On the number of generators of the module of derivations and multiplicity of certain rings

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Abstract

The present paper studies the module of derivations of certain rings and the multiplicity of certain rational surface singularities. We also consider some relation between these two for 2-dimensional quotient singularities. We conjecture a relation between the multiplicity of a rational surface singularity and the order of the divisor class group of the singularity and verify the same for several cases.
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Introduction

Throughout this paper let \( k \) denote an algebraically closed field of characteristic zero.

We study the generators of the module of derivations of certain rings. We obtain bounds on the minimum number of generators of the module of derivations in case of certain rings. We use a result of Herzog and Kühl [HK87] which implies that for a 2-dimensional hypersurface ring \( R \), \( \mu(\text{Der}R) \) is even, where \( \mu(\text{Der}R) \) denotes the minimal number of generators of \( \text{Der}R \) as an \( R \)-module.

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We discuss a conjecture of H. Krämer on a minimal generating set of $\text{Der} R$ for a 2-dimensional normal quasihomogeneous hypersurface. We give an equivalent condition for this conjecture in terms of the some ideals related to the Jacobian ideal.

We consider the module of derivations of the ring of invariants of finite subgroups of $GL_2(k)$. We obtain a bound on $\mu(\text{Der} R)$ which is similar to the one obtained by Herzog and Kühl for maximal Cohen–Macaulay modules:

**Theorem 0.1.** Let $R = k[Y_1, Y_2]^G = k[x_1, \ldots, x_n]$ where $G \subset GL_2(k)$ is a finite group having no non-trivial pseudo-reflections. Suppose there exists $i$ such that $x_i$ is not a zero divisor in the associated graded ring of $R$. Then

$$\mu(\text{Der} R) \leq 2e(\hat{R}) + 1$$

where $\hat{R}$ is an $m$-adic completion of $R$.

We consider the multiplicity of 2-dimensional rational singularities. We have obtained the following result which proves the conjecture below stating an inequality between multiplicity and the order of the divisor class group for any rational surface singularity.

**Theorem 0.2.** Let $(V, p)$ denote a germ of a 2-dimensional rational singularity. If $(V, p)$ is a quotient singularity or the multiplicity of $(V, p)$ is 3 or 4, then the multiplicity is at most equal to the order of its divisor class group.

We have also proved this inequality for rational quasihomogeneous singularities with divisor class group of order 2.

**Conjecture.** For any rational surface singularity the multiplicity cannot exceed the order of the divisor class group of the corresponding complete local ring.

We have given a sufficient condition on the fundamental cycle of the singularity for this conjecture to be true.

**Theorem 0.3.** Let $(V, p)$ be a rational singularity and $\pi : \tilde{V} \to V$ be a resolution of the singularity. Let $\mathcal{Z}$ be the divisor defined by $m_p \mathcal{O}_{\tilde{V}}$ which is the fundamental cycle. Write $\mathcal{Z} = \sum_{i=1}^{r} m_i C_i$ where $C_i$’s are the exceptional irreducible components of $\pi^{-1}(p)$. If $m_i = 1$ for some $i$ then

$$-3^2 \leq \left| \det(C_i \cdot C_j) \right|$$

where $(C_i \cdot C_j)$ denotes the intersection matrix of $C_i$’s.

Using this result and a bound on $\mu(\text{Der} R)$ mentioned in Theorem 0.1, we get another bound on $\mu(\text{Der} R)$.

**Corollary 0.4.**

$$\mu(\text{Der} R) \leq 2 \left| \frac{G}{[G,G]} \right| + 1,$$

provided there exists $x_i$ as mentioned in Theorem 0.1.
We have proved a general upper bound on the multiplicity of rings of invariants of finite groups acting on \( n \)-dimensional polynomial rings.

**Theorem 0.5.** Let \( G \subset GL_n(k) \) be a finite group acting on \( k[X_1,\ldots,X_n] \) and let \( R = k[X_1,\ldots,X_n]^G \) be the ring of invariants. Then the multiplicity \( e(R) \) at its irrelevant maximal ideal is \( \leq |G|^{n-1} \).

We give an example of a subgroup of \( GL_n(k) \) such that this bound is best possible. However for a general subgroup of \( GL_n(k) \) this bound appears to be large. The following result gives a bound on the number of generators of the module of derivations of the ring of invariants, in case of a finite subgroup of \( GL_2(k) \) acting on \( k[X, Y] \).

**Theorem 0.6.** Let \( G \subset GL_2(k) \) be a finite group acting linearly on \( k[X, Y] \) and let \( R = k[X, Y]^G \). Then the module of \( k \)-derivations of \( R \) is generated by at most \( 2|G| + 1 \) elements.

1. \( \mu(\text{Der } R) \)

1.1. Motivation

The basic motivation to study the number of generators of the module of derivations comes from a result of D.P. Patil and B. Singh [PS90,Pat89]. They have studied certain properties of the module of derivations and found the exact number of generators for the module of derivations of a plane curve.

**Proposition 1.1 (Patil–Singh).** Let \( f \in k[[X, Y]] \) be an irreducible power series of multiplicity \( \geq 2 \). Let \( R = k[[X, Y]]_{(f)} \). Then \( \mu(\text{Der } R) = 2 \).

A short proof of this result has been given in [Wag06].

In view of this result, the first author has raised the following question in 3-dimension.

**Question 1.2 (Gurjar).** Let \( R = k[[X,Y,Z]]_{(f)} \) be a normal domain. Does there exist a universal bound for the number of generators of the module of derivations? Are 4 generators enough?

Assuming that the Krämer’s conjecture (Conjecture 1.6) is true, we can ask an equivalent question:

\[
\text{Is } \mu((f, f_X, f_Y) : (f_Z)) = 3? 
\]

**Remark.** This question has an affirmative answer if \( R \) is quasihomogeneous. But in general, answer to this question is negative.

Consider

\[
f = X^3 - 3XYZ - Y^3 - X^2Z^2 + Y^3Z - Z^6 \in \mathbb{C}[X, Y, Z].
\]
Then the computations using a computer algebra system SINGULAR [GPS04] have shown that a minimal generating set of the quotient ideal requires 5 elements (this was obtained using the SINGULAR command \texttt{minbase}):\
\[ \mu \left( (f, f_x, f_y) : (f_Z) \right) = 5. \]

Then we can see that\
\[ \mu \left( \text{Der} \left( \frac{C[X, Y, Z]}{(f)} \right) \right) = 6. \]

Further restricting to the rings of invariants of finite subgroups of $GL_2(k)$ a similar question can be asked.

\textbf{Question 1.3 (Gurjar).} Let $G \subset GL_2(k)$ be a finite subgroup. Let $R = k[X, Y]^G$ be the ring of invariants.

Is then $\mu(\text{Der} R) \leq 4$?

This question has an affirmative answer for $G \subset SL_2(k)$ and for cyclic subgroups of $GL_2(k)$. For the proof we refer the reader to [Wag06].

In case of cyclic subgroups of $GL_2(k)$ it has been proved that $\mu(\text{Der} R) = 4$ and the explicit generators are also given [Wag06]. In view of this result, we ask a general question.

\textbf{Question 1.4.} Let $G \subset GL_n(k)$ be a finite cyclic subgroup acting on the polynomial ring $k[X_1, \ldots, X_n]$. Does there exist a (universal) bound for the minimum number of generators of the module of derivations of the ring of invariants?

Later we will give a bound which is in terms of $|G|$.

\subsection*{1.2. Krämer’s conjecture}

\textbf{Definition 1.5 (Regular derivations).} Let $R = \frac{k[X, Y, Z]}{(f)}$ where $f$ is an irreducible power series. We know that any derivation $\delta \in \text{Der} R$ can be written as $(a, b, c) \in \mathbb{R}^3$ such that $af_x + bf_y + cf_z = 0 \in R$. Then the following are derivations:

\begin{align*}
\Delta_X &= (0, -f_z, f_y), \\
\Delta_Y &= (-f_z, 0, f_x), \\
\Delta_Z &= (-f_y, f_x, 0). \\
\end{align*}

These derivations are called as the Regular or Natural derivations of $R$.

\textbf{Notations.} The following ideals of $k[X, Y, Z]$ will be used in the sequel:

\begin{align*}
I_X &= (f, f_y, f_z) : (f_x), \\
I_Y &= (f, f_x, f_z) : (f_y), \\
I_Z &= (f, f_x, f_y) : (f_z). \\
\end{align*}

(\texttt{**})
Remark. Let \( R = \frac{\mathbb{C}[X,Y,Z]}{(f)} \) be normal. If \((f_X, f_Y)\) do not form a regular sequence in \( R \) then it can be proved that there exists an irreducible power series \( g(X, Y) \) such that \( f \in (Z, g^2) \). The converse of this also holds.

If the variable \( X, Y, Z \) are sufficiently general then \( f_X, f_Y \) form a regular sequence in \( R \).

In the following discussions, we will assume that the variables are sufficiently general, so that the pairs \((f_X, f_Y), (f_Y, f_Z)\) and \((f_X, f_Z)\) are all regular sequences modulo \((f)\).

The ideal \( I^X/(f) \) is the ideal generated by all \( \delta(x) \) as \( \delta \) varies over \( \text{Der} \, R \). Since \( f_Y, f_Z \) form a regular sequence in \( R \), we see easily that \( \text{Der}(R) \) is generated by the derivations of the type \((0, b, c)\) and those of the type \((a, b, c)\) with \( a \in I^X/(f) \). This observation will be used in the following proof.

In [Krä69], H. Krämer made a conjecture about these derivations.

Conjecture 1.6 (Krämer). Let \( R \) be as above. Assume that \( R \) is normal. Then one of the regular derivations \( \Delta_X, \Delta_Y, \Delta_Z \) belongs to a minimal generating set of \( \text{Der} \, R \).

The ideals mentioned in (**) play an important role for an equivalent condition for Krämer’s conjecture to be true.

Proposition 1.7. Let \( f \in k[X, Y, Z] \) be an irreducible power series. Let \( R = \frac{k[X, Y, Z]}{(f)} \) be normal. Then Krämer’s conjecture is true if and only if \( f \in mI \) where \( I \) is one of \( I^X, I^Y \) or \( I^Z \) defined in (**) above for general variables \( X, Y, Z \).

Proof. Without loss of generality we can assume \( I = I^Z = (f, f_X, f_Y) : (f_Z) \). Suppose \( f \in mI \). Since the ideal \((f, f_X, f_Y)\) is irreducible in \( k[X, Y, Z] \), by results from [ZS75a, Chapter IV, §16] the ideal \( I \) is irreducible. In this situation, using Watanabe’s result [Wat73], we get that \( \mu(I) \) is odd. Therefore \( \mu(I/(f)) \) is odd. It is easy to see that \( \text{Der} \, R \) is a maximal Cohen–Macaulay module over \( R \) of rank 2. Hence by a result of Herzog and Külhl [HK87], \( \mu(\text{Der} \, R) \) is even. Now it follows that \( \Delta_Z \) is in a minimal generating set of \( \text{Der} \, R \).

Conversely, suppose \( \Delta_Z \) is in a minimal generating set of \( \text{Der} \, R \). Therefore using a theorem of Herzog and Külhl [HK87], we get that \((f_X, f_Y) : (f_Z)\) is generated by odd number of elements. Now \((f, f_X, f_Y) : (f_Z)\) is irreducible ideal and hence it is generated by odd number of elements. Therefore \( f \) cannot be in the minimal generating set of \( I_Z \) by Watanabe’s result.

Remark. In his paper [Krä69] H. Krämer has shown that an affirmative answer to the Conjecture 1.6 gives an affirmative answer to Zariski–Lipman conjecture when the ring is hypersurface in 3 variables.

A simpler proof of this implication has been given in [Wag06].

1.3. Bound on \( \mu(\text{Der} \, R) \) for the ring of invariants

In this section we give bounds for the number of generators of the derivation module for the ring of invariants of finite subgroups of \( GL_2(k) \) in terms of the multiplicity of the ring.

We quote a lemma by E. Matlis.
Lemma 1.8. (See Matlis [Mat73].) Let \((R, \mathfrak{m})\) be a pure 1-dimensional geometric local ring. Let \(e(R)\) denote the multiplicity of \(R\). Then any ideal in \(R\) can be generated by \(e(R)\) elements.

Theorem 1.9. Let \(G \subseteq \text{GL}_2(k)\) be a finite subgroup. Let \(R = k[X, Y]_G = k[x_1, \ldots, x_n]\) be the corresponding ring of invariants where \(x_i\)'s are homogeneous. Let \(\hat{R}\) denote the completion of \(R\) with respect to the irrelevant maximal ideal of \(R\). Suppose for some \(x_i, \overline{x_i}\) is not a zero divisor in the associated graded ring of \(R\). Then,

\[
\mu(\text{Der } R) \leq 2e(\hat{R}) + 1.
\]

Proof. Let \(\delta \in \text{Der } R\). Write \(\delta\) as an \(n\)-tuple \((a_1, \ldots, a_n)\). Define a map

\[
\phi : \text{Der } R \rightarrow R, \quad (a_1, \ldots, a_n) \mapsto a_1.
\]

Now \(\ker(\phi)\) is the set of all tuples \((0, a_2, \ldots, a_n)\). Thus an element of \(\ker(\phi)\) can be thought of as a derivation of \(R/(x_1)\). Now \(\dim R/(x_1) = 1\). Using similar arguments as that in the proof of Theorem 1.10 below, any derivation \(\delta \in \text{Der}(R/(x_1))\) is uniquely determined by its value on some element, say \(H_2 \in R/(x_1)\). Therefore \(\text{Der}(R/(x_1))\) is isomorphic to an ideal \(J\) in \(R/(x_1)\),

\[
\therefore \ker(\phi) \subset \text{Der}(R/(x_1)) \cong J \subset R/(x_1),
\]

\[
\therefore \ker(\phi) \cong J \subset R/(x_1).
\]

Therefore using the Lemma 1.8 we have,

\[
\mu(\ker(\phi)) \leq e(R/(x_1)). \tag{I}
\]

Suppose \(\overline{x_i}\) is not a zero divisor in the associated graded ring of \(R\) for some \(i\), say \(i = 1\). From the theory of multiplicity we have \(e(R/(x_1)) = e(\hat{R})\) [ZS75b, Chapter VIII, §10]. Let \(I\) be the ideal of all \(a_1\)'s which occur as \(\delta\) varies in \(\text{Der } R\). Then using the Euler derivation of \(R\), we get that \(x_1 \in I\).

Hence \(I/(x_1) \subset R/(x_1)\). As in the proof of Theorem 1.10, \(\hat{R}/(x_1) \supset k[H_2]\) with degree of extension \(= e(\hat{R})\).

\[
\therefore \mu(I/(x_1)) \leq e(\hat{R}/(x_1)) = e(\hat{R}),
\]

\[
\therefore \mu(I) \leq e(\hat{R}) + 1. \tag{II}
\]

Therefore from (I) and (II),

\[
\mu(\text{Der } R) \leq 2e(\hat{R}) + 1. \quad \square
\]

Using this result and results from the papers of O. Riemenschneider [Rie77] and [Rie81], we get a better bound on \(\mu(\text{Der } R)\). Also, using this theorem and Theorem 2.4 we get a better bound for \(\mu(\text{Der } R)\).

Here is a result which gives bound on \(\mu(\text{Der } R)\) in terms of the order of the group \(G\).
Theorem 1.10. Let $G \subset \text{GL}_2(k)$ be a finite group acting linearly on $k[X_1, X_2]$ and let $R = k[X_1, X_2]^G$. Then the module of derivations of $R$ is generated by at most $2|G| + 1$ elements.

Before proceeding to the proof of this theorem, we mention a result by E. Noether.

Proposition 1.11. (See E. Noether [NS02].) Let $G \subset \text{GL}_n(k)$ be a finite group acting linearly on $k[X_1, \ldots, X_n]$. Let $R$ be its ring of invariants. Then $R$ is generated by homogeneous elements $F_1, \ldots, F_N$ such that $\deg F_i \leq |G|$.

Proof of Theorem 1.10. Let $F_1, \ldots, F_n$ be a minimal homogeneous system of generators of $R$ (Proposition 1.11). Then $R = k[y_1, \ldots, y_n]$ with $y_i$ corresponding to $F_i$, for some homogeneous prime ideal $P$. Now $R$ has the Euler derivation

$$\delta_0 = \alpha_1 y_1 \frac{\partial}{\partial y_1} + \alpha_2 y_2 \frac{\partial}{\partial y_2} + \cdots + \alpha_n y_n \frac{\partial}{\partial y_n},$$

where $\alpha_i > 0$ are suitable integers. We write any derivation $\delta$ of $R$ as an $n$-tuple $(a_1, \ldots, a_n)$, where $a_i = \delta(y_i)$. For any such $\delta$ there exists an $r \in R$ such that $\delta - r\delta_0 = (a_1', \ldots, a_n')$, where $a_i'$ does not involve $y_1$. Let $I$ denote the set of all such $a_i'$'s which are the first coordinates of derivations of $R$. Then the image of $I$ under the homomorphism $R \to R/(y_1)$ is an ideal in $R/(y_1)$. As in the proof of Theorem 2.1 later, we can find a general linear combination of $F_1, \ldots, F_N$, say $H_2$, such that $\hat{R}$ is integral over $k[F_1, H_2]$. Then $\hat{R}$ (being Cohen–Macaulay) is a free $k[F_1, H_2]$-module. Hence $\hat{R}$ is a free $k[F_1, H_2]$-module. Hence $\frac{\hat{R}}{F_1 \hat{R}}$ is a free $k[H_2]$-module.

In the proof of Theorem 2.1 we will see that the degree of the extension $\hat{R}/k[F_1, H_2]$ is at most $|G|$. Hence the ideal $I$, considered as a $k[H_2]$-module, has at most $|G|$ generators. This gives $|G| + 1$ generators for the ideal generated by $a_1$'s.

The submodule of the derivations of $R$ such that $a_1 = 0$ induces derivations of $\hat{R}/(F_1)$. This ring is a free module over $k[H_2]$ of rank at most $|G|$. From this it is easy to see that any derivation of $\hat{R}/(F_1)$ is uniquely determined by its value on $H_2$. Consequently, the derivation module $\text{Der}(\hat{R}/(F_1))$ is isomorphic to the ideal generated by $\Delta(H_2)$ as $\Delta$ varies over all the $k$-derivations of this 1-dimensional ring. Hence this derivation module, as a $k[H_2]$-module, is free of rank $\leq |G|$. Putting together the two sets of generators we see that $\text{Der} \hat{R}$ can be generated by at most $2|G| + 1$ derivations.

Remark. In view of Theorem 2.1 below, Theorem 1.9 is stronger than 1.10, provided the hypothesis of Theorem 1.9 is satisfied.

Note that the hypothesis of 1.9 is not always satisfied. A counter example for the same can be found using some rings of invariants described by O. Riemenschneider [Rie81].

2. Multiplicity of rational surface singularities

2.1. Multiplicity of the rings of invariants

This section deals with the multiplicity of the ring of invariants of finite subgroups of $\text{GL}_n(k)$. 
Theorem 2.1. Let $G \subset GL_n(k)$ be a finite group acting on $k[X_1, \ldots, X_n]$ and let $R = k[X_1, \ldots, X_n]^G$ be the ring of invariants. Then the multiplicity of $R$ at its irrelevant maximal ideal is $\leq |G|^{n-1}$.

Below we give an example of a subgroup of $GL_n(k)$ such that this bound is best possible. But for a general subgroup of $GL_n(k)$ this bound appears to be large.

Example. Let $G$ be the cyclic group of order $m$ acting on $k[X_1, \ldots, X_n]$ through scalar matrices. Then the ring of invariants is generated by all monomials of degree $m$. Using the proof of Theorem 2.1 we get that the multiplicity of the ring of invariants is precisely $|G|^{n-1}$.

Proof. Let $F_1, \ldots, F_N$ be homogeneous generators of $R$ with $\deg F_i \leq |G|$ (Proposition 1.11). Since $R$ has dimension $n$, we can find $n$ general linear combinations

$$g_1 = a_{11}F_1 + \cdots + a_{1N}F_N, \quad \ldots, \quad g_n = a_{n1}F_1 + \cdots + a_{nN}F_N$$

such that $k[g_1, \ldots, g_n] \subset \hat{R}$ is an integral extension of degree $e$, where $e$ is the multiplicity of $R$ at its irrelevant maximal ideal $m$ and $\hat{R}$ denotes the completion of $R$ with respect to $m$.

For the degree of the extension we have,

$$k[X_1, \ldots, X_n]/k[g_1, \ldots, g_n] = \ell\left(\frac{k[X_1, \ldots, X_n]}{(g_1, \ldots, g_n)}\right) \leq |G|^n.$$

The inequality in this equation follows since the length, $\ell(\frac{k[X_1, \ldots, X_n]}{(g_1, \ldots, g_n)})$ is the intersection multiplicity of $n$ hypersurfaces $\{g_i = 0\}$ at the origin. Since degree of each $g_i$ is at most $|G|$, using Bezout’s theorem, their total intersection multiplicity at origin is at most $|G|^n$.

Now the degree of the extension $k[X_1, \ldots, X_n]/\hat{R}$ is $|G|$.

Therefore it follows that the degree of the extension $\hat{R}/k[g_1, \ldots, g_n]$ is at most $|G|^{n-1}$.

2.2. An inequality between the multiplicity and the order of the divisor class group

While studying the multiplicity of the ring of invariants and its divisor class group, denoted by $D_G(R)$, E. Brieskorn in his paper [Bri68] has given a complete classification of finite subgroups of $GL_2(k)$. In the same paper he has also given the corresponding divisor class groups for these rings of invariants together with the multiplicities. Using this classification many interesting facts have been observed. This classification and the calculation of the divisor class group and the multiplicity helps us to visualize the result:

$$e(R) \leq |D_G(R)|$$

where $R$ is the ring of invariant of a finite subgroup of $GL_2(\mathbb{C})$. In fact, we have proved this result in a more general situation and the proof does not depend on the above mentioned classification. Note that by Mumford’s result [Mum61], $D_G(R) \cong H_1(V - p, \mathbb{Z})$ and $|D_G(R)| = |\det(C_i \cdot C_j)|$ where $(C_i \cdot C_j)$ denotes the intersection matrix of the resolution of singularities.

This observation gives rise to a natural conjecture:
Conjecture 2.2. For any rational surface singularity, we always have

\[ e(R) \leq |D_G(R)|. \]

Later in the paper, we have verified this conjecture for all quotient singularities (except for \( E_8 \) singularity), rational singularities with multiplicity 3 and 4 and certain quasihomogeneous rational singularities.

Before we state the main result of this section, we first recall a few results by M. Artin.

Let \((V, p)\) be a rational singularity of dimension 2. Let \(\pi: \tilde{V} \to V\) be a resolution of singularity. Let \(C_i, i = 1, \ldots, r\), be exceptional irreducible components of \(\pi^{-1}(p)\). Let \(\mathcal{Z}\) be a divisor with the property that for any \(C_i\), \(\mathcal{Z} \cdot C_i \leq 0\) and for any other divisor \(\mathcal{E}\) with this property, \(\mathcal{Z} \leq \mathcal{E}\). Then \(\mathcal{Z}\) is called the fundamental cycle of \(V\).

Artin has proved the existence of such a divisor \(\mathcal{Z}\) and that it is effective.

**Theorem 2.3** (M. Artin [Art66]). Let \((V, p)\) be a rational singularity. Let \(\mathcal{Z}\) be the fundamental cycle. Let \(e = e(V)\) be the multiplicity of \((V, p)\). Then

\[ -\mathcal{Z}^2 = e. \]

Let \(\pi: \tilde{V} \to V\) be a resolution of singularity. Then \(m_p\mathcal{O}_{\tilde{V}}\) is an invertible ideal which defines \(\mathcal{Z}\).

We use this result to prove the following general result.

**Theorem 2.4.** Let \((V, p)\) be a germ of a rational surface singularity and let \(\mathcal{Z} = \sum_{i=1}^{r} m_i C_i\) where \(C_i\)'s are the exceptional irreducible components of \(\pi^{-1}(p)\). If \(m_i = 1\) for some \(i\) then

\[ -\mathcal{Z}^2 \leq |\det(C_i \cdot C_j)|, \]

where \((C_i \cdot C_j)\) denotes an intersection matrix of \(C_i\)'s.

**Proof.** Let \(\mathcal{Z} = \sum_{i=1}^{r} m_i C_i\) with \(m_1 = 1\). Let \(|\det(C_i \cdot C_j)| = \delta\). Then there exists a divisor \(\mathcal{E}\) with rational coefficients supported on \(\bigcup_{i} C_i\) such that \(\mathcal{E} \cdot C_1 = -\delta\) and \(\mathcal{E} \cdot C_i = 0\) for \(i > 1\).

**Claim.** \(\mathcal{E}\) is an effective divisor with integral coefficients.

Write \(\mathcal{E} = \sum_j a_j C_j\), then we have the equations,

\[ \mathcal{E} \cdot C_1 = -\delta, \]

\[ \mathcal{E} \cdot C_i = 0 \quad \text{for} \quad i > 1. \]

Thus,

\[ a_1 \cdot C_1 \cdot C_1 + a_2 \cdot C_2 \cdot C_1 + \cdots + a_n \cdot C_n \cdot C_1 = -\delta, \]

\[ \vdots \]

\[ a_1 \cdot C_1 \cdot C_i + a_2 \cdot C_2 \cdot C_i + \cdots + a_n \cdot C_n \cdot C_i = 0. \]
We see by Cramer’s rule that $E$ has integer coefficients. It is a well-known fact that since the intersection matrix $(C_i \cdot C_j)$ is negative definite, $E$ is effective. Now

$$E \cdot Z = -\delta.$$ 

Write $E = Z + E'$. Then $E' \geq 0$ since $Z$ is the fundamental cycle,

$$\therefore (Z + E') \cdot Z = -\delta,$$

$$\therefore Z^2 + Z \cdot E' = -\delta,$$

but,

$$Z \cdot E' \leq 0.$$

$$\therefore \delta \geq -Z^2 = \text{multiplicity of } \mathcal{O}(V,p) \ldots \text{using Artin’s result.} \quad \Box$$

**Fact.** (See [Bri68].) For a quotient singularity, any curve with self intersection less than $-2$, occurs with multiplicity 1 in $Z$.

**Corollary 2.5.** For any quotient singularity, other than $E_8$, we have

$$e \leq |\det(C_i \cdot C_j)|.$$ 

**Corollary 2.6.** If $(V, p)$ is a rational triple point, then

$$3 = e \leq |\det(C_i \cdot C_j)|.$$ 

**Proof.** Since $(V, p)$ is a rational triple point, $Z^2 = -3$. Assume that $V \rightarrow V$ is the minimal resolution of singularities. By Artin’s result, $Z^2 + Z \cdot K = -2$ and hence $Z \cdot K = 1$, where $K$ is the canonical divisor. Since $K \cdot C_i \geq 0$ for every $i$, by adjunction formula there exists a unique $(-3)$-curve (i.e. self intersection of this curve is $-3$) and all other curves are $(-2)$-curves (i.e. self intersection of this curve is $-2$) in $\text{Support}(Z)$ and the $(-3)$-curve occurs with multiplicity 1 in $Z$. □

**Corollary 2.7.** For a quotient singularity, other than $E_8$, we have,

$$\mu(\text{Der } R) \leq 2 \left| \frac{G}{[G,G]} \right| + 1,$$

provided there exists $x_i$ as mentioned in Theorem 1.9.

**Proof.** By Mumford’s theorem, $|\det(C_i \cdot C_j)| = |\frac{G}{[G,G]}|$. Therefore applying Theorems 1.9 and 2.4 we get the required inequality. □
Remark. Using Theorem 1.9, we examine a few examples. Consider the associated graded ring of the ring of invariants in each case. Let $I$ be the set of defining equations of the ring of invariants, given by O. Riemenschneider [Rie77]. Then using the result of J. Wahl we have,

$$\langle \text{LT}(I) \rangle = \langle \text{LT}(f) \mid f \text{ is a generator of } I \text{ given in [Rie77]} \rangle,$$

where $\langle \text{LT}(I) \rangle$ denotes the leading ideal of $I$ and $\text{LT}(f)$ denotes the leading term of $f$.

In many cases, some $x_i$ will be a non-zero divisor in the associated graded ring, in which case Theorem 1.9 gives a better bound on $\mu(\text{Der } R)$.

2.3. Bound on multiplicity using gluing of rational trees

We use some terminology and results about weighted trees from [TT04].

Definition 2.8 (Valency of vertex). Let $R$ be a weighted tree. We define the valency of a vertex $E_i$ to be the number of vertices joined to $E_i$. It is denoted by $v_R(E_i)$.

If $R$ is the dual graph of a resolution of a rational surface singularity then we call $R$ a rational tree.

If $R$ corresponds to a rational double point then we abbreviate it by RDP.

Definition 2.9 (Gluing of trees). Let $\Gamma_1$ and $\Gamma_2$ be two weighted trees. The weighted tree $\Gamma$ obtained by attaching a vertex of $\Gamma_1$ and a vertex of $\Gamma_2$ by an edge is called the gluing tree of $\Gamma_1$ and $\Gamma_2$ at these vertices.

Remark. In case of dual graphs of rational surface singularities, the weight of a vertex is equal to negative of the self intersection of the corresponding curve.

Theorem 2.10 (Spivakovsky [TT04, Proposition 3.3]). If $R$ is a rational tree, then for any vertex $E_i$ of $R$ of weight $w_i$,

$$w_i + 1 \geq v_R(E_i).$$

Theorem 2.11. (See Tráng and Tosun [TT04, Corollary 4.1].) If the gluing of the rational trees $R_1$ and $R_2$ at $E_1$ and $F_1$ is rational, then the coefficient of $E_1$ (respectively $F_1$) in the fundamental cycle of $R_1$ (respectively $R_2$) is 1.

Theorem 2.12. (See Tráng and Tosun [TT04, Lemma 5.4].) Suppose $R$ is a rational tree of multiplicity $m$ with weights of every vertex is $\geq 2$. Assume that $R$ contains a unique vertex of weight $\geq 3$. Then the number of branch points is $\leq m - 2$.

Definition 2.13 (Generalized quotient singularity). Let $(V, p)$ be a quotient singularity of dimension 2 and $\pi: \tilde{V} \to V$ be a resolution of singularity. Let $G$ denote the dual graph of the exceptional divisor. Let $R$ denote the dual graph obtained from $G$ by reducing the self intersections of the irreducible components of $G$ arbitrarily. Then $R$ is called as a generalized quotient singularity.
Remark. Note that a generalized quotient singularity is a rational singularity. It is easy to see that the dual graph of a generalized quotient singularity is a subgraph of a resolution of a quotient singularity.

Corollary 2.14. If $R$ is a sandwiched singularity or a generalized quotient singularity then the Conjecture 2.2 is true.

Proof. By definition, a sandwiched singularity corresponds to a subgraph of a resolution of a smooth point. Now using Theorem 2.11, there exists at least one curve in the fundamental cycle of a sandwich singularity (or a generalized quotient singularity) with coefficient one. Now apply Theorem 2.4. □

An example of a sandwiched singularity

Let $R$ be a negative definite weighted tree of non-singular rational curves such that there is only one branch point and all the weights are $\geq 2$. Assume that there are $n$ linear branches meeting the branch point and the branch point has self intersection $-b \leq -n - 1$.

We claim that this is a sandwiched singularity, hence a rational singularity. To see this, we start with a $(-1)$-curve $C_0$ on a smooth projective rational surface $X$. We will construct $R$ starting from $C_0$ by a sequence of blowups. Choose $b - 1$ points $P_1, \ldots, P_{b-1}$ on $C_0$. Blowup $X$ at $P_1, \ldots, P_{b-1}$ and let $X_1$ be the surface obtained and let $C'_0$ be the proper transform of $C_0$. Then $C'_0^2 = -b$. Let the exceptional divisors be $E_1, \ldots, E_{b-1}$. Now blowing up suitable points on $E_1, \ldots, E_n$ we can create a suitable tree of $\mathbb{P}^1$’s which contains the proper transform of $C_0$ and $n$ arbitrary linear chains of non-singular rational curves with prescribed self intersections of the irreducible components $\leq -2$ meeting this proper transform. It is clear that $R$ is a sandwiched singularity. Thus by Corollary 2.14 we see that the Conjecture 2.2 is true.

It is easy to construct a quasihomogeneous rational singularity whose dual graph is $R$.

Remark. (See [GM99, Lemma 2.7].) Any proper connected subgraph of a singular fiber of a $\mathbb{P}^1$-fibration on a smooth projective surface defines a rational singularity.

Corollary 2.15. Let $F_0$ be a singular fiber of a $\mathbb{P}^1$-fibration on a smooth projective surface $\phi : X \to B$. Let $\Delta$ be obtained by removing at least two irreducible components from $F_0$. Using above remark we get that $\Delta$ contracts to a rational singularity $(V, p)$. Then using Theorem 2.11, we see that Conjecture 2.2 is true for $(V, p)$.

2.4. Verification of Conjecture 2.2 for multiplicity 4 case

Let $(V, p)$ be a rational singularity with multiplicity 4. Let $R$ be the corresponding rational tree.

In this section we verify the result $4 = e \leq |DG|$ for $V$.

Let $\pi : \tilde{V} \to V$ be the minimal resolution of singularities.

Fundamental cycles of multiplicity 4 surface singularities

Let $3$ be the fundamental cycle. Write $3 = \sum a_i C_i$. Therefore

$$3^2 = \left( \sum a_i C_i \right)^2 = -4.$$
We will assume that \( a_j \geq 2 \) for all \( j \) and arrive at a contradiction to prove the conjecture in this case.

Since \( V \) is a rational singularity, for the canonical divisor \( K \) we have,

\[
3^2 + K \cdot 3 = -2,
\]

\[
K \cdot 3 = 2,
\]

\[
K \cdot \left( \sum a_i C_i \right) = 2.
\]

The possible cases for this to happen are as follows:

(I) \( K \cdot C_i = 2 \) for some \( i \) with \( a_i = 1 \),

\( K \cdot C_j = 0 \) for \( j \neq i \).

(II) \( K \cdot C_i = 1 \) for some \( i \) with \( a_i = 1 \),

\( K \cdot C_j = 1 \) with \( a_j = 1, j \neq i \),

\( K \cdot C_k = 0 \) for \( k \neq i, j \).

(III) \( K \cdot C_i = 1 \) for some \( i \) with \( a_i = 2 \),

\( K \cdot C_j = 0 \) for \( j \neq i \).

Now in Cases (I) and (II), using Theorem 2.4, we get the required inequality,

\[ 4 = e \leq |D_G|. \]

Consider Case (III): We have \( K \cdot C_i = 1 \) and \( a_i = 2 \). Now by adjunction formula, for \( j \neq i \), we have,

\[
K \cdot C_j + C_j^2 = -2,
\]

therefore, \( C_j^2 = -2 \);

\[
K \cdot C_i + C_i^2 = -2,
\]

therefore, \( C_i^2 = -3 \).

Thus there exists exactly one \((-3)\)-curve and the other curves are \((-2)\)-curves. The \((-3)\)-curve occurs with multiplicity 2 in \( \mathfrak{3} \).

**Corollary 2.16.** The \((-3)\)-curve can meet at most 4 other curves.

**Proof.** Follows from Theorem 2.10. \( \Box \)

First we will prove that the \((-3)\)-curve cannot meet 4 other curves. For, if this happens, then since \( a_j \geq 2 \) for all \( j \) by assumption and the \((-3)\)-curve occurs with multiplicity 2 in \( \mathfrak{3} \), we get \( 3 \cdot C_i > 0 \).

Suppose \( C_i \) meets exactly 3 curves, say \( C_1, C_2, C_3 \). Then note that \( a_i = 2 \) for \( i = 1, 2, 3 \), for if at least one of the \( a_1, a_2, a_3 \) is bigger than 2, say \( a_1 > 2 \), then \( 3 \cdot C_i > 0 \) since \( a_i = 2 \). This is not possible. Now suppose there exists a curve \( C_{11} \) which meets \( C_1 \). Hence,
$0 \geq 3 \cdot C_1 = (2C_i + 2C_1 + a_{11}C_{11} + \cdots) \cdot C_1$

$\quad = 2C_i C_1 + 2C_1^2 + a_{11}C_1 C_{11} + \cdots$

$\quad = 2 - 4 + a_{11} + \alpha \quad \text{where} \quad \alpha \geq 0.$

Now since $a_{11} \geq 2$, we get that $\alpha = 0$ and $a_{11} = 2$, i.e. $C_1$ is not a branch point (i.e. $C_1$ meets only $C_i$ and $C_{11}$). Now suppose $C_{11}$ meets $C_{12}$. Using similar arguments, we get that $a_{12} = 2$ and $C_{12}$ is not a branch point.

Thus from this we see that the chain starting from $C_1$, $C_2$ or $C_3$ is a linear chain (i.e. does not contain a branch point) and coefficient of every curve is 2. Hence the fundamental cycle can be written as

$$3 = 2 \cdot \sum C_j = 2 \cdot \tilde{3}$$

which is a contradiction to the fact the $\tilde{3}$ is a fundamental cycle. Therefore $C_i$ can meet at most 2 curves.

Suppose $C_i$ meets 2 curves, say $C_1$, $C_2$. Then both $C_1$, $C_2$ are part of RDPs, i.e. $C_i$ meets 2 RDPs.

**Corollary 2.17.** Number of branch points in $\mathcal{R}$ is at most 2.

**Proof.** Follows from Theorem 2.12. $\square$

If $\mathcal{R}$ has no branch point, then $\mathcal{R}$ defines a quotient singularity. So we can assume that $\mathcal{R}$ has at least one branch point.

This in turn implies that the $(-3)$-curve meets two RDPs so that $\mathcal{R}$ has one or two branch points.

Then some simple calculations for the determinant of these trees verify that the determinant of $\mathcal{R}$ is at least 4. This verifies the conjecture in all cases for a rational singularity of multiplicity 4.

**Corollary 2.18.** Let $(V, p)$ be a rational singularity with multiplicity 4. Then

$$4 = e \leq |\det(C_i \cdot C_j)|.$$

2.5. Quasihomogeneous rational trees with determinant 2

In this section we verify that if $(V, p)$ is a quasihomogeneous rational surface singularity with divisor class group of $(V, p)$ of order 2 then the multiplicity of $(V, p)$ is 2.

First we recall some results due to W. Neumann [Neu83].

Let $\mathcal{R}$ be a rational tree with determinant 2 corresponding to the minimal resolution of a quasihomogeneous rational singularity $(V, p)$, $\pi : \tilde{V} \to V$. Then by Mumford’s theorem,

$$H_1(V - p, \mathbb{Z}) \cong \mathbb{Z}/(2).$$

**Fact.** For a quasihomogeneous rational singularity the corresponding rational tree has the following form. There is only one branch point, say $C_0$, with self intersection $-b \leq -2$ and there are $n$
linear chains meeting \( C_0 \). Let the absolute values of the determinants of these chains be \( \alpha_i \). Let \( \beta_i \) denote the absolute values of the determinants of the linear subchains obtained by removing the curves meeting \( C_0 \).

Then the absolute value of the determinant of \( R \) is

\[
\Delta = \left( b - \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} \right) \cdot \alpha_1 \cdot \alpha_2 \cdots \alpha_n
\]

which is by assumption 2. Note that \( \beta_i < \alpha_i \) and \( \gcd (\alpha_i, \beta_i) = 1 \).

Theorem 2.10 implies \( n \leq b + 1 \).

**Theorem 2.19.** (See Neumann [Neu83].) Let \( \pi : (V', p') \rightarrow (V, p) \) be the maximal abelian cover. \( V' \) is an affine surface with a good \( \mathbb{C}^* \)-action such that the morphism is finite and étale outside \( p \). Then \( V' \) is defined by a Brieskorn complete intersection:

\[
V' = \{ z \in \mathbb{C}^n \mid a_{i1}z_1^{\alpha_1} + \cdots + a_{in}z_n^{\alpha_n} = 0, \ 1 \leq i \leq n - 2, \ \alpha_1 \leq \cdots \leq \alpha_n \}
\]

where the coefficients \( a_{ij} \in \mathbb{C} \) and any \((n - 2) \times (n - 2)\) minor of the matrix \((a_{ij})\) is non-zero. Since the coefficients \( a_{ij} \) are general, the equations become

\[
F_1: \quad a_{11}z_1^{\alpha_1} + \cdots + a_{1n}z_n^{\alpha_n} = 0,
F_2: \quad a_{22}z_2^{\alpha_2} + \cdots + a_{2n}z_n^{\alpha_n} = 0,
\]

\[
\vdots
F_{n-2}: \quad a_{n-2,n-2}z_{n-2}^{\alpha_{n-2}} + a_{n-2,n-1}z_{n-1}^{\alpha_{n-1}} + a_{n-2,n}z_n^{\alpha_n} = 0.
\]

We can assume that \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \). Note that the lowest degree terms of these equations form a regular sequence in the power series ring \( \mathbb{C}[z_1, \ldots, z_n] \). Therefore the multiplicity of \( V' \) is the product of their degrees \( \alpha_1 \alpha_2 \cdots \alpha_{n-2} \).

Let \( A \) be the normal graded coordinate ring of \( V \). Let \( X = (V - p)/\mathbb{C}^* \).

**Theorem 2.20.** (See K.-i. Watanabe [Wat81].) There exists an exact sequence

\[
0 \rightarrow \mathbb{Z} \xrightarrow{\theta} \text{cl}(X) \rightarrow \text{cl}(A) \rightarrow \text{coker}(\Phi) \rightarrow 0
\]

where \( \theta \) is given by \( 1 \mapsto \alpha D \) where \( \alpha = \text{lcm}(\alpha_1, \ldots, \alpha_n) \). Here \( D = D_0 + x_1 + \cdots + x_n - \sum_{i=1}^{n} \left( \frac{\beta_i}{\alpha_i} \right)x_i \) where \( D_0 \) is a suitable divisor on \( X \) and \( x_1, \ldots, x_n \) are the points in \( X \) corresponding to the \( n \) linear chains. Then \( \deg(D_0 + x_1 + \cdots + x_n) = b = -C_0^2 \). The map \( \Phi \) is given by

\[
\Phi : \mathbb{Z} \rightarrow \bigoplus_{i=1}^{n} \mathbb{Z}/(\alpha_i \mathbb{Z}),
\]

\[
1 \mapsto (\overline{\beta_1}, \overline{\beta_2}, \ldots, \overline{\beta_n}).
\]
Now from the above exact sequence, we get that if the coker(Φ) = (0) then \( \bigoplus \mathbb{Z}/(\alpha_i)\mathbb{Z} \) is generated by \( (\beta_1, \ldots, \beta_n) \). This is possible only if all \( \alpha_i \)'s are pairwise coprime. In this case \( \deg \alpha D = 2 \).

Suppose coker(Φ) \( \neq (0) \), then coker(Φ) = \( \mathbb{Z}/(2) \). But then \( |\bigoplus \mathbb{Z}/(\alpha_i)\mathbb{Z}| = \Pi_i \alpha_i \) and \( \text{ord}(\beta_1, \ldots, \beta_n) \) in this group is \( \text{lcm}(\alpha_1, \ldots, \alpha_n) = \frac{1}{2} \Pi_i \alpha_i \).

First we verify Conjecture 2.2 for \( n = 3, 4 \) and then give arguments for general \( n \).

Assume \( n = 3 \).

Therefore using Theorem 2.19, \( V' \) is given by a single equation:

\[ z_1^{\alpha_1} + z_2^{\alpha_2} + z_3^{\alpha_3} = 0. \]

Then the canonical divisor is given by

\[ K_{V'} = \frac{dz_2 \wedge dz_3}{\alpha_1 z_1^{\alpha_1-1}}. \]

Now the group \( \mathbb{Z}/(2) \) is acting on \( \mathbb{C}[z_1, z_2, z_3] \). The action is

\[ z_1 \mapsto \pm z_1, \]
\[ z_2 \mapsto \pm z_2, \]
\[ z_3 \mapsto \pm z_3. \]

**Case I.** coker(Φ) = (0): Therefore all \( \alpha_i \)'s are pairwise coprime.

**Case I.1.** All \( \alpha_i \)'s are odd.

Then note that the all the three signs should be same, otherwise the hypersurface is not preserved under the \( \mathbb{Z}/(2) \)-action. Also note that all positive signs give a trivial action. Therefore the action should necessarily be:

\[ z_1 \mapsto -z_1, \]
\[ z_2 \mapsto -z_2, \]
\[ z_3 \mapsto -z_3. \]

Then the canonical divisor \( K_{V'} \) is invariant under this action and hence \( K_V \) is trivial. But \( K_V \) is trivial if and only if \( V \) is an RDP, which is not possible since all \( \alpha_i \)'s are odd.

**Case I.2.** One of the \( \alpha_i \)'s is even, say \( \alpha_1 \), and \( \alpha_2, \alpha_3 \) are odd.

But then the action would be: (since two negatives and one positive will not preserve the hypersurface)
\[ z_1 \mapsto -z_1, \]
\[ z_2 \mapsto z_2, \]
\[ z_3 \mapsto z_3. \]

Note that this is a pseudo-reflection irrespective of the parity of \( \alpha_i \)'s.

**Case II.** coker(\( \Phi \)) = \( \mathbb{Z}/(2) \):

**Case II.1.** \( \alpha_1, \alpha_2 \) are even and \( \alpha_3 \) is odd.

Then the action would be:

\[ z_1 \mapsto -z_1, \]
\[ z_2 \mapsto -z_2, \]
\[ z_3 \mapsto z_3. \]

Then the canonical divisor \( K_{V'} \) is 

\[ K_{V'} = \frac{dz_1 \wedge dz_2}{\alpha_3 z_3^{\alpha_3-1}} \]

is invariant under \( \mathbb{Z}/(2) \)-action. Hence \( K_V \) is trivial. But then \( V \) is an RDP. We have already verified the conjecture in case of RDPs.

**Case II.2.** All \( \alpha_i \)'s are even.

**Claim.** This case cannot occur.

**Proof.** Suppose \( \alpha_i = 2\alpha'_i \) for \( i = 1, 2, 3 \).

\[ \Phi : \mathbb{Z} \to \left( \mathbb{Z}/(2\alpha'_1) \mathbb{Z} \oplus \mathbb{Z}/(2\alpha'_2) \mathbb{Z} \oplus \mathbb{Z}/(2\alpha'_3) \mathbb{Z} \right) = G, \]

\[ 1 \mapsto (\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = \bar{\beta}. \]

Now \(|G| = 8\alpha'_1\alpha'_2\alpha'_3\) and \(|\bar{\beta}| = \text{lcm}(\alpha_1, \alpha_2, \alpha_3) = 2\alpha'_1\alpha'_2\alpha'_3\). Therefore \(|G|/|\bar{\beta}| = 4 \neq 2. \]

**Remark.** The conjecture can also be verified in case of \( n = 3 \) using results of T. Okuma [Oku04].

**Assume** \( n = 4 \).

The Brieskorn complete intersection is given by the following equations:

\[ \begin{align*}
F_1: & \quad a_1z_1^{\alpha_1} + a_2z_2^{\alpha_2} + a_3z_3^{\alpha_3} + a_4z_4^{\alpha_4} = 0, \\
F_2: & \quad b_1z_1^{\alpha_1} + b_2z_2^{\alpha_2} + b_3z_3^{\alpha_3} + b_4z_4^{\alpha_4} = 0
\end{align*} \]

with \( a_i, b_i \in \mathbb{C} \).
The group $\mathbb{Z}/(2)$ is acting on $\mathbb{C}[z_1, z_2, z_3, z_4]$ via the action $z_i \mapsto \pm z_i$, $i = 1, 2, 3, 4$. The canonical divisor is given by

$$K_V' = \frac{dz_3 \wedge dz_4}{|J(F_1, F_2)(z_1, z_2)|} = \frac{dz_3 \wedge dz_4}{\gamma (z_1^{\alpha_1-1} z_2^{\alpha_2-1})}$$

for some non-zero constant $\gamma$.

Now as in the previous case, the conjecture can be verified by making the following subcases.

**Case I.** $\text{coker}(\Phi) = (0)$:

**Case I.1.** All $\alpha_i$'s are odd.

Then for the $\mathbb{Z}/(2)$-action, all the signs should be same. All positive signs will give the trivial action. Suppose the action is $z_i \mapsto -z_i$ for $i = 1, 2, 3, 4$. Then the canonical divisor $K_V'$ is invariant under $\mathbb{Z}/(2)$-action. Hence $K_V$ is trivial. Therefore $V$ is an RDP which is not possible since all $\alpha_i$'s are odd.

**Case I.2.** $\alpha_1$ is even and $\alpha_2, \alpha_3, \alpha_4$ are odd.

But then the action would be (since two negatives and two positive will not preserve the hypersurface):

- $z_1 \mapsto -z_1$,
- $z_2 \mapsto z_2$,
- $z_3 \mapsto z_3$,
- $z_4 \mapsto z_4$.

But this is a pseudo-reflection.

**Case II.** $\text{coker}(\Phi) = \mathbb{Z}/(2)$:

Suppose $\alpha_1, \alpha_2$ are even and $\alpha_3, \alpha_4$ are odd. Let $\alpha_1 = 2\alpha'_1$ and $\alpha_2 = 2\alpha'_2$. Then the action is:

- $z_1 \mapsto -z_1$,
- $z_2 \mapsto -z_2$,
- $z_3 \mapsto z_3$,
- $z_4 \mapsto z_4$.

But the canonical divisor $K_V'$ is invariant under $\mathbb{Z}/(2)$-action. Hence $K_V$ is trivial. Therefore $V$ is an RDP, for which we have already verified the conjecture.

As mentioned earlier, the case when three or more $\alpha_i$'s are even cannot occur.

Now we give the arguments for the general case.
The equations for the Brieskorn complete intersection are as in Theorem 2.19. Then the canonical divisor $K_{V'}$ is given as:

$$K_{V'} = \frac{dz_1 \wedge dz_2}{|J(F_1, ..., F_{n-2})|}.$$ 

As before the subcases to be considered are as follows:

**Case I.** $\text{coker}(\Phi) = (0)$: All $\alpha_i$’s are pairwise coprime.

**Case I.1.** All $\alpha_i$’s are odd. Therefore all signs should be same. If all signs are positive then we get the trivial action. Suppose the action is $z_i \mapsto -z_i$, for all $i$. Then the canonical divisor $K_{V'}$ is invariant under $\mathbb{Z}/(2)$-action. Hence $K_V$ is trivial. Therefore $V$ is an RDP which is not possible since all $\alpha_i$’s are odd.

**Case I.2.** $\alpha_1$ is even and $\alpha_i$’s are odd for $i \neq 1$. But this gives a pseudo-reflection.

**Case II.** $\text{coker}(\Phi) = \mathbb{Z}/(2)$:

Suppose $\alpha_1, \alpha_2$ are even and $\alpha_i$’s are odd for $i \neq 1, 2$. Then the canonical divisor $K_{V'}$ is invariant under $\mathbb{Z}/(2)$-action. Hence $K_V$ is trivial. Therefore $V$ is an RDP, for which we have already verified the conjecture.

The case when three or more $\alpha_i$’s are even cannot occur as in Case II.2 for $n = 3$.

This verifies the conjecture for all quasihomogeneous rational singularities with $\text{cl}(A) = \mathbb{Z}/(2)$.

### 2.6. An interesting example

The authors would like to thank M. Tosun for providing the following interesting example.

Consider a tree obtained by gluing of two $D_5$ type singularities at the small end to a vertex of weight 3. Then this defines a rational surface singularity with all the coefficients in the fundamental cycle $\geq 2$. The determinant of this tree is $-8$ and the multiplicity of the singular point is 4.

```
-2 -2 -2 -3 -2 -2 -2
-2
-2
-2
```

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