N}$:

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{\Phi\eta} & (Y/(-/\Phi)\pi_0)U \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle \Phi N\tilde{N}} & \\ & \mathbf{A} & \end{array}$$

$\Phi\eta$ is an object of $\mathbf{gph}/Y/(-/\Phi)\pi_0$, and hence $-/\Phi\eta$ is defined.

Proposition 1.3. $(-/\Phi\eta)\pi_0 \simeq 1$, *where* $\mathbf{A}^{\mathrm{op}} \xrightarrow{1} \mathbf{set}$ *is the functor with constant value* 1.

Proof. A direct calculation is easy, however, knowing the result for the unit of $M \dashv \tilde{M}$ [3] and observing that $\eta S = S\eta$, the proposition follows immediately from diagram (*) with $\mathbf{A} = Y/(-/\Phi)\pi_0$. ■

Given a morphism $\Phi N \xrightarrow{\tau} \Psi N (= (-/\Phi)\pi_0 \xrightarrow{\tau} (-/\Psi)\pi_0)$ in $\mathbf{set}^{\mathbf{A}^{\mathrm{op}}}$,

$$\begin{array}{ccc} \Phi N & \xrightarrow{\tau} & \Psi N \\ {\scriptstyle \Phi\eta N}\downarrow & & \downarrow{\scriptstyle \Psi\eta N} \\ \Phi N\tilde{N}N & \xrightarrow{\tau\tilde{N}N} & \Psi N\tilde{N}N \end{array}$$

commutes by naturality since $\eta N = (N\varepsilon)^{-1}$.

Remark 1.4. Using Proposition 1.3 we conclude that to any $(-/\Phi)\pi_0 \xrightarrow{\tau} (-/\Psi)\pi_0$ we may associate a diagram:

$$\begin{array}{ccccc} \mathbf{I} & \xrightarrow{\Lambda} & \mathbf{B}U & \xleftarrow{\Gamma} & \mathbf{J} \\ & {\scriptstyle\Phi}\searrow & \downarrow{\scriptstyle \Theta U} & \swarrow{\scriptstyle\Psi} & \\ & & \mathbf{A} & & \end{array}$$

with $\Theta \in |\mathbf{cat}/\mathbf{A}|$ and $(-/\Gamma)\pi_0 \simeq 1$, such that $\Lambda N(\Gamma N)^{-1} = \tau$, for we may take $\Gamma = \Psi\eta$ above.

2. Free colimits

To any graph $\mathbf{I}: E \overset{s}{\underset{t}{\rightrightarrows}} V$ we may associate a new graph $\bar{\mathbf{I}}: \bar{E} \overset{\bar{s}}{\underset{\bar{t}}{\rightrightarrows}} V$ where $(\bar{s}, \bar{t})$ is the kernel pair of the coequalizer of (s, t). This defines a triple $(\bar{-})$ on **gph**. We define $\overline{\mathbf{gph}}_0$ to be the full subcategory of the Kleisli category [2] for $(\bar{-})$ determined by the objects of $\mathbf{gph}_0$.

Definition 2.1. $\mathbf{gph}_0$ is *stable* if for every $\mathbf{D} \in |\mathbf{gph}_0|$ and every diagram $\mathbf{D} \xrightarrow{\Delta} \overline{\mathbf{gph}}_0$ the graph $\Delta\Sigma$ defined below is in $\mathbf{gph}_0$. $\Delta\Sigma$ has as vertices all pairs (D, I) where

$D \in |\mathbf{D}|$ and $I \in |D\Delta|$, and edges: $(D, I) \xrightarrow{(D,\alpha)} (D, J)$ where $I \xrightarrow{\alpha} J$ is an edge in $D\Delta$, $(D, I) \xrightarrow{(\delta, I)} (D', I')$ where $D \xrightarrow{\delta} D'$ is an edge in $\mathbf{D}$ and $I(\delta\Delta) = I'$.

For each $D \in |\mathbf{D}|$ there is a morphism of graphs $D\Delta \xrightarrow{Di} \Delta\Sigma$ given by $(I \xrightarrow{\alpha} J) \mapsto ((D, I) \xrightarrow{(D,\alpha)} (D, J))$.

Henceforth, assume that $\mathbf{I}_0 : \emptyset \rightrightarrows \{0\} \in |\mathbf{gph}_0|$. Define $\tilde{\mathbf{A}}$ to be the full subcategory of $\mathbf{set}^{\mathbf{A}^{op}}$ determined by the functors $(-/\Phi)\pi_0$ where $\Phi \in |\mathbf{gph}_0/\mathbf{A}|$. Letting $\mathbf{I}_0 \xrightarrow{\ulcorner A \urcorner} \mathbf{A}$ denote the diagram determined by $A \in |\mathbf{A}|$, we have full embeddings:

$$\begin{array}{ccc} \mathbf{A} & \overset{H}{\rightarrowtail} & \tilde{\mathbf{A}} \\ & \searrow \quad \swarrow & \\ & \mathbf{set}^{\mathbf{A}^{op}} & \end{array}$$

where $AH = (-/\ulcorner A \urcorner)\pi_0 = [-, A]$.

Theorem 2.2. *If* $\mathbf{gph}_0$ *is stable then* $\tilde{\mathbf{A}}$ *has all* $\mathbf{gph}_0$ *colimits.*

Proof. Let $\mathbf{D} \xrightarrow{Y} \tilde{\mathbf{A}}$ be a diagram in $\tilde{\mathbf{A}}$ where $\mathbf{D} \in |\mathbf{gph}_0|$. By definition of $\tilde{\mathbf{A}}$ we can write

$$(D \xrightarrow{\delta} D') \mapsto ((-/DY)\pi_0 \xrightarrow{\delta Y} (-/D'Y)\pi_0)$$

and the latter, by Remark 1.4, is induced by a diagram of the form:

$$\begin{array}{ccccc} (DY)_0 & \xrightarrow{\Lambda_\delta} & \mathbf{B}_\delta U & \xleftarrow{\Gamma_\delta} & (D'Y)_0 \\ & {\scriptstyle DY}\searrow & \downarrow{\scriptstyle \Theta_\delta U} & \swarrow{\scriptstyle D'Y} & \\ & & \mathbf{A} & & \end{array}$$

where $(-/\Gamma_\delta)\pi_0 \simeq 1$. That is, for $B \in |\mathbf{B}_\delta|$, B/Γ_δ is non-empty and connected. Hence, for each $D \xrightarrow{\delta} D'$ in $\mathbf{D}$ and each $I \in |(DY)_0|$ we may choose a vertex $I(\delta\Delta)$ of $(D'Y)_0$ and a morphism $I\Lambda_\delta \xrightarrow{\delta_I} (I(\delta\Delta))\Gamma_\delta$. If $I \xrightarrow{\alpha} J$ is an edge in $(DY)_0$, the existence of a "connection"

$$\begin{array}{c} I\Lambda_\delta \xrightarrow{\alpha\Lambda_\delta} J\Lambda_\delta \\ {\scriptstyle \delta_I}\swarrow \quad \swarrow \quad \searrow \quad \searrow{\scriptstyle \delta_J} \\ (I(\delta\Delta))\Gamma_\delta \xleftarrow{\alpha_0\Gamma_\delta}\rightarrow I_1'\Gamma_\delta \longleftrightarrow \cdots \longleftrightarrow I_n'\Gamma_\delta \xleftarrow{\alpha_n\Gamma_\delta}\rightarrow (J(\delta\Delta))\Gamma_\delta \end{array}$$

shows that for each δ in **D** the choices made determine a morphism of graphs $(DY)_0 \xrightarrow{\delta\Delta} (D'Y)_0$ and hence we have a diagram $\mathbf{D} \xrightarrow{\Delta} \mathbf{gph}_0$. Construct $\Delta\Sigma$, in $\mathbf{gph}_0$ by hypothesis, and define $\Delta\Sigma \xrightarrow{\check{Y}} \mathbf{A}$ by:

$$(D,I) \xrightarrow{(D,\alpha)} (D,J)) \mapsto (I(DY) \xrightarrow{\alpha(DY)} J(DY)),$$

$((D,I) \xrightarrow{(\delta,I)} (D', I(\delta\Delta))) \mapsto (I\Lambda_\delta(\Theta_\delta U) \xrightarrow{\delta_I(\Theta_\delta U)} I(\delta\Delta)\Gamma_\delta(\Theta_\delta U))$. Then, $(-/\check{Y})\pi_0$ is a colimit of Y. We have

$$\begin{array}{ccc} (DY)_0 & \xrightarrow{Di} & \Delta\Sigma \\ & {\scriptstyle D'Y}\searrow \quad \swarrow{\scriptstyle \check{Y}} & \\ & \mathbf{A} & \end{array}$$

for all $D \in |\mathbf{D}|$ and hence $\langle(-/DY)\pi_0 \to (-/\check{Y})\pi_0\rangle_{D\in|\mathbf{D}|}$. To see that this is actually a limiting co-cone observe that the choices made in the construction provide an explicit description of the $(-/DY)\pi_0 \xrightarrow{\delta Y} (-/D'Y)\pi_0$:

$$[A \xrightarrow{x} (I)DY] \xmapsto{(A)\delta Y} [A \xrightarrow{x} (I)DY \xrightarrow{\delta_I(\Theta_\delta U)} (I(\delta\Delta))D'Y)]$$

where the square brackets denote "equivalence class of". A direct verification of the claim is now immediate. ∎

Theorem 2.3. *If* $\mathbf{gph}_0$ *is stable then* $\tilde{\mathbf{A}}$ *is the free* $\mathbf{gph}_0$ *colimit completion of* $\mathbf{A}$.

Proof. Let **C** be any category with $\mathbf{gph}_0$ colimits and $\mathbf{A} \xrightarrow{F} \mathbf{C}$ any functor. Direct calculation shows that for all $\mathbf{I} \xrightarrow{\Phi} \mathbf{A}$, $(-/\Phi)\pi_0 \simeq (\Phi Y)\varinjlim \simeq (\Phi H)\varinjlim$. Moreover, any $[-,A] \xrightarrow{\bar{x}} (-/\Phi)\pi_0$ factors as $[-,A] \xrightarrow{[-,x]} [-,I\Phi] \xrightarrow{Ii} (-/\Phi)\pi_0$ for some $I \in |\mathbf{I}|$ where the second morphism is an injection into the colimit. The factorization is not unique, however, any two factorizations are "connected", and $\bar{x}\tilde{F} = xF(Ij)$, where $I\Phi F \xrightarrow{Ij} (\Phi F)\varinjlim$ is an injection, is well defined. Hence, there exists a $\mathbf{gph}_0$ colimit preserving functor, $\tilde{F}$, unique up to isomorphism, such that

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{C} \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle \tilde{F}} & \\ & \tilde{\mathbf{A}} & \end{array}$$

commutes. ∎

For n a regular infinite cardinal, $\mathbf{gph}_n$ as defined earlier is stable, hence, the free $\mathbf{gph}_n$ colimit completion of **A** is the full subcategory of $\mathbf{set}^{\mathbf{A}^{op}}$ determined by $\mathbf{gph}_n$ colimits of representables.

Remark 2.4. If **A** is only locally small (for every pair of objects $A, A' \in |\mathbf{A}|$, $[A, A'] \in |\mathbf{set}|$), Theorem 2.3 remains valid for small $\mathbf{gph}_0$ if we define $\tilde{\mathbf{A}}$ by: $|\tilde{\mathbf{A}}| = |\mathbf{gph}_0/\mathbf{A}|$, $[\Phi, \Psi] = [I\Phi, J\Psi]\varinjlim_{J} \varprojlim_{I}$ for $\mathbf{I} \xrightarrow{\Phi} \mathbf{A}$, $\mathbf{J} \xrightarrow{\Psi} \mathbf{A}$. When **A** is small this agrees with the previous description, up to equivalence.

3. Applications

As further examples of stable $\mathbf{gph}_0$ we mention; all graphs, discrete graphs, connected graphs, and graphs with one vertex.

For the remainder of the paper we assume that $\mathbf{gph}_0'$ is small and **A** and **C** are locally small. Let $\mathbf{gph}_0$ now denote a small, stable, full subcategory of **gph** containing $\mathbf{gph}_0'$. (There is a minimal such $\mathbf{gph}_0$ but we make no use of it here.) If for any category **C**, **C** has $\mathbf{gph}_0'$ colimits implies **C** has $\mathbf{gph}_0$ colimits, then the free $\mathbf{gph}_0$ colimit completion of **A** is also the free $\mathbf{gph}_0'$ colimit completion of **A**.

Let $\mathbf{gph}_0'$ be determined by the graphs (denoted graphically) $\{.,. \rightrightarrows .\}$. Define $\mathbf{gph}_0$ to consist of those finite graphs **I**, with vertices labelled $n, \ldots, 2, 1$; $n \geq 1$, such that:
1) $i \xrightarrow{e} j$ in **I** implies $i > j$.
2) For all $i \in |\mathbf{I}|$, $i > 1$, there exists a chain:

$$i \xrightarrow{e_m} i_m \xrightarrow{e_{m-1}} i_{m-1} \to \cdots \to i_2 \xrightarrow{e_1} i_1 = 1, m \geq 1.$$

Lemma 3.1. *If* **C** *has* $\mathbf{gph}_0'$ *colimits (coequalizers) then* **C** *has* $\mathbf{gph}_0$ *colimits.*

Proof. An induction on the number of vertices of objects of $\mathbf{gph}_0$ shows this to be true. ■

Theorem 3.2. $\mathbf{gph}_0$ *is stable and hence, if* $\tilde{\mathbf{A}}$ *is constructed from it as above,* $\tilde{\mathbf{A}}$ *is the free category with coequalizers on* **A**.

Proof. For any $\mathbf{D} \xrightarrow{\Delta} \overline{\mathbf{gph}_0}$, $\mathbf{D} \in |\mathbf{gph}_0|$, $\Delta\Sigma$ is clearly finite. If **D** has vertices labelled $n, \ldots, 1$ and $i\Delta$ has vertices $n_i, \ldots, 1$, then by ordering the vertices, (i, j), of $\Delta\Sigma$ lexicographically, with preference given to the first component, condition 1) is met. If (i, j) is any vertex, then either $i = 1$ or there is a path $i = i_{m+1} \xrightarrow{e_m} i_m \to \cdots \to i_1 = 1$ in **D**. In the first case 2) is met immediately and in the second we have a path $(i, j) = (i_{m+1}, j_{m+1}) \to (i_m, j_m) \to \cdots \to (1, j_1)$, where $j_k = j_{k+1}(e_k\Delta)$, which reduces the problem to the first case. ■

We remark that the notion of "stable" is somewhat coarse in that a category **C** may have $\mathbf{gph}_0'$ colimits yet fail to have $\mathbf{gph}_0$ colimits for any stable $\mathbf{gph}_0$ containing $\mathbf{gph}_0'$. For let $\mathbf{gph}_0' = \{., . \to .\}$. Then any category has $\mathbf{gph}_0'$ colimits, but $\begin{array}{ccc} \cdot & \to & \cdot \\ \downarrow & & \downarrow \\ \cdot & \to & \cdot \end{array}$ is an object in any $\mathbf{gph}_0$ and (commutativity being undefined) such colimits are nontrivial. A deeper analysis of this subject should perhaps involve "graphs with commutativity relations" and a more subtle definition of $\Delta\Sigma$ (Section 2).

References

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