NEARNESS AND QUASI-UNIFORM STRUCTURES

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The concept of a locally right symmetric quasi-uniform space is introduced. Every symmetric topological space admits a locally right symmetric quasi-uniform structure. It is shown that this property characterizes those quasi-uniform spaces for which the collection of quasi-uniform covers forms a nearness structure with the same closure operator. With this link, various concepts and results in these two areas are compared.

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Introduction

Uniform structures can be defined in terms of covers or entourages. Generalizing the cover definition one obtains nearness spaces; a generalization of the entourage definition yields quasi-uniform structures. Once we have a quasi-uniform space we can consider the collection of quasi-uniform covers. This paper characterizes those quasi-uniform spaces for which the collection of quasi-uniform covers is a nearness structure with the same topological closure operator. The characterizing property is called locally right symmetric.

The definition of the closure operator in a nearness space is essentially different from its definition in a quasi-uniform space. This provides much of the fascination and difficulty in attempting to compare these two structures.

Nearness structures and quasi-uniform structures agree if they are uniform structures. It is the point of this paper that they can be compared in a much broader area; every locally right symmetric quasi-uniform structure generates, via the collection of quasi-uniform covers, a nearness structure with the same closure operator. Moreover, every symmetric topological space admits a locally right symmetric quasi-uniform structure.

Using this link, it is possible to compare how various concepts have evolved in these two areas. Additionally, it is possible to translate certain results already obtained for nearness spaces over to locally right symmetric quasi-uniform spaces, and in some cases extend the result to any quasi-uniform space.
2. Preliminaries

Let $X$ be a set; then $\mathcal{P}^n(X)$ will denote the power set of $\mathcal{P}^{n-1}(X)$ for each natural number $n$ and $\mathcal{P}^0(X) = X$. Let $\xi$ be a subset of $\mathcal{P}^2(X)$ and $\mathcal{A}$ and $\mathcal{B}$ subsets of $\mathcal{P}(X)$. Let $A$ and $B$ be subsets of $X$. Then the following notation is used:

1. $\mathcal{A}$ is near means $\mathcal{A} \in \xi$.
2. $\text{cl}_\xi \mathcal{A} = \{x \in X : \{\{x\}, \mathcal{A}\} \in \xi\}$.
3. $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

**Definition 2.1.** Let $X$ be a set and $\xi \subseteq \mathcal{P}^2(X)$. Then $(X, \xi)$ is called a nearness space provided:

1. (N1) $\bigcap \mathcal{A} \neq \emptyset$ implies $\mathcal{A} \in \xi$.
2. (N2) If $\mathcal{A} \in \xi$ and for each $B \in \mathcal{B}$ there exists $A \subseteq \mathcal{A}$ with $A \subseteq \text{cl}_\xi B$, then $\mathcal{B} \in \xi$.
3. (N3) If $\mathcal{A} \subseteq \xi$ and $\mathcal{B} \subseteq \xi$ then $\mathcal{A} \vee \mathcal{B} \subseteq \xi$.
4. (N4) $\emptyset \in \mathcal{A}$ implies $\mathcal{A} \subseteq \xi$.

Given a nearness space $(X, \xi)$, the operator $\text{cl}_\xi$ is a closure operator on $X$. Hence there exists a topology associated with each nearness space in a natural way. This topology is denoted by $t(\xi)$. This topology is symmetric (Recall that a topology is symmetric provided $x \in \{y\}$ implies $y \in \{x\}$.) Conversely, given any symmetric topological space $(X, t)$ there exists a compatible nearness structure $\xi$, given by $\xi_t = \{\mathcal{A} \subseteq \mathcal{P}(X) : \bigcap \mathcal{A} \neq \emptyset\}$. To say that a nearness structure $\xi$ is compatible with a topology $t$ on a set $X$ means that $t = t(\xi)$.

**Definition 2.2.** Let $(X, \xi)$ be a nearness space.

1. Each $\xi$-maximal element is called a cluster.
2. $(X, \xi)$ is complete if each cluster has a nonempty adherence.
3. $(X, \xi)$ is ultrafilter complete if each near ultrafilter converges.
4. $(X, \xi)$ is concrete if each near collection is contained in a cluster.
5. $(X, \xi)$ is topological provided $\mathcal{A} \in \xi$ implies $\bigcap \mathcal{A} \neq \emptyset$.
6. $(X, \xi)$ is totally bounded provided $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{A}$ has the finite intersection property implies $\mathcal{A} \subseteq \xi$.

Let $(X, \xi)$ be a nearness space. Set

$$\mu = \{\mathcal{C} \subseteq \mathcal{P}(X) : \{X - C : C \in \mathcal{C}\} \notin \xi\}.$$ 

$\mu$ is called the collection of uniform covers of the nearness space $(X, \xi)$. A nearness space can be defined in terms of the uniform covers [9], and denoted by $(X, \mu)$. The interior operator is defined by $\text{int}_\mu(A) = \{x : \{X - \{x\}, A\} \in \mu\}$.

**Definition 2.3.** Let $X$ be a set. A quasi-uniform structure $\mathcal{U}$ on $X$ is a filter on $X \times X$ satisfying:

1. $\{(x, x) : x \in X\} \subseteq U$ for each $U \in \mathcal{U}$.
2. For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. 

Let $\mathcal{U}$ be a quasi-uniform structure on $X$. Set

$$\mathcal{P}(X) \times \mathcal{P}(X) = \{\mathcal{A} \times \mathcal{B} : \mathcal{A} \subseteq \mathcal{P}(X), \mathcal{B} \subseteq \mathcal{P}(X)\}.$$ 

$\mathcal{P}(X) \times \mathcal{P}(X)$ is a uniformity on $X$. For each $U \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$, let $\mathcal{U}(U)$ be the collection of all $\mathcal{A} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ such that $\mathcal{A} \subseteq U$. Then $\mathcal{U}(U)$ is a quasi-uniform structure on $X$. 

Let $\mathcal{P}(X) \times \mathcal{P}(X)$ be a quasi-uniform structure on $X$. Set

$$\mathcal{P}(X) \times \mathcal{P}(X) = \{\mathcal{A} \subseteq \mathcal{P}(X) \times \mathcal{P}(X) : \mathcal{A} \subseteq \mathcal{U}(U)\}.$$ 

The interior operator is defined by $\text{int}_{\mathcal{U}}(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \subseteq \mathcal{A}, \mathcal{B} \subseteq \mathcal{U}(U)\}$.
Each quasi-uniform structure generates a topology \( t(\mathcal{U}) = \{ O \subseteq X : \text{if } x \in O \text{ then there exists } U \in \mathcal{U} \text{ such that } U[x] \subseteq O \} \). It follows that \( \mathbb{A} = \bigcap \{ U^{-1}[A] : U \in \mathcal{U} \} \) and \( \text{int}(A) = \{ x : \text{there exists } U \in \mathcal{U} \text{ such that } U[x] \subseteq A \} \). A quasi-uniform structure \( \mathcal{U} \) is compatible with a topology \( t \) on \( X \) provided \( t = t(\mathcal{U}) \).

Let \( (X, t) \) be a topological space. A \( Q \)-cover of \( X \) is an open cover \( C \) such that \( A(C, x) \in t \) for each \( x \in X \), where \( A(C, x) = \bigcap \{ C : x \in C \in C \} \). A topological space is orthocompact if every open cover of \( X \) has a \( Q \)-cover refinement. Let \( \gamma \) be a collection of \( Q \)-covers of \( X \) such that if \( x \in O \in t \), then there exists \( C \in \gamma \) such that \( A(C, x) \subseteq O \). For each \( C \in \gamma \), set
\[
U(C) = \bigcup \{ \{ x \} \times A(C, x) : x \in X \} \quad \text{and} \quad \mathcal{S} = \{ U(C) : C \in \gamma \}.
\]
Then \( \mathcal{S} \) is a transitive base for a compatible quasi-uniformity \( \mathcal{U}(\gamma) \) for \( (X, t) \). \( \mathcal{U}(\gamma) \) is called the covering quasi-uniformity for \( (X, t) \) with respect to \( \gamma \). It is shown in [7] that if \( \gamma \) is the collection of all \( Q \)-covers, point-finite open covers, locally finite open covers, finite open covers, then \( \mathcal{U}(\gamma) \) is the finite transitive, point-finite covering, locally finite covering, Pervin quasi-uniformity, respectively, for \( (X, t) \) and is denoted by \( FT, PF, LF, \) and \( P \), respectively. The Pervin quasi-uniformity \( P \) can also be defined as the quasi-uniformity generated by the subbase \( \{ S(O) : O \in t \} \) where \( S(O) = (O \times O) \cup ((X - O) \times X) \).

3. Results

**Definition 3.1.** Let \( (X, \mathcal{U}) \) be a quasi-uniform space.

1. \((X, \mathcal{U})\) is called **locally left symmetric** if for each \( x \in X \) and \( U \in \mathcal{U} \) there exists \( V \in \mathcal{U} \) such that \( V^{-1}[V[x]] \subseteq U[x] \).
2. \((X, \mathcal{U})\) is called **locally right symmetric** if for each \( x \in X \) and \( U \in \mathcal{U} \) there exists \( V \in \mathcal{U} \) such that \( V[V^{-1}[x]] \subseteq U[x] \).

Locally left symmetric, previously called locally symmetric, has been studied in [10] and [6]. Theorem 3.3 shows that locally right symmetric is precisely the condition needed for the collection of all quasi-uniform covers to form a nearness structure generating the same underlying topology. The abundance of locally right symmetric quasi-uniform structures is indicated by Theorem 3.5. The following lemma follows from basic results in [10].

**Lemma 3.1.** Let \( (X, \mathcal{U}) \) be a quasi-uniform space.

1. If \((X, \mathcal{U})\) is locally left symmetric, then \( t(\mathcal{U}) \) is symmetric.
2. If \((X, \mathcal{U})\) is locally right symmetric, then \( t(\mathcal{U}) \) is symmetric.

**Definition 3.2.** Let \( X \) be a set and \( \mu \) and \( \xi \) be subsets of \( \mathcal{P}^2(X) \). Set

1. \( \xi(\mu) = \{ A \subseteq \mathcal{P}(X) : \{ X - A : A \in \mathcal{A} \} \in \mu \} \).
2. \( \mu(\xi) = \{ C \subseteq \mathcal{P}(X) : \{ X - C : C \in \mathcal{C} \} \in \xi \} \).
Let $(X, \mathcal{U})$ be a quasi-uniform structure. A natural way to attempt to generate a nearness structure is to consider the collection of all quasi-uniform covers of $X$. Or one might choose to say that $\mathcal{A}$ is a near collection provided $\bigcap \{U^{-1}[A]: A \in \mathcal{A}\} \neq \emptyset$ for each $U \in \mathcal{U}$. As we shall see, these two methods are equivalent and, moreover, they generate a nearness structure with $t(\mathcal{U})$ as the underlying topology if and only if $\mathcal{U}$ is locally right symmetric.

**Definition 3.3.** Let $(X, \mathcal{U})$ be a quasi-uniform space. Set

1. $\xi(\mathcal{U}) = \{\mathcal{A} \subset \mathcal{P}(X): \bigcap \{U^{-1}[A]: A \in \mathcal{A}\} \neq \emptyset$ for each $U \in \mathcal{U}\}$.
2. $\mu(\mathcal{U}) = \{C \subset \mathcal{P}(X):$ there exists $U \in \mathcal{U}$ such that $\{U[x]: x \in X\}$ refines $C\}$.

A nearness structure $\xi$ is said to be *compatible* with a quasi-uniform structure $\mathcal{U}$ provided $\xi = t(\mathcal{U})$.

$\mu(\mathcal{U})$ is simply the collection of all $\mathcal{U}$-quasi-uniform covers on $X$. The following lemma provides the desired relationship between $\xi(\mathcal{U})$ and $\mu(\mathcal{U})$.

**Lemma 3.2.** Let $(X, \mathcal{U})$ be a quasi-uniform space. Then

1. $\xi(\mathcal{U}) = \xi(\mu(\mathcal{U}))$.
2. $\mu(\mathcal{U}) = \mu(\xi(\mathcal{U}))$.

By Lemma 3.2, it follows that a nearness structure $\xi$ is compatible with $\mathcal{U}$ if $\mu(\xi) = \mu(\mathcal{U})$.

Let $(X, \mathcal{U})$ be a quasi-uniform space. The closure of $A \subset X$ is given by

$$cl(A) = \{U^{-1}[A]: U \in \mathcal{U}\}.$$ But the closure of $A$ with respect to $\xi(\mathcal{U})$ yields

$$cl_{\xi(\mathcal{U})}(A) = \{x: U^{-1}[x] \cap U^{-1}[A] \neq \emptyset \text{ for each } U \in \mathcal{U}\}.$$ A similar difference occurs for the interior operators. For quasi-uniform spaces, $x \in int(A)$ provided there exists $U \in \mathcal{U}$ with $U[x] \subset A$. The nearness space analogue is

$$int_{\mu(\mathcal{U})}(A) = \{x: \text{there exists } U \in \mathcal{U} \text{ such that } \{U[t]: t \in X\} \text{ refines } \{A, X - \{x\}\}\}.$$ For an arbitrary quasi-uniform space $(X, \mathcal{U})$, $cl_{\xi(\mathcal{U})}$ need not be a closure operator; it is, however, a Čech closure operator. Similarly, $int_{\mu(\mathcal{U})}$ need not be an interior operator. Thus, there is a distinct difference in the basic definitions of the closure operators for quasi-uniform spaces and nearness spaces. Also; $\xi(\mathcal{U})$ is in general not a nearness structure but rather a semi-nearness structure. We now show that these two "closure" operators, or equivalently the two "interior" operators, agree if and only if the quasi-uniform space $(X, \mathcal{U})$ is locally right symmetric. In this case, $\xi(\mathcal{U})$ is a nearness structure.
Theorem 3.3. Let \((X, \mathcal{U})\) be a quasi-uniform space. The following statements are equivalent.

1. \((X, \mathcal{U})\) is locally right symmetric.
2. For \(A \subset X\),
   \[\text{int}(A) = \{x : \text{there exists } U \in \mathcal{U} \text{ such that } \{U[t] : t \in X\} \text{ refines } \{X - \{x\}, A\}\}\]
3. For \(A \subset X\),
   \[\text{cl}(A) = \{x : U^{-1}[x] \cap U^{-1}[A] \neq \emptyset \text{ for each } U \in \mathcal{U}\}\]

Proof. (1) \(\Rightarrow\) (2). Let \(A \subset X\). Set
   \[\text{int}_{\mathcal{U}(\mathcal{U})}(A) = \{x : \text{there exists } U \in \mathcal{U} \text{ such that } \{U[t] : t \in X\} \text{ refines } \{X - \{x\}, A\}\}\]

Easily, \(\text{int}_{\mathcal{U}(\mathcal{U})}(A) \subset \text{int}(A)\). Let \(x \in \text{int}(A)\). Then there exists \(U \in \mathcal{U}\) with \(U[x] \subset A\). By (1), there exists \(V \in \mathcal{U}\) with \(V \circ V^{-1}[x] \subset U[x]\).

Claim: \(\{V[t] : t \in X\} \text{ refines } \{X - \{x\}, A\}\). Let \(t \in X\). If \(x \notin V[t]\), then \(V[t] \subset X - \{x\}\). Suppose \(x \in V[t]\). Let \(p \in V[t]\), then \(t \in V^{-1}[x]\) and \(p \in V \circ V^{-1}[x]\) and thus \(p \in U\). Hence \(V[t] \subset A\) and therefore \(\text{int}(A) \subset \text{int}_{\mathcal{U}(\mathcal{U})}(A)\).

(2) \(\Rightarrow\) (3). Let \(A \subset X\). Set
   \[\text{cl}_{\mathcal{U}(\mathcal{U})}(A) = \{x : U^{-1}[x] \cap U^{-1}[A] \neq \emptyset \text{ for each } U \in \mathcal{U}\}\]

Easily \(\text{cl}(A) \subset \text{cl}_{\mathcal{U}(\mathcal{U})}(A)\). Let \(x \in \text{cl}(A)\). Then, by (2),
   \[x \in \text{int}(X - \text{cl}(A)) = X - \text{cl}(A) = \text{int}_{\mathcal{U}(\mathcal{U})}(X - \text{cl}(A))\]

Hence, there exists \(V \in \mathcal{U}\) such that \(\{V[t] : t \in X\} \text{ refines } \{X - \{x\}, X - \text{cl}(A)\}\). If \(x \in \text{cl}_{\mathcal{U}(\mathcal{U})}(A)\) there exists \(p \in V^{-1}[x] \cap V^{-1}[A]\). Thus, there exists \(a \in A\) with \((a, p) \in V^{-1}\) and \((x, p) \in V^{-1}\). Hence \((p, a) \in V\) and \((p, x) \in V\). Therefore, \(V[p] \not\subset X - \{x\}\) and \(V[p] \not\subset X - \text{cl}(A)\) and we have a contradiction. Hence, \(x \in \text{cl}_{\mathcal{U}(\mathcal{U})}(A)\) and it follows that \(\text{cl}(A) = \text{cl}_{\mathcal{U}(\mathcal{U})}(A)\).

(3) \(\Rightarrow\) (1). Let \(U \in \mathcal{U}\) and \(x \in X\). Suppose \(V \circ V^{-1}[x] \cap (X - U[x]) \neq \emptyset\) for each \(V \in \mathcal{U}\).

Claim: \(V^{-1}[x] \cap V^{-1}[X - U[x]] \neq \emptyset\). Let \(t \in (V \circ V^{-1}[x]) \cap (X - U[x])\). Then \(t \in V \circ V^{-1}[x]\) and \(t \in X - U[x]\). There exists \(p \in V^{-1}[x]\) and \(t \in V[p]\). Hence the claim holds and by (3) it follows that \(x \in \text{cl}(X - U[x]) = \bigcap \{W^{-1}[X - U[x]] : W \in \mathcal{U}\}\). Therefore, \(x \in U^{-1}[X - U[x]]\) and thus there exists \(s \in U[x]\) with \(x \in U^{-1}[s]\) which is impossible. Hence \((X, \mathcal{U})\) is locally right symmetric.

Theorem 3.4. Let \((X, \mathcal{U})\) be a quasi-uniform space. The following are equivalent.

1. \((X, \mathcal{U})\) is locally right symmetric.
2. \((X, \mathcal{E}(\mathcal{U}))\) is a nearness space and \(\text{cl}(A) = \text{cl}_{\mathcal{E}(\mathcal{U})}(A)\) for each \(A \subset X\).
3. \((X, \mu(\mathcal{U}))\) is a nearness space and \(\text{int}(A) = \text{int}_{\mu(\mathcal{U})}(A)\) for each \(A \subset X\).
Proof. (1) ⇒ (2). By Theorem 3.3, \( \text{cl}(A) = \text{cl}_{\xi(\mathcal{U})}(A) \) for each \( A \subseteq X \).

Claim: \( \xi(\mathcal{U}) \) is a nearness structure. Axioms (N1), (N3) and (N4) are easy to verify. Suppose \( A \in \xi(\mathcal{U}) \) and for each \( B \in \mathcal{B} \) there exists \( A \in \mathcal{A} \) such that \( A \subseteq \text{cl}_{\xi(\mathcal{U})}(B) \). Let \( U \in \mathcal{U} \) and \( V \in \mathcal{U} \) with \( V \circ V \subseteq \mathcal{U} \). Then there exists \( t \in \bigcap \{ V^{-1}[A]: A \in \mathcal{A} \} \).

Now
\[
\bigcap \{ V^{-1}[A]: A \in \mathcal{A} \} \subseteq \bigcap \{ V^{-1}[\text{cl}_{\xi(\mathcal{U})}(B)]: B \in \mathcal{B} \}
\]
\[
= \bigcap \{ V^{-1} [\text{cl}(B)]: B \in \mathcal{B} \}
\]
\[
= \bigcap \{ V^{-1} \bigcap \{ W^{-1}[B]: W \in \mathcal{U} \}: B \in \mathcal{B} \}
\]
\[
\subseteq \bigcap \{ V^{-1} \circ V^{-1}[B]: B \in \mathcal{B} \}
\]
\[
= \bigcap \{ U^{-1}[B]: B \in \mathcal{B} \}
\]
Therefore, \( \mathcal{B} \in \xi(\mathcal{U}) \) and hence \( \xi(\mathcal{U}) \) is a nearness structure.

(2) ⇒ (1) follows by Theorem 3.3. The equivalence of (2) and (3) follows in a similar manner using Lemma 3.2 and Theorem 3.3.

We now show that many of the quasi-uniform spaces that are studied are indeed locally right symmetric.

Theorem 3.5. Let \( (X, t) \) be a symmetric topological space. Then the fine transitive, point-finite covering, locally finite covering, and the Pervin quasi-uniformities are each locally right symmetric.

Corollary. Every symmetric topological space admits a compatible locally right symmetric quasi-uniformity.

Proof. Let \( \mathcal{P} \) denote the Pervin quasi-uniformity. Let \( A \subseteq X \). By Theorem 3.3, it suffices to show \( x \notin \text{cl}(A) \) implies \( W^{-1}[x] \cap W^{-1}[A] = \emptyset \) for some \( W \in \mathcal{P} \). Suppose \( x \notin \text{cl}(A) \). If \( \text{cl}(A) = \emptyset \) we are through. Suppose \( y \in \text{cl}(A) \). Then \( x \notin \text{cl}(y) \) and since \( X \) is symmetric \( y \notin \text{cl}(x) \). Thus, there exists an open set \( O_y \) such that \( y \in O_y \) and \( x \notin O_y \).

Set \( Q = \bigcup \{ O_y: y \in \text{cl}(A) \} \). Let \( O = X - \text{cl}(A) \) and set
\[
U = (O \times O) \cup ((X - O) \times X)
\]
and
\[
V = (Q \times Q) \cup ((X - Q) \times X).
\]

Let \( W = U \cup V \). The \( W \in \mathcal{P} \) and
\[
W = [(O \cap Q) \times (O \cap Q)] \cup [(O - Q) \times O] \cup [(Q - O) \times Q].
\]

Suppose \( t \in W^{-1}[x] \cap W^{-1}[A] \). Then for some \( a \in A \) we have \( (t, x) \in W \) and \( (t, a) \in W \). But this is impossible; since either \( t \in O \cap Q \) which implies \( x \in Q \), but \( x \notin Q \); or \( t \in O - Q \) which implies \( a \in O \) but \( a \notin O \); or \( t \in Q - O \) which implies \( x \in Q \) but \( x \notin Q \). Hence \( W^{-1}[x] \cap W^{-1}[A] = \emptyset \).
Since each of the other quasi-uniformities stated in this theorem are covering quasi-uniformities generated by collections of covers containing all finite open covers a similar argument shows that they are also locally right symmetric.

We now have a bridge from locally right symmetric quasi-uniform spaces to nearness spaces. \( \xi(\mathcal{U}) \) can be thought of in a natural way as the underlying nearness structure. In the following section we use this bridge to compare concepts and results obtained for quasi-uniform spaces and nearness spaces.

Before we turn to that task, we note that a number of interesting questions arise at this point. Easily two quasi-uniformities \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) may have the same underlying nearness structure; that is, \( \xi(\mathcal{U}_1) = \xi(\mathcal{U}_2) \). These might be called nearness equivalent and then studied. A nearness structure \( \xi \) might be called quasi-uniformizable provided there existed a quasi-uniform structure \( \mathcal{U} \) such that \( \xi = \xi(\mathcal{U}) \). It would be of interest to find necessary and sufficient conditions characterizing the quasi-uniformizable nearness structures. The following example shows that not all nearness structures are quasi-uniformizable.

**Example 3.1.** Let \((X, t)\) be a \( T_1 \) topological space that is not preorthocompact. Fletcher and Lindgren provide such an example and the necessary definitions in [6]. They also show that if a topological space admits a Lebesgue quasi-uniformity then it must be preorthocompact. Let \( \mu \) be the nearness structure generated by the collection of all open covers of \( X \). Then \( \mu \) is topological and since \((X, t)\) does not admit a Lebesgue quasi-uniformity \( \mu \) can not be generated by a quasi-uniform structure.

### 4. Corresponding results

**Definition 4.1.** Let \((X, \mathcal{U})\) be a quasi-uniform space.

1. \((X, \mathcal{U})\) is called totally bounded if for each \( U \in \mathcal{U} \) there exists a finite collections of subsets of \( X \), \( F_1, \ldots, F_n \), such that \( X = \bigcup F_i \) and \( F_k \times F_k \subset U \) for \( 1 \leq k \leq n \).

2. \((X, \mathcal{U})\) is called (countably) pre-compact if for each \( U \in \mathcal{U} \) there exists a (countable) finite set \( F \subset X \) such that \( X = U[F] \).

3. A filter \( \mathcal{F} \) on \( X \) is called Cauchy if for each \( U \in \mathcal{U} \) there exists \( x \in X \) with \( U[x] \in \mathcal{F} \).

4. A filter \( \mathcal{F} \) on \( X \) is called a weak Cauchy filter if \( \bigcap \{U^{-1}[F]: F \in \mathcal{F}\} \neq \emptyset \) for each \( U \in \mathcal{U} \).

5. \((X, \mathcal{U})\) is called complete if each Cauchy filter has a nonempty adherence; or equivalently, if each Cauchy ultrafilter converges.

6. \((X, \mathcal{U})\) is called Lebesgue if each open cover is a \( \mathcal{U} \)-quasi-uniform cover.

We now consider several concepts in a quasi-uniform space and their equivalent counter-parts in a nearness space in the sense made precise by the following theorem.
Theorem 4.1. Let \((X, \mathcal{U})\) be a locally right symmetric quasi-uniform space. Then the following pairs of statements are equivalent.

(A) \(\mathcal{F}\) is a weak Cauchy filter in \((X, \mathcal{U})\).  
(A') \(\mathcal{F}\) is a near filter in \((X, \xi(\mathcal{U}))\).
(B) \(\mathcal{F}\) is a Cauchy ultrafilter in \((X, \mathcal{U})\).  
(B') \(\mathcal{F}\) is a near ultrafilter in \((X, \xi(\mathcal{U}))\).
(C) \((X, \mathcal{U})\) is complete.  
(C') \((X, \xi(\mathcal{U}))\) is ultrafilter complete.
(D) \((X, \mathcal{U})\) is pre-compact.  
(D') \((X, \xi(\mathcal{U}))\) is totally bounded.
(E) \((X, \mathcal{U})\) is Lebesgue.  
(E') \((X, \xi(\mathcal{U}))\) is topological.

The following results for a nearness space are found in Carlson [3] and [4] and are stated here for the convenience of the reader.

Theorem A. Let \((X, \xi)\) be a nearness space. Then

1. The underlying topology is compact if and only if \(\xi\) is ultrafilter complete and totally bounded.
2. The underlying topology is countably compact if and only if \(\xi\) is countably totally bounded and the closure of every near filter has the countable intersection property.
3. The underlying topology is Lindelöf if and only if \(\xi\) is countably bounded and every near filter with the countable intersection property clusters.
4. If the underlying topology is Hausdorff then it is H-closed if and only if \(\xi\) is open ultrafilter complete and open totally bounded.

By making use of the results obtained in this paper and changing terminology where necessary it is clear that these results can be translated at once to statements about locally right symmetric quasi-uniform spaces. Moreover; they provide the motivation to determine if the corresponding results hold for quasi-uniform spaces in general; that is, do the results hold in the absence of the locally right symmetric condition?

Statements (1) and (4) are already well-known theorems for quasi-uniform spaces. The first statement with totally bounded replaced by pre-compact was proved by Sieber and Pervin [11], and was the motivation for the corresponding result for nearness spaces [3]. Statement (4) in slightly different terminology was proved by Fletcher and Naimpally [5]. Before we show that statements (2) and (3) also hold for quasi-uniform spaces we state the following lemma.

Lemma 4.2. Let \((X, \mathcal{U})\) be a quasi-uniform space. The following pairs of statements are equivalent.

(A) \((X, \mathcal{U})\) is pre-compact.
(A') If \(A \subset \mathcal{P}(X)\) and \(\bigcap \{U^{-1}[A]: A \in \mathcal{A}\} = \emptyset\) for some \(U \in \mathcal{U}\), then there exists a finite \(\mathcal{B} \subset \mathcal{A}\) with \(\bigcap \mathcal{B} = \emptyset\)
(B) \((X, \mathcal{U})\) is countably pre-compact.
(B') If \(\mathcal{A} \subset \mathcal{P}(X)\) and \(\bigcap \{U^{-1}[A]: A \in \mathcal{A}\} = \emptyset\) for some \(U \in \mathcal{U}\), then there exists a countable \(\mathcal{B} \subset \mathcal{A}\) with \(\bigcap \mathcal{B} = \emptyset\).
Theorem 4.3. Let \((X, \mathcal{U})\) be a quasi-uniform space. Then the underlying topology is Lindelöf if and only if

1. \(\mathcal{U}\) is countably pre-compact, and
2. every weak Cauchy filter with the countable intersection property clusters.

Proof. Suppose \(t(\mathcal{U})\) is Lindelöf. Let \(U \in \mathcal{U}\). Then \(\{\text{int } U[x]: x \in X\}\) is an open cover of \(X\) and hence there exists a countable subcover. Thus \(\mathcal{U}\) is countably pre-compact. Let \(\mathcal{F}\) be a weak Cauchy filter with the countable intersection property. Suppose \(\text{adh } \mathcal{F} = \emptyset\), then \(\{X - \tilde{F}: F \in \mathcal{F}\}\) is an open cover and there exists a countable subcover, but this is a contradiction. Hence \(\mathcal{F}\) clusters.

Suppose the conditions on \(\mathcal{U}\) hold. Let \(\mathcal{O} = \{O_\alpha: \alpha \in \Omega\}\) be an open cover of \(X\) with no countable subcover. Then \(\{X - O_\alpha: \alpha \in \Omega\}\) is a subbase for a filter \(\mathcal{F}\) which has the countable intersection property. Since \(\mathcal{U}\) is countably pre-compact it follows, by Lemma 6.4, that \(\mathcal{F}\) is a weak Cauchy filter. Then by (2), \(\mathcal{F}\) clusters, which is a contradiction. Hence \((X, t(\mathcal{U}))\) is Lindelöf.

Theorem 4.4. Let \((X, \mathcal{U})\) be a quasi-uniform space. Then the underlying topology is countably compact if and only if

1. if \(\mathcal{A}\) is a countable collection of subsets of \(X\) for which there exists \(U \in \mathcal{U}\) such that \(\bigcap\{U^{-1}[A]: A \in \mathcal{A}\} = \emptyset\), then there exists a finite \(\mathcal{B} \subset \mathcal{A}\) with \(\bigcap \mathcal{B} = \emptyset\), and
2. the closure of every weak Cauchy filter has the countable intersection property.

Proof. Assume that \(t(\mathcal{U})\) is countably compact. Let \(\mathcal{A}\) be a countable collection of subsets of \(X\) such that there exists \(U \in \mathcal{U}\) with \(\bigcap\{U^{-1}[A]: A \in \mathcal{A}\} = \emptyset\). Then \(\bigcap \{\tilde{A}: A \in \mathcal{A}\} = \emptyset\). Then \(\{X - \tilde{A}: A \in \mathcal{A}\}\) is a countable open cover of \(X\) and there exists a finite subcollection \(\{A_i: 1 \leq i \leq n\}\) such that \(\bigcap_{1}^{n} \tilde{A}_i = \emptyset\). Then \(\bigcap \{A_i: 1 \leq i \leq n\} = \emptyset\). Hence \(\mathcal{U}\) satisfies condition (1). If \(t(\mathcal{U})\) is countably compact then the closure of every filter has the countable intersection property and thus (2) holds.

Suppose the conditions on \(\mathcal{U}\) hold. Let \(\mathcal{O} = \{O_i: i \in N\}\) be a countable open cover of \(X\). Suppose \(\mathcal{O}\) does not have a finite subcover. Then \(\mathcal{I} = \{X - O_i: i \in N\}\) has the finite intersection property: let \(\mathcal{F}\) denote the filter generated by \(\mathcal{I}\). Then \(\text{adh } \mathcal{F} = \emptyset\). Then \(\mathcal{F}\) does not have the countable intersection property and hence \(\mathcal{F}\) is not a weak Cauchy filter. Thus there exists \(U \in \mathcal{U}\) with \(\bigcap \{U^{-1}[F]: F \in \mathcal{F}\} = \emptyset\). Hence there exists finite \(\mathcal{B} \subset \mathcal{F}\) such that \(\bigcap \mathcal{B} = \emptyset\). This implies that there exists a finite subcollection of \(\mathcal{F}\) with an empty intersection. But this is a contradiction. Therefore \(t(\mathcal{U})\) is countably compact.

References


