Convergence of Solutions for a Class of Neutral Difference Equations

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Abstract—Consider the neutral difference equation

\[ \nabla(x_n - cx_{n-k}) = f(n, x_n, x_{n-k}), \]

where \( k \) is a positive integer, \( 0 < c < 1 \), \( \nabla \) denotes the backward difference operator \( \nabla y_n = y_n - y_{n-1} \), \( f : N \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous and decreasing with respect to the second argument. We give the results that solutions of the equation convergent to constants. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Consider the difference equation

\[ \nabla(x_n - cx_{n-k}) = f(n, x_n, x_{n-k}), \]

where \( k \) is a positive integer, \( 0 \leq c < 1 \), \( \nabla \) denotes the backward difference operator \( \nabla y_n = y_n - y_{n-1} \), \( f : N \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous and \( f(n, u, v) \) is decreasing with respect to \( u \in \mathbb{R} \).

Some special cases of equation (1) have been discussed in the literature, for example, in [1,2], the second author and Yu, respectively, considered the equation

\[ \nabla(x_n - cx_{n-k}) = -F(x_n) + G(x_{n-k}) \]

and the special case where \( c = 0 \) in equation (2), where \( k \) is a positive integer, \( 0 \leq c < 1 \), \( f \) and \( G \) are continuous, and \( F \) is increasing on \( \mathbb{R} \), and that if \( F(y) - G(y) \) does not change sign for all \( y \in \mathbb{R} \), then every bounded solution of equation (2) tends to a constant as \( n \rightarrow \infty \); if \( F(y) = G(y) \) for all \( y \in \mathbb{R} \), then every solution of equation (2) tends to a constant as \( n \rightarrow \infty \). Recently, in [3], we considered the special case where \( c = 0 \) in equation (1) and gave the results of asymptotic behavior of solutions of equation (1).

In this paper, we consider the case where \( 0 < c < 1 \) in equation (1). For this case, equation (1) is a neutral difference equation, and it can be viewed as a discrete analogue of the neutral delay differential equation

\[ \frac{d}{dt}[x(t) - cx(t - r)] = f(t, x(t), x(t - r)), \quad 0 < c < 1, \quad r > 0. \]

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The problem of the convergence of solutions of equation (3) has been investigated in [4], and some special cases of equation (3) also have been considered for this problem by many authors (see, e.g., [5-8]).

By a solution of equation (1), we mean a sequence \( \{x_n\} \) of real numbers which is defined for \( n \geq n_0 - k + 1 \) and satisfies equation (1) for \( n = n_0 + 1, n_0 + 2, \ldots \), and initial conditions

\[ x_{n_0 - j} = a_j, \quad j = 0, 1, \ldots, k - 1, \tag{4} \]

where \( a_j (j = 0, 1, \ldots, k - 1) \) are some arbitrary real constants.

Throughout this paper, we shall assume that \( g : \mathbb{R}^2 \to \mathbb{R} \) is continuous, \( g(u, v) \) is decreasing with respect to \( u \) and increasing with respect to \( v \), and \( g(u, u) \equiv 0 \), for all \( u \in \mathbb{R} \), the real sequence \( \{p_n\} \) and \( \{q_n\} \) satisfy:

\[ \sum_{i=1}^{\infty} |p_i| < \infty, \quad 0 < q_n < M, \]

where \( M \) is a positive constant.

Let \( \{x_n\} (n = n_0 - k + 1, n_0 - k + 2, \ldots) \) be a solution of equation (1). We define the set

\[ N_i = \{ik, ik + 1, \ldots, (i + 1)k\}, \quad I_{n_0} = \{n_0, n_0 + 1, \ldots, n_0 + 2k\} \]

and the sequences \( \{A_i\} \) and \( \{B_i\} \):

\[ A_i = \max_{n \in N_i} \max\{x_n, y_n\}, \quad B_i = \min_{n \in N_i} \min\{x_n, y_n\}, \]

where \( y_n = x_n - cx_{n-k}/1 - c \) and \( i > -1 + n_0/k \) is a nonnegative integer. For the sake of convenience, throughout this paper, we shall use the above notations.

Our aim in this paper is to prove that if either \( f(n, u, v) < 0 \) for \( u > v \) and \( f(n, u, v) \leq q_n g(u, v) + p_n \) for \( u \leq v \) or \( f(n, u, v) > 0 \) for \( u < v \) and \( f(n, u, v) \geq q_n g(u, v) + p_n \) for \( u \geq v \), then every bounded solution of equation (1) tends to a constant as \( n \to \infty \); if \( (u - v)f(n, u, v) < 0 \) for \( u \neq v \) and either \( f(n, u, v) \leq q_n g(u, v) + p_n \) for \( u \leq v \) or \( f(n, u, v) > q_n g(u, v) + p_n \) for \( u \geq v \), then every solution of equation (1) tends to a constant as \( n \to \infty \). Our results as special cases include the all results in [1].

## 2. LEMMAS

In this section, we will establish several important lemmas which will be used to prove our main results in Section 3.

**LEMMA 1.** For any \( a_j \in \mathbb{R} (j = 0, 1, \ldots, k - 1) \), the initial value problem (1) and (4) has a unique solution.

**PROOF.** Since \( f(n, u, v) \) is continuous on \( N \times \mathbb{R}^2 \) and decreasing with respect to \( u \in \mathbb{R} \), by setting \( \phi(n, u, v) = u - f(n, u, v) \), where \( n \) and \( v \) are independent of \( u \), \( \phi(n, u, v) \) is also continuous and increasing with respect to \( u \in \mathbb{R} \), and it is clear that \( \lim_{u \to -\infty} \phi(n, u, v) = -\infty \) and \( \lim_{u \to \infty} \phi(n, u, v) = \infty \). Therefore, exists a function \( \phi^{-1}(n, z, v) \), the inverse function of \( z = \phi(n, u, v) \), for all \( z \in \mathbb{R} \).

By (1), we have

\[ \phi(n, x_n, x_{n-k}) = cx_{n-k} + x_{n-1} - cx_{n-1-k}, \]

which implies that

\[ x_n = \phi^{-1}(n, cx_{n-k} + x_{n-1} - cx_{n-1-k}, x_{n-k}), \]

for \( n = n_0 + 1, n_0 + 2, \ldots \).

Therefore, by the method of steps, we find that the initial value problem (1) and (4) has a unique solution. The proof is complete.
LEMMA 2. If $f(n, u, v) < 0$ for $u > v$, then $A_i \geq A_{i+1}$ for all nonnegative integer $i$ with $i > -1 + n_0/k$.

PROOF. Suppose the contrary. Then there exists an integer $m > -1 + n_0/k$ such that $A_m < A_{m+1}$, that is
\[ \max_{n \in N_m} \{x_n, y_n\} < \max_{n \in N_{m+1}} \max \{x_n, y_n\}. \]

Let $a_{m+1} \in N_{m+1}$, such that
\[ \max \{x_{a_{m+1}}, y_{a_{m+1}}\} = A_{m+1}. \]

Then we have only two cases to consider.

CASE 1. $A_{m+1} = x_{a_{m+1}} \geq y_{a_{m+1}} = x_{a_{m+1}} - cx_{a_{m+1}-k}/1 - c$. In this case, we have $x_{a_{m+1}} \leq x_{a_{m+1}-k}$, and hence
\[ A_{m+1} = x_{a_{m+1}} \leq x_{a_{m+1}-k} \leq \max_{n \in N_m} \{x_n\} \leq A_m, \]
which is contrary to the assumption $A_m < A_{m+1}$.

CASE 2. $A_{m+1} = y_{a_{m+1}} = x_{a_{m+1}} - cx_{a_{m+1}-k}/1 - c > x_{a_{m+1}}$. In this case, we have $x_{a_{m+1}} > x_{a_{m+1}-k}$, which implies $f(n, x_{a_{m+1}}, x_{a_{m+1}-k}) < 0$. On the other hand, in view of equation (1), we have
\[
\begin{align*}
  f(n, x_{a_{m+1}}, x_{a_{m+1}-k}) &= \nabla(x_{a_{m+1}} - cx_{a_{m+1}-k}) \\
  &= (x_{a_{m+1}} - cx_{a_{m+1}-k}) - (x_{a_{m+1}-1} - cx_{a_{m+1}-1-k}) \\
  &= (1 - c) \left[ A_{m+1} - \frac{x_{a_{m+1}-1} - cx_{a_{m+1}-1-k}}{1 - c} \right] \\
  &\geq (1 - c) \left[ A_{m+1} - \max_{n \in N_m \cup N_{m+1}} \{x_n, y_n\} \right] \\
  &= 0.
\end{align*}
\]

This is a contradiction, and so there exists no integer $m$ such that $A_m < A_{m+1}$. This completes the proof.

Similarly, we can prove.

LEMMA 3. If $f(n, u, v) > 0$ for $u < v$, then $B_i < B_{i+1}$ for all nonnegative integer $i$ with $i > -1 + n_0/k$.

LEMMA 4. (See [1].) Let $c$ be a constant with $0 < c < 1$, $\{x_n\} (n = n_0 - k + 1, n_0 - k + 2, \ldots)$ be a bounded sequence with
\[ \lim_{n \to \infty} (x_n - cx_{n-k}) = \alpha \]
exists. Then $\lim_{n \to \infty} x_n$ exists and $\lim_{n \to \infty} x_n = \alpha/1 - c$. The proof of Lemma 4 can be found in [1].

LEMMA 5. Consider the ordinary difference equation
\[ u_n - u_{n-1} = g(u_n, k), \quad n = n_0 + 1, n_0 + 2, \ldots \] (5)
and the initial condition
\[ u_{n_0} = a, \] (6)
where $K$ and $a$ are any constants. Then the initial value problem (5),(6) exists a unique solution $\{u_n(n_0, k)\} (n = n_0, n_0 + 1, \ldots)$, and for any given positive integer $m$, $\lambda(K) = u_{n_0+m}(n_0, k)$ is independent of $n_0$, and $\lambda(K)$ is continuous with respect to $K$.

PROOF. Set $\psi(y, k) = y - g(y, k)$. By using the similar argument as in the proof of Lemma 1, there exists a function $\psi^{-1}(z, k)$, the inverse function of $\psi(y, k)$, for all $z \in R$. Thus, by using the method of steps, it is easy to know that the initial value problem (5),(6) has a unique solution
\[ u_n(n_0, k) = \psi^{-1}(u_{n-1}(n_0, k), k), \quad n = n_0 + 1, n_0 + 2, \ldots \]
(7)
and it is easy to see that (7) implies $\lambda(K) = u_{n_0+m}(n_0, k)$ is independent of $n_0$. 


Now we prove that \( \lambda(K) \) is continuous with respect to \( K \). Otherwise, there exists constants \( K_0 \in \mathbb{R} \) and \( \varepsilon_0 > 0 \), and the sequence \( \{ K_i \} \) with \( K_i \to K_0 \) as \( i \to \infty \), such that

\[ |\psi(K_i) - \psi(K_0)| \geq \varepsilon_0 > 0, \]

that is

\[ |u_{n_0+m}(n_0, k_i) - u_{n_0+m}(n_0, k_0)| \geq \varepsilon_0 > 0. \]

Without loss of generality, we may assume \( K_0 - 1 \leq K_i \leq K_0 + 1 \). Then

\[ u_{n_0+1}(n_0, k_i) = u_{n_0}(n_0, k_i) = u_{n_0+1}(n_0, k_0 + 1) - g(\{u_{n_0+1}(n_0, k_0 + 1), k_0 + 1\}) \leq u_{n_0+1}(n_0, k_0 + 1) - g(\{u_{n_0+1}(n_0, k_0 + 1), k_i\}), \]

which implies \( u_{n_0+1}(n_0, k_i) \leq u_{n_0+1}(n_0, k_0 + 1) \).

Furthermore, using induction, we can prove

\[ u_n(n_0, k_i) \leq u_n(n_0, k_0 + 1), \quad \text{for} \quad n = n_0, n_0 + 1, \ldots \] (9)

Similarly, we can prove

\[ u_n(n_0, k_i) \geq u_n(n_0, k_0 - 1), \quad \text{for} \quad n = n_0, n_0 + 1, \ldots \]

which together with (9) implies

\[ u_n(n_0, k_0 - 1) \leq u_n(n_0, k_i) \leq u_n(n_0, k_0 + 1), \quad \text{for} \quad n = n_0, n_0 + 1, \ldots \] (10)

It follows that \( \{u_{n_0+1}(n_0, k_i)\}_{i=1,2,\ldots} \) are bounded and so there exists a convergent subsequence \( \{u_{n_0+1}(n_0, k_i^{(1)})\}_{i=1,2,\ldots} \) with \( u_{n_0+1}(n_0, k_i^{(1)}) \to u_{n_0+1} \) as \( i \to \infty \), where \( \{K_i^{(1)}\} \subset \{K_i\} \). By (10), \( \{u_{n_0+2}(n_0, k_i^{(1)})\}_{i=1,2,\ldots} \) are also bounded, so there exists a convergent subsequence \( \{u_{n_0+2}(n_0, k_i^{(2)})\}_{i=1,2,\ldots} \) with \( u_{n_0+2}(n_0, k_i^{(2)}) \to u_{n_0+2} \) as \( i \to \infty \), where \( \{K_i^{(2)}\} \subset \{K_i^{(1)}\} \). Generally, consider the sequence \( \{u_{n_0+l}(n_0, k_i^{(l-1)})\}_{i=1,2,\ldots}(1 \leq l \leq m) \), we can choose a convergent subsequence \( \{u_{n_0+l}(n_0, k_i^{(l)})\}_{i=1,2,\ldots} \) with \( u_{n_0+l}(n_0, k_i^{(l)}) \to u_{n_0+l} \) as \( i \to \infty \), where \( \{K_i^{(l)}\} \subset \{K_i^{(l-1)}\} \). Thus, we get some sequences \( \{K_i^{(m)}\} \subset \{K_i^{(m-1)}\} \subset \cdots \subset \{K_i^{(1)}\} \subset \{K_i\} \). It is clear that \( \lim_{i \to \infty} u_n(n_0, k_i^{(m)}) = v_n \) for \( n = n_0, n_0 + 1, \ldots, n_0 + m \).

On the other hand, we have

\[ u_n(n_0, k_i^{(m)}) - u_{n-1}(n_0, k_i^{(m)}) = g\left(u_n\left(n_0, k_i^{(m)}\right), k_i^{(m)}\right), \quad n = n_0, n_0 + 1, \ldots, n_0 + m. \]

Let \( i \to \infty \). Then

\[ v_n - v_{n-1} = g(u_n, k_0), \quad \text{for} \quad n = n_0, n_0 + 1, \ldots, n_0 + m, \]

which implies that \( \{v_n\}(n = n_0, n_0 + 1, \ldots, n_0 + m) \) are some terms of the solution of the initial value problem (5),(6) for \( K = K_0 \). But \( \{u_n(n_0, k_0)\} \) are also the solution of the initial value problem (5),(6) for \( K = K_0 \), according to the uniqueness of solution for the initial value problem (5),(6), we have

\[ v_n = u_n(n_0, k_0), \quad n = n_0, n_0 + 1, \ldots, n_0 + m. \]

Therefore,

\[ \lim_{i \to \infty} u_n(n_0, k_i^{(m)}) = u_n(n_0, k_0), \quad n = n_0, n_0 + 1, \ldots, n_0 + m, \]

which is contrary to (8). This completes the proof of Lemma 5.
Lemma 6. Let $m$ be a given positive integer, \( \{u_n(n_0, \epsilon)\} \) \((n = n_0, n_0 + 1, \ldots)\) be the solution of the initial value problem

\[
    u_n - u_{n-1} = g(u_n, a + \epsilon), \quad n = n_0 + 1, n_0 + 2, \ldots, u_{n_0} = a < A, \tag{11}
\]

where $A$ is a constant and $\epsilon$ is a parameter with $0 \leq \epsilon \leq 1$. Then there exists a positive constant $\mu$ independent of $n_0$ and $\epsilon$ such that

\[
    A + \epsilon - u_n(n_0, \epsilon) \geq \mu > 0,
\]

for $n = n_0, n_0 + 1, \ldots, n_0 + m$.

Proof. Since $K = A + \epsilon$ is continuous with respect to $\epsilon$, it follows from Lemma 5 that

\[
    \mu(\epsilon) = A + \epsilon - u_{n_0+m}(n_0, \epsilon)
\]

is continuous with respect to $\epsilon$ and is independent of $n_0$.

By (11), we have

\[
    u_{n_0+1}(n_0, \epsilon) - g(u_{n_0+1}(n_0, \epsilon), A + \epsilon) = u_{n_0}(n_0, \epsilon) = a < A + \epsilon = A + \epsilon - g(A + \epsilon, a + \epsilon),
\]

which implies $u_{n_0+1}(n_0, \epsilon) < A + \epsilon$. Furthermore, using induction, we can prove

\[
    u_n(n_0, \epsilon) < A + \epsilon, \quad \text{for } n = n_0, n_0 + 1, \ldots \tag{12}
\]

It follows that

\[
    \mu(\epsilon) = A + \epsilon - u_{n_0+m}(n_0, \epsilon) > 0. \tag{13}
\]

Let $\mu = \min_{0 \leq \epsilon \leq 1} \mu(\epsilon)$. Then it is certain that $\mu > 0$ and $\mu$ is independent of $n_0$ and $\epsilon$.

In view of (11) and (12), for all $n = n_0 + 1, n_0 + 2, \ldots$, we have

\[
    u_n(n_0, \epsilon) - u_{n-1}(n_0, \epsilon) = g(u_n(n_0, \epsilon), a + \epsilon) > g(A + \epsilon, a + \epsilon) = 0.
\]

That is

\[
    u_n(n_0, \epsilon) > u_{n-1}(n_0, \epsilon), \quad \text{for } n = n_0 + 1, n_0 + 2, \ldots,
\]

and hence,

\[
    u_{n_0+m}(n_0, \epsilon) \geq u_n(n_0, \epsilon), \quad \text{for } n = n_0, n_0 + 1, \ldots, n_0 + m,
\]

which together with (13) implies

\[
    A + \epsilon - u_n(n_0, \epsilon) \geq A + \epsilon - u_{n_0+m}(n_0, \epsilon) \geq \mu > 0,
\]

for $n = n_0, n_0 + 1, \ldots, n_0 + m$. This completes the proof of Lemma 6.

By an analogous argument, we can prove the following.

Lemma 7. Let $m$ be a given positive integer, \( \{u_n(n_0, \epsilon)\} \) \((n = n_0, n_0 + 1, \ldots)\) be the solution of the initial value problem

\[
    u_n - u_{n-1} = g(u_n, a - \epsilon), \quad n = n_0 + 1, n_0 + 2, \ldots, u_{n_0} = b > A, \tag{14}
\]

where $A$ is a constant and $\epsilon$ is a parameter with $0 \leq \epsilon \leq 1$. Then there exists a positive constant $\nu$ independent of $n_0$ and $\epsilon$ such that

\[
    u_n(n_0, \epsilon) - (A - \epsilon) \geq \nu > 0, \quad \text{for } n = n_0, n_0 + 1, \ldots, n_0 + m.
\]
3. MAIN RESULTS

In this section, we will state and prove the main results of this paper.

**THEOREM 1.** Assume that

(i) \( \displaystyle u > v \) implies \( \displaystyle f(n, u, v) < 0 \);
(ii) \( \displaystyle f(n, u, v) \leq q_n g(u, v) + p_n \) for \( u \leq v \).

Then every bounded solution of equation (1) tends to a constant as \( n \to \infty \).

**PROOF.** Let \( \{x_n\} \ (n = n_0 - k + 1, n_0 - k + 2, \ldots) \) be a bounded solution of equation (1). By Lemma 2, we know that the sequence \( \{A_i\} \) has finite limits as \( i \to \infty \) and let

\[
\lim_{i \to \infty} A_i = A. \tag{15}
\]

Set \( L = \lim_{n \to \infty} \inf y_n \) and \( S = \lim_{n \to \infty} \sup y_n \), where \( y_n = x_n - cx_{n-k}/1 - c \). It is easy to see that

\[
-\infty < L < S < A < \infty.
\]

In the following we are going to prove \( L = S \).

Assume \( L < S \). Then we take a constant \( D \) such that \( L < D < S \leq A \). Obviously, for any large positive integer \( N^* > 0 \), we may choose integers \( n^*, n_0, \) and constant \( \mu_1 > 0 \), such that

\[
\begin{align*}
n^* &> N^*, \\
n_0 &\in N^*, \quad \text{and} \quad 1/p_i < \mu_1.
\end{align*}
\]

It is clear that

\[
I_{n_0} = \{n_0, n_0 + 1, \ldots, n_0 + 2k\} \subset N_n \cup N_{n+1} \cup N_{n+2},
\]

and so \( n - k \in N_{n-1} \cup N_n \cup N_n, \) for all \( n \in I_{n_0} \).

In view of Lemma 2, we have

\[
A_{n-1} \geq A_n \geq A_{n+1} \geq A_{n+2}.
\]

For all \( n \in I_{n_0} \), we have

\[
x_{n-k} \leq \max_{n \in N_n \cup N_n \cup N_n+1} \max\{x_n, y_n\} = A_{n-1}.
\]

Set \( A_{n-1} = A + \epsilon_n \). Then

\[
x_{n-k} \leq A_{n-1} = A + \epsilon_n, \quad \text{for all } n \in I_{n_0} \tag{17}
\]

and by (15) and Lemma 2, without loss of generality, we may assume \( 0 \leq \epsilon_n \leq 1 \). Now we define

\[
x_n = y_n - \frac{1}{1-c} \sum_{i=n_0}^{n} |p_i|.
\]

If \( x_n \leq x_{n-k} \), then \( y_n = x_n - cx_{n-k}/1 - c \leq x_n \). By (17), we have

\[
x_n \leq y_n \leq A + \epsilon_n, \quad \text{for all } n \in I_{n_0},
\]

which yields

\[
g(x_n, a + \epsilon_n) \geq g(A + \epsilon_n, a + \epsilon_n) = 0, \quad \text{for all } n \in I_{n_0}.
\]
Thus, by (1) we have

\[
\nabla z_n = \nabla y_n - \frac{|p_n|}{1 - c} \\
= \frac{1}{1 - c} \left[ f(n, x_n, x_{n-k}) - |p_n| \right] \\
\leq \frac{1}{1 - c} g_n g(x_n, x_{n-k}) \leq \frac{1}{1 - c} g_n g(z_n, x_{n-k}) \\
\leq \frac{1}{1 - c} g_n g(z_n, a + \epsilon_n) \\
\leq \frac{1}{1 - c} M g(z_n, a + \epsilon_n^*), \quad \text{for all } n \in I_{n_0}.
\]

If \( x_n > x_{n-k} \), then \( f(n, x_n, x_{n-k}) < 0 \), which implies

\[
\nabla z_n = \nabla y_n - \frac{|p_n|}{1 - c} = \frac{1}{1 - c} \left[ f(n, x_n, x_{n-k}) - |p_n| \right] < 0.
\]

On the other hand, for \( n \in I_{n_0} \) we have

\[
z_n \leq y_n \leq \max_{n \in N_n} \max \{ x_n, y_n \} = A + \epsilon_n^*.
\]

which implies

\[
\frac{1}{1 - c} M g(z_n, a + \epsilon_n^*) \geq \frac{1}{1 - c} M g(A + \epsilon_n^*, a + \epsilon_n^*) = 0,
\]

and hence, \( \nabla z_n < 1 / (1 - c) M g(z_n, a + \epsilon_n^*) \).

Therefore, for \( n \in I_{n_0} \), we always have

\[
z_n - z_{n-1} = \nabla z_n \leq \frac{M}{1 - c} g(z_n, a + \epsilon_n^*). \tag{18}
\]

Now let \( \{ u_n(n_0, \epsilon_n^*) \} \) be the solution of the initial value problem

\[
u_n - u_{n-1} = \frac{M}{1 - c} g(u_n, a + \epsilon_n^*), \quad n = n_0 + 1, n_0 + 2, \ldots, \tag{19}
\]

\[u_{n_0} = D.
\]

Notice that \( z_{n_0} \leq y_{n_0} < D \leq A \). It follow from Lemma 6 that there exists a constant \( \mu > 0 \) such that

\[
(A + \epsilon_n^*) - u_n(n_0, \epsilon_n^*) \geq \mu > 0, \quad \text{for all } n \in I_{n_0}, \tag{20}
\]

and \( \mu \) is independent of \( n_0 \) and \( \epsilon_n^* \).

By comparing (18) and (19), it is easy to prove that

\[
z_n \leq u_n(n_0, \epsilon_n^*), \quad \text{for all } n \in I_{n_0}.
\]

Therefore, it follows from (20) that

\[
A_n^* - z_n = (A + \epsilon_n^*) - z_n \\
\geq (A + \epsilon_n^*) - u_n(n_0, \epsilon_n^*) \\
\geq \mu > 0, \quad \text{for all } n \in I_{n_0}.
\]
That is
\[ y_n = z_n + \frac{1}{1 - c} \sum_{i=n_0}^{n} |p_i| \leq A_{n^* - 1} - \mu + \mu_1, \quad \text{for all } n \in \mathbb{N}_0. \]

We may take \( \mu_1 > 0 \), such that \( \mu_1 \leq \mu/2 \), then
\[ y_n \leq A_{n^* - 1} - \frac{\mu}{2}, \quad \text{for all } n \in \mathbb{N}_0. \] (21)

Again since \( N_{n^* + 1} = \{(n^* + 1)k, (n^* + 1)k + 1, \ldots, (n^* + 2)k\} \subset \mathbb{N}_0 \), let \( a_{n^* + 1} \in N_{n^* + 1} \) such that
\[ \max\{x_{a_{n^* + 1}}, y_{a_{n^* + 1}}\} = A_{n^* + 1}. \]

Then there are two possible cases to consider.

CASE A. \( x_{a_{n^* + 1}} \leq y_{a_{n^* + 1}} \). In this case, by (21) we have
\[ A_{n^* + 2} \leq A_{n^* + 1} = y_{a_{n^* + 1}} \leq A_{n^* - 1} - \frac{\mu}{2}, \]
that is
\[ A_{n^* - 1} - A_{n^* + 2} \geq \frac{\mu}{2} > 0. \] (22)

CASE B. \( x_{a_{n^* + 1}} > y_{a_{n^* + 1}} \). In this case, \( x_{a_{n^* + 1}} = A_{n^* + 1} \geq A_{n^* + 2} \). From (17), we obtain
\[ y_{a_{n^* + 1}} = \frac{x_{a_{n^* + 1}} - ce_{a_{n^* + 1}} - k}{1 - c} \geq \frac{A_{n^* + 2} - cA_{n^* - 1}}{1 - c}. \]

Substituting the above inequality into (21), we get
\[ A_{n^* - 1} - \frac{A_{n^* + 2} - cA_{n^* - 1}}{1 - c} \geq \frac{\mu}{2} > 0, \]
that is
\[ A_{n^* - 1} - A_{n^* + 2} \geq (1 - c)\frac{\mu}{2} > 0. \] (23)

Combining (22) and (23), we obtain
\[ A_{n^* - 1} - A_{n^* + 2} \geq (1 - c)\frac{\mu}{2} > 0. \] (24)

Notice that \( n^* \geq N^* \) and \( N^* \) may be sufficiently large. It is obvious that the inequality (24) contradicts \( \lim_{n \to \infty} A_i = A \), and hence \( L = S \). That is,
\[ \lim_{n \to \infty} \frac{x_n - ce_{n - k}}{1 - c} = L = S \]
exists and is finite. Since \( 0 < c < 1 \), it follows from Lemma 4 that \( \lim_{n \to \infty} x_n = L \). This complete the proof of Theorem 1.

By an analogous argument, we can prove the following.

THEOREM 2. If
(i) \( u < v \) implies \( f(n, u, v) > 0 \),
(ii) \( f(n, u, v) \geq q_n g(u, v) + p_n \) for \( u \geq v \),
then every bounded solution of equation (1) tends to a constant as \( n \to \infty \).
THEOREM 3. Assume that

(i) \((u - v)f(n, u, v) < 0\) for \(u \neq v\);
(ii) either \(f(n, u, v) \leq q_n g(u, v) + p_n\) for \(u \leq v\); or \(f(n, u, v) \geq q_n g(u, v) + p_n\) for \(u \geq v\).

Then every solution of equation (1) tends to a constant as \(n \to \infty\).

PROOF. By Lemma 2 and 3, it is easy to see that every solution of equation (1) is bounded if (i) holds, and hence it follows from Theorems 2 and 3 that every solution of equation (1) tends to a constant as \(n \to \infty\). This completes the proof of Theorem 3.

In the following, we consider the neutral delay difference equation

\[
\nabla(x_n - c x_{n-k}) = q_n(-F(x_n) + G(x_{n-k})) + p_n,
\]

(25)

where \(F, G : R \to R\) are continuous, and \(F\) is increasing on \(R\). Set \(g(u, v) = -F(u) + F(v)\). By using Theorems 1-3, we get the following.

COROLLARY 1. If \(F(u) \geq G(u)\) and \(p_n \leq 0\), then each bounded solution of equation (25) tends to a constant as \(n \to \infty\).

COROLLARY 2. If \(F(u) \leq G(u)\) and \(p_n \geq 0\), then each bounded solution of equation (25) tends to a constant as \(n \to \infty\).

COROLLARY 3. If \(F(u) \equiv G(u)\) and \(p_n = 0\), then each solution of equation (25) tends to a constant as \(n \to \infty\).

REMARK. Note that [1] given Corollaries 1-3 only in case \(q_n = 1, p_n = 0\). Our results, obviously, improve and extend those of [1].

REFERENCES