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Convergence of Solutions for a Class of Neutral Difference Equations

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Abstract—Consider the neutral difference equation

$$\nabla(x_n - cx_{n-k}) = f(n, x_n, x_{n-k}),$$

where k is a positive integer, $0 < c < 1$, ∇ denotes the backward difference operator $\nabla y_n = y_n - y_{n-1}$, $f : N \times R^2 \to R$ is continuous and decreasing with respect to the second argument. We give the results that solutions of the equation convergent to constants. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Convergence, Neutral difference equation.

1. INTRODUCTION

Consider the difference equation

$$\nabla(x_n - cx_{n-k}) = f(n, x_n, x_{n-k}), \tag{1}$$

where k is a positive integer, $0 \le c < 1, \nabla$ denotes the backward difference operator $\nabla y_n = y_n - y_{n-1}, f : N \times R^2 \to R$ is continuous, and $f(n, u, v)$ is decreasing with respect to $u \in R$.

Some special cases of equation (1) have been discussed in the literature, for example, in [1,2], the second author and Yu, respectively, considered the equation

$$\nabla(x_n - cx_{n-k}) = -F(x_n) + G(x_{n-k}) \tag{2}$$

and the special case where $c = 0$ in equation (2), where k is a positive integer, $0 \le c < 1$, f and G are continuous, and F is increasing on R, and that if $F(y) - G(y)$ does not change sign for all $y \in R$, then every bounded solution of equation (2) tends to a constant as $n \to \infty$; if $F(y) \equiv G(y)$ for $y \in R$, then every solution of equation (2) tends to a constant as $n \to \infty$. Recently, in [3], we considered the special case where $c = 0$ in equation (1) and gave the results of asymptotic behavior of solutions of equation (1).

In this paper, we consider the case where $0 < c < 1$ in equation (1). For this case, equation (1) is a neutral difference equation, and it can be viewed as a discrete analogue of the neutral delay differential equation

$$\frac{d}{dt}[x(t) - cx(t-r)] = f(t, x(t), x(t-r)), \qquad 0 < c < 1, \quad r > 0. \tag{3}$$

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The problem of the convergence of solutions of equation (3) has been investigated in [4], and some special cases of equation (3) also have been considered for this problem by many authors (see, e.g., [5–8]).

By a solution of equation (1), we mean a sequence $\{x_n\}$ of real numbers which is defined for $n \geq n_0 - k + 1$ and satisfies equation (1) for $n = n_0 + 1,\ n_0 + 2, \dots$, and initial conditions

$$x_{n_0 - j} = a_j, \qquad j = 0, 1, \dots, k-1, \tag{4}$$

where $a_j (j = 0, 1, \dots, k-1)$ are some arbitrary real constants.

Throughout this paper, we shall assume that $g : R^2 \to R$ is continuous, $g(u,v)$ is decreasing with respect to u and increasing with respect to v, and $g(u,u) \equiv 0$, for all $u \in R$, the real sequence $\{p_n\}$ and $\{q_n\}$ satisfy:$\sum_{i=1}^{\infty} |p_i| < \infty,\ 0 < q_n < M$, where M is a positive constant.

Let $\{x_n\}$ $(n = n_0 - k + 1, n_0 - k + 2, \dots)$ be a solution of equation (1). We define the set

$$N_i = \{ik, ik+1, \dots, (i+1)k\}, \qquad I_{n_0} = \{n_0, n_0 + 1, \dots, n_0 + 2k\}$$

and the sequences $\{A_i\}$ and $\{B_i\}$:

$$A_i = \max_{n \in N_i} \max\{x_n, y_n\}, \qquad B_i = \min_{n \in N_i} \min\{x_n, y_n\},$$

where $y_n = x_n - cx_{n-k}/1 - c$ and $i > -1 + n_0/k$ is a nonnegative integer. For the sake of convenience, throughout this paper, we shall use the above notations.

Our aim in this paper is to prove that if either $f(n,u,v) < 0$ for $u > v$ and $f(n,u,v) \leq q_n g(u,v) + p_n$ for $u \leq v$ or $f(n,u,v) > 0$ for $u < v$ and $f(n,u,v) \geq q_n g(u,v) + p_n$ for $u \geq v$, then every bounded solution of equation (1) tends to a constant as $n \to \infty$; if $(u-v)f(n,u,v) < 0 (u \neq v)$ and either $f(n,u,v) \leq q_n g(u,v) + p_n$ for $u \leq v$ or $f(n,u,v) \geq q_n g(u,v) + p_n$ for $u \geq v$, then every solution of equation (1) tends to a constant as $n \to \infty$. Our results as special cases include the all results in [1].

2. LEMMAS

In this section, we will establish several important lemmas which will be used to prove our main results in Section 3.

LEMMA 1. *For any $a_j \in R (j = 0, 1, \dots, k-1)$, the initial value problem (1) and (4) has a unique solution.*

PROOF. Since $f(n,u,v)$ is continuous on $N \times R^2$ and decreasing with respect to $u \in R$, by setting $\phi(n,u,v) = u - f(n,u,v)$, where n and v are independent of $u, \phi(n,u,v)$ is also continuous and increasing with respect to $u \in R$, and it is clear that $\lim_{u \to -\infty} \phi(n,u,v) = -\infty$ and $\lim_{u \to \infty} \phi(n,u,v) = \infty$. Therefore, exists a function $\phi^{-1}(n,z,v)$, the inverse function of $z = \phi(n,u,v)$, for all $z \in R$.

By (1), we have

$$\phi(n, x_n, x_{n-k}) = cx_{n-k} + x_{n-1} - cx_{n-1-k},$$

which implies that

$$x_n = \phi^{-1}(n, cx_{n-k} + x_{n-1} - cx_{n-1-k}, x_{n-k}),$$

for $n = n_0 + 1, n_0 + 2, \dots$.

Therefore, by the method of steps, we find that the initial value problem (1) and (4) has a unique solution. The proof is complete.

LEMMA 2. *If $f(n,u,v) < 0$ for $u > v$, then $A_i \geq A_{i+1}$ for all nonnegative integer i with $i > -1 + n_0/k$.*

PROOF. Suppose the contrary. Then there exists an integer $m > -1 + n_0/k$ such that $A_m < A_{m+1}$, that is

$$\max_{n \in N_m} \{x_n, y_n\} < \max_{n \in N_{m+1}} \max\{x_n, y_n\}.$$

Let $a_{m+1} \in N_{m+1}$, such that

$$\max\{x_{a_{m+1}}, y_{a_{m+1}}\} = A_{m+1}.$$

Then we have only two cases to consider.

CASE 1. $A_{m+1} = x_{a_{m+1}} \geq y_{a_{m+1}} = x_{a_{m+1}} - cx_{a_{m+1}-k}/1 - c$. In this case, we have $x_{a_{m+1}} \leq x_{a_{m+1}-k}$, and hence

$$A_{m+1} = x_{a_{m+1}} \leq x_{a_{m+1}-k} \leq \max_{n \in N_m} \{x_n\} \leq A_m,$$

which is contrary to the assumption $A_m < A_{m+1}$.

CASE 2. $A_{m+1} = y_{a_{m+1}} = x_{a_{m+1}} - cx_{a_{m+1}-k}/1 - c > x_{a_{m+1}}$. In this case, we have $x_{a_{m+1}} > x_{a_{m+1}-k}$, which implies $f(n, x_{a_{m+1}}, x_{a_{m+1}-k}) < 0$. On the other hand, in view of equation (1), we have

$$\begin{aligned} f(n, x_{a_{m+1}}, x_{a_{m+1}-k}) &= \nabla(x_{a_{m+1}} - cx_{a_{m+1}-k}) \\ &= (x_{a_{m+1}} - cx_{a_{m+1}-k}) - (x_{a_{m+1}-1} - cx_{a_{m+1}-1-k}) \\ &= (1-c)\left[A_{m+1} - \frac{x_{a_{m+1}-1} - cx_{a_{m+1}-1-k}}{1-c}\right] \\ &\geq (1-c)\left[A_{m+1} - \max_{n \in N_m \bigcup N_{m+1}} \{x_n, y_n\}\right] \\ &= 0. \end{aligned}$$

This is a contradiction, and so there exists no integer m such that $A_m < A_{m+1}$. This completes the proof.

Similarly, we can prove.

LEMMA 3. *If $f(n,u,v) > 0$ for $u < v$, then $B_i < B_{i+1}$ for all nonnegative integer i with $i > -1 + n_0/k$.*

LEMMA 4. *(See [1].) Let c be a constant with $0 < c < 1$, $\{x_n\}(n = n_0 - k + 1, n_0 - k + 2, \dots)$ be a bounded sequence with*

$$\lim_{n \to \infty} (x_n - cx_{n-k}) = \alpha$$

exists. Then $\lim_{n \to \infty} x_n$ exists and $\lim_{n \to \infty} x_n = \alpha/1 - c$. The proof of Lemma 4 can be found in [1].

LEMMA 5. *Consider the ordinary difference equation*

$$u_n - u_{n-1} = g(u_n, k), \qquad n = n_0 + 1,\ n_0 + 2, \dots \tag{5}$$

and the initial condition

$$u_{n_0} = a, \tag{6}$$

where K and a are any constants. Then the initial value problem (5),(6) exists a unique solution $\{u_n(n_0, k)\}$ $(n = n_0, n_0 + 1, \dots)$, and for any given positive integer m, $\lambda(K) = u_{n_0+m}(n_0, k)$ is independent of n_0, and $\lambda(K)$ is continuous with respect to K.

PROOF. Set $\psi(y,k) = y - g(y,k)$. By using the similar argument as in the proof of Lemma 1, there exists a function $\psi^{-1}(z,k)$, the inverse function of $\psi(y,k)$, for all $z \in R$. Thus, by using the method of steps, it is easy to know that the initial value problem (5),(6) has a unique solution

$$\begin{gathered} u_n(n_0, k) = \psi^{-1}(u_{n-1}(n_0, k), k), \qquad n = n_0 + 1,\ n_0 + 2, \dots, \\ u_{n_0} = a, \end{gathered} \tag{7}$$

and it is easy to see that (7) implies $\lambda(K) = u_{n_0+m}(n_0, k)$ is independent of n_0.

Now we prove that $\lambda(K)$ is continuous with respect to K. Otherwise, there exists constants $K_0 \in R$ and $\epsilon_o > 0$, and the sequence $\{K_i\}$ with $K_i \to K_0$ as $i \to \infty$, such that

$$|\psi(K_i) - \psi(K_0)| \geq \epsilon_0 > 0,$$

that is

$$|u_{n_0+m}(n_0, k_i) - u_{n_0+m}(n_0, k_0)| \geq \epsilon_0 > 0. \tag{8}$$

Without loss of generality, we may assume $K_0 - 1 \leq K_i \leq K_0 + 1$. Then

$$\begin{aligned} u_{n_0+1}(n_0, k_i) - g(u_{n_0+1}(n_0, k_i), k_i) &= u_{n_0}(n_0, k_i) = a \\ &= u_{n_0+1}(n_0, k_0 + 1) - g(u_{n_0+1}(n_0, k_0 + 1), k_0 + 1) \\ &\leq u_{n_0+1}(n_0, k_0 + 1) - g(u_{n_0+1}(n_0, k_0 + 1), k_i), \end{aligned}$$

which implies $u_{n_0+1}(n_0, k_i) \leq u_{n_0+1}(n_0, k_0 + 1)$.

Furthermore, using induction, we can prove

$$u_n(n_0, k_i) \leq u_n(n_0, k_0 + 1), \qquad \text{for } n = n_0,\ n_0 + 1, \ldots. \tag{9}$$

Similarly, we can prove

$$u_n(n_0, k_i) \geq u_n(n_0, k_0 - 1), \qquad \text{for } n = n_0,\ n_0 + 1, \ldots,$$

which together with (9) implies

$$u_n(n_0, k_0 - 1) \leq u_n(n_0, k_i) \leq u_n(n_0, k_0 + 1), \qquad \text{for } n = n_0,\ n_0 + 1, \ldots. \tag{10}$$

It follows that $\{u_{n_0+1}(n_0, k_i)\}_{i=1,2,\ldots}$ are bounded and so there exists a convergent subsequence $\{u_{n_0+1}(n_0, k_i^{(1)})\}_{i=1,2,\ldots}$ with $u_{n_0+1}(n_0, k_i^{(1)}) \to v_{n_0+1}$ as $i \to \infty$, where $\{K_i^{(1)}\} \subset \{K_i\}$. By (10), $\{u_{n_0+2}(n_0, k_i^{(1)})\}_{i=1,2,\ldots}$ are also bounded, so there exists a convergent subsequence $\{u_{n_0+2}(n_0, k_i^{(2)})\}_{i=1,2\ldots}$ with $u_{n_0+2}(n_0, k_i^{(2)}) \to v_{n_0+2}$ as $i \to \infty$, where $\{K_i^{(2)}\} \subset \{K_i^{(1)}\}$. Generally, consider the sequence $\{u_{n_0+l}(n_0, k_i^{(l-1)})\}_{i=1,2,\ldots}(1 \leq l \leq m)$, we an choose a convergent subsequence $\{u_{n_0+l}(n_0, k_i^{(l)})\}_{i=1,2,\ldots}$ with $u_{n_0+l}(n_0, k_i^{(l)}) \to v_{n_0+l}$ as $i \to \infty$, where $\{K_i^{(l)}\} \subset \{K_i^{(l-1)}\}$. Thus, we get some sequences:$\{K_i^{(m)}\} \subset \{K_i^{(m-1)}\} \subset \cdots \subset \{K_i^{(1)}\} \subset \{K_i\}$. It is clear that $\lim_{i\to\infty} u_n (n_0, k_i^{(m)}) = v_n$ for $n = n_0, n_0 + 1, \ldots, n_0 + m$.

On the other hand, we have

$$u_n\left(n_0, k_i^{(m)}\right) - u_{n-1}\left(n_0, k_i^{(m)}\right) = g\left(u_n\left(n_0, k_i^{(m)}\right), k_i^{(m)}\right), \qquad n = n_0,\ n_0 + 1, \ldots, n_0 + m.$$

Let $i \to \infty$. Then

$$v_n - v_{n-1} = g(v_n, k_0), \qquad \text{for } n = n_0,\ n_0 + 1, \ldots, n_0 + m,$$

which implies that $\{v_n\}(n = n_0, n_0 + 1, \ldots, n_0 + m)$ are some terms of the solution of the initial value problem (5),(6) for $K = K_0$. But $\{u_n(n_0, k_0)\}$ are also the solution of the initial value problem (5),(6) for $K = K_0$, according to the uniquence of solution for the initial value problem (5),(6), we have

$$v_n = u_n(n_0, k_0), \qquad n = n_0,\ n_0 + 1, \ldots, n_0 + m.$$

Therefore,

$$\lim_{i\to\infty} u_n\left(n_0, k_i^{(m)}\right) = u_n(n_0, k_0), \qquad n = n_0,\ n_0 + 1, \ldots, n_0 + m,$$

which is contrary to (8). This completes the proof of Lemma 5.

LEMMA 6. *Let m be a given positive integer, $\{u_n(n_0,\epsilon)\}$ $(n = n_0, n_0+1,\dots)$ be the solution of the initial value problem*

$$u_n - u_{n-1} = g(u_n, a+\epsilon), \qquad n = n_0+1,\ n_0+2,\dots, u_{n_0} = a < A, \tag{11}$$

where A is a constant and ϵ is a parameter with $0 \le \epsilon \le 1$. Then there exists a positive constant μ independent of n_0 and ϵ such that

$$A + \epsilon - u_n(n_0,\epsilon) \ge \mu > 0,$$

for $n = n_0, n_0+1,\dots, n_0+m$.

PROOF. Since $K = A+\epsilon$ is continuous with respect to ϵ, it follows from Lemma 5 that

$$\mu(\epsilon) = A + \epsilon - u_{n_0+m}(n_0,\epsilon)$$

is continuous with respect to ϵ and is independent of n_0.

By (11), we have

$$u_{n_0+1}(n_0,\epsilon) - g(u_{n_0+1}(n_0,\epsilon), A+\epsilon) = u_{n_0}(n_0,\epsilon) = a < A+\epsilon = A+\epsilon - g(A+\epsilon, a+\epsilon),$$

which implies $u_{n_0+1}(n_0,\epsilon) < A+\epsilon$. Furthermore, using induction, we can prove

$$u_n(n_0,\epsilon) < A+\epsilon, \qquad \text{for } n = n_0,\ n_0+1,\dots. \tag{12}$$

It follows that

$$\mu(\epsilon) = A + \epsilon - u_{n_0+m}(n_0,\epsilon) > 0. \tag{13}$$

Let $\mu = \min_{0\le\epsilon\le1} \mu(\epsilon)$. Then it is certain that $\mu > 0$ and μ is independent of n_0 and ϵ.

In view of (11) and (12), for all $n = n_0+1,\ n_0+2,\dots$, we have

$$u_n(n_0,\epsilon) - u_{n-1}(n_0,\epsilon) = g(u_n(n_0,\epsilon), a+\epsilon) > g(A+\epsilon, a+\epsilon) = 0.$$

That is

$$u_n(n_0,\epsilon) > u_{n-1}(n_0,\epsilon), \qquad \text{for } n = n_0+1,\ n_0+2,\dots,$$

and hence,

$$u_{n_0+m}(n_0,\epsilon) \ge u_n(n_0,\epsilon), \qquad \text{for } n = n_0,\ n_0+1,\dots,n_0+m,$$

which together with (13) implies

$$A+\epsilon - u_n(n_0,\epsilon) \ge A+\epsilon - u_{n_0+m}(n_0,\epsilon) \ge \mu > 0,$$

for $n = n_0, n_0+1,\dots,n_0+m$. This completes the proof of Lemma 6.

By an analogous argument, we can prove the following.

LEMMA 7. *Let m be a given positive integer, $\{u_n(n_0,\epsilon)\}$ $(n = n_0, n_0+1,\dots)$ be the solution of the initial value problem*

$$\begin{aligned} u_n - u_{n-1} = g(u_n, a-\epsilon), \qquad n = n_0+1, \quad n_0+2,\dots,\\ u_{n_0} = b > A, \end{aligned} \tag{14}$$

where A is a constant and ϵ is a parameter with $0 \le \epsilon \le 1$. Then there exists a positive constant v independent of n_0 and ϵ such that

$$u_n(n_0,\epsilon) - (A-\epsilon) \ge v > 0, \qquad \textit{for } n = n_0,\ n_0+1,\dots,n_0+m.$$

3. MAIN RESULTS

In this section, we will state and prove the main results of this paper.

THEOREM 1. *Assume that*

(i) $u > v$ *implies* $f(n,u,v) < 0$;

(ii) $f(n,u,v) \le q_n g(u,v) + p_n$ *for* $u \le v$.

Then every bounded solution of equation (1) tends to a constant as $n \to \infty$.

PROOF. Let $\{x_n\}$ $(n = n_0 - k + 1, n_0 - k + 2, \dots)$ be a bounded solution of equation (1). By Lemma 2, we know that the sequence $\{A_i\}$ has finite limits as $i \to \infty$ and let

$$\lim_{i\to\infty} A_i = A. \tag{15}$$

Set $L = \lim_{n\to\infty} \inf y_n$ and $S = \lim_{n\to\infty} \sup y_n$, where $y_n = x_n - cx_{n-k}/1 - c$. It is easy to see that

$$-\infty < L \le S \le A < \infty.$$

In the following we are going to prove $L = S$.

Assume $L < S$. Then we take a constant D such that $L < D \le S \le A$. Obviously, for any large positive integer $N^* > 0$, we may choose integers n^*, n_0, and constant $\mu_1 > 0$, such that $n^* \ge N^*$, $n_0, \in N_{n^*}$, and

$$y_{n_0} < D, \qquad \frac{1}{1-c}\sum_{i=n_0}^{n} |p_i| < \mu_1. \tag{16}$$

It is clear that

$$I_{n_0} = \{n_0, n_0+1, \dots, n_0+2k\} \subset N_{n^*} \bigcup N_{n^*+1} \bigcup N_{n^*+2},$$

and so $n - k \in N_{n^*-1} \bigcup N_{n^*} \bigcup N_{n^*+1}$, for all $n \in I_{n_0}$.

In view of Lemma 2, we have

$$A_{n^*-1} \ge A_{n^*} \ge A_{n^*+1} \ge A_{n^*+2}.$$

For all $n \in I_{n_0}$, we have

$$x_{n-k} \le \max_{n\in N_{n^*-1}\bigcup N_{n^*}\bigcup N_{n^*+1}} \max\{x_n, y_n\} = A_{n^*-1}.$$

Set $A_{n^*-1} = A + \epsilon_{n^*}$. Then

$$x_{n-k} \le A_{n^*-1} = A + \epsilon_{n^*}, \qquad \text{for all } n \in I_{n_0} \tag{17}$$

and by (15) and Lemma 2, without loss of generality, we may assume $0 \le \epsilon_{n^*} \le 1$. Now we define

$$z_n = y_n - \frac{1}{1-c}\sum_{i=n_0}^{n} |p_i|.$$

If $x_n \le x_{n-k}$, then $y_n = x_n - cx_{n-k}/1 - c \le x_n$. By (17), we have

$$z_n \le y_n \le A + \epsilon_{n^*}, \qquad \text{for all } n \in I_{n_0},$$

which yields

$$g(z_n, a + \epsilon_{n^*}) \ge g(A + \epsilon_{n^*}, a + \epsilon_{n^*}) = 0, \qquad \text{for all } n \in I_{n_0}.$$

Thus, by (1) we have

$$\begin{aligned}\nabla z_n &= \nabla y_n - \frac{|p_n|}{1-c}\\ &= \frac{1}{1-c}[f(n,x_n,x_{n-k}) - |p_n|]\\ &\le \frac{1}{1-c} q_n g(x_n,x_{n-k}) \le \frac{1}{1-c} q_n g(z_n,x_{n-k})\\ &\le \frac{1}{1-c} q_n g(z_n, a+\epsilon_{n^*})\\ &\le \frac{1}{1-c} M g(z_n, a+\epsilon_{n^*}), \qquad \text{for all } n \in I_{n_0}.\end{aligned}$$

If $x_n > x_{n-k}$, then $f(n,x_n,x_{n-k}) < 0$, which implies

$$\begin{aligned}\nabla z_n &= \nabla y_n - \frac{|p_n|}{1-c}\\ &= \frac{1}{1-c}[f(n,x_n,x_{n-k}) - |p_n|] < 0.\end{aligned}$$

On the other hand, for $n \in I_{n_0}$ we have

$$\begin{aligned} z_n &\le y_n\\ &\le \max_{n \in N_{n^*-1} \bigcup N_{n^*} \bigcup N_{n^*+1} \bigcup N_{n^*+2}} \max\{x_n, y_n\}\\ &= A_{n^*-1} = A + \epsilon_{n^*}\end{aligned}$$

which implies

$$\frac{1}{1-c} M g(z_n, a+\epsilon_{n^*}) \ge \frac{1}{1-c} M g(A+\epsilon_{n^*}, a+\epsilon_{n^*}) = 0,$$

and hence, $\nabla z_n < 1/1 - cMg(z_n, a+\epsilon_{n^*})$.

Therefore, for $n \in I_{n_0}$, we always have

$$z_n - z_{n-1} = \nabla z_n \le \frac{M}{1-c} g(z_n, a+\epsilon_{n^*}). \tag{18}$$

Now let $\{u_n(n_0, \epsilon_{n^*})\}$ be the solution of the initial value problem

$$\begin{aligned} u_n - u_{n-1} &= \frac{M}{1-c} g(u_n, a+\epsilon_{n^*}), \qquad n = n_0+1,\ n_0+2, \dots,\\ u_{n_0} &= D.\end{aligned} \tag{19}$$

Notice that $z_{n_0} \le y_{n_0} < D \le A$. It follow from Lemma 6 that there exists a constant $\mu > 0$ such that

$$(A+\epsilon_{n^*}) - u_n(n_0, \epsilon_{n^*}) \ge \mu > 0, \qquad \text{for all } n \in I_{n_0}, \tag{20}$$

and μ is independent of n_0 and ϵ_{n^*}.

By comparing (18) and (19), it is easy to prove that

$$z_n \le u_n(n_0, \epsilon_{n^*}), \qquad \text{for all } n \in I_{n_0}.$$

Therefore, it follows from (20) that

$$\begin{aligned} A_{n^*-1} - z_n &= (A+\epsilon_{n^*}) - z_n\\ &\ge (A+\epsilon_{n^*}) - u_n(n_0, \epsilon_{n^*})\\ &\ge \mu > 0, \qquad \text{for all } n \in I_{n_0}.\end{aligned}$$

That is

$$y_n = z_n + \frac{1}{1-c}\sum_{i=n_0}^{n} |p_i|$$
$$\le A_{n^*-1} - \mu + \mu_1, \qquad \text{for all } n \in I_{n_0}.$$

We may take $\mu_1 > 0$, such that $\mu_1 \le \mu/2$, then

$$y_n \le A_{n^*-1} - \frac{\mu}{2}, \qquad \text{for all } n \in I_{n_0}. \tag{21}$$

Again since $N_{n^*+1} = \{(n^*+1)k, (n^*+1)k+1, \dots, (n^*+2)k\} \subset I_{n_0}$, let $a_{n^*+1} \in N_{n^*+1}$ such that

$$\max\{x_{a_{n^*+1}}, y_{a_{n^*+1}}\} = A_{n^*+1}.$$

Then there are two possible cases to consider.

CASE A. $x_{a_{n^*+1}} \le y_{a_{n^*+1}}$. In this case, by (21) we have

$$A_{n^*+2} \le A_{n^*+1} = y_{a_{n^*+1}} \le A_{n^*-1} - \frac{\mu}{2},$$

that is

$$A_{n^*-1} - A_{n^*+2} \ge \frac{\mu}{2} > 0. \tag{22}$$

CASE B. $x_{a_{n^*+1}} > y_{a_{n^*+1}}$. In this case, $x_{a_{n^*+1}} = A_{n^*+1} \ge A_{n^*+2}$. From (17), we obtain

$$y_{a_{n^*+1}} = \frac{x_{a_{n^*+1}} - cx_{a_{n^*+1}-k}}{1-c} \ge \frac{A_{n^*+2} - cA_{n^*-1}}{1-c}.$$

Substituting the above inequality into (21), we get

$$A_{n^*-1} - \frac{A_{n^*+2} - cA_{n^*-1}}{1-c} \ge \frac{\mu}{2} > 0,$$

that is

$$A_{n^*-1} - A_{n^*+2} \ge (1-c)\frac{\mu}{2} > 0. \tag{23}$$

Combining (22) and (23), we obtain

$$A_{n^*-1} - A_{n^*+2} \ge (1-c)\frac{\mu}{2} > 0. \tag{24}$$

Notice that $n^* \ge N^*$ and N^* may be sufficiently large. It is obvious that the inequality (24) contradicts $\lim_{i\to\infty} A_i = A$, and hence $L = S$. That is,

$$\lim_{n\to\infty} \frac{x_n - cx_{n-k}}{1-c} = L = S$$

exists and is finite. Since $0 < c < 1$, it follows from Lemma 4 that $\lim_{n\to\infty} x_n = L$. This complete the proof of Theorem 1.

By an analogous argument, we can prove the following.

THEOREM 2. *If*

(i) $u < v$ *implies* $f(n,u,v) > 0$,
(ii) $f(n,u,v) \ge q_n g(u,v) + p_n$ *for* $u \ge v$,

then every bounded solution of equation (1) tends to a constant as $n \to \infty$.

THEOREM 3. *Assume that*

(i) $(u-v)f(n,u,v) < 0$ *for* $u \neq v$;

(ii) *either* $f(n,u,v) \leq q_n g(u,v) + p_n$ *for* $u \leq v$; *or* $f(n,u,v) \geq q_n g(u,v) + p_n$ *for* $u \geq v$.

Then every solution of equation (1) tends to a constant as $n \to \infty$.

PROOF. By Lemma 2 and 3, it is easy to see that every solution of equation (1) is bounded if (i) holds, and hence it follows from Theorems 2 and 3 that every solution of equation (1) tends to a constant as $n \to \infty$. This completes the proof of Theorem 3.

In the following, we consider the neutral delay difference equation

$$\nabla(x_n - cx_{n-k}) = q_n(-F(x_n) + G(x_{n-k})) + p_n, \tag{25}$$

where $F, G : R \to R$ are continuous, and F is increasing on R. Set $g(u,v) = -F(u) + F(v)$. By using Theorems 1–3, we get the following.

COROLLARY 1. *If* $F(u) \geq G(u)$ *and* $p_n \leq 0$, *then each bounded solution of equation (25) tends to a constant as* $n \to \infty$.

COROLLARY 2. *If* $F(u) \leq G(u)$ *and* $p_n \geq 0$, *then each bounded solution of equation (25) tends to a constant as* $n \to \infty$.

COROLLARY 3. *If* $F(u) \equiv G(u)$ *and* $p_n = 0$, *then each solution of equation (25) tends to a constant as* $n \to \infty$.

REMARK. Note that [1] given Corollaries 1–3 only in case $q_n = 1$, $p_n = 0$. Our results, obviously, improve and extend those of [1].

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