MATHEMATICS

FLEXIBLE HEXAGONS

BY

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1. In a recently published *Note on flexible hexagons* H. A. LAUWERIER [1] has proved the theorem: a spatial hexagon the sides and angles of which are given is flexible if and only if its opposite elements are equal. In what follows we make some remarks on the paper, the first of which is of a negative character: the criterion, though sufficient, is shown to be not necessary. There are classes of (non-singular) flexible hexagons the opposite elements of which are not equal.

2. The problem stated by Lauwerier arose from structural organic chemistry and dealt with the question whether the carbon skeleton of a certain molecule is flexible provided the six sides and the six angles are fixed. We remark that it is equivalent to a well-known problem of *kinematics* which asks for the degree of mobility of a certain spatial mechanism. If P_1, P_2, ... P_6 are the vertices of the hexagon in cyclic order and if b_{ij} denotes the distance P_iP_j, the sides $b_{i,\,i+1}$ are fixed, and as the angles are fixed the same is true for the six shorter diagonals $b_{i,\,i+2}$. But that means that P_i are the vertices of an *octahedron* O, the twelve edges of which are given. The mechanism just mentioned consists of eight rigid bodies corresponding with the faces of O; two adjacent bodies are permitted to rotate relatively one to another about their common edge and the question arises whether the whole structure is flexible or rigid.

It is obvious that an octahedron is "in general" rigid. This follows for instance from a theorem of Legendre according to which a polyhedron is "determined" by its edges. In spatial kinematics Grübler's formula would not give a positive number of degrees of freedom for the mechanism under consideration. But these tests do not exclude the flexibility of octahedra the edges of which satisfy certain conditions.

As far as we know the first flexible octahedron has been discovered by BRICARD [2] who answered a question which was asked by Stephanos. Such structures are now well-known in kinematic literature (*octaèdres articulés*, *wackelige Achtflache*) and models may be seen in kinematic museums. BLASCHKE [3] studied the configuration but he restricted himself to *infinitesimal* displacements (for which the problem has relations to affine geometry).

It was BRICARD who solved the problem completely in a very clearly written memoir of 1897 [4], in which all flexible octahedra were listed systematically. He proved that there exist *three* types. The first and most symmetric type is that considered by Lauwerier: opposite edges are equal. The second type is again characterized by six equalities for the twelve edges but the scheme is less symmetric, a representative of this type being given by

$$(1)\quad \begin{cases} b_{12}=b_{45},\ b_{13}=b_{34},\ b_{15}=b_{24}, \\ b_{16}=b_{46},\ b_{23}=b_{35},\ b_{26}=b_{56}. \end{cases}$$

The *third* type can not be described in such an easy way but Bricard has given a rather simple geometric construction for it. For this type yields that in each of the six tetrahedral vertices opposite angles are equal (or supplementary).

3. It is interesting to compare Lauwerier's attractive approach with Bricard's method, because both encounter essentially the same hitch. The latter introduces three variables u, v, w, each being the tangent of half a dihedral angle of the octahedron and derives three biquadratic equations

$$F_1(v, w)=0,\ F_2(w, u)=0,\ F_3(u, v)=0.$$

If the octahedron is flexible they must have an infinity of solutions. Discussing the conditions to be satisfied for this it is seen to be essential to know whether one or more F_i are reducibel (to be written as the product of two polynomials of lesser degree). If this is *not* the case the first type is the only answer, the other hypothesis leads to types two and three. Lauwerier, in a more elegant way, choses the three main diagonals P_1P_4, P_2P_5 and P_3P_6 as *his* variables. His starting-point is a 7×7 determinant M which gives the volume of a simplex in five-dimensional space S_5 as a function of the edges. If $M=0$ the six vertices are in a S_4, if its rank is five they are in a S_3. If $x=b_{14}^2$, $y=b_{25}^2$, $z=b_{36}^2$ six biquadratic equations

$$G_1(y, z)=H_1(y, z)=0,\ G_2(z, x)=H_2(z, x)=0,\ G_3(x, y)=H_3(x, y)=0$$

are derived. The condition that there is an infinity of solutions implies that for instance G_1 and H_1 are linearly dependent, *provided they are irreducible.* On the strength of a short (and unconvincing) argument this is supposed to be always the case. At that moment the road to the types two and three was closed. It may be verified without difficulty that for the octahedron [1] LAUWERIER's equation (2.1) (p. 331) *is* reducible: the left-hand side is the product of two quadratic equations.

4. We restrict ourselves in what follows to octahedra of the first type, so that Lauwerier's theory is valid. Making use of the notations $a_{ij} = b_{ij}^2$ and

$$(2) \quad \begin{cases} a_{23} = a_{56} = b_1, \; a_{35} = a_{62} = c_1 \\ a_{34} = a_{61} = b_2, \; a_{46} = a_{13} = c_2 \\ a_{45} = a_{12} = b_3, \; a_{51} = a_{24} = c_3 \end{cases}$$

he shows that a solution (x, y, z) of his conditions must satisfy the two equations

$$(3) \qquad D_1 \equiv xyz - \Sigma x(b_1 - c_1)^2 + 2\Pi(b_1 - c_1) = 0$$

$$(4) \qquad D_3 \equiv \Sigma(x - x_r)(y - y_r) = 0$$

where $x_r = -b_1 - c_1 + b_2 + c_2 + b_3 + c_3$ and cyclic for y_r and z_r and that moreover $x = x_r$, $y = y_r$, $z = z_r$ is a solution. The point (xyz) is therefore either a point of intersection of (3) and (4) the locus of which is a space curve Γ of degree six or it is the vertex of the quadratic cone (4).

The conclusion is: there is a continuous set of flexible positions and moreover an isolated position, which is rigid.

We make two remarks on this.

If in the matrix M we substitute $x = x_r$, $y = y_r$, $z = z_r$ the minor of M obtained by cancelling the third and the sixth rows and columns is seen to be zero. This implies that $P_1P_2P_4P_5$ are in one plane and these points are therefore the vertices of a parallelogram. The same holds for $P_2P_3P_5P_6$ and $P_3P_4P_6P_1$.

Hence the main diagonals pass through a point, the centre of the octahedron, the opposite faces of which are parallel. In short: rigid octahedra are the affine images of a *regular* octahedron.

Hence they are *convex* and the rigidity is in accordance with a classical theorem of Cauchy which states that a convex polyhedron is always rigid. This implies that flexible octahedra (of all types) are nonconvex.

Our second remark deals with the reality of the configuration. It is obvious that an octahedron (with equal opposite elements) the six edges b_i and c_i of which are given does not necessarily exist because a system of inequalities for the edges has to be satisfied. But *if* such an octahedron exists this does not imply that there is a continuous set of flexible octahedra and *moreover* an isolated rigid one. In order to show this we give an example. We make use of a theorem of Mannheim stating that an octahedron of the first type has an axis of symmetry l, which intersects the three main diagonals orthogonally. If we introduce a Cartesian frame OXYZ, with OZ along l and OX along P_1P_4, then

$$P_1 = (d_1, 0, 0),\ P_2 = (d_2 \cos \varphi_2, d_2 \sin \varphi_2, h_2),\ P_3 \equiv (d_3 \cos \varphi_3, d_3 \sin \varphi_3, h_3),$$
$$P_4 = (-d_1, 0, 0),\ P_5 = (-d_2 \cos \varphi_2, -d_2 \sin \varphi_2, h_2),$$
$$P_6 = (-d_3 \cos \varphi_3, -d_3 \sin \varphi_3, h_3)$$

are the vertices of a flexible octahedron of the first type and therefore a continuous set of such octahedra exists. We have $b_1 = d_2^2 + d_3^2 - 2d_2d_3 \cos(\varphi_2 - \varphi_3) + (h_2 - h_3)^2$, etc. and therefore $b_1 + c_1 = 2d_2^2 + 2d_3^2 + 2(h_2 - h_3)^2$, etc., so that $x_r = 4(d_1^2 + h_2h_3)$.

No rigid octahedron exists therefore if, for instance, $h_2h_3 < -d_1^2$. The discussion on the reality of the octahedra if b_i and c_i are given seems in general rather complicated. For the existence of the rigid one we need not only $x_r \geqslant 0, y_r \geqslant 0, z_r \geqslant 0$, but moreover $b_1 c_1 \geqslant (b_2 + c_2 - b_3 - c_3)^2$ and two similar inequalities. For that of the flexible ones we have to study the curve Γ. It has always an infinity of real points (x, y, z) because a generator of the quadratic cone (4) has at least one real point in common with the cubic surface (3). If such a point corresponds however with a real octahedron the conditions $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and moreover a set of inequalities such as $\sqrt{x} \leqslant \sqrt{b_2} + \sqrt{c_2}$ must be satisfied.

5. We shall prove that the curve Γ the points of which correspond with the positions of a flexible octahedron, is a curve of genus *one*. Indeed, consider the transformation

$$x = x_r + \frac{1}{u_1}, \; y = y_r + \frac{1}{u_2}, \; z = z_r + \frac{1}{u_3} \tag{5}$$

It is easily seen that (3) is equivalent to an equation of the *third* degree in u_i, and (4) is transformed into

$$u_1 + u_2 + u_3 = 0 \tag{6}$$

Hence by the birational transformation (5) the curve Γ corresponds with a plane cubic, which has in general no double points. It follows from this that the squares of the three main diagonals of a set of flexible octahedra may be written as *elliptic* functions of a parameter *t*. The same holds then for the cosines of the variable angles of the configuration, such as $\angle P_1P_2P_4$. If we consider the trihedral angle $P_2(P_1P_3P_4)$ we see that the sides $\angle P_1P_2P_3$ and $P_3P_2P_4$ are fixed; therefore by means of the cosine law of spherical trigonometry we are able to express the cosine of the dihedral angle on the edge P_2P_3 (and thus on any edge) by an elliptic function of *t*. The same holds for the square of the distance of any two main diagonals and the cosine of their angle. The invariant of the plane cubic (or its equivalent: the modulus of the elliptic functions) depends on b_i and c_i.

If in the equations (3) and (4) we introduce a fourth coordinate w in order to make them homogeneous it is easily seen that Γ has three double points $D_1 = (1000)$, $D_2 = (0100)$ and $D_3 = (0010)$, because these are double points of the cubic surface (3). The projecting cone of Γ with its centre at D_1 is of the fourth order and its intersection with $x = 0$ a plane quartic Γ_1 with double points at D_2 and D_3 and thus a curve of genus *one*. This confirms that Γ itself is of genus one. The equation of Γ_1 is obviously $G_1=0$ (or, what is the same thing, $H_1=0$) where G_1 is the biquadratic form mentioned above. It follows from this that $G_1=0, G_2=0$ and $G_3=0$ have the same invariant. In other words $G_1=0$ considered as a quadratic function of y and $G_2=0$ considered as one of x have the same discriminant, which is polynomial of the fourth degree in z. This may be verified by direct calculation and is in accordance with a theorem shown by Bricard in his memoir and applied by him to *his* set of three biquadratic equations.

It would be interesting to investigate for a flexible octahedron the motion of a link with respect to another. That will not be done here. We remark only that for such a study Bricard's method seems more appropriate then the approach considered in this paper.

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