EMBEDDING OF REGULAR SEMIGROUPS IN WREATH PRODUCTS

R.J. WARNE

Department of Mathematics, University of Alabama, Birmingham and Department of Mathematics, University of California, Berkeley, USA

Communicated by J. Rhodes
Received 17 November 1982

The purpose of this paper is to embed various classes of regular semigroups in wreath products of more special classes of semigroups. If \( S_1 \) and \( S_2 \) are semigroups, then \( S_2 * S_1 \) denotes the wreath product of \( S_1 \) by \( S_2 \) (i.e. \( p_1 : S_2 * S_1 \to S_1 \) where \( p_1 \) is the projection surmorphism) and \( S_1 \leq S_2 \) means there exists an isomorphism of \( S_1 \) into \( S_2 \). \( \textsf{\Lambda} \), \( \textsf{L} \), \( \textsf{R} \), \( \textsf{J} \), and \( \textsf{D} \) will denote Green's relations on an arbitrary semigroup \( S \) while \( E(S) \) (\( T(S) \)) will denote the set of idempotents (union of the maximal subgroups) of \( S \). A semigroup \( S \) is termed combinatorial if each of its maximal subgroups is a singleton. In Section 1, we consider regular semigroups \( S \) admitting a given congruence relation \( \rho \) such that \( e\rho \) is a left group for all \( e \in E(S) \). We term \( (S, \rho) \) a left regular pair. If \( (S, \rho) \) is a left regular pair, we show

\[
S \leq (\ker \rho) \cap (S/\rho) \cap (\ker \rho) \cap (S/\rho)
\]

where \( \ker \rho \) denotes the subsemigroup of \( S \) generated by \( \ker \rho = \bigcup (ep : e \in E(S)) \), and \( (\ker \rho) \) is \( (\ker \rho) \) with identity "1" appended (Theorem 1.7). Theorem 1.7 is first specialized to obtain structure theorems for \( \textsf{R} \)-compatible inverse semigroups, inverse semigroups, \( \textsf{R} \)-unipotent semigroups (each \( \textsf{R} \)-class of \( S \) contains precisely one idempotent) and natural \( \textsf{R} \)-unipotent semigroups (\( T(S) \) is a semilattice of left groups). If \( S \) is an \( \textsf{R} \)-compatible inverse semigroup, \( \rho = \textsf{R} \) and \( (\ker \rho) = E(S) \) (Theorem 1.16). If \( S \) is an inverse semigroup, \( \rho \) is the maximum idempotent separating congruence on \( S \) and \( \ker \rho \) is a semilattice \( E(S) \) of groups (Theorem 1.9 and Remark 1.10). If \( S \) is an \( \textsf{R} \)-unipotent semigroup, \( \rho \) is the minimum inverse semigroup congruence on \( S \) and \( \ker \rho = E(S) \) (Theorem 1.12). If \( S \) is a natural \( \textsf{R} \)-unipotent semigroup, \( \rho \) is the smallest combinatorial inverse semigroup congruence on \( S \) and \( \ker \rho = T(S) \) (Theorem 1.17). Theorem 1.17 may be refined to obtain

\[
S \leq (E(S)) \cap ((T(S)/\nu) \cap (S/\nu))
\]

where \( \nu(t) \) is the smallest inverse semigroup (combinatorial inverse semigroup) congruence on \( T(S) \) (Theorem 1.19). Next we assume that \( S/\rho \) is an \( \textsf{D} \)-unipotent
semigroup (this implies $\ker \varrho$ is a union of groups on which $\mathcal{L}$ is a congruence relation (Proposition 1.20)). In this case, we show

$$S \leq (W^1) \circ ((\ker \varrho / \mathcal{L})^1) \circ (S / \varrho)^1 \quad \text{and} \quad S / \varrho \leq (S / \varrho / \delta)^1 \circ (\ker \varrho / \mathcal{L})^1$$

where $W$ is a lower partial chain of $\mathcal{L}$-classes of $\ker \varrho$, $\delta$ is the smallest inverse semigroup congruence on $S / \varrho$, and "$\circ$" denotes the reverse of "$\circ$" (Theorem 1.26). We specialize Theorem 1.26 to obtain structure theorems for generalized $\mathcal{L}$-unipotent semigroups ($E(S)$ is a subsemigroup on which $\mathcal{L}$ is a congruence relation) and left natural regular semigroups ($T(S)$ is a subsemigroup on which $\mathcal{L}$ is a congruence relation). If $S$ is a generalized $\mathcal{L}$-unipotent semigroup, $\varrho = \nu \cap \mathcal{L}$ where $\nu$ is the smallest inverse semigroup congruence on $S$ and $\ker \varrho = E(S)$ (Theorem 1.28). If $S$ is a left natural regular semigroup, $\varrho$ is the largest congruence on $S$ contained in the $\mathcal{L}$-unipotent semigroups, generalized $\mathcal{L}$-unipotent semigroups, and natural $\mathcal{L}$-unipotent semigroups and left natural regular semigroups are considerably simpler than those previously obtained for these classes of semigroups in [16], [17], and [22] respectively.

In Section 2, we embed various classes of regular semigroups $S$ in wreath products containing the bicyclic semigroup $C$ or extended bicyclic semigroup $C^*$ ($C^* = I \times I$, where $I$ is the set of integers, under the multiplication $(m, n)(p, q) = (m + p - \min(n, p), n + q - \min(n, p))$). Let $N$ denote the set of natural numbers. Let $\mathcal{N}(I)$ denote $N(I)$ under the reverse of the usual order. $E(S)$ is always assumed to be under its natural order ($e \vartriangleleft f$ if $ef = fe = e$). Let $S$ be a bisimple inverse semigroup, let $n$ be a positive integer, and let $\mathcal{N}[n]$ denote the cartesian product of $\mathcal{N}$ with itself $n$ times. If $E(S) \cong \mathcal{N}[n]$ ($I \times I \times \mathcal{N}[n]$) under the lexicographic order, $S$ is termed an $\omega^n$-bisimple ($\omega^n$I-bisimple semigroup). If $E(S) \cong I$, $S$ is termed an $I$-bisimple semigroup. If $S$ is an $\omega^n$-bisimple semigroup, then $S \leq T(S) \circ C$ for $n = 1$ and $S \leq T(S) \circ C^n \circ C$ for $n > 1$ where $C$ is $C$ with constant transformations appended and $C^n \circ C^{n-1} = C \circ \cdots \circ C$ ($n - 1$ times) (Theorem 2.9). If $S$ is an $I$-bisimple semigroup, $S \leq T(S) \circ C^*$ (Theorem 2.10). If $S$ is an $\omega^n$I-bisimple semigroup, $S \leq T(S) \circ C^n \circ C^*$ (Theorem 2.13). Let $S$ be a simple regular semigroup. If $E(S) \cong I$, $S$ is termed a simple $I$-regular semigroup. If $S$ is a simple $I$-regular semigroup, $S \leq T(S) \circ k \circ C^*$ where $k = \{0, 1, 2, \ldots, k - 1\}$ ($k$, a positive integer) with $ij = \max\{i, j\}$ for $i, j \in k$ ($k$ is the number of $\mathcal{L}$-classes of $S$) (Theorem 2.14). Let $A = \mathcal{N} \times Y$, where $Y$ is a semilattice with greatest element, be under the lexicographic order. An inverse semigroup $S$ is termed an $\omega Y$-inverse semigroup if $E(S) = A$ and $(n, y) \vartriangleleft (m, z)$ if and only if $y = z$. If $S$ is an $\omega Y$-inverse semigroup, $S \leq T(S) \circ Y \circ C$ (Theorem 2.17). A regular semigroup $S$ is termed an $\omega Y$-$\mathcal{L}$-unipotent semigroup if $E(S)$ is a semilattice $A$ of left zero semigroups $\{E_{(n, y)}, (n, y) \in A\}$ and $e_{(n, y)} \circ e_{(m, z)}$ for $e_{(n, y)} \in E_{(n, y)}$, $e_{(m, z)} \in E_{(m, z)}$ if and only if $y = z$. If $S$ is an $\omega Y$-$\mathcal{L}$-unipotent semigroup,

$$S \leq ((E(S))^1) \circ (G^1) \circ \mathcal{Y} \circ C^1$$
where \( G \) is a semilattice \( A \) of groups (Theorem 2.16). A regular simple semigroup \( S \) is termed a simple \( \omega-R \)-unipotent semigroup if \( E(S) \) is a semilattice \( \mathcal{N} \) of left zero semigroups. \( S \) is simple \( \omega-R \)-unipotent if and only if \( S \) is \( \omega Y-R \)-unipotent with \( Y \) a finite chain (Remark 2.18). A regular bisimple semigroup \( S \) is termed a generalized \( \omega-L \)-unipotent bisimple semigroup if \( E(S) \) is a semilattice \( \mathcal{N} \) of rectangular bands and \( \mathcal{L} \) is a congruence on \( E(S) \). If \( S \) is a generalized \( \omega-L \)-unipotent bisimple semigroup, then

\[ S \leq (W^{1})^{1} \circ \left( (E(S)/\mathcal{L})^{1} \right)^{1} \circ (S/\mathcal{Q})^{1} \]

where \( W \) is a semilattice \( \mathcal{N} \) of left zero semigroups and \( \mathcal{Q} \) is the smallest \( \mathcal{L} \)-unipotent congruence on \( S \) and

\[ S/\mathcal{Q} \leq C^{1} \circ \left( G^{1} \right)^{1} \circ \left( (E(S)/\mathcal{L})^{1} \right)^{1} \]

where \( G \) is a semilattice \( \mathcal{N} \) of groups (Theorem 2.19). The structure theorems given here for \( \omega^n \)-bisimple and \( \omega^n I \)-bisimple semigroups, \( I \)-bisimple semigroups, simple \( I \)-regular semigroups, \( \omega Y \)-inverse and \( \omega Y-R \)-unipotent semigroups, and generalized \( \omega-L \)-unipotent bisimple semigroups are considerably simpler than those previously obtained for these classes of semigroups in [10], [11], [14], [19], and [20] respectively.

In Section 3, we apply the Rhodes expansion and the derived semigroup to obtain a structure theorem for orthodox semigroups (regular semigroups \( S \) such that \( E(S) \) is a subsemigroup), and a structure theorem for a class of bisimple inverse monoids. Let us first briefly describe the Rhodes expansion of an arbitrary semigroup \( S \). Let \( S \) be a semigroup. If \( a, b \in S \), \( a \in b \) means \( a \cup S a \subseteq b \cup S b \) and \( a \in b \) means \( a \subseteq b \) but \( a \notin b \). Let

\[ S = \{ (s_{n} \subseteq \cdots \subseteq s_{1}) : 1, 2, \ldots, n \in \mathbb{N} \setminus \{0\} \text{ and } s_{i} \in S \text{ for } i = 1, 2, \ldots, n \} \]

under the product

\[ (s_{n} \subseteq \cdots \subseteq s_{1}) \cdot (t_{m} \subseteq \cdots \subseteq t_{1}) = (\text{red}(s_{n} t_{m} \subseteq \cdots \subseteq s_{1} t_{m} \subseteq t_{m}) \subseteq t_{m-1} \cdots \subseteq t_{1}) \]

where "\text{red}" deletes the "right most" of two \( \mathcal{L} \)-equivalent elements. Then, \( (\hat{S}, \cdot) \) is a semigroup. \( \hat{S} \) is termed the Rhodes expansion of \( S \) after its inventor, John Rhodes.

Define \((s_{n} \subseteq \cdots \subseteq s_{1}) \eta = s_{n} \). Then, \( \eta \) is a homomorphism of \( \hat{S} \) onto \( S \) such that \( e \eta^{-1} \in E(\hat{S}) \) for all \( e \in E(S) \) (Theorem 3.1). Let \( \varphi \) be a homomorphism of \( S \) onto a semigroup \( T \). Then, \((s_{n} \subseteq \cdots \subseteq s_{1}) \hat{\varphi} = \text{red}(s_{n} \varphi \subseteq \cdots \subseteq s_{1} \varphi) \) defines a homomorphism of \( \hat{S} \) onto \( \hat{T} \) (Theorem 3.1). The derived semigroup \( D(\varphi) \) of \( \varphi \) is \((S \times T') \cup \{0, \cdot \}) \) where \( T' \) is the monoid generated by \( T' \), \( 0 \in S \times T' \), and \((s_{1}, t_{1})(s_{2}, t_{2}) = (s_{1} s_{2}, t_{1}) \) or \( 0 \) according to whether \( t_{2} = t_{1}(s_{1} \varphi) \) or \( t_{2} \neq t_{1}(s_{1} \varphi) \) and \((s_{1}, t_{1}) 0 = 0 (s_{1}, t_{1}) = 0 \) or \( 0 = 0 \). The derived semigroup was invented by Bret Tilson [25].

Let \( S \) be an orthodox semigroup. Then,

\[ S \hookrightarrow \hat{S} \leq D(\hat{\varphi}) \circ (\hat{S}/\hat{\mathcal{Q}}) \]
where $S$ is an orthodox semigroup with $E(S)/\mathcal{J} \cong E(S)/\mathcal{J}$, where $\mathcal{J}$ is the smallest inverse semigroup congruence on $S$. $D(\mathcal{J})$ is an aperiodic (combinatorial and torsion) semigroup whose set of regular elements is an aperiodic orthodox subsemigroup, and $(S/\mathcal{J})$ is an $\mathcal{Y}$-unipotent semigroup. Furthermore,

$$(S/\mathcal{J}) \cong ((S/\mathcal{J})/\mathcal{J}) \circ (E(S/\mathcal{J}))$$

where $\mathcal{J}$ is the smallest inverse semigroup congruence on $S$, $E((S/\mathcal{J})/\mathcal{J}) \cong E(S)/\mathcal{J}$ and $E((S/\mathcal{J}))$ is a semilattice $E(S)/\mathcal{J}$ of right zero semigroups (Theorem 3.4). Let $S$ be a bisimple inverse monoid with right unit subsemigroup $P$ and group of units $U$. There exists a one-to-one correspondence $V \rightarrow \mathcal{Q}^V$ between the right normal divisors of $P$ (subgroups $V$ of $P$ such that $aV \subseteq Va$ for all $a \in P$) and the idempotent separating congruences of $S$ (see Remark 1.11). If $N = \{ u \in U \mid hu = h \text{ for some } h \in P \}$ is a right normal divisor of $P$, then $S \leq \ker \mathcal{Q}^N \circ S/\mathcal{Q}^N$ where $\ker \mathcal{Q}^N$ is a semilattice $E(S)$ of groups and $S/\mathcal{Q}^N = D(\mathcal{J}) \circ S/\mathcal{Q}^N/\mathcal{J}$ where $\mathcal{J}$ is the smallest group congruence on $S/\mathcal{Q}^N$ and $D(\mathcal{J})$ is a combinatorial inverse semigroup such that $E(D(\mathcal{J}))$ is a direct sum of copies of $E(S)^0(E(S)$ with a zero appended) (Theorem 3.5). We give necessary and sufficient conditions for an $\omega^\omega$-bisimple semigroup to obey the hypothesis of Theorem 3.5 (Example 3.7). We term $S$ a strict bisimple inverse monoid if $P$ is left cancellative and the principal right ideals of $P$ are linearly ordered (by set theoretic inclusion). If $S$ is a strict bisimple inverse monoid, $S \leq E(S) \circ S/\mathcal{Q}$ where $\mathcal{Q}$ is the smallest group congruence on $S$ (Theorem 3.8). Examples of strict bisimple inverse monoids are given (Examples 3.9 and 3.10). The structure theorem for strict bisimple inverse monoids given here is considerably simpler than the one previously obtained for this class of semigroups in [24].

In Section 4, we consider regular semigroups $S$ such that $T(S)$ is a subsemigroup, $T(S)/\mathcal{J}$ has a greatest element, and $eT(S)e$ is an inverse semigroup for all $e \in E(T(S))$. We term such semigroups standard regular semigroups. Let $\theta$ be a homomorphism of a semigroup $S$ onto a semigroup $W$. Following Rhodes [4, 5], we term $\theta$ an $\mathcal{H}$-epimorphism if $s_1 \theta = s_2 \theta$ implies $s_1 \mathcal{H} s_2$ and we term $\theta$ a $\mathcal{Y}(\mathcal{Y})$-epimorphism if $\theta$ is one-to-one on the $\mathcal{Y}$-classes of $S$. We term $\theta$ an $E$-epimorphism if $e \theta : \subseteq E(S)$ for all $e \in E(W)$. We write

$$S \subset X$$

if there exists a homomorphism of a subsemigroup of the semigroup $X$ onto the semigroup $S$ which is an $\mathcal{H}$-epimorphism, a $\mathcal{Y}(\mathcal{Y})$-epimorphism, and an $E$-epimorphism. If $S$ is a standard regular semigroup,

$$S \subset E^1 \circ G^1 \circ V \circ J$$

where $E$ is a semilattice $T(S)/\mathcal{J}$ of left zero semigroups, $G$ is a semilattice $T(S)/\mathcal{J}$ of groups, $V$ is a combinatorial inverse semigroup with semilattice $T(S)/\mathcal{J}$, and $J$ is the semigroup of constant transformations of a set $J$ (Theorem 4.8). The structure theorem for standard regular semigroups given here is considerably simpler than those previously obtained for this class of semigroups in [21] and [23].
We next introduce and clarify the terminology and notation which will be used throughout the paper.

In this paragraph we are using the notation and definitions of Eilenberg [2] unless otherwise specified. A transformation semigroup is a pair \((Q, S)\) where \(Q\) is a set and \(S\) is a subsemigroup of \(F(Q)\) the semigroup of all functions of \(Q\) into \(Q\) under composition. Here \(qs\) will denote the image of \(q\) under \(s\). \((S, Q)\) will denote the dual of \((Q, S)\) (i.e., if \(s \in S\) and \(q \in Q\), we write \(sq\) as the image of \(q\) under \(s\)). If \(S\) is a semigroup, \(S^1 = S \cup \{1 \in S\}\) under the multiplication \(s_1 \cdot s_2 = s_1 s_2\) (product in \(S\)) if \(s_1, s_2 \in S\), \(s \cdot 1 = 1 \cdot s = s\) for all \(s \in S\), and \(1 \cdot 1 = 1\). A monoid is a semigroup \(S\) with two sided identity element (i.e. there exists \(e \in S\) such that \(es = se = s\) for all \(s \in S\)). If \(S\) is a semigroup, we define \(S' = S\) if \(S\) is a monoid and \(S' = S^1\), otherwise. We write \(S = (S', S)\) where the image of \(x \in S^1\) under \(s \in S\) is \(xs\) (product in \(S^1\)). If \(Q\) is a set and \(q \in Q\), we define \(xq = q\) for all \(x \in Q\) and let \(Q = \{q : q \in Q\}\). We also write \(Q = (Q, \tilde{Q})\) and \((Q, S) = (Q, SU \tilde{Q})\). If \(S\) is a monoid, we let \(S' = (S, S \cup \tilde{S})\). Let \((Q, S)\) and \((P, T)\) be transformation semigroups. Let \(W = S^P \times T\) where \(S^P\) is the semigroup of all functions of \(P\) into \(S\) under pointwise multiplication \(*\) of functions (i.e. if \(f, g \in S^P\), \(x(f \ast g) = xfxg\) for all \(x \in P\)). We define a product in \(W\) by \((f, t)(g, u) = (f \ast g, tu)\) where \(x(f \ast g) = (xt)g\) for all \(x \in P\) and juxtaposition (of elements in \(T\)) is product in \(T\). \(W\) is easily shown to be a semigroup. We call \(W\) the wreath product of \((P, T)\) by \((Q, S)\) and write \((Q, S) \circ (P, T) = W\). This notation differs slightly from that used by Eilenberg in [2]. Eilenberg defines \((Q, S) \circ (P, T) = (Q \times P, W)\) where \((q, n)(f, t) = (q(nf), pt)\) for \((q, n) \in Q \times P\) and \((f, t) \in W\). \((Q \times P, W)\) is then a transformation semigroup with composition of maps in \(W\) given by the product above. In this paper, we will then replace \(\circ\) by \(\circ\)'. If \((T, P)\) and \((S, Q)\) are the duals of \((P, T)\) and \((Q, S)\) respectively, we write \((T, P) \circ (S, Q) = T \times S\) where \(P\) is the dual of \(S\) (i.e. if \(p \in P\) and \(f \in S\), the set of functions of \(P\) into \(S\), the image of \(p \under f\) under \(f\) is written as \(fp\)) under the product \((u, g)(t, f) = (ut, g _ f)\) where \(g _ f x = g(tx)\) for \(x \in P\). Let \((Q, S)'' = (Q, S) \circ \ldots \circ (Q, S)(n\) times). (Use same notation for \(\circ\).)

We will use the following basic definitions and notation from Clifford and Preston [1] and/or Krohn, Rhodes, and Tilson [4]: Green's relations (\(\delta, \rho, \gamma, \lambda, \vee, \wedge\), and \(\chi\)), \(\delta\)-class, regular element, regular semigroup, simple semigroup, bisimple semigroup, inverse, inverse semigroup, equivalent definitions of inverse semigroup, left zero semigroup, rectangular band, left group, idempotent, natural partial order of idempotents, semilattice, completely simple semigroup, semilattice of semigroups, right unit subsemigroup, group of units, maximal subgroup, bicyclic semigroup, maximal group homorphic image of semigroup, and equivalence between a semilattice and commutative idempotent semigroup.

Let \(\varrho\) be a congruence relation on a semigroup \(S\). \(\varrho\) will also denote the natural homomorphism of \(S\) onto \(S/\varrho\). If \(a \in S\), \(a\varrho\) will denote the \(\varrho\)-class of \(S\) containing \(a\) and \(\ker \varrho = \bigcup \{eq : e \in E(S)\}\). If \(S/\varrho\) is a type \(A\) semigroup, \(\varrho\) is called a type \(A\) semigroup congruence on \(S\). For example, if \(S/\varrho\) is an inverse semigroup, \(\varrho\) is called an inverse semigroup congruence on \(S\). A congruence \(\varrho\) is termed an idempotent
separating congruence if \( e\varphi f, e, f \in \mathcal{E}(S) \) implies \( e = f \). A semigroup \( S \) is termed \( \varphi \)-compatible if \( \varphi \) is a congruence on \( S \).

A semigroup \( S \) is termed a union of groups if \( S \) is the union of its subgroups. Clifford [1] has shown that a semigroup \( S \) is a union of groups if and only if \( \mathcal{J} \) is a congruence on \( S \) and \( S \) is a semilattice \( S/\mathcal{J} \) of completely simple semigroups. We will use this result without explicit mention.

Let \( S \) be a semigroup. If \( x \in S \), \( V(x) \) will denote the collection of inverses of \( x \). If \( x \in S \) implies there exist distinct positive integers \( r \) and \( s \) such that \( x^r = x^s \), \( S \) is termed a torsion semigroup. If \( \varphi \) is a homomorphism \( S \) onto a semigroup \( T \), we write \( S \rightarrow \varphi T \). If \( \varphi \) is also one-to-one, we write \( S \cong T \). If \( X \) is a subset of \( S \), \(<X>\) denotes the subsemigroup of \( S \) generated by \( X \).

If \( X \) is a set, \( X^{[n]} \), \( n \) a positive integer, will denote the cartesian product of \( X \) with itself \( n \) times. If \( (X_i, \prec_i) \) \((i = 1, 2, \ldots, n) \) is a finite collection of \( (\text{partially}) \) ordered sets, we say \( X = X_1 \times \cdots \times X_n \) is under the lexicographic order \( (\prec) \) if \((x_1, x_2, \ldots, x_n) \prec (y_1, y_2, \ldots, y_n) \) means \( x_1 \prec_y y_1 \) or \( x_1 = y_1 \) and \( x_2 \prec y_2 \) or \( \ldots \), or \( x_i = y_i \) for \( 1 \leq i \leq k \) and \( x_k \nprec y_k \) or \( \ldots \), \( x_n = y_n \).

If a semigroup \( S \) is a semilattice \( [N^{[n]}] \) (\( I \times [N^{[n]}] \)) of semigroups, we say \( S \) is an \( \omega^n \)-\( (\omega^n I) \)-chain of these semigroups. If \( S \) is a semilattice \( I \) of semigroups, \( S \) is termed an \( I \)-chain of these semigroups.

Let \( W \) be a partial groupoid which is a union of a collection of pairwise disjoint subsemigroups \((T_y : y \in Y)\) where \( Y \) is a semilattice. If \( a \in T_y \), \( b \in T_z \), and \( y \nprec z \) (in \( Y \)) imply \( ab \) is defined \((in W)\) and \( ab \in T_{\varphi} \) and if \( z \nprec v \) and \( c \in T_v \), imply \( (ab)c = a(bc) \), we term \( W \) a lower partial chain \( Y \) of the semigroups \((T_y : y \in Y)\).

A regular semigroup \( S \) is termed a natural regular semigroup if \( T(S) \) is a subsemigroup. A regular semigroup \( S \) is termed \( \varphi \)-unipotent if each \( \varphi \)-class of \( S \) contains precisely one idempotent.

Let \( S^0 = SU \cup 0 \) under the multiplication \( s_1 \cdot s_2 = s_1 s_2 \) (product in \( S \)) if \( s_1, s_2 \in S \), \( s \cdot 0 = 0 \cdot s = 0 \) for all \( s \in S \), and \( 0 \cdot 0 = 0 \). \( S^0 \) is termed \( S \) with zero appended.

Let \((X_b : b \in B)\) be a collection of pairwise disjoint semigroups. Adjoin a common zero, \( 0 \), to each \( X_b \) and let \( X = \bigcup(X^0_b : b \in B) \) under the multiplication \( x \cdot y = xy \) (product in \( X^0_b \)) if \( x, y \in X^0_b \) and \( x \cdot y = 0 \) if \( x \in X^0_b \) and \( y \in X^0_c \) with \( b \neq c \). \( X \) is termed to be direct sum of the semigroups \((X^0_b : b \in B)\).

1. Left regular pairs

In Section 1, we first give a structure theorem for left regular pairs \((S, \varphi)\) (Theorem 1.7). Theorem 1.7 is then specialized to give structure theorems for \( \varphi \)-compatible inverse semigroups (Theorem 1.16), inverse semigroups (Theorem 1.9 and Remark 1.10), \( \varphi \)-unipotent semigroups (Theorem 1.12), and natural \( \varphi \)-unipotent semigroups (Theorem 1.17 and Theorem 1.19).

A left regular pair \((S, \varphi)\) is termed a left \( \varphi \)-unipotent pair if \( S/\varphi \) is an \( \varphi \)-unipotent semigroup. We give a structure theorem for left \( \varphi \)-unipotent pairs (Theorem 1.26).
We specialize Theorem 1.26 to obtain structure theorems for generalized \(\mathcal{L}\)-uni-
po*ent semigroups (Theorem 1.28) and left natural regular semigroups (Theorem
1.1).

Theorem 1.7 will be a consequence of Lemmas 1.1-1.6 (below). In Lemmas
1.1-1.5, \((S, \varrho)\) will be a left regular pair.

**Lemma 1.1.** For each \(s \in S/\varrho\) select precisely one \(u_s \in \varrho^{-1}\) and precisely one
\(s' \in V(s)\). Then, every element of \(S\) may be uniquely expressed in the form \(g_{ss'}u_s\) where
\(g_{ss'} \in (ss')\varrho^{-1}\).

**Proof.** Let \(x \in S\). Then, \(x \in \varrho^{-1}\) for some \(s \in S/\varrho\). Let \(x' \in V(x)\). Thus,
\(x' \varrho = s \in \varrho^{-1}\). Since \((x'x)\varrho = (x'xu_s)\varrho = ss'\), \(x'\) and \(x'xu_s\) are both contained
in the left group \((ss')\varrho^{-1}\). Hence, there exists \(y \in (ss')\varrho^{-1}\) such that
\(y(x'xu_s)u_s = x'\). Thus, \(x = x(x') = (xxy'xu_s)u_s\). Let \(xxy'xu_s = g_{ss'} \in (ss')\varrho^{-1}\). Hence, every
element of \(S\) may be expressed in the form \(g_{ss'}u_s\). Uniqueness is easily verified.

**Remark 1.1.** Lemmas 1.2-1.5 and Theorems 1.7 and 1.8 are valid for any regular
semigroup \(S\) satisfying the conclusion of Lemma 1.1.

**Lemma 1.2.** There exists a mapping \(r \rightarrow \alpha_r\) of \(S/\varrho\) into \(F(\ker \varrho)\), the full transfor-
mation semigroup on \(\ker \varrho\), where \(\alpha_r\) is defined by \(u_r g_{\varrho} = (g_{\varrho} \alpha_r) u_{\varrho}\) where \(q_{\varrho} \in \varrho^{-1}\),
\(q \in E(S/\varrho)\). \(q_{\varrho} \alpha_r \in ((rq)(rq))\varrho^{-1}\). We have:
(a) \((q_{\varrho})^{-1} \alpha_r \subseteq ((rq)(rq))\varrho^{-1}\) where \(q \in E(S/\varrho)\).
(b) \((g, h) \alpha_r = (g_{\varrho} \alpha_r) (h_{\varrho} \alpha_r)\) for \(g, h \in \varrho^{-1}\). \(h, y \in \varrho^{-1}\), and \(x, y\), and
\(xy \in E(S/\varrho)\).

**Proof.** The first part of the lemma and (a) follow directly from Lemma 1.1. To
obtain (b), we apply Lemma 1.1 to the following:
\((g, h) \alpha_r u_{xy} = (u_r g_{xy}) h_r = (g_{\varrho} \alpha_r) (u_{rx} h_r) = (g_{\varrho} \alpha_r) (h_{\varrho} \alpha_r) u_{xy}\).

**Lemma 1.3.** There exists a mapping \((r, s) \rightarrow f_{rs}\) of \((S/\varrho)^2\) into \(\ker \varrho\) where \(f_{rs}\) is
defined by \(u_r u_s = f_{rs} u_s\), where \(f_{rs} \in ((rs)(rs))\varrho^{-1}\) such that
(a) \(f_{rs} f_{st} = f_{rs} (r_{rs} f_{st} t_{rs} t_{rs})\) for all \(s, t, z \in S/\varrho\).
(b) \((g_{zz} \alpha_t) = (g_{zz} \alpha_t) (f_{zz} \alpha_t) (f_{zz} \alpha_t)\) for all \(s, t, z \in S/\varrho\) and \(g_{zz} \in (zz')\varrho^{-1}\).

**Proof.** The first part of the lemma is immediate from Lemma 1.1. To obtain (a)
and (b), apply Lemma 1.1 to the following strings of equalities respectively.
\(f_{rs} f_{st} u_{stc} = f_{rs} (u_{st} u_{tc}) = (f_{rs} u_{st}) u_{tc} = (u_{rs} u_{st}) u_{tc} = (u_{rs} f_{st} u_{tc}) u_{tc}\)
\(f_{rs} g_{zz} \alpha_t u_{stc} = f_{rs} (u_{rs} g_{zz} \alpha_t) = (f_{rs} u_{st}) g_{zz} (u_{rs} u_{st}) g_{zz} = u_{rs} (u_{rs} g_{zz} \alpha_t)\)
\(= g_{zz} \alpha_t (u_{rs} g_{zz} \alpha_t) u_{stc} = g_{zz} \alpha_t (u_{rs} g_{zz} \alpha_t) u_{stc}\).
Lemma 1.4. \((g_{ss'}u_s)(h_{tt'}u_t) = g_{ss'}(h_{tt'}\alpha_s)f_{ss't't}u_{st}\) where \(g_{ss'} \in (ss')^{-1}\) and \(h_{tt'} \in (tt')^{-1}\).

Proof. Using Lemmas 1.2 and 1.3,
\[
(g_{ss'}u_s)(h_{tt'}u_t) = g_{ss'}(h_{tt'}\alpha_s)u_{st}u_t = g_{ss'}(h_{tt'}\alpha_s)f_{tt'u_t}u_{st}.
\]
\[
\square
\]

Lemma 1.5. Let \((S, \rho)\) be a left regular pair. Then, there exists a homomorphism \(\phi\) of \(S\) into \(\langle \ker \rho \rangle \circ S/\rho\). For \(g_{rr} \in (rr')^{-1}\), \((g_{rr} u_r)\phi = (v, r)\). If \(S/\rho\) has no identity, \(zv = g_{rr} (z \rho)_{rr'}r \) for \(z \in S/\rho\) and \(1v = g_{rr}\). If \(S/\rho\) has an identity \(zv = g_{rr} \alpha_{z}(z \rho)_{rr'}r \) for \(z \in S/\rho\).

Proof. Using Lemma 1.1, \(\phi\) defines a mapping of \(S\) into \(\langle \ker \rho \rangle \circ S/\rho\). We next show \(\phi\) is a homomorphism. For \(h_{ss'} \in (ss')^{-1}\), \((h_{ss'}u_s)\phi = (\delta, s)\) where \(z\delta = h_{ss'} \alpha_{z}f_{ss'}s\) and \(1\delta = h_{rr'}\). (We will assume \(S/\rho\) has no identity. The other case is simpler.) Using Lemma 1.4,
\[
((g_{rr'}u_r)(h_{ss'}u_s))\phi = (g_{rr'}(h_{ss'}\alpha_r)f_{ss's's}u_{rs})\phi = (\theta, rs)
\]
where
\[
\theta = g_{rr'} (h_{ss'}\alpha_r)f_{ss's's}
\]
and
\[
z\theta = (g_{rr'}(h_{ss'}\alpha_r)f_{ss's's})\alpha_{z}(z \rho)_{z^*z}(z \rho)_{z^*z}
\]
for \(z \in S/\rho\).

However,
\[
(g_{rr'}u_r)\phi(h_{ss'}u_s)\phi = (v, r)(\delta, s) = (v \circ \rho, \delta, rs).
\]
Hence,
\[
1(v \circ \rho, \delta) = (1v)(r\delta) = g_{rr'}(h_{ss'}\alpha_r)f_{ss's's} = \theta
\]
and
\[
z(v \circ \rho, \delta) = z\theta = g_{rr'} \alpha_{z}(z \rho)_{rr'}(h_{ss'} \alpha_{z})(z \rho)_{rr'}\alpha_{z}(z \rho)_{rr'}s
\]
for \(z \in S/\rho\).

We next show \(z\theta = z(v \circ \rho, \delta)\) for all \(z \in S/\rho\). Using Lemma 1.2 twice, Lemma 1.3(a) and Lemma 1.3(b),
\[
z\theta = (g_{rr'}(h_{ss'}\alpha_r)f_{ss's's})\alpha_{z}(z \rho)_{z^*z}(z \rho)_{z^*z}
\]
\[
= (g_{rr'}(h_{ss'}\alpha_r))\alpha_{z}(f_{ss's's})(z \rho)_{z^*z}(z \rho)_{z^*z}
\]
\[
= (g_{rr'} \alpha_{z})(h_{ss'} \alpha_{z}f_{ss's's})\alpha_{z}(z \rho)_{z^*z}(z \rho)_{z^*z}
\]
\[
= (g_{rr'} \alpha_{z})(h_{ss'} \alpha_{z}f_{ss's's})\alpha_{z}(z \rho)_{z^*z}s
\]
\[
= (g_{rr'} \alpha_{z})(f_{rr'}z_{rr'})(h_{ss'} \alpha_{z})f_{ss's's}
\]
\[
= z(v \circ \rho, \delta).
\]

Lemma 1.6. Let \(A, B,\) and \(D\) be semigroups. If \(A \cong B\), then \(D \circ A \cong D \circ B\). If \(A \cong B\), then \(A \circ D \cong B \circ D\).

Proof. Let \(\beta\) be an isomorphism of \(B'\) onto \(A'\). We define an isomorphism \(\xi\) of \(D^A \times A\) into \(D^B \times B\) as follows: \((f, a)\xi = (f', a\beta^{-1})\) where \(bf' = (b\beta)f\) for \(b \in B'\).
Let $\delta$ be an isomorphism of $A$ into $B$. We define an isomorphism $\nu$ of $A^D \times D$ into $B^D \times D$ as follows: $(f, d)\nu = (f', d')$ where $xf' = (xf)\delta$ for $x \in D$.

**Theorem 1.7.** Let $(S, \varrho)$ be a left regular pair. Then,

\[ S \leq \langle \ker \varrho \rangle^1 \circ (S/\varrho)^1. \]

If $S$ has an identity, the superscript "1" in $(\ast)$ may be deleted.

**Proof.** Let $(S, \varrho)$ be a left regular pair. Define a congruence $\varrho^*$ on $S^1$ by $\varrho^* = (1, 1) \cup \varrho$. Then, $e \varrho^* = eq$ for all $e \in E(S)$ and $1 \varrho^* = 1$. Hence, $(S^1, \varrho^*)$ is a left regular pair, $1 \varrho^*$ is the identity of $S^1/\varrho^*$, $u_{1 \varrho^*} = 1$ (notation of Lemma 1.1), $\langle \ker \varrho^* \rangle \equiv \langle \ker \varrho \rangle^1$, and $S^1/\varrho^* \equiv (S/\varrho)^1$. Let $\phi$ be the homomorphism of $S^1$ into $(\ker \varrho^*) \circ S^1/\varrho^*$ given by Lemma 1.5. Suppose $(g_{rr}, u_r)\phi = (h_{ss}, u_s)\phi$. Hence, $(v, r) = (\delta, s)$ where $v$ and $\delta$ are given in the statement and proof of Lemma 1.5. Let $1 \varrho^* = e$. Thus, using Lemmas 1.2 and 1.3,

\[ g_{rr}u_r = u_e g_{rr}u_r = g_{rr}a_e u_{rr}u_r = g_{rr}a_\delta f_{rr}, u_r \]

Thus, $S^1 \leq \langle \ker \varrho^* \rangle \circ S^1/\varrho^*$. Thus, using Lemma 1.6, $S \leq S^1 \leq \langle \ker \varrho \rangle^1 \circ (S/\varrho)^1$. If $S$ has an identity $f$, let $e = f_0$ and $u_e = f$ in the string of equalities above.

A left regular pair $(S, \varrho)$ is termed a left orthodox pair if $S/\varrho$ is an orthodox semigroup.

**Theorem 1.8.** Let $(S, \varrho)$ be a left orthodox pair. Then

\[ S \leq \langle \ker \varrho \rangle^1 \circ (S/\varrho)^1. \]

If $S$ has an identity, the superscript "1" in $(\ast)$ may be deleted.

**Proof.** Use Theorem 1.7 and Lallement's lemma [3, Lemma 4.6, p. 52].

We will use Theorem 1.3 to obtain structure theorems for inverse semigroups (Theorem 1.9), $\mathcal{H}$-unipotent semigroups (Theorem 1.12), $\mathcal{H}$-compatible inverse semigroups (Theorem 1.16), and natural $\mathcal{H}$-unipotent semigroups (Theorems 1.17 and 1.19). In each case, we construct a left orthodox pair. Theorem 1.8 then gives the structure theorem.

A semigroup $S$ is termed $\mathcal{H}$-unipotent if each $\mathcal{H}$-class of $S$ contains precisely one idempotent. A semigroup $S$ is $\mathcal{H}$-unipotent if and only if $S$ is regular and $E(S)$ is a semilattice of left zero semigroups [16, Proposition 5]. Using [21, Lemma 2.13], the natural inverse semigroups are precisely the $\mathcal{H}$-compatible inverse semigroups (i.e. the inverse semigroups on which $\mathcal{H}$ is a congruence). A regular semigroup $S$ is natural $\mathcal{H}$-unipotent if and only if $T(S)$ is a semilattice $T(S)/\mathcal{H}$ of left groups.

**Theorem 1.9.** Let $S$ be an inverse semigroup and $\varrho$ be any idempotent separating
congruence on $S$. Then, $(S, \varrho)$ is a left orthodox pair, $\ker \varrho$ is a semilattice $E(S)$ of groups, and $E(S/\varrho) \cong E(S)$.

**Proof.** Note that $eg$ is a group for all $e \in E(S)$.

**Remark 1.10.** Let $S$ be an inverse (bisimple inverse) semigroup and let $\varrho$ denote the maximum idempotent separating congruence on $S$. Then, a characterization of $S/\varrho$ has been given [3, Theorem 4.10, p. 145] ([3, Theorem 6.4 p. 151]).

**Remark 1.11.** Let $S$ be a bisimple inverse monoid with right unit subsemigroup $P$ and group of units $U$. A subgroup $V$ of $P$ is called a right normal divisor of $P$ if and only if $aV \subseteq Vb$ for all $a \in P$. By [9, Theorem 2], there exists a one-to-one correspondence $V \rightarrow V'$ (for $a, b \in P$, $g'_{ab} = \{a^{-1}vb : v \in V\}$) between the right normal divisors of $P$ and the idempotent separating congruences on $S$. If $M = \{g \in U \mid xg \in Ux$ for all $x \in P\}$, $g^M$ is the maximum idempotent separating congruence on $S$.

**Theorem 1.12.** Let $S$ be an $\mathcal{H}$-unipotent semigroup and $\varrho$ be the minimum inverse semigroup congruence on $S$. Then $(S, \varrho)$ is a left orthodox pair, $\ker \varrho = E(S)$, and $E(S/\varrho) \cong E(S)/\mathcal{F}$.

**Proof.** Utilize [16, remark after Proposition 5] or [24, Remark 4.1] and Theorem 1.8.

**Remark 1.13.** For any orthodox (in particular, $\mathcal{H}$-unipotent) semigroup $S$, $v = \{(x, y) \in S^2 : V(x) = V(y)\}$ is a description of the minimum inverse semigroup congruence on $S$ [3, Theorem 1.12, p. 190 (due to T.E. Hall)].

**Remark 1.14.** If an $\mathcal{H}$-unipotent semigroup $S$ is a union of groups (hence, a semilattice $E(S)/\mathcal{F}$ of left groups), $S/\varrho$ is a semilattice $E(S)/\mathcal{F}$ of groups.

The following theorem when combined with Theorem 1.8 will yield Theorems 1.16 and 1.17 (below).

**Theorem 1.15** [22, Propositions 1.24, 1.19, 1.21, and 1.25]. Let $S$ be a natural regular semigroup. Then $t = \{(a, b) \in S \times S : aa'bb' \in T(S) \text{ and } a'a'b'b \in T(S)\}$ for some $a' \in V(a)$ and $b' \in V(b)$ is the minimum combinatorial inverse semigroup congruence on $S$. The $t$-classes of $S$ containing idempotents are precisely the $\mathcal{F}$-classes of $T(S)$. Hence $E(S/t) \cong T(S)/\mathcal{F}$. $t = \mathcal{H}$ if and only if $S$ is an inverse semigroup.

**Theorem 1.16.** Let $S$ be an $\mathcal{H}$-compatible inverse semigroup and $\varrho = \mathcal{H}$. Then $(S, \varrho)$ is a left orthodox pair, $\ker \varrho = T(S)$, and $E(S/\varrho) \cong E(S)$. 
Remark 1.16. Using Lemmas 1.1 and 1.5, the superscript "1" in the statement of Theorem 1.8 may be removed in the case of Theorem 1.16.

Theorem 1.17. Let $S$ be a natural $\mathcal{R}$-unipotent semigroup and let $q = t$ (notation of Theorem 1.15). Then, $(S, e)$ is a left orthodox pair, $\ker q = T(S)$, and $E(S, q) \equiv T(S)/\mathcal{R}$.

To establish Theorem 1.19, we will need the following lemma.

Lemma 1.18. Let $S, T, X$, and $Y$ be semigroups and let $R$ be a transformation semigroup. Then, $S \leq T \circ R$ and $T \leq X \circ Y$ imply $S \leq X \circ Y \circ R$.

Proof. Let $R = (Q, W)$. Since $T \leq X \circ Y$, there exists an isomorphism $\lambda$ of $T$ into $X \circ Y = X^* \times Y$. We define a mapping $\theta$ of $T \circ R = TQ \times W$ into $(X \circ Y) \circ R = (X^* \times Y)^2 \times W$ as follows: for $(f, w) \in TQ \times W$, $(f, w) \theta = (f', w)$ where $qf' = (qf)\lambda$ for $q \in Q$. It is easily checked that $\theta$ is an isomorphism of $T \circ R$ into $X \circ Y \circ R$. Thus, $S \leq X \circ Y \circ R$.

Theorem 1.19. Let $S$ be a natural $\mathcal{R}$-unipotent semigroup. Then

$$S \leq (E(S)^1)^1 \circ ((T(S)/v)^1)^1 \circ (S/t)^1$$

where $E(S)$ is a semilattice of left zero semigroups, $v$ is the minimum inverse semigroup congruence on $T(S)$ (hence, $T(S)/v$ is a semilattice of groups), and $t$ is the minimum combinatorial inverse semigroup congruence on $S$.

Proof. Utilize Theorem 1.17, Theorem 1.12, Remarks 1.13 and 1.14 and Lemma 1.18.

We note that a semigroup $S$ is $\mathcal{R}$-unipotent if and only if $S$ is regular and $E(S)$ is a semilattice of right zero semigroups [16, Proposition 5].

Proposition 1.20. Let $(S, q)$ be a left $\mathcal{R}$-unipotent pair. Then $\ker q$ is a union of groups on which $\mathcal{R}$ is a congruence relation and $E(S/q) \equiv \ker q / \mathcal{R}$.

Proof. Since $S/q$ is an $\mathcal{R}$-unipotent semigroup, $E(S/q)$ is a semilattice $Y$ of right zero semigroups ($X_y : y \in Y$). Let $T_y = \bigcup (eq^{-1} : e \in X_y)$. Then, $\ker q$ is the semilattice $Y$ of completely simple semigroups ($T_y : y \in Y$). Let $L_e$ denote the $\mathcal{R}$-class of $e \in E(\ker q)$. Thus, $L_e = eq$ for each $e \in E(\ker q)$. Hence, $\mathcal{R}$ is a congruence relation on $\ker q$. Furthermore, $L_e \to eq$ ($e \in E(\ker q)$) defines an isomorphism of $\ker q / \mathcal{R}$ onto $E(S/q)$.

We "decompose" unions of groups on which $\mathcal{R}$ is a congruence relation (Theorem 1.24) and, hence ker q of Proposition 1.20. This will yield a "finer decom-
position” of left $\mathcal{L}$-unipotent pairs (Theorem 1.76) than was given for left orthodox pairs in Theorem 1.8. We will need the concept of “wreath product associated with a partial chain of semigroups”. Let $W$ be a lower partial chain $Y$ of the semigroups $(T_y : y \in Y)$.

Let $X$ be a semilattice $Y$ of semigroups $(X_y : y \in Y)$. Let $W \circ X = W^X \times X$. For brevity if $a \in T_y$ and $b \in T_z$ with $y \geq z$, we write $a \geq b$ immediately below. If $(v, s)(\delta, t) \in W \circ X$ and $x v \geq x s \delta$ for all $x \in X^*$, let $(v, s)(\delta, t) = (v \circ g_s \delta, st)$. Then, $(v, s)(\delta, t)$ is well defined and $(v, s)(\delta, t) \in W \circ X$. If $(\beta, z) \in W \circ X$ and $x \beta \geq (x t) \beta$ for all $x \in X^*$, then $(v, s)(\delta, t)(\beta, z) = (v, s)((\delta, t)(\beta, z))$. If $\phi$ is a mapping of a semigroup $S$ into $W \circ X$ such that $a \phi b \phi$ is defined for all $a, b \in S$ and $a \phi b \phi = (a b) \phi$, $\phi$ is termed a homomorphism of $S$ into $W \circ X$. If $\phi$ is also one-to-one, then $\phi$ is termed an isomorphism and we write $S \leq W \circ X$.

Theorem 1.24 will be a consequence of the next three lemmas.

Lemma 1.21 [18]. Let $S$ be a union of groups on which $\mathcal{L}$ is a congruence relation. Let $X = S / \mathcal{L}$. Hence, $X$ is a semilattice $Y$ of right zero semigroups $(X_y : y \in Y)(Y = S / \mathcal{L})$. For each $y \in Y$, select $e_y \in X_y$ and let $T_y = e_y \mathcal{L}^{-1}$. Then, $W$ is a lower partial chain $Y$ of the left groups $(T_y : y \in Y)$. If $s \in X_y$, let $s' = y$. For each $s \in X$, select $u_s \in s \mathcal{L}^{-1}$. Then every element of $S$ may be uniquely expressed in the form $x = g_s u_s$ where $g_s \in T_y$. There exists a mapping $r \rightarrow \alpha_r$ of $X$ into $F(W)$, the full transformation semigroup on $W$, where $\alpha_r$ is defined by $u_s g_s = (g_s, \alpha_r) u_s$ where $g_s \in T_y$. Let $Q \in W_{(e_y)}$. Furthermore, $(g_s, u_s)(h_t, u_t) = g_s (h_t, u_t) f_{s, t, u_s, u_t}$ where $g_s \in T_y$ and $h_t \in T_t$. We have

(a) $T_{Q} \alpha_r \subseteq T_{(e_y)}$.
(b) $(g_s, h_{t'}) \alpha_r = (g_s, \alpha_r)(h_s, \alpha_{t'})$ if $\beta \leq \alpha$ where $g_s \in T_y$ and $h_{t'} \in T_{t'}$.
(c) $f_{s, t, u, z} f_{s, t, u, r} = f_{s, t, u, r} \alpha_r f_{s, t, u, r}$ for all $s, t, z \in X$.
(d) $f_{s, t, u} \alpha_r = f_{s, t, u} \alpha_r f_{s, t, u, r}$ for all $s, t, z \in X$ and $g_z \in T_z$.

Lemma 1.22. We use the notation of Lemma 1.21. Let $(h_t, u_t) \phi = (\theta, t)$ where $h_t \in T_t$. If $X$ has an identity, $x h_t, h_t, u_t \phi = f_{s, t, u} \alpha_r f_{s, t, u, r}$ for $x \in X$. If $X$ has no identity, $x \theta = h_t, \alpha_r f_{s, t, u} \alpha_r f_{s, t, u} \alpha_r f_{s, t, u, r}$ for $x \in X$ and $1 \theta = h_t, \alpha_r f_{s, t, u} \alpha_r f_{s, t, u}$.

Proof. We will assume $X$ has no identity (the other case is simpler). Let $g_S, u_s, h_t, u_t \in S$ with $g_s \in T_y$ and $h_t \in T_t$. Then, using Lemma 1.21,

$$(g_s, u_s)(h_t, u_t) \phi = (g_s, h_t, u_t) f_{s, t, u_s, u_t} \phi = (\theta, t)$$

where

$1 \theta = g_s, h_t, u_t \phi = (v, s)(\delta, t) - (v \circ g_s \delta, st) \quad \text{for} \ x \in X$.

However,

$$(g_s, u_s)(h_t, u_t) \phi = (v, s)(\delta, t) - (v \circ g_s \delta, st)$$

where

$1 \delta = h_t, \alpha_r f_{s, t, u} \alpha_r f_{s, t, u}$.
Embedding of regular semigroups in wreath products

Hence
\[ (v \circ g_s \delta) = v(s) = g_s(h_t, \alpha_s, f_{se_t, t}, \cdot) = 1 \theta \]
and
\[ x(v \circ g_s \delta) = x(v(s)) = (g_s(h_t, \alpha_s, f_{se_t, t}))(h_t, \alpha_s, f_{se_t, t}). \]

We proceed to show \( x(\theta) = x(v \circ g_s \delta) \). Using Lemma 1.21(b) twice, Lemma 1.21(c) and Lemma 1.21(d),

If \( W \) is a lower partial chain of semigroups, \( W^1 = W \cup \{ 1 \in W \} \) under the multiplication "\( \circ \)" where \( x \circ y = xy \) (juxtaposition denotes the given partial product in \( W \)) if \( x, y \in W, 1 \circ w = w \circ 1 = w \) for all \( w \in W \) and \( 1 \circ 1 = 1 \).

Lemma 1.23. In the notation of Lemma 1.21, \( S \leq S^1 \leq W^1 \circ X^1 \).

Proof. We show the \( \phi \) of Lemma 1.22 is an isomorphism of \( S^1 \) into \( W^1 \circ X^1 \). We regard \( S^1 / \mathcal{Y} = X^1 \) as a semigroup with identity in definition of \( \phi \). In the notation of Lemma 1.21, \( u_1 = 1 \). Using the notation of Lemma 1.22, suppose \( (g_s, u_3) \mathcal{Y} = (\theta, s) = (\delta, t) = (h_t, u_t) \mathcal{Y} \) where \( g_s, u_3 \in T_s, h_t, u_t \in T_t, \) and \( \theta, \delta \in (W^1 \circ X^1) \mathcal{Y} \). Hence, \( s = t \) and \( 1 \theta = 1 \delta \). Thus, using Lemmas 1.21 and 1.22, \( g_s, u_3 = u_1 g_s = u_3 = g_s, u_1 u_e, u_3 = g_s, u_1, u_3 = (1 \delta) u_3 = (1 \delta) u_3 = h_3, u_1, u_3 = h_3, u_3 \). Apply Lemma 1.21.

Theorem 1.24. Let \( S \) be a union of groups on which \( \mathcal{Y} \) is a congruence relation. Then,
\[ S \leq W^1 \circ (S / \mathcal{Y})^1 \]
where \( W \) is a lower partial chain \( S / \mathcal{Y} \) of \( \mathcal{Y} \)-classes of \( S \) (each \( \mathcal{Y} \)-class of \( S \) contains precisely one of these \( \mathcal{Y} \)-classes).

Remark 1.24'. Each \( \mathcal{Y} \)-class of \( S \) is a left group.

To state Theorem 1.26, we will need the following definition. Let \( W \) be a lower partial chain \( Y \) of the semigroups \( (W_y : y \in Y) \), \( X \) be a semilattice \( Y \) of semigroups \( (X_y : y \in Y) \), and \( R \) be a semigroup. Let \( W \circ X \circ R = (W \times X \times X)^R \times R \). Let \( \phi \) be a one-to-one mapping of a semigroup \( S \) into \( W \circ X \circ R \). If, for all \( a, b \in S, a \phi b \phi \)'s defined in \( W \circ X \circ R \) and \( a \phi b \phi = (ab) \phi \), \( \phi \) is termed an isomorphism of \( S \) into \( W \circ X \circ R \) and we then write \( S \leq W \circ X \circ R \).

The following lemma combined with Theorem 1.24, Lemma 1.23, Remark 1.24', Proposition 1.20, Theorem 1.8, and the duals of Theorems 1.8, 1.12, 1.17, and 1.6 will yield Theorem 1.26 below.
Lemma 1.25. Let $S, T,$ and $R$ be semigroups. Let $W$ be a lower partial chain $Y$ of the semigroups $(W_y : y \in Y)$ and let $X$ be a semilattice $Y$ of the semigroups $(X_y : y \in Y)$. Suppose $S \leq R$ and $T \leq W \circ X$. Then, $S \leq W \circ X \circ R$.

Proof. Since $T \leq W \circ X$, there exists an isomorphism $\lambda$ of $T$ into $W \times X$. We define a mapping $\phi$ of $T \circ R = T^R \times R$ into $W \circ X \circ R = (W^X \times X)^R \times R$ as follows: $(f, r) \phi = (f', r)$ where $r' = (rf)^\lambda$ for $x \in R^\lambda$. It is straightforward to verify $\phi$ is an isomorphism.

Theorem 1.26. Let $(S, \varrho)$ be a left $\gamma$-unipotent pair. Then,

$$S \leq (W^1) \circ ((\ker \varrho \gamma)^1) \circ (S/\varrho)^1$$

where $W$ is a lower partial chain $\ker \varrho \gamma$ of $\gamma$-classes (each $\gamma$-class being a left group) of the union of groups $\ker \varrho$ (each $\gamma$-class of $\ker \varrho$ contains precisely one of these $\gamma$-classes), $\ker \varrho \gamma$ is a semilattice $\ker \varrho / \gamma$ of right zero semigroups, and

$$S/\varrho \leq (S/\varrho / \delta)^1 (\ker \varrho \gamma)^1$$

where $\delta$ is the smallest inverse semigroup congruence on $S/\varrho$. If $S/\varrho$ is combinatorial, $\delta$ is the smallest combinatorial inverse semigroup congruence on $S/\varrho$. In either case, $E(S/\varrho / \delta) = \ker \delta / \gamma$.

We will next apply Theorem 1.26 to obtain structure theorems for generalized $\gamma$-unipotent semigroups (Theorem 1.28) and left natural regular semigroups (Theorem 1.31). In each case we construct a left unipotent pair. Theorem 1.26 then gives the structure theorem.

The following theorem when combined with Theorem 1.26 will yield Theorem 1.28.

Theorem 1.27 ([17, Theorem 2 and its proof and Lemma 3] and [22, Remark 3.4]). Let $S$ be a generalized $\gamma$-unipotent semigroup. Then, $\varrho = \nu \cap \gamma$, where $\nu$ is the smallest inverse semigroup congruence on $S$, is the smallest $\gamma$-unipotent congruence on $S$, $\ker \varrho = E(S)$, and $e \varrho$ is the $\gamma$-class of $E(S)$ containing $e$ for all $e \in E(S)$.

Theorem 1.28. Let $S$ be a generalized $\gamma$-unipotent semigroup and let $\varrho = \nu \cap \gamma$ (notation of Theorem 1.27). Then $(S, \varrho)$ is a left $\gamma$-unipotent pair and $\ker \varrho = E(S)$.

The following two lemmas will yield Theorem 1.31 (below).

Lemma 1.29 [22, §3, p. 698]. Let $S$ be a left natural regular semigroup. Let $\lambda = \{(x, y) \in S \times S : (sx, sy) \in t \cap \gamma$ for all $s \in S\}$ ($t$ is given in Theorem 1.15). Then, $\lambda$ is a combinatorial $\gamma$-unipotent congruence on $S$. 

Lemma 1.30. Let $S$ and $\lambda$ be as in Proposition 1.29. Then, for $e \in E(S)$, $e\lambda = l_e$, the $\gamma$-class of $T(S)$ containing $e$, and $\ker \lambda = T(S)$.

Proof. We show $l_e = e\lambda$ for $e \in E(S)$. Let $x \in l_e$. Thus, since $\mathcal{L}$ is congruence on $\mathcal{L}(S)$, $(s^{-1}sx, s^{-1}se) \in \mathcal{L}(T(S))$ for $s \in S$ and $s^{-1}e \in V(s)$. Hence, there exist $y, u \in T(S)$ such that $ys^{-1}sx = s^{-1}se$ and $us^{-1}se = s^{-1}sx$. Thus, $(sys^{-1})sx = se$ and $(usu^{-1})se = sx$. Thus, $(sx, se) \in \mathcal{L}$ for all $s \in S'$. Hence, since $(sx, se) \in \mathcal{L}$ for all $s \in S'$ by virtue of Theorem 1.15, $x \in e\lambda$. Let $x \in e\lambda$. Hence, $(x, e) \in \mathcal{L}$ and $(x, e) \in \mathcal{L}$ by virtue of Theorem 1.15, $x \in T(S)$. Let $x^{-1} \in V(x) \cap T(S)$. Hence, $(x^{-1}x, e) \in \mathcal{L}$, $(ex^{-1})x = e$ and $xe = x$. Thus, $x \in l_e$.

Theorem 1.31. Let $S$ be a left natural regular semigroup. Then $(S, \lambda)$ (notation of Lemma 1.29) is a left $\mathcal{L}$-unipotent pair and $\ker \lambda = T(S)$.

2. Some generalizations of the bicyclic semigroup

In this section, we describe the structure of various classes of regular semigroups as subsemigroups of wreath products containing the bicyclic semigroup $C$ or extended bicyclic semigroup $C^*$ as a factor. We give structure theorems for the following classes of semigroups: $\omega^n$-bisimple semigroups (Theorem 2.9), $I$-bisimple semigroups (Theorem 2.11), $\omega^I$-bisimple semigroups (Theorem 2.13), simple $I$-regular semigroups (Theorem 2.14), $\omega I$-inverse semigroups (Theorem 2.15), simple $\omega I$-inverse semigroups (Theorem 2.16), simple $\omega I$-unipotent semigroups (Remark 2.18), and generalized $\omega I$-unipotent bisimple semigroups (Theorem 2.19).

We first give some results on transformation semigroups (Lemma 2.1, Remark 2.3, and Lemmas 2.4-2.6) which will be utilized in this section. Lemma 2.1 will be used in the proofs of Theorems 2.9, 2.13, and 2.17.

Lemma 2.1. Let $(Q, S)$ and $(P, T)$ be transformation semigroups. Suppose there exists a bijection $\varphi$ of $P$ onto $Q$ such that for each $s \in S$ there exists $t \in T$ such that $\varphi s = t\varphi$. Define $s\delta = t$. Then, $\delta$ defines an isomorphism of $S$ into $T$.

Definition 2.2. If $(Q, S)$ and $(P, T)$ are transformation semigroups satisfying the conditions of Lemma 2.1, we write $(Q, S) \preccurlyeq (P, T)$.

Remark 2.3. If $(Q, S)$, $(P, T)$, and $(R, W)$ are transformation semigroups and $(Q, S) \preccurlyeq (P, T)$ and $(P, T) \preccurlyeq (R, W)$, then $(Q, S) \preccurlyeq (R, W)$.

Lemma 2.4 will be used in the proof of Lemma 2.8 and Lemma 2.5 will be used in the proof of Lemma 2.8, Theorem 2.9, and Theorem 2.17.
Lemma 2.4. Let \((Q, S)\) and \((P, T)\) be transformation semigroups. Then, \((Q, S) \leq (P, T)\) implies \((Q, S) \leq (P, T)\).

Proof. There exists a bijection \(\varphi\) of \(P\) onto \(Q\) such that \(s \in S\) implies there exists \(t \in T\) such that \(\varphi s = t\varphi\). If \(\tilde{q} \in \tilde{Q}\), then \(\varphi \tilde{q} = \tilde{t}\varphi\) where \(\tilde{p} \varphi = q\).

Lemma 2.5. Let \((P, T)\), \((W, R)\), and \((Q, S)\) be transformation semigroups. Then, \((P, T) \leq (W, R)\) implies \((Q, S) \circ' (P, T) \leq (Q, S) \circ' (W, R)\) and \((P, T) \circ' (Q, S) \leq (W, R) \circ' (Q, S)\).

Proof. Since \((P, T) \leq (W, R)\), there exists a 1-1 map \(\varphi\) of \(W\) onto \(P\) such that for \(t \in T\) there exists \(r \in R\) such that \(\varphi t = r \varphi\). Note that \((Q, S) \circ' (P, T) = (Q \times P, S \times T)\) and \((Q, S) \circ' (W, R) = (Q \times W, S \times R)\). Define \((q, w) = (q, w) \delta\) for \((q, w) \in Q \times W\). Then, \(\delta\) is a 1-1 map of \(Q \times W \) onto \(P \times P\). Let \((f, t) \in S \times T\) and define \(wq = (w \varphi) f\) for \(w \in W\). Then \(g \in S \times W\) and \(\delta(f, t) = (g, r) \delta\). Thus, \((Q, S) \circ' (P, T) = (Q, S) \circ' (W, R)\). Note that \((P, T) \circ' (Q, S) = (P \times Q, T \times S)\) and \((W, R) \circ' (Q, S) = (W \times Q, R \times S)\). Define \((w, q) = (w \varphi, q)\) for \((w, q) \in W \times Q\). Then, \(\lambda\) is a 1-1 map of \(W \times Q \) onto \(P \times P\). Let \((f, s) \in T \times S\). Define \((q, g) = \delta(f, s)\). Then, \((P, T) \circ' (Q, S) = (W, R) \circ' (Q, S)\).

Lemma 2.6 will be utilized in the proof of Lemma 2.8.

Lemma 2.6. \((Q, S) \circ' (P, T) \leq (Q, S) \circ' (P, T)\).

Proof. Note that \((Q, S) \circ' (P, T) = (Q \times P, S \times T)\) and \((Q, S) \circ' (P, T) = (Q \times P, (S \cup P) \times T)\). In the notation of Lemma 2.1, let \(\varphi\) denote the identity transformation of \(Q \times P\). If \(s = (f, a) \in S \times P\), let \(t = (f, a)\). If \(s = (\tilde{q}, \tilde{p}) \in \tilde{Q} \times \tilde{P}\), let \(t = (q, p)\) where \(xg = q\) for all \(x \in P\). Then, \(t \in \tilde{Q} \times \tilde{P}\). In both cases, \(s \varphi = t\varphi\).

Lemma 2.7 is utilized in the proof of Lemma 2.8 and Theorem 2.17. Lemma 2.8 is utilized in the proof of Theorem 2.9 and its proof is utilized in the proof of Theorem 2.13.

We will need the following notation and definitions. Let \(X\) be an arbitrary semigroup. Define \(X \odot C(X \odot C^*)\) to be \(X \times C(X \times C^*)\) under the multiplication \((a, (m, n)) = ab\) or \(ab\) according to whether \(n > p, p > n, r = p\). Define \(C_1 = C, C_2 = C_1 \odot C, \ldots, C_n = C_{n-1} \odot C\). Define \(C^* = C^*\) and \(C_n = C_{n-1} \odot C^*\) for \(n \geq 2\).

Lemma 2.7. Let \(X\) be a semigroup with identity. Then, \(X \odot C \leq X \circ' C\).

Proof. Note that \(X \circ' C = (X, X \cup \tilde{X}) \circ' (C, C) = (X \times C, (X \cup \tilde{X}) \circ C)\). In the nota-
Embedding of regular semigroups in wreath products

Let \( s = (h, (p, q)) \in X \times C \), let \( t = (f, (p, q)) \) where \((m, n) = (1, \alpha) \), or \( h \) according to whether \( n > p \), \( p > n \), or \( n = p \). Then, \( t \in \langle X \cup \hat{X} \rangle \times C \) and \( \phi s = t\phi \).

Lemma 2.8. \( C_n \triangleleft \overline{C}^{n-1} \circ C \) for \( n \geq 2 \).

**Proof.** Lemma 2.8 is valid for \( n = 2 \) by Lemma 2.7. Assume

\[
C_k \triangleleft \overline{C}^{k-1} \circ C \quad \text{for } k \geq 2.
\]

\[
C_{k+1} = C_k \circ C \triangleleft \overline{C}_k \circ C \quad \text{(Lemma 2.7),}
\]

\[
\overline{C}_k \triangleleft \overline{C}^{k-1} \circ C \triangleleft C_k \quad \text{(Lemma 2.4),}
\]

\[
\overline{C}^{k-1} \circ C \triangleleft C \quad \text{(Lemma 2.6 and Lemma 2.5),}
\]

\[
\overline{C}_k \triangleleft \overline{C}_k \circ C \quad \text{(Remark 2.3),}
\]

\[
C_{k+1} \triangleleft \overline{C}_k \circ C \quad \text{(Remark 2.3).}
\]

We next establish structure theorems for \( \omega^n \)-bisimple semigroups (Theorem 2.9) and \( I \)-bisimple semigroups (Theorem 2.10).

**Theorem 2.9.** Let \( S \) be an \( \omega^n \)-bisimple semigroup. Then,

\[
S \leq H \circ C \quad \text{for } n = 1.
\]

\[
S \leq H \circ \overline{C}^{n-1} \circ C \quad \text{for } n \geq 2
\]

where \( H \) is an \( \omega^n \)-chain of isomorphic groups (the \( \omega^n \)-chain of the maximal subgroups of \( S \)) and \( C \) is the bicyclic semigroup.

**Proof.** Let \( S \) be an \( \omega^n \)-bisimple semigroup. Then, by [8, Theorem 2.3], \( \mathcal{X} \) is a congruence relation on \( S \) and \( S/\mathcal{X} \equiv C_n \). Thus, using Theorem 1.16, Remark 1.16', Theorem 1.8 and Lemma 1.6, \( S \leq H \circ C_n \). Since \( C_n \triangleleft \overline{C}^{n-1} \circ C \) (\( n \geq 2 \)) by Lemma 2.8, \( H \circ C_n \triangleleft H \circ C^{n-1} \circ C \) by Lemma 2.5. Thus, using Lemma 2.1 and Definition 2.2, \( S \leq H \circ \overline{C}^{n-1} \circ C \) (\( n \geq 2 \)). The maximal subgroups of \( S \) are isomorphic since \( S \) is bisimple [1, Theorem 2.20].

**Theorem 2.10.** Let \( S \) be an \( I \)-bisimple semigroup. Then,

\[
S \leq H \circ C^*
\]

where \( H \) is an \( I \)-chain of isomorphic groups (the \( I \)-chain of the maximal subgroups \( S \)) and \( C^* \) is the extended bicyclic semigroup. \( S \) is \( \mathcal{X} \) compatible and \( S/\mathcal{X} \equiv C^* \).

**Proof.** Let \( S \) be an \( I \)-bisimple semigroup. By [11, Corollary 1.3], \( \mathcal{X} \) is a congru-
Lemma 2.11 is needed for the proof of Lemma 2.12.

**Lemma 2.11.** Let \((Q, S)\) and \((P, T)\) be transformation semigroups. Suppose there exists a partial surjection \(\varphi\) of \(P\) onto \(Q\) such that for each \(s \in S\) there exists \(t \in T\) such that \(\varphi s \subseteq t\varphi\) (if \(\varphi t\) is defined, \(\varphi s = (pt)\varphi\)). Then, \(T' = \{t \in T : \varphi s \subseteq t\varphi\text{ for some } s \in S\}\) is a subsemigroup of \(T\). For \(t \in T'\), define \(t\delta = s\) where \(\varphi s \subseteq t\varphi\). Then, \(\delta\) is a homomorphism of \(T'\) onto \(S\).

Lemma 2.12 is used in the proof of Theorems 2.13 and 2.14.

**Lemma 2.12.** Let \(X\) be a semigroup with identity. Then, there exists an isomorphism \(\lambda\) of \(X \circ C^*\) into \((X \cup \tilde{X})^\ell \times C^*\) such that \((a, (m, n))(b, (p, q))\lambda = (a, (m, n))(b, (p, q))\) (product in \(X \circ C^*\)) and \((x, 1)((b, (p, q))\lambda) = (b, (p, q))\) for all \(x \in X\).

**Proof.** We will use the notation of Lemma 2.11. Define a partial surjection \(\varphi\) of \(X \times (C^*)^1\) onto \((X \circ C^*)^1\) as follows: \((a, (m, n))\varphi = (a, (m, n))\) for \((a, (m, n)) \in X \times C^*\) and \((1, 1)\varphi = 1\). Let \(s = (b, (p, q)) \in X \circ C^*\). Define \(\theta \in (X \cup \tilde{X})^\ell (C^*)^1\) as follows: \((m, n)\theta = 1, b\) or \(\tilde{b}\) according to whether \(n > p\), \(n = p\), or \(p > n\) and \(1\theta = \tilde{b}\). Let \(t = (\theta, (p, q))\). Then, \(\varphi s \subseteq t\varphi\). Define \((\theta, (p, q))\delta = (b, (p, q))\). Then, by Lemma 2.11, \(\delta\) defines a homomorphism \(\delta\) of \(T'\) onto \(X \circ C^*\). Since it is easily seen that \(\delta\) is 1-1, \(\delta\) is an isomorphism of \(T'\) onto \(X \circ C^*\). Let \(\lambda = \delta^{-1}\). Then, \(\lambda\) is an isomorphism of \(X \circ C^*\) into \(\tilde{X} \circ C^*\). The last two equalities given in the lemma are easily checked.

We next establish structure theorems for \(\omega^nI\)-bisimple semigroups (Theorem 2.13) and simple \(I\)-regular semigroups (Theorem 2.14).

**Theorem 2.13.** Let \(S\) be an \(\omega^nI\)-bisimple semigroup. Then

\[ S \leq H \circ \tilde{C}_n \circ C^* \]

where \(H\) is an \(\omega^nI\)-chain of isomorphic groups (the \(\omega^nI\)-chain of the maximal subgroups of \(S\)), \(C\) is the bicyclic semigroup, and \(C^*\) is the extended bicyclic semigroup.

**Proof.** Let \(S\) be an \(\omega^nI\)-bisimple semigroup. Using [10, Theorem 4.6] (see also [13, Theorem 1.4]), \(\mathcal{Y}\) is a congruence relation on \(S\) and \(S/\mathcal{Y} \cong C_{n+1}\). Thus, using Theorem 1.16, Remark 1.16', Theorem 1.8, and Lemma 1.6, \(S \leq H \circ C_{n+1}\). Note that

\[ H \circ C_{n+1} = H^{(C_{n+1} \times (C^*)^1)} \times (C \circ C^*) \]

and

\[ H \circ \tilde{C}_n \circ C^* = H^{C_{n+1} \times (C^*)^1} \times (C \circ C^*). \]
For \((f, r) \in H^{C^{[n]} \times C^*})^{U1} \times (C_n \odot C^*),\) define \((f, r)\phi = (f', r\lambda)\) where \(\lambda\) is the isomorphism of \(C_n \odot C^*\) into \(\hat{C}_n \odot C^*\) given by Lemma 2.12 and where
\[
\begin{align*}
(x, (m, n))f' &= (x, (m, n))f & \text{for } (x, (m, n)) \in C^{[n]} \times C^*, \\
(x, 1)f' &= 1f & \text{for } (x, 1) \in C^{[r]} \times \{1\}.
\end{align*}
\]

Using Lemma 2.12, it is straightforward to verify that \(\phi\) is an isomorphism of \(H \odot C_{n+1}^*\) into \(H \odot \hat{C}_n \odot C^*.\) Thus, \(S \leq H \odot \hat{C}_n \odot C^*.\) By the proof of Lemma 2.8, \(\hat{C}_n \leq \hat{C}_n.\) Thus, using Lemma 2.5, \(H \odot \hat{C}_n \leq H \odot \hat{C}_n \odot C^*.\) Hence, using Lemma 2.5 and Lemma 2.1, \(H \odot \hat{C}_n \odot C^* \leq H \odot \hat{C}_n \odot C^*.\) Thus, \(S \leq H \odot \hat{C}_n \odot C^*.\)

**Theorem 2.14.** Let \(S\) be a simple \(I\)-regular semigroup. Then,
\[
S \leq H \odot \hat{C}_n \odot C^*
\]

where \(H\) is the \(I\)-chain of the maximal subgroups of \(S\) and \(k = \{0, 1, 2, \ldots, k - 1\}\) \((k\ \text{a positive integer})\) with \(ij = \max\{i, j\}\) for \(i, j \in k\). \(S\) has \(k\) \(I\)-classes.

**Proof.** Let \(S\) be a simple \(I\)-regular semigroup. Using [14, Theorem 1.1] and [15, Remark 1.2], \(\mathcal{R}\) is a congruence on \(S\) and \(S/\mathcal{R} \cong k \odot C^*\) where \(k\) is the number of \(I\)-classes of \(S\). Thus, using Theorem 1.16, Remark 1.16', Theorem 1.8 and Lemma 1.6, \(S \leq H \odot (k \odot C^*).\) Note that
\[
H \odot (k \odot C^*) = H^{(k \odot C^*)'} \times (k \odot C^*) = H^{(k \times C^*)^{U1}} \times (k \odot C^*)
\]
and
\[
H \odot (k \odot C^*) = H^{k \times (C^*)'} \times (k \odot C^*).
\]

Let \(\lambda\) denote the isomorphism of \(k \odot C^*\) into \((k \odot C^*)\) given by Lemma 2.12. For \((f, r) \in H \odot (k \odot C^*),\) define \((f, r)\phi = (f', r\lambda)\) where
\[
\begin{align*}
(x, (m, n))f' &= (x, (m, n))f & \text{for } (x, (m, n)) \in k \times C^*, \\
(x, 1)f' &= 1f & \text{for } (x, 1) \in k \times \{1\}.
\end{align*}
\]

Using Lemma 2.12, it is straightforward to verify that \(\phi\) is an isomorphism of \(H \odot (k \odot C^*)\) into \(H \odot \hat{C}_n \odot C^*.\) Hence, \(S \leq H \odot \hat{C}_n \odot C^*.\)

Let \(\Lambda = N \times Y,\) where \(N\) is the natural numbers and \(Y\) is a semilattice with greatest element, under the lexicographic order. We term \(\Lambda\) an \(\omega Y\)-semilattice.

A regular semigroup \(S\) is termed \(\omega Y\)-\(\delta\)-unipotent if
(1) \(E(S)\) is an \(\omega Y\)-semilattice of left zero semigroups \(\{ E_{(m, \delta)} : (n, \delta) \in N \times Y \},\) and
(2) \(e_{(m, \delta)} \prec e_{(m, \lambda)}\) where \(e_{(m, \delta)} \in E_{(m, \delta)}\) and \(e_{(m, \lambda)} \in E_{(m, \lambda)}\) if and only if \(\delta = \lambda.\)

To prove our structure theorem for \(\omega Y\)-\(\delta\)-unipotent semigroups (Theorem 2.16), we will need the following lemma.

**Lemma 2.15.** Let \(X\) be a semigroup with identity. Then, there exists an isomorphism \(\lambda\) of \((X \odot C)^{1}\) into \(X \odot C^* = (N \cup \hat{X})^C \times C^*\) such that \((a, (m, n)))((b, (p, q)))\lambda =
\((a, (m, n)) ((b, (p, q))) \) (product in \(X \circ C\)), \((x, 1) ((b, (p, q)) \lambda) = (b, (p, q))\), \((a, m, n) (1 \lambda) = (a, (m, n))\) and \((x, 1) (1 \lambda) = (x, 1)\).

**Proof.** We will use the notation of Lemma 2.11. Define a partial surjection \(\varphi\) of \(X \times C \downarrow\) onto \((X \circ C) \downarrow\) as follows: \((a, (m, n)) \varphi = (a, (m, n))\) for \((a, (m, n)) \in X \times C\) and \((1, 1) \varphi = 1\). Let \(s = (b, (p, q)) \in X \circ C\). Define \(\theta \in (X \cup X) \circ C\) as in the proof of Lemma 2.12. Let \(t = (\theta, (p, q)).\) Then, \(\varphi s \subseteq t \varphi\). If \(s = 1\), let \(t = (v, 1)\) where \(zv = 1\) for all \(z \in C\). Then, \(\varphi s \subseteq t \varphi\). Define \((\theta, (p, q)) \delta = (b, (p, q))\) and \((v, 1) \delta = 1\). Let \(\lambda = \delta^{-1}\) and proceed as in the proof of Lemma 2.12.

**Theorem 2.16.** Let \(S\) be an \(\omega Y\)-\(\mathfrak{R}\)-unipotent semigroup. Then,

\[ S \leq \left( (E(S)) \downarrow \circ (G) \downarrow \right) \circ (Y \circ C) \downarrow \]

where \(G\) is an \(\omega Y\)-semilattice of groups and \(C\) is the bicyclic semigroup.

**Proof.** Let \(S\) be an \(\omega Y\)-\(\mathfrak{R}\)-unipotent semigroup. Using [19, Theorem 3.3, Note 3.4, Proposition 3.5, and Proposition 3.6], \(S\) is a natural \(\mathfrak{R}\)-unipotent semigroup, \(S/\mathfrak{T} \equiv Y \circ C\), and \(T(S)\) is an \(\omega Y\)-semilattice of left groups. Hence, using Theorem 1.17, Theorem 1.8, and Lemma 1.6, \(S \leq (T(S)) \downarrow \circ (Y \circ C) \downarrow\).

We define a mapping \(\phi\) of \((T(S)) \downarrow \circ (Y \circ C) \downarrow\) into \((T(S)) \downarrow \circ (Y \circ C) \downarrow\) as follows. For \((f, z) \in ((T(S)) \downarrow \circ (Y \circ C) \downarrow)\), let \((f, z) \phi = (f', z \lambda)\) where \(\lambda\) is the isomorphism of \((Y \circ C) \downarrow\) into \(Y \circ C \downarrow\) given by Lemma 2.15 and where \((y, c) f' = (y, c) f\) for \((y, c) \in Y \times C\) and \((y, 1) f' = 1 f\) for \((y, 1) \in Y \times \{1\}\). Using Lemma 2.15, it is straightforward to verify that \(\phi\) is an isomorphism. Hence, \(S \leq (T(S)) \downarrow \circ (Y \circ C) \downarrow\). Using Theorem 1.12, Remark 1.13, Remark 1.14, and Theorem 1.8, \((T(S)) \downarrow \leq ((E(S)) \downarrow \circ (G) \downarrow) \downarrow\) where \(G\) is an \(\omega Y\)-semilattice of groups \((G = T(S))/\nu\) where \(\nu\) is the smallest inverse semigroup congruence on \(T(S)\). Thus, using Lemma 1.6, \(S \leq ((E(S)) \downarrow \circ (G) \downarrow) \circ (Y \circ C) \downarrow\). \(\square\)

A regular semigroup \(S\) is termed an \(\omega Y\)-inverse semigroup if (1) \(E(S)\) is an \(\omega Y\)-semilattice and (2) \((m, \delta) \subseteq (n, \lambda)\) if and only if \(\delta = \lambda\).

**Theorem 2.17.** Let \(S\) be an \(\omega Y\)-inverse semigroup. Then

\[ S \leq H \circ (Y \circ C) \]

where \(H\) is the \(\omega Y\)-semilattice of maximal subgroups of \(S\) and \(C\) is the bicyclic semigroup.

**Proof.** Let \(S\) be an \(\omega Y\)-inverse semigroup. Using [19, Corollary 2.2 and Theorem...
3.3], \# is a congruence on \( S \) and \( S/\# = Y \odot C \). Using Theorem 1.16, Remark 1.16', Theorem 1.8 and Lemma 1.6, \( S \subseteq H \circ (Y \odot C) \). By Lemma 2.7, \( Y \odot C \trianglelefteq \bar{Y} \circ' C \). Hence, by Lemma 2.5, \( H \circ'(Y \odot C) \trianglelefteq H \circ'(\bar{Y} \circ' C) \). Thus, using Definition 2.2 and Lemma 2.1, \( H \circ (Y \odot C) \trianglelefteq H \circ \bar{Y} \circ C \). Hence, \( S \subseteq H \circ \bar{Y} \circ C \).

**Remark 2.18.** Let \( S \) be a regular semigroup. \( S \) is termed an \( \omega \)-\( \mathcal{R} \)-unipotent semigroup if \( E(S) \) is an \( \omega \)-chain of left zero semigroups. \( S \) is a simple \( \omega \)-\( \mathcal{R} \)-unipotent semigroup if and only if \( S \) is an \( \omega Y \)-\( \mathcal{R} \)-unipotent semigroup with \( Y \) the finite chain \( 0 > 1 > \cdots > d - 1 \) where \( d \) is a positive integer [19, Theorem 7.7]. \( S \) is bisimple if and only if \( |Y| = 1 \) [19, Theorem 7.11].

A regular bisimple semigroup \( S \) is termed a generalized \( \omega \)-\( \mathcal{L} \)-unipotent bisimple semigroup if \( E(S) \) is an \( \omega \)-chain of rectangular bands \( (E_n : n \in \mathbb{N}) \) and \( \mathcal{L} \) is congruence on \( E(S) \).

We close the section by giving a structure theorem for generalized \( \omega \)-\( \mathcal{L} \)-unipotent bisimple semigroups (Theorem 2.19).

**Theorem 2.19.** Let \( S \) be a generalized \( \omega \)-\( \mathcal{L} \)-unipotent bisimple semigroup. Then,

\[
S \subseteq (W^1) \circ ((E(S)/\mathcal{L})^1) \circ (S/Q)^1
\]

where \( W \) is an \( \omega \)-chain of left zero semigroups \( (W_n : n \in \mathbb{N}) \) \((W_n \text{ is an } \mathcal{L} \text{-class of } E_n \text{ for } n \in \mathbb{N})\), \( E(S)/\mathcal{L} \) is an \( \omega \)-chain of right zero semigroups, and \( Q \) is the smallest \( \mathcal{L} \)-unipotent congruence on \( S \) \((Q \text{ is given by Theorem } 1.27)\). Furthermore,

\[
S/Q \subseteq C^1 \mathcal{L} (G^1)^{1 \mathcal{L}} ((E(S)/\mathcal{L})^1)^1
\]

where \( C \) is the bicyclic semigroup and \( G \) is an \( \omega \)-chain of groups. \( S/Q \) is a natural \( \mathcal{L} \)-unipotent semigroup, \( C \equiv S/Q/t \) and \( G = T(S/Q)/\nu \) where \( t \) and \( \nu \) are given by Theorem 1.15 and Remark 1.13 respectively.

**Proof.** Let \( S \) be a generalized \( \omega \)-\( \mathcal{L} \)-unipotent bisimple semigroup. Then, using Theorem 1.27, \( S/Q \) (notation of Theorem 1.27) is a bisimple \( \omega \)-\( \mathcal{L} \)-unipotent semigroup with \( E(S/Q) \cong E(S)/\mathcal{L} \). Using [19, Note 3.4, Proposition 3.5, Proposition 3.6, and Theorem 3.3], \( S/Q \) is a natural \( \mathcal{L} \)-unipotent semigroup with \( S/Q/t \equiv C \). Hence, using the duals of Theorem 1.17, Theorem 1.8, Lemma 1.6, Theorem 1.12, Remarks 1.14 and 1.13 and Lemma 1.6 and the fact \( E(S/Q) \equiv E(S)/\mathcal{L} \),

\[
S/Q \subseteq C^1 \mathcal{L} (G^1)^{1 \mathcal{L}} ((E(S)/\mathcal{L})^1)^1.
\]

Using [20, Lemma 1.11], for each \( n \in \mathbb{N} \), we may select \( f_n \in E_n \) such that \( f_n f_m = f_{\max(n,m)} \). In the statement of Lemma 1.21, let \( e_n = f_n \mathcal{L} \). Then, \( W \) in the statement of Theorem 1.21 (and, hence, in the statement of Theorems 1.28 and 1.26) becomes an \( \omega \) chain of left zero semigroups \( (W_n : n \in \mathbb{N}) \) where \( W_n = e_n \mathcal{L}^{-1} \). Apply Theorem 1.28 and Theorem 1.26.
3. Application of the Rhodes expansion and the derived semigroup to the structure theory

The purpose of this section is to apply the Rhodes expansion and the derived semigroup to establish a structure theorem for orthodox semigroups (Theorem 3.4) and to use the derived semigroup to establish a structure theorem for a class of bisimple inverse monoids (Theorem 3.5).

We first describe the Rhodes expansion $\bar{S}$ of an arbitrary semigroup $S$ and the derived semigroup of a semigroup homomorphism. Let $S_+ = \{(s_n, \ldots, s_1) : s_i \in S \text{ for } 1 \leq i \leq n \}$ and $s_n \subset \cdots \subset s_1$. If $x = (s_n, \ldots, s_1), \ y = (t_m, \ldots, t_1) \in S_+$, define $xy = (s_n t_m, \ldots, s_1 t_m, t_m, \ldots, t_1)$. Then, $S_+$ is a semigroup under this multiplication. If $a = (s_n, \ldots, s_1) \in S_+$ and $s_k \supset s_{k+1}$ for some $1 \leq k \leq n-1$, delete $s_k$ to obtain $a_1 \in S_+$ and denote the deletion by $a \rightarrow a_1$. Perform $a \rightarrow a_1 \rightarrow \cdots \rightarrow a_p$ where $a_p = (s_n, s_n, \ldots, s_n)$ with $s_n \subset s_n \subset \cdots \subset s_n$ (such an $a_p$ is termed an irreducible element of $S_+$). Write $a_0 = a g$ and $a \sim b$ if $a g = b g$ for $a, b \in S_+$. The equivalence relation $\sim$ is a congruence relation on $S_+$. Let $\bar{S} = S_+ / _\sim$. $\bar{S}$ may be treated as the set of irreducible elements of $S$ with the multiplication $ab = (ab) g$. Let $\phi$ be a homomorphism of a semigroup $S$ into a semigroup $T$. The derived semigroup $D(\phi)$ (notation due to Rhodes [6]) will be defined as follows:

$$D(\phi) = \{(t, s, t') : t \in T^*, s \in S, t' = t(s\phi)\} \cup \{0\}$$

under the multiplication

$$(t_1, s_1, t_1')(t_2, s_2, t_2') = \begin{cases} (t_1, s_1 s_2, t_2) & \text{if } t_1 = t_2, \\ 0 & \text{otherwise} \end{cases}$$

and $(t, s, t') 0 = 0 (t, s, t') = 0 0 = 0$. $D(\phi)$ under the above multiplication is a semigroup. See [25], [2, pp. 367–369] and [6] for further details on the Rhodes expansion and [25], [2, p. 356] and [6] for the derived semigroup.

We next give the properties of the Rhodes expansion and the derived semigroup which will be used in the proofs of Theorem 3.4 and/or Theorem 3.5. We first collect some of the fundamental properties of the Rhodes expansion and the derived semigroup in Theorem 3.1. Theorem 3.1(a), 3.1(d), 3.1(e), and 3.1(i) were established by Rhodes in [6]. Theorem 3.1(b), 3.1(c), 3.1(f), 3.1(g), 3.1(h), and 3.1(j) are contained in Chapter 12 (written by Tilson) of Eilenberg [2] for the finite case (these results were first established by Tilson in [25]) but are also valid for the infinite case with the same proofs.

Before stating Theorem 3.1, we must introduce some terminology and notation. Let $S$ be a semigroup and let $s \in S$. Then, $S_s = \{s' \in S : ss' = s\}$. Let $\phi$ be a homomorphism of $S$ onto a semigroup $T$. Following Krohn, Rhodes, and Tilson [4], we term $\phi$ a $\gamma$-epimorphism if $\phi$ is one-to-one on subgroups of $S$.

**Theorem 3.1.** Let $\phi$ be a homomorphism of a semigroup $S$ onto a semigroup $T$. Then

(a) $S \leq D(\phi) \leq T$. 

Embedding of regular semigroups in wreath products

199

(b) \( \phi \) is a homomorphism of \( \hat{S} \) onto \( \hat{T} \). If \( \varphi \) is a \( \gamma \)-epimorphism of \( S \) onto \( T \), then \( \phi \) is a \( \gamma \)-epimorphism of \( \hat{S} \) onto \( \hat{T} \).

(c) If \( \varphi \) is a \( \gamma \)-epimorphism and \( T_i \) is combinatorial for all \( t \in T \), then \( D(\varphi) \) is a combinatorial semigroup.

(d) Let \( \hat{s} = (s_n, \ldots, s_1) \in \hat{S} \). Then, \( x^{n+1} = x^n \) for all \( x \in \hat{S} \).

(e) Let \( \hat{s} \in \hat{S} \). Then, each \( \beta \)-class of \( \hat{S} \) is a singleton.

(f) \( E(\hat{S}) = \{(e, s_{n-1}, \ldots, s_1) \in \hat{S} : e \in E(S)\} \).

(g) \((s_n, \ldots, s_1)\varphi(t_m, \ldots, t_1) \) (in \( \hat{S} \)) if and only if \( n = m \) and \( s_n \varphi t_m \) (in \( S \)) and \( s_i = t_i \) for \( 1 \leq i \leq n - 1 \).

(h) \((s_n, \ldots, s_1)n = s_n \) defines a homomorphism of \( \hat{S} \) onto \( S \).

(i) Let \( S \) be a regular semigroup. Then, if \( a, b \in \hat{S} \), \( \eta \# b \eta \) (in \( S \)) if and only if \( a \# b \) (in \( \hat{S} \)).

(j) Let \( D(\varphi)_i = \{(t, s, t) \in D(\varphi)\} \). Then \((t, s, t) \rightarrow s \) defines an isomorphism of \( D(\varphi)_i \) onto \( T_i \varphi^{-1} \).

Theorem 3.2. Let \( S \) be a semigroup admitting a congruence \( \varrho \) such that \( e \varrho^{-1} \subseteq E(S) \) for all \( e \in E(S/\varrho) \). Then,

\[
S \leftarrow \eta \quad \hat{S} \subseteq D(\hat{\varrho}) \circ (\hat{S}/\hat{\varrho})
\]

where \((s_n, \ldots, s_1)\eta = s_n \) defines an \( E \)-epimorphism of \( \hat{S} \) onto \( S \) and \( D(\hat{\varrho}) \) is an aperiodic semigroup. If \( S \) is a regular, \( \eta \# b \eta \) (in \( S \)) if and only if \( a \# b \) (in \( \hat{S} \)).

Proof. Using Theorem 3.1(h) and 3.1(f), \( \eta \) has the required properties. Using Theorem 3.1(b), 3.1(e) and 3.1(c), \( D(\hat{\varrho}) \) is a combinatorial semigroup. We next show that \( D(\hat{\varrho}) \) is a torsion semigroup. Using Theorem 3.1(f), \( e \hat{\varrho}^{-1} \subseteq E(\hat{S}) \) for all \( e \in E(\hat{S}/\hat{\varrho}) \). Let \((i, \hat{s}, i_1) \in D(\hat{\varrho}) \). If \( i \neq i_1 \), \((i, \hat{s}, i_1)^2 = 0 \). If \( i = i_1 \),

\[
(i, \hat{s}, i) \in D(\hat{\varrho}) \setminus \{ (i, \hat{s}, i_1) : s \in \hat{S} \text{ such that } i(\hat{s} \hat{\varrho}) = i_1 \}.
\]

By Theorem 3.1(i), \( D(\hat{\varrho})_i \) is isomorphic to \((\hat{S}/\hat{\varrho})_i \hat{\varrho}^{-1} \) by the mapping \((i, \hat{s}, i) \rightarrow \hat{s} \). Hence, \( \hat{s} \hat{\varrho} \in (\hat{S}/\hat{\varrho})_i \). Thus, using Theorem 3.1(d), \( \hat{s}^k \hat{\varrho} \in E((\hat{S}/\hat{\varrho})) \) for some positive integer \( k \). Hence, \( \hat{s}^k \in E(\hat{S}) \) and, thus \((i, \hat{s}, i)^k \in E(D(\hat{\varrho})) \). Finally apply Theorem 3.1(a). The last sentence of the theorem is just Theorem 3.1(i).

Proposition 3.3. Let \( S \) be a regular semigroup.

(a) \( \hat{S} \) is a regular semigroup (also established by Rhodes [6]). If \( S \) is an orthodox semigroup, \( \hat{S} \) is an orthodox semigroup with \( E(\hat{S})/\varrho \equiv E(S)/\varrho \). If \( S \) is an inverse semigroup, then \( \hat{S} \) is an \( \mathcal{Z} \)-unipotent semigroup such that \( E(\hat{S}) \) is a semilattice \( E(S) \) of right zero semigroups.

(b) Let \( \varphi \) be a homomorphism of \( S \) into a semigroup \( T \). Then, \((t, s, t(\varphi s)) \) is a regular element of \( D(\varphi) \) if and only if there exists \( s^{-1} \in V(s) \) such that \( t(\varphi ss^{-1}) = t \).

If \( S \) is an orthodox semigroup, \( D(\varphi)_{reg} \), the set of regular elements \( D(\varphi) \), is an orthodox subsemigroup of \( D(\varphi) \). If \( T \) is a group, \( E(D(\varphi)) \) is a direct sum of copies of \( E(S) \).

\[E(S)\]
Proof. (a) Let \((s_n, \ldots, s_1) \in \hat{S}\). Since \(S\) is a regular semigroup, \(s_n \neq e\) for some \(e \in E(S)\). Hence, \((s_n, s_{n-1}, \ldots, s_1) \neq (e, s_{n-1}, \ldots, s_1)\) by Theorem 3.1(g) and \((e, s_{n-1}, \ldots, s_1) \in E(S)\) by Theorem 3.1(f). Hence, \(\hat{S}\) is a regular semigroup. Let \(S\) be an orthodox semigroup. Then, \(E(S)\) is a semilattice \(Y = E(S)/\hat{\gamma}\) of rectangular bands \((E_y : y \in Y)\). For \(y \in Y\), let \(X_y = \bigcup (e \gamma^{-1} : e \in E_y)\) (notation of Theorem 3.2). Using Theorem 3.1(f) and a straightforward calculation, \(X_y\) is a rectangular band for each \(y \in Y\) and \(E(S)\) is the semilattice \(Y\) of rectangular bands \((X_y : y \in Y)\). Finally, let \(S\) be an inverse semigroup, and, for \(y \in E(S)\), let \(E_y = y \gamma^{-1}\). Then, \(E(\hat{S})\) is the semilattice \(E(S)\) of right zero semigroups \((E_y : y \in E(S))\). Hence, \(\hat{S}\) is an \(\gamma\)-unipotent semigroup by [16, Proposition 5].

(b) To establish the second sentence of (b), just note that \((t, s, t(s\gamma)) \in D(\phi)_{\text{reg}}\) if and only if \((t(s\gamma), s^{-1}, t) \in V((t, s, t(s\gamma)))\) for some \(s^{-1} \in V(s)\). Let \(S\) be an orthodox semigroup and let \(x = (t_1, s_1, t_1(s_1 \phi))\), \(y = (t_2, s_2, t_2(s_2 \phi)) \in D(\phi)_{\text{reg}}\). Thus, \(t_1(s_1 s_1^{-1}) \phi = t_1\) and \(t_2(s_2 s_2^{-1}) \phi = t_2\) for some \(s_1^{-1} \in V(s_1)\) and \(s_2^{-1} \in V(s_2)\). If \(t_1(s_1 \phi) = t_2\), then

\[
t_1(s_1 s_2 s_2^{-1} s_1^{-1}) \phi = t_2(s_2 s_2^{-1} s_1^{-1}) \phi - t_2(s_1^{-1}) \phi = t_1(s_1 s_1^{-1}) \phi = t_1.
\]

Since \(S\) is an orthodox semigroup, \(s_2^{-1} s_1^{-1} \in V(s_1, s_2)\) by a result of Reilly and Scheiblich [3, Chapter 6, Theorem 1]. Thus, \((t, s, t(s\gamma)) \in D(\phi)_{\text{reg}}\). If \(t_1(s_1 \phi) \neq t_2\), \(x y = 0 \in D(\phi)_{\text{reg}}\). If \((t, s, t(s\gamma)) \in D(\phi)_{\text{reg}}\), then

\[(t(s\gamma), s^{-1}, t) \in D(\phi)_{\text{reg}} \cap V(t, s, t(s\gamma)).\]

By a straightforward calculation, \(E(D(\phi)) = \{(t, s, t) : t \in T, t(s\gamma) = t, s \in E(S)\}\) and \(E(D(\phi))\) is a subsemigroup. Clearly, \(E(D(\phi)_{\text{reg}}) = E(D(\phi))\). Let \(T\) be a group. For \(t \in T\), let \(E_t - \{(t, s, t) : s \in E(S)\}\) Then, \((t, s, t) \mapsto s\) defines an isomorphism of \(E_t\) onto \(E(S)\). Furthermore, \(E(D(\phi))\) is the direct sum of \((E_0^t : t \in T)\).

Theorem 3.4. Let \(S\) be an orthodox semigroup. Then,

\[
S \stackrel{\eta}{\longrightarrow} \hat{S} \cong D(\hat{\phi}) \circ (S(\hat{\phi}))
\]

where \((s_n, \ldots, s_1) \eta = s_n\) defines an E-epimorphism of \(\hat{S}\) onto \(S\) such that \(a \gamma b\) \(\eta\) (in \(S\)) if and only if \(a \gamma b\) \(\eta\) (in \(\hat{S}\)). \(\hat{S}\) is an orthodox semigroup with \(E(\hat{S})/\hat{\gamma} = E(S)/\gamma\). \(\phi\) is the smallest inverse semigroup congruence on \(S\), \(D(\phi)\) is an aperiodic semigroup such that \((D(\phi))_{\text{reg}}\), the set of regular elements of \(D(\phi)\), is an orthodox aperiodic subsemigroup of \(D(\phi)\), and \((S(\hat{\phi}))\) is an \(\gamma\)-unipotent semigroup. Furthermore,

\[
(S(\hat{\phi})) \cong ((S(\hat{\phi}))/\delta)^{\text{1}} (E((S(\hat{\phi})))^1)
\]

where \(\delta\) is the smallest inverse semigroup congruence on \((S(\hat{\phi}))\), \(E((S(\hat{\phi}))/\delta) \cong E(S)/\gamma\), and \((E((S(\hat{\phi})))\) is a semilattice \(E(S)/\gamma\) of right zero semigroups.

Proof. Let \(S\) be an orthodox semigroup. Let \(\phi\) be the smallest inverse semigroup congruence on \(S\). Then, \(\phi\) is given by Remark 1.13. Suppose \(a \phi = e \phi\) for \(e \in E(S)\).

i.e. \(V(a) = V(e)\). Then, \(a^2 \phi = e \phi\) or \(V(a^2) = V(e)\). Hence, \(a^2 = (aea)(ae a) = a(ea^2) a = ace - a\). Thus, we may apply Theorem 3.2 to obtain (1). Next, apply Proposition
Embedding of regular semigroups in wreath products

3.3(a), Theorem 3.2, Proposition 3.3(b), Proposition 3.3(a), the duals of the Theorems 1.12 and 1.8 to obtain (2), the dual of Theorem 1.12, Proposition 3.3(a), [3, Theorem 1.10, p. 189], and Proposition 3.3(a).

Before stating and proving Theorem 3.5, it will be helpful to summarize the construction of the maximal group homomorphic image of a bisimple inverse monoid S (see [7, §2] for details). Let P denote the right unit subsemigroup of S. Define \( \eta(a, b) \) if there exists \( h \in P \) such that \( ha = hb \). Then, \( \eta \) is a congruence relation on \( P \). Let \( a \to \overline{a} \) denote the natural homomorphism of \( P \) onto \( P/\eta = \overline{P} \). \( \overline{P} \) may be embedded in a group \( G \). The mapping \( a^{-1}b \to \overline{a}^{-1}\overline{b} \) is a homomorphism of \( S \) onto \( G \) and \( G \) is thereby a maximal group homomorphic image of \( S \). Furthermore, \( \overline{a}^{-1}\overline{b} = \overline{c}^{-1}\overline{d} \) implies there exists \( x, y \in \overline{P} \) such that \( x\overline{a} = y\overline{c} \) and \( x\overline{b} = y\overline{d} \).

**Theorem 3.5.** Let \( S \) be a bisimple inverse monoid with right unit subsemigroup \( P \) and group of units \( U \). If \( N = \{ u \in U : hu = h \) for some \( h \in P \} \) is a right normal divisor of \( P \), then

(1) \( S \leq \ker \varrho^N \circ S/\varrho^N \) (notation of Remark 1.11) where \( \varrho^N \) is the idempotent separating congruence on \( S \) corresponding to \( N \), \( \ker \varrho^N \) is a semilattice \( E(S) \) of groups, and \( E(S/\varrho^N) = E(S) \). Furthermore,

\[
S/\varrho^N \leq D(\delta) \circ S/\varrho^N/\delta
\]

where \( \delta \) is the smallest group congruence on \( S/\varrho^N \) and \( D(\delta) \) is a combinatorial inverse semigroup such that \( E(D(\delta)) \) is a direct sum of \( \delta \)-copies of \( E(S) \).

**Proof.** Use Remark 1.11 and Theorems 1.9 and 1.8 to obtain (1). We first show that \( hu = h \) where \( h \in P_N \), the right unit subsemigroup of \( S/\varrho^N \) and \( u \in U_N \), the group of units of \( S/\varrho^N \), imply \( u = e \), the identity of \( S/\varrho^N \). From this, we will deduce that \( \delta \) is a \( \gamma \)-epimorphism. Then, on application of Theorems 3.1(a), 3.1(c), and Proposition 3.3(b) will conclude the proof. Since \( E(S) = \{ k^{-1}k : k \in P \} \), \( h^{-1}hu = h^{-1}h = (k^{-1}k)\varrho^N \) for some \( k \in P \). Using the fact \( \varrho^N \) is an idempotent separating congruence, \( z\varrho^N = u \) for some \( z \in U \). Thus, \( k^{-1}kz\varrho^N = k^{-1}k \). Hence, using Remark 1.11, \( k^{-1}kz = k^{-1}vk \) for some \( v \in N \). Thus, \( k \varrho^N = k \). Since \( v \in N \), there exists \( h \in P \) such that \( hu = h \). Hence, \( k \varrho^N = k \). Since \( z \in N \) and \( u = e \). Next, we show \( \delta \) is a \( \gamma \)-epimorphism. In the paragraph preceding Theorem 3.5, let \( S = S/\varrho^N \), \( P = P_N \) and \( \phi : a^{-1}b \to a^{-1}\overline{b} \). Let \( \delta = \phi^{-1} \circ \phi \). Let \( W \) be a subgroup of \( S/\varrho^N \) and let \( a^{-1}b, c^{-1}d \in W \) where \( a, b, c, d \in P_N \). Thus \( a^{-1}b, c^{-1}d \in W \). Hence, \( a = uc \) and \( b = vd \) for some \( u, v \in U_N \). Suppose \( (a^{-1}b)\varrho^N = (c^{-1}d)\varrho^N \). Hence, there exist \( x, y \in P_N \) such that \( xa = yc \) and \( xb = yd \). Thus, there exist \( f, g \in P_N \) such that \( fx = fy \) and \( gxv = gy \). Hence, \( x\overline{u} = y\overline{v} \). Hence, \( a = c^{-1}d \).
**Remark 3.6.** For any bisimple inverse monoid, $N$ (notation of Theorem 3.5) is a normal subgroup of $U$, the group of units of $S$.

We next give a class of examples of the bisimple inverse monoids given by Theorem 3.5.

**Example 3.7.** Let $S = (G, C_n, v_1, v_2, \ldots, v_n, t_1, t_2, \ldots, t_{\varphi(n)})$ where $\varphi(n) = \frac{1}{2}n(n - 1)$ ($n$, a positive integer) be an $\omega^n$-bisimple semigroup ([10 Theorem 2.3] and notation of [12, p. 258]). Then using [12, Lemma 3.1], $N$ (notation of Theorem 3.5) is a right normal divisor if and only if $N$ is a $v_1 - v_2 - \cdots - v_n$ invariant subgroup of $G$ (i.e. $Nv_i \subseteq N$ for $i = 1, 2, \ldots, n$). Using [10, Theorem 1.4], $N$ is trivial if $v_1$ is a monomorphism.

We close this section by giving a structure theorem for a class of bisimple inverse monoids where $N$ is trivial (Theorem 3.8) and give two examples (Examples 3.9 and 3.10) of this class of semigroups.

Let $S$ be a bisimple inverse monoid and let $P$ denote the right unit subsemigroup of $S$. If (1) $P$ is left cancellative and (2) the principal right ideals of $P$ are linearly ordered (by set theoretic inclusion), we termed $S$ a strict bisimple inverse monoid in [24].

**Theorem 3.8.** Let $S$ be a strict bisimple inverse monoid. Then,

$$S \cong E(S) \cdot S/\varnothing$$

where $\varnothing$ is the smallest group congruence on $S$.

**Proof.** Utilize [24, Lemma 5.1] and Remark 1.1'.

**Example 3.9.** Let $F$ be the positive part of an ordered field. Let $F^* = F \setminus \{0\}$ and $S = F \times F^* \times F$ under the multiplication

$$(b, c, d)(y, w, z) = \begin{cases} \left\{ \frac{y - d + bc}{c}, wc, z \right\} & \text{if } y \geq d, \\ (b, wc, (d - y)w + z) & \text{if } d \geq y. \end{cases}$$

$S$ is a strict bisimple inverse monoid with right unit subsemigroup $P = F^* \times F$ under the multiplication $(a, b)(c, d) = (ac, bc + d)$. Since the only right normal divisor of $P$ is the identity, the identity congruence is the only idempotent separating congruence on $S$ by Remark 1.11. Thus, $S$ is fundamental in the sense of [3, p. 141]. However, since $(F^*, +) = U$, the group of units of $S$, $S$ is not combinatorial.

**Example 3.10.** Let $S = A \times A$ ($A$ is the non-negative reals or rationals) under the
multiplication \((a, b) \cdot (c, d) = (c + c - \min(bc), b + d - \min(h, c))\). Then, \(S\) is a strict bisimple inverse monoid.

### 4. Standard regular semigroups

A regular semigroup \(S\) is termed standard regular if \(T(S)\) is a subsemigroup, \(T(S)/\mathcal{J}\) has a greatest element, and \(eT(S)e\) is an inverse semigroup for all \(e \in E(T(S))\). If \(S\) is a standard regular semigroup, we show (Theorem 4.8) that

\[
S \prec \bigoplus_{\mathcal{J} \in \mathcal{E}(\mathcal{J})} E^1 \circ G^1 \circ V \circ J
\]

where \(E\) is a standard semilattice \(T(S)/\mathcal{J}\) of left zero semigroups, \(G\) is a semilattice \(T(S)/\mathcal{J}\) of groups, \(V\) is a combinatorial inverse semigroup with semilattice \(T(S)/\mathcal{J}\), and \(J\) is a set.

A regular semigroup \(S\) is a standard regular semigroup if and only if \(T(S)\) is semigroup, \(T(S)/\mathcal{J}\) has a greatest element, and \(e, f, g \in E(T(S))\), \(e \geq f\) and \(e \geq g\) imply \(fg = gf\). In the remainder of this section, \(S\) will denote a standard regular semigroup.

Let \(Y = T(S)/\mathcal{J}\) and let \(y_0\) denote the greatest element of \(Y\). Let \(\{e_{y}: y \in Y\}\) denote the set of structure homomorphisms of \(T(S)\) [21, §1]. Let \(E_y = E(T_y)\). Select and fix \(e_{y_0} \in E_{y_0}\). For each \(y \in Y\), define \(e_y = e_{y_0} \zeta_{y_0/y}\). Let \(S_0 = e_{y_0} S e_{y_0}\). Let \(I_y, J_y\) denote the set of idempotents of the \(\mathcal{J}\)-class of \(T_y\) containing \(e_y\) and \(J_y\) denote the set of idempotents of the \(\mathcal{J}\)-class of \(T_y\) containing \(e_y\). Let \(I = \bigcup (I_y : y \in Y)\) and \(J = \bigcup (J_y : y \in Y)\).

Lemmas 4.1–4.6 below will be used in the proof of Theorem 4.7 which when combined with Theorem 1.8 and Remark 1.11 will yield Theorem 4.8.

**Lemma 4.1** [21, Lemma 2.2, left-right duals of Lemmas 2.4 and 2.5, and Lemma 2.15]. \(y \rightarrow e_y\) defines an isomorphism of \(Y\) onto \(E(S_0)\). \(I\) is a standard semilattice \(Y\) of left groups \((I_y : y \in Y)\) and \(J\) is a standard semilattice \(Y\) of right zero semigroups \((J_y : y \in Y)\). There exists a homomorphism \(\phi\) of \(S_0\) onto a combinatorial inverse semigroup \(V\) where \(E(V) = Y\) and, if \(h_c = c e^{-1}\), \(\{h_c : c \in V\}\) is the collection of \(x\)-classes of \(S_0\).

**Remark 4.2.** For each \(c \in V\), select a representative element \(v_c \in h_c\). For \(y \in Y\), let \(v_y = e_y\) (by proof of [21, Lemma 2.15], \(e_y \in h_y\)). Using Lemma 4.1, \(v_y v_{y'} = e_{y+y'}\) and \(v_{y'}^{-1} v_y = e_{y-y'}\). For \(s \in V\) and \(u \in I\), define \(u \theta_s = v_s uu_{s^{-1}}\). For \(y \in Y\), let \(H_y\) denote the maximal subgroup of \(S\) containing \(e_y\).

**Lemma 4.3.** Every element of \(S\) may be uniquely expressed in the form \(iv_c j\) where \(i \in I_{c^{-1}}\) and \(j \in J_{c^{-1}}\). If \(c, d \in V\), \(v_c v_d = g(c, d) v_{cd}\) where \(g\) is a function of \(V \times V\) into \(H = \bigcup (H_y : y \in Y)\) such that \(g(c, d) \in F_{(cd)(cd)^{-1}}\). For \(s \in V\), \(\theta_s \in \text{End } I\) and \(I_r \theta_s \subseteq H_{S_{rs^{-1}}}\).

**Proof.** The first sentence of the lemma is the left-right dual of [21, Lemma 2.16].
The second (third) sentence of the lemma is contained in the left-right dual of [21, Lemma 2.17] ([21, Lemma 2.18]).

Lemma 4.4. For \( r, s \in V \), \( e_r \theta_s = e_{rs} \).

Proof. Utilize Lemma 4.3.

Lemma 4.5. If \( j \in J_y \) and \( i \in I_z \), then \( ji \in H_{yz} \).

Proof. Since \( j \triangleright e_y, j \xi_y \triangleright e_y, \xi_y \triangleright e_{yz} \). Similarly, \( i \triangleright e_z \) implies \( i \xi_z \triangleright e_{yz} \). Thus, \( ji = e_y \triangleright ji e_{yz} \) and, hence \( ji \in H_{yz} \).

Lemma 4.6. Let \( \alpha, \beta \in V \), \( i \in I_{aa} \), \( j \in J_y \) where \( y \geq \alpha^{-1} \alpha, u \in I_{\beta \beta} \), and \( z \in J_{\delta} \) with \( \delta \geq \beta^{-1} \beta \). Then

\[
(i_{\alpha,j})(uv_{\beta}z) = i(v_{\alpha}(ju)u_{\beta}v_{\alpha}^{-1})u_{\alpha \beta}e_{(\alpha \beta)}^{-1}(\alpha \beta)z
\]

where \( v_{\alpha}(ju)u_{\beta}v_{\alpha}^{-1} \in H_{(\alpha \beta)(\alpha \beta)} \).

Proof. Using Lemmas 4.1, 4.3, and 4.5 and Remark 4.2,

\[
(i_{\alpha,j})(uv_{\beta}z) = iv_{\alpha}(e_{\alpha}^{-1}a(ju))e_{\alpha}^{-1}a_{\beta}e_{\alpha}^{-1}a_{\beta}z
= iv_{\alpha}e_{\alpha}^{-1}a(ju)e_{\alpha}^{-1}a_{\beta}z = iv_{\alpha}e_{\alpha}^{-1}a(ju)v_{\alpha}^{-1}v_{\beta}z
= iv_{\alpha}e_{\alpha}^{-1}a(ju)g(\alpha, \beta)v_{\beta}v_{\alpha}^{-1}.
\]

Using Lemma 4.3, \( (e_{\alpha}^{-1}a(ju))g(\alpha, \beta) \in H_{(\alpha \beta)(\alpha \beta)} \).

However, since, using Lemma 4.3 and Remark 4.2, \( g(\alpha, \beta) = v_{\alpha}v_{\beta}v_{\alpha}^{-1} \),

\[
(e_{\alpha}^{-1}a(ju))g(\alpha, \beta) = v_{\alpha}e_{\alpha}^{-1}a(ju)v_{\alpha}^{-1}v_{\alpha}v_{\beta}v_{\alpha}^{-1} = v_{\alpha}(ju)v_{\beta}v_{\alpha}^{-1}.
\]

Remark 4.6'. Using the above proof and Lemma 4.1, \( v_{\alpha}(ju)v_{\beta}v_{\alpha}^{-1} \in I_{\alpha \beta \beta}^{-1} \) for \( u \in I_{\beta \beta} \) and \( j \in J_y \).

Theorem 4.7. Let \( S \) be a standard regular semigroup. Then,

\[
S < I \circ V \circ J
\]

where \( I \) is a standard semilattice \( T(S)/\gamma \) of \( \gamma \)-classes of \( T(S) \) (each such \( \gamma \)-class is a left group), \( V \) is a combinatorial inverse semigroup with semilattice \( T(S)/\gamma \); \( (V \cong e_{\gamma} \mathcal{S} e_{\gamma} \gamma /\gamma \) where \( e_{\gamma} = e_{\gamma} \) is contained in largest \( \gamma \)-class of \( T(S) \)), and \( J \) is a set.

Proof. First note that \( I \circ V \circ J = I^1 \times J \times I^J \times I \) under the multiplication

\[
(f_1, g_1, z_1)(f_2, g_2, z_2) = (f_1 \triangleright g_1, z_1) f_2, g_1 \triangleright g_2, z_2).
\]
For $\beta \in V$ and $u \in I_{\beta \beta^{-1}}$, using Remark 4.6', define $f_{(u, \beta)} \in I^V \times J$ as follows: $(\alpha, j)f_{(u, \beta)} = v_{\alpha}(ju)v_{\beta}v_{\alpha \beta}^{-1}$ if $(\alpha, j) \neq (\gamma_0, e_{\gamma_0})$ and $(\gamma_0, e_{\gamma_0})f_{(u, \beta)} = u$ and define $g_{\beta} \in V^J$ by $jg_{\beta} = \beta$ for $j \in J$. Let

$$A = \{(f_{(u, \beta)}, g_{\beta}, z) : z \in J_v \text{ with } v \geq \beta^{-1} \beta\}.$$

We next show that $A$ is a subsemigroup of $I \circ V \circ J$. Let $(f_{(i, a)}, g_{\alpha}, i, f_{(u, \beta)}, g_{\beta}, z) \in A$. Then,

$$(f_{(i, a)}, g_{\alpha}, i)(f_{(u, \beta)}, g_{\beta}, z) = (f_{(i, a)} \circ g_{(\alpha, i)}f_{(u, \beta)}, g_{\alpha} \circ g_{\beta}g_{\beta}, z)$$

We first show

$$f_{(i, a)} \circ g_{(\alpha, i)}f_{(u, \beta)} = f_{(i \circ a \nu(Ju)v_{\beta}v_{\alpha}^{-1}, a \beta)}.$$

(Note $v_{\alpha}(ju)v_{\beta}v_{\alpha \beta}^{-1} \in H_{(\alpha \beta)(\alpha \beta)}$ by Lemma 4.6.) For $(v, k) \in V \times J$, using Remark 4.2 and Lemmas 4.1, 4.4 and 4.5,

$$v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1} = \nu_{\alpha}(e_{\alpha \beta}^{-1})v_{\alpha \beta}^{-1}v_{\alpha \beta} = \nu_{\alpha}(e_{\alpha \beta}^{-1})v_{\alpha \beta}^{-1}v_{\alpha \beta}$$

while, using Remark 4.2 and Lemma 4.1,

$$v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1} = v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1}v_{\alpha \beta}v_{\alpha \beta}^{-1} = v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1}v_{\alpha \beta}v_{\alpha \beta}^{-1}$$

Furthermore,

$$v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1} = v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1}v_{\alpha \beta}v_{\alpha \beta}^{-1} = v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1}v_{\alpha \beta}v_{\alpha \beta}^{-1}$$

while

$$v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1} = v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1}v_{\alpha \beta}v_{\alpha \beta}^{-1} = v(ki)\nu_{\alpha \beta}v_{\alpha \beta}^{-1}v_{\alpha \beta}v_{\alpha \beta}^{-1}$$

For $x \in J$, $x(g_{\alpha} \circ g_{\beta} = \alpha \beta = x(g_{\alpha \beta})$. Thus, $g_{\alpha} \circ g_{\beta} = g_{\alpha \beta}$. Hence, if

$$(f_{(i, a)}, g_{\alpha}, i)(f_{(u, \beta)}, g_{\beta}, z) \in A,$$

then

$$(f_{(i, a)}, g_{\alpha}, i)(f_{(u, \beta)}, g_{\beta}, z) - (f_{(i \circ a \nu(Ju)v_{\beta}v_{\alpha}^{-1}, a \beta)}, g_{\alpha \beta}, z).$$

Define $(f_{(i, a)}, g_{\alpha}, i)\varphi = i\nu_{\alpha}(e_{\alpha \beta}^{-1})$ for $(f_{(i, a)}, g_{\alpha}, i) \in A$.

Thus, $\varphi$ is a well defined mapping of $A$ into $S$. 
Using (*) Lemmas 4.6 and 4.1,
\[(f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{z})\varphi = i(v_{a}(ju)u_{\beta}v_{a}^{-1})v_{\alpha}e_{(a\beta)^{-1}(a\beta)}z\]
\[= (v_{a}(e_{a^{-1}(a\beta)})u_{\beta}(e_{\beta^{-1}\beta}z))\]
\[= (f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{z})\varphi\]
Hence, \(\varphi\) is a homomorphism.

Suppose \((f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{z})\varphi\). Then, \(i(v_{a}(e_{a^{-1}(a\beta)}) = uv_{a}(e_{\beta^{-1}\beta}z)\). Hence, using Lemma 4.3, \(i = u, \alpha = \beta, \) and \(e_{a^{-1}(a\beta)} = e_{(a\beta)^{-1}(a\beta)}z\).

Using (*), Remark 4.2, and Lemma 4.1,
\[(f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{z}) - (f_{i(a)}, g_{a}, \tilde{z})\]
and
\[(f_{i(a)}, g_{a}, \tilde{z})(f_{u(\beta)}, g_{\beta}, \tilde{z}) = (f_{i(a)}, g_{a}, \tilde{f})\]
Hence, \((f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{z})(e \in A)\). Thus, \(\varphi\) is a \(R\)-epimorphism of \(A\) onto \(S\).

Suppose \((f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{k})\). Then, \(j = k\). Hence \((f_{i(a)}, g_{a}, \tilde{f})(f_{u(\beta)}, g_{\beta}, \tilde{k})\) implies \((f_{i(a)}, g_{a}, \tilde{f}) = (f_{i(a)}, g_{a}, \tilde{k})\). Thus, \(\varphi\) is a \(v\)-\(R\)-epimorphism of \(A\) onto \(S\).

Using Lemmas 4.1 and 4.5 and Remark 4.2 and a straightforward calculation, if \(iv_{a}(e_{a^{-1}(a\beta)})(i \in J_{\delta} \text{ with } \delta \geq a^{-1}a\) and \(i \in I_{a\alpha}^{-1}\) is an idempotent, then \(v_{a}(j_{i}) = v_{a}\) and \(a \in Y\). Thus, using (*), \(\varphi\) is an \(E\)-epimorphism.

**Theorem 4.8.** Let \(S\) be a standard regular semigroup. Then,
\[S \subseteq E \circ G \circ V \circ J\]
where \(E\) is a standard semilattice \(T(S) / \gamma\) of left zero semigroups, \(G\) is a semilattice \(T(S) / \gamma\) of groups, \(V\) is a combinatorial inverse semigroup with semilattice \(T(S) / \gamma\) \((V \equiv e_{\gamma} \circ Se_{\gamma} / \gamma\) where \(e_{\gamma} = e_{\gamma} \) is contained in the greatest \(\gamma\)-class of \(T(S))\), and \(J\) is a set.

**Proof.** Utilize Theorem 4.7, Theorem 1.12, Remark 1.14, Theorem 1.8 and Lemma 1.6.

**Remark 4.9.** If \(S\) in Theorem 4.8 is a union of groups, \(V = T(S) / \gamma\).

**Acknowledgement**

I had the good fortune to attend the exciting classes and seminars of John Rhodes at Berkeley from 1980 to 1982. These courses and seminars together with numerous conversations with Professor Rhodes motivated the present paper.
References