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The Paley–Wiener Theorem and the local Huygens' principle for compact symmetric spaces: The even multiplicity case ☆

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ABSTRACT

We prove a Paley–Wiener theorem for a class of symmetric spaces of the compact type, in which all root multiplicities are even. This theorem characterizes functions of small support in terms of holomorphic extendability and exponential type of their (discrete) Fourier transforms. We also provide three independent new proofs of the strong Huygens' principle for a suitable constant shift of the wave equation on odd-dimensional spaces from our class.

INTRODUCTION

In the context of spherical harmonic analysis, the compactness of a symmetric space U/K is reflected by the discreteness of its dual space, which is the set of irreducible K-spherical unitary representations of U. The same set parametrizes the set of (elementary) spherical functions. Thus, the spherical Fourier transforms of K-invariant functions on U/K are functions on a discrete set. Likewise, the formula for spherical inversion, which recovers a sufficiently regular function on the symmetric space in terms of spherical functions, is given by a series. This structural

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discreteness can be overcome for functions with "small support", by relating them to functions on the tangent space $T_x(U/K)$ at some point $x \in U/K$. (Observe that it suffices to consider the special case $x = x_0 = \{K\}$ as any other point x can be achieved by a translation $x = gx_0$.) This procedure can be easily illustrated in the Euclidean setting: consider a smooth function $f : S^1 \to \mathbb{C}$, where S^1 denotes the unit circle. View f as a periodic functions on $\mathbb{R}$ by $t \mapsto f(e^{it})$ and assume that f has small support, say in $[-R, R] + 2\pi\mathbb{Z}$, where $0 < R < \pi$. We can then regard f as a smooth function on the real line with support in $[-R, R]$ by setting it equal to 0 outside of the fundamental period $[-\pi, \pi)$. By the classical Paley–Wiener theorem on $\mathbb{R}$, the Fourier transform of f is an entire function of exponential type R. It therefore provides a holomorphic extension of the Fourier transform of f as a function on S^1. Likewise, if F is a holomorphic function on $\mathbb{C}$ of exponential type R, $0 < R < \pi$, then the inversion formula for the continuous Fourier transform gives a function f_1 with support in $[-R, R]$, and we can define a function f on S^1 by $f(e^{it}) = f_1(t)$.

The possibility of characterizing central smooth functions with "small support" on compact Lie groups by means of the entire extension and exponential growth of their Fourier transform was first proved by Gonzalez in [10]. In this paper we extend the local Paley–Wiener theorem to all compact symmetric spaces U/K with even multiplicities: the K-invariant smooth functions on U/K with "small support" will be characterized in terms of holomorphic extendability and exponential growth of their spherical Fourier transform. Moreover, the exponential growth of the transformed function will be linked to the size of the support of the function on the symmetric space. The given characterization relies on the fact that the spherical functions on a compact symmetric space extend holomorphically to the complexified symmetric space. Their restrictions to the noncompact dual symmetric space G/K are in turn spherical functions on G/K. This allows us to use known information on the spherical functions on G/K and classical Fourier analysis on the Lie algebra of a maximal abelian subspace of $\mathfrak{q} \simeq T_{x_0}(U/K)$. In particular, the classical Paley–Wiener theorem is used to obtain the required holomorphic extension of the compact spherical Fourier transform of a K-invariant function on U/K with a small support.

Properties of holomorphic extendability for spherical functions on symmetric spaces have been the objects of intensive recent study, with different approaches and perspectives. See, e.g., [19], [27], and [23]. The situation which we consider in this paper corresponds to symmetric spaces with even multiplicities. It is rather special because of the existence of shift operators providing explicit formulas for the spherical functions by relating them to exponential functions [22]. These shift operators are suitable multiples of Opdam's shift operators. The multiplying factor has been chosen so, as to cancel the singularities of the coefficients of Opdam's shift operators along the walls of the Weyl chambers. The resulting operators are differential operators with holomorphic coefficients. Hence, we can read off the properties of holomorphic extendability of the spherical functions directly from these formulas. Furthermore, the shift operators allow us, as mentioned above, to reduce several problems in harmonic analysis on symmetric spaces of even multiplicities to the corresponding problems in Euclidean harmonic analysis.

Our proof depends heavily on the assumption that all root multiplicities are even, and it is not possible to generalize it to obtain local Paley–Wiener type theorems for general compact symmetric spaces. On the other hand, the proof can be modified to hold for arbitrary root systems with positive even-valued multiplicity functions which are not geometric. This avenue is further explored in [5].

The relation between spherical transforms on compact and noncompact symmetric spaces investigated in this paper also yields a representation of smooth functions with "small support" on the compact space as integrals of spherical functions of the noncompact dual. These integral formulas are the key ingredient for studying the solutions of the wave equation on Riemannian symmetric spaces of the compact type. From exponential estimates for the solutions, we deduce in Section 4 that the strong Huygens' principle is valid on these spaces.

The *(strong) Huygens' principle* states that, in odd dimensions, the light at time t_0 at a location x influences at later times t_1 only those locations which have distance exactly $t_1 - t_0$ from x. Hence, if a wave is supported in the sphere $\{x \mid \|x\| \leqslant R\}$ at the initial time 0, then it will be supported in the annulus $\{x \mid t - R \leqslant \|x\| \leqslant t + R\}$ at time t. In particular, at times $t > R$, the wave will vanish inside the sphere $\{x \mid \|x\| < t - R\}$.

Several different authors have proved the validity of Huygens' principle on odd dimensional Riemannian symmetric spaces with even multiplicities of either the noncompact or the compact type. Here "light" is to be interpreted as a solution of a suitable wave equation, obtained by a certain constant shift of the d'Alembertian. Their proofs use a variety of different methods. The first results in this direction were given by Helgason [12,15], see also [17], who proved Huygens' principle for symmetric spaces G/K for which either G is complex or $G = \mathrm{SO}_0(n,1)$, and for compact groups. In the general case of odd dimensional Riemannian symmetric spaces of the noncompact type with even multiplicities, the validity of Huygens' principle was stated without proof by Solomantina [28]. A proof by Radon transform methods was provided by Ólafsson and Schlichtkrull [25]. An independent proof was obtained by Helgason [16] by means of his Fourier transform. In [6] the authors proved an exponential decay property for solutions of the wave equation with compactly supported initial data. This method implied another independent proof of the Huygens' principle for odd dimensional symmetric spaces with even multiplicities; see [6]. Finally, a completely different approach based on Heckman–Opdam's shift operators and explicit formulas for the fundamental solutions was provided by Chalykh and Veselov in [8]. The formulas of Chalykh and Veselov give the fundamental solution of the wave equation in polar coordinates. By replacing hyperbolic functions with their trigonometric counterparts, one can also deduce formulas for the fundamental solutions of the wave equation on compact symmetric spaces. These formulas will be valid for small values of time. Using this argument, Chalykh and Veselov state that Huygens' principle holds also on Riemannian symmetric spaces of the compact type with even multiplicities.

In the context of Riemannian symmetric spaces, Huygens' principle has been much less studied for the compact type than for the noncompact type. In [26], Ørsted used conformal properties of wave operators and of Lorentzian spaces covered by $\mathbb{R} \times S^{2n+1}$ to establish Huygens' principle for the wave, Dirac, and Maxwell equations on S^{2n+1}. His proof makes it clear that analogues will be valid for other linear differential operators with suitable hyperbolicity and conformal properties. A different proof for the wave equation on the odd sphere S^{2n+1} were given by Lax and Phillips [21]. Branson [2] extended the Lax–Phillips proof to an infinite class of hyperbolic equations on the odd sphere. Helgason proved Huygens' principle for the compact group case, see [15]. Finally, Branson and Ólafsson [4] proved that the local Huygens' principle for a compact symmetric space U/K is valid if and only if Huygens' principle holds for the non-compact dual space G/K.

In this article we provide three independent new proofs of a local version of the strong Huygens' principle for compact symmetric spaces U/K with even multiplicities. One of these methods comes from exponential estimates for the smooth solutions of K-invariant Cauchy problems for the modified wave equations on U/K. These estimates are obtained by methods similar to those introduced for the noncompact setting in [6]. It is nevertheless important to mention that the use of the shift operators indeed reduces the proof of the of Huygens' principle on Riemannian symmetric spaces of either type (compact or noncompact) to the validity of the same principle in the Euclidean setting. The proof presented in this paper is therefore easier than that in [6].

Another proof of the local strong Huygens' principle is in the spirit of the paper of Chalykh and Veselov [8]. The formulas for the spherical functions proved in Theorem 2.9 permit us to derive an explicit formula for the solution of the wave equation corresponding to a given smooth initial condition. The tools for writing down these formulas appear in the proof of the local Paley–Wiener theorem. An essential property in our argument is that our shift operators, which link spherical functions to exponential functions, have regular (indeed analytic) coefficients. This fact was not proved in [8].

Our paper is organized as follows. In Section 1 we recall some structure theory of Riemannian symmetric spaces of the compact type. The spherical functions and spherical representations are introduced in Section 2. Theorem 2.9 proves the existence of differential shift operators. These provide explicit formulas for the spherical functions on compact symmetric spaces. The main theorem in this paper is the local Paley–Wiener theorem, which is stated and proved in Section 3. The integral formula for functions with "small support" is given by Corollary 3.16. Finally, Section 4 contains the proofs of the local strong Huygens' principle on Riemannian symmetric spaces of the compact type.

1. SYMMETRIC SPACES

1.1. Compact symmetric spaces

In this section we recall some facts about compact symmetric spaces. We use [13, Chapter VII], and [29, Chapter II], as standard references.

Let U be a connected compact Lie group with center Z and Lie algebra $\mathfrak{u}$. Denote by $\mathfrak{z}$ the center of $\mathfrak{u}$. Then $\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{u}'$, where $\mathfrak{u}' := [\mathfrak{u}, \mathfrak{u}]$ is semisimple. Let $\exp : \mathfrak{u} \to U$ be the exponential map. If $\mathfrak{z} \neq \{0\}$, then we set $\Gamma_0 := \{X \in \mathfrak{z} \mid \exp X = e\}$, where e denotes the identity of U. Then Γ_0 is a full rank lattice in $\mathfrak{z}$ and $T := \mathfrak{z}/\Gamma_0$ is isomorphic to the identity connected component of Z. We will from now on write $T = Z_0$. Denote by U' the analytic subgroup of U with Lie algebra $\mathfrak{u}'$. Then U' is semisimple with finite center and $U = TU' \simeq T \times_F U'$ where $F = T \cap U'$ is a finite central subgroup of U. We will for simplicity assume that F is trivial. Thus $U \simeq T \times U'$.

Let $\tau : U \to U$ be a non-trivial analytic involution. Set $U^\tau := \{u \in U \mid \tau(u) = u\}$, and define K be the identity connected component of U^τ. Then U/K is a connected compact symmetric space (also called Riemannian symmetric space of the compact type). The derived involution of τ on $\mathfrak{u}$ will be denoted by the same letter τ. Thus $\tau(\exp(X)) = \exp(\tau(X))$ for all $X \in \mathfrak{u}$.

Let $\mathfrak{k}$ denote the Lie algebra of K. We shall assume that $\mathfrak{k} \cap \mathfrak{z} = \{0\}$. Then

$$\mathfrak{k} = \mathfrak{u}^\tau := \left\{X \in \mathfrak{u} \mid \tau(X) = X\right\} \subset \mathfrak{u}'.$$

Set

$$\mathfrak{q} := \left\{X \in \mathfrak{u} \mid \tau(X) = -X\right\}.$$

Then $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ and $\mathfrak{z} \subseteq \mathfrak{q}$.

For a real vector space V we denote by V^* its dual and by $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$ its complexification. If V is a Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$ and $W \subseteq V$ is a subspace, then $W^\perp$ denotes the orthogonal complement of W in V. We identify W^* with the space $\{f \in V^* \mid f|_{W^\perp} = 0\}$. The complex linear extension to $V_\mathbb{C}$ of a linear map $\varphi : V \to V$ will be denoted by the same symbol φ. For $\lambda \in V^*$ define $h_\lambda \in V$ by $\lambda(H) = \langle H, h_\lambda \rangle$. For $\lambda \neq 0$ we set $H_\lambda := 2\langle h_\lambda, h_\lambda \rangle^{-1} h_\lambda$. Then $\lambda(H_\lambda) = 2$. Finally we define an inner product on V^* by

$$\langle \lambda, \mu \rangle := \langle h_\lambda, h_\mu \rangle = \lambda(h_\mu) = \mu(h_\lambda).$$

Recall that the Killing form κ on $\mathfrak{u}$ is negative definite on $\mathfrak{u}'$. Fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{z}$ and define a U-invariant inner product on $\mathfrak{u}$ by

$$\langle Z_1 + X_1, Z_2 + X_2 \rangle := \langle Z_1, Z_2 \rangle - \kappa(X_1, X_2), \quad Z_1, Z_2 \in \mathfrak{z},\ X_1, X_2 \in \mathfrak{u}'.$$

Let $\mathfrak{b} \subseteq \mathfrak{q}$ be a maximal abelian subspace and set $\mathfrak{b}_1 := \mathfrak{b} \cap \mathfrak{u}'$. Then

$$\mathfrak{b} = \mathfrak{z} \oplus \mathfrak{b}_1.$$

Set $\mathfrak{a} := i\mathfrak{b} \subseteq \mathfrak{u}_\mathbb{C}$ and $\mathfrak{a}_1 = i\mathfrak{b}_1$. Then, by restriction, $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathfrak{a}$, and hence we can apply the above notational conventions to $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$. In particular, $H_\lambda \in \mathfrak{a}$ is well defined for all nonzero $\lambda \in \mathfrak{a}^*$.

For $\alpha \in \mathfrak{b}_\mathbb{C}^* = \mathfrak{a}_\mathbb{C}^*$ let

$$\mathfrak{u}_\mathbb{C}^\alpha := \left\{X \in \mathfrak{u}_\mathbb{C} \mid \forall H \in \mathfrak{b}\colon\ [H, X] = \alpha(H)X\right\}$$

and set $m_\alpha := \dim_{\mathbb{C}} \mathfrak{u}_{\mathbb{C}}^\alpha$. If $\mathfrak{u}_{\mathbb{C}}^\alpha \neq \{0\}$, then α is called a *root* and m_α is its *multiplicity*. We denote by Δ the set of roots and by $W = W(\Delta)$ the corresponding Weyl group. Recall that W is generated by the reflections s_α with $\alpha \in \Delta$. Here $s_\alpha(H) := H - \alpha(H)H_\alpha$. If $\alpha \in \Delta$, then $\mathfrak{u}_{\mathbb{C}}^\alpha \subseteq \mathfrak{u}'_{\mathbb{C}}$, $\alpha|_{\mathfrak{z}_{\mathbb{C}}} = 0$, and $\alpha \in i\mathfrak{b}_1^* = \mathfrak{a}_1^*$. Hence α is real valued on $\mathfrak{a}$ and $\alpha|_{\mathfrak{z}} = 0$. Choose $X \in \mathfrak{a}$ so that $\alpha(X) \neq 0$ for all roots α. Then $\Delta^+ := \{\alpha \in \Delta \mid \alpha(X) > 0\}$ is a set of *positive roots*. We denote by Σ the corresponding set of simple roots.

1.2. Integration on U/K

We now fix our normalization of measures. If L is a locally compact Hausdorff topological group, then dl denotes a left invariant (Haar) measure on L. When L is a compact group we normalize dl so that the volume of L is 1. In this case, if M is a closed (and hence compact) subgroup of L, then we normalize the invariant measure $d(lM)$ on L/M so that L/M has volume 1. We then have for all $f \in L^1(L/M)$ and $g \in L^1(L)$:

$$\int_{L/M} f(lM)\, d(lM) = \int_L f(lM)\, dl = \int_L (f \circ \pi)(l)\, dl$$

and

$$\int_L g(l)\, dl = \int_{L/M} \int_M g(lm)\, dm\, d(lM),$$

where $\pi : L \to L/M$ is the canonical projection $l \mapsto lM$.

Let $B := \exp(\mathfrak{b})$ and $B_1 := \exp(\mathfrak{b}_1) = B \cap U'$. Then $U = KBK = TKB_1K$. In particular, denoting by x_0 the point $\{K\} \in U/K$, then $KB \cdot x_0 = T(KB_1) \cdot x_0 = U/K$. Set $M = Z_K(B)$ and define $\Psi : K/M \times B \to U/K$ by $\Psi(kM, b) := kb \cdot x_0$. Then Ψ is smooth and surjective. Furthermore,

$$\left|\det(d\Psi_{(kM,\exp(H))})\right| = \prod_{\alpha \in \Delta^+} \left|\sin \alpha(H)\right|^{m_\alpha} =: \delta\big(\exp(H)\big). \tag{1.1}$$

This proves the following integration formula (cf., e.g., [14, Theorem 5.10]).

Lemma 1.1. *There exists a constant $c > 0$ such that for all $f \in C(U/K)$ we have*

$$\int_{U/K} f(uK)\, d(uK) = c \int_{K/M} \int_B f(kb \cdot x_0) \delta(b)\, db\, d(kM).$$

2. SPHERICAL FUNCTIONS AND SPHERICAL REPRESENTATIONS

In this section we recall some necessary facts about spherical functions and spherical representations. We refer to [14, Chapter V], as standard reference. Our main result in this section is Theorem 2.9. It states that, if m_α is even for all $\alpha \in \Delta$, then there exists a differential operator D on B with analytic coefficients and a rational function Q on $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{b}_{\mathbb{C}}^*$ such that

$$\delta(b)\psi_\mu(b) = Q(\mu)D\Big(\sum_{w\in W} b^{w(\mu+\rho)}\Big).$$

Here ψ_μ is the spherical function corresponding to the spherical representation with highest weight μ, the function δ is as in (1.1), and

$$(2.1) \qquad \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha \in \mathfrak{a}^*.$$

The differential operator D will be constructed explicitly from Opdam's shift operator and in addition the rational function Q will be explicitly determined. It is a holomorphic extension (and a trivial extension to T) of the differential shift operator constructed in [22] for the noncompact symmetric case with even multiplicities.

2.1. Spherical representations

Let (π, V) be an irreducible unitary representation of U. Let

$$V^K := \{v \in V \mid \forall k \in K\colon \pi(k)v = v\}.$$

We say that π is *spherical* if $V^K \neq \{0\}$. In this case $\dim V^K = 1$. We denote by $\widehat{U}$ the set of equivalence classes of irreducible unitary representations of U and by $\widehat{U_K}$ the subset of equivalence classes of irreducible K-spherical representations. We shall use the same notation for a given unitary representation and for the corresponding equivalence class in $\widehat{U}$.

If $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and $b = \exp(H) \in B$, then we write $b^\lambda = e^{\lambda(H)}$ provided this is well defined. The same notation will be adopted for elements in the complexification of B. Let $\pi \in \widehat{U}$. As T is central in U, it follows by Schur's Lemma that π is of the form

$$\pi(tu) = t^\lambda \pi'(u), \quad t \in T,\ u \in U',$$

where λ is some element of $i\mathfrak{z}^*$ and $\pi' = \pi|_{U'}$.

If $\mathfrak{z} \neq \{0\}$, then we let $\Gamma_0 := \{X \in \mathfrak{z} \mid \exp(X) = e\}$, as before. Then

$$i\Gamma_0^* := \{\lambda \in i\mathfrak{z}^* \mid \forall H \in \Gamma_0\colon \lambda(H) \in 2\pi i\mathbb{Z}\} \simeq \widehat{T},$$

where the isomorphism is given by $\lambda \mapsto \chi_\lambda$ and $\chi_\lambda(t) := t^\lambda$. Note that, if we do not assume $T \cap U' = \{e\}$, then we have to impose the additional condition that $\pi'(t) = t^\lambda \mathrm{id}$ for all $t \in T \cap U'$.

Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{u}$ containing $\mathfrak{b}$. Set $\mathfrak{c}_1 := \mathfrak{c}\cap\mathfrak{u}'$. We say that $\mu \in i\mathfrak{c}^*$ is an *extremal weight* of an irreducible representation π of U if μ is the highest weight of π with respect to some ordering in $i\mathfrak{c}^*$. We fix an ordering on $i\mathfrak{z}^*$, then we extend it to $i\mathfrak{b}^*$ by using the lexicographic ordering on $\mathfrak{a}_1^*$, and we finally extend it to an ordering in $i\mathfrak{c}^*$. If π is an irreducible representation of U, then $\mu(\pi) \in i\mathfrak{c}^*$ denotes the highest weight of π with respect to this ordering. Similarly, if $\sigma \in \widehat{U}'$, then $\mu(\sigma) \in i\mathfrak{c}_1^*$ denotes the highest weight of σ. Notice that, in this notation, we have

$$\mu(\pi)|_{\mathfrak{c}_1} = \mu(\pi|_{U'}).$$

For $\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ and $\alpha \in \Delta$ let

$$\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{1}{2}\lambda(H_\alpha). \tag{2.2}$$

Denote by $\Lambda_K^+ = \Lambda_K^+(U)$ the set of highest weights of spherical representations of U. Then we have $\Lambda_K^+(U) = i\Gamma_0^* \oplus \Lambda_K^+(U')$. Here we employ the notation $\oplus$ to indicate that an element of $\Lambda_K^+(U)$ can be written in a unique way as a sum of an element of Γ_0^* and an element of $\Lambda_K^+(U')$. If $\mu \in \Lambda_K^+(U)$, then π_μ denotes the corresponding spherical representation and w_μ a K-invariant vector in the space of π_μ satisfying $\|w_\mu\| = 1$.

Theorem 2.1. *Let (π, V) be an irreducible representation of U. Then the following holds*:

(1) *If π is spherical then $\mu(\pi) \in i\Gamma_0^* \oplus \mathfrak{a}_1^*$ and*

$$\frac{\langle \mu(\pi), \alpha \rangle}{\langle \alpha, \alpha \rangle} =: \mu(\pi)_\alpha \in \mathbb{N}_0 \tag{2.3}$$

for all $\alpha \in \Delta^+$. Here $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

(2) *Let $\mu \in i\Gamma_0^* \oplus \mathfrak{a}_1^*$ so that $\mu_\alpha \in \mathbb{Z}$ for all $\alpha \in \Delta$. If U' is simply connected, then there exists a unique spherical representation π with extremal weight μ.*

(3) *If U' is simply connected then $\Lambda_K^+(U) = i\Gamma_0^* \oplus \{\mu \in \mathfrak{a}_1^* \mid \forall \alpha \in \Delta^+\colon\ \mu_\alpha \in \mathbb{N}_0\}$.*

2.2. Spherical functions

Recall the following definition.

Definition 2.2. Let G be a locally compact Hausdorff topological group and $K \subset G$ a compact subgroup. A continuous function $\varphi : G \to \mathbb{C}$ is said to be *spherical* if φ is K-bi-invariant, is not identically 0, and satisfies the identity

$$\int_K \varphi(xky)\, dk = \varphi(x)\varphi(y)$$

for all $x, y \in G$.

For $\mu \in \Lambda_K^+(U)$ define $\psi_\mu : U \to \mathbb{C}$ by

$$\psi_\mu(u) = \big(\pi_\mu(u) w_\mu, w_\mu\big), \tag{2.4}$$

where $(\cdot,\cdot)$ denotes the inner product in the space of π_μ for which this representation is unitary. Then ψ_μ is a spherical function on U and every spherical function on U is of the form ψ_μ for some $\mu \in \Lambda_K^+(U)$. Notice that, with $\lambda := \mu|_{\mathfrak{z}} \in \Gamma_0^*$ and $\mu' := \mu|_{\mathfrak{b}_1} \in \Lambda_K^+(U')$, we have

$$\psi_\mu(tu') = t^\lambda\big(\pi_{\mu'}(u')w_\mu, w_\mu\big) = t^\lambda \psi_{\mu'}(u'), \quad t \in T,\ u' \in U'. \tag{2.5}$$

Since π_μ is unitary, (2.4) implies the following lemma.

Lemma 2.3. *Let $\mu \in \Lambda_K^+(U)$. Then*

$$\overline{\psi_\mu(u)} = \psi_\mu\big(u^{-1}\big)$$

for all $u \in U$.

Let $\iota : U \to U_{\mathbb{C}}$ be the universal complexification of U. Hence, if L is a complex Lie group and $\varphi : U \to L$ is a Lie group homomorphism, then there exists a holomorphic homomorphism $\varphi_{\mathbb{C}} : U_{\mathbb{C}} \to L$ such that $\varphi_{\mathbb{C}} \circ \iota = \varphi$. As U is compact, it follows that there exists a faithful representation $\pi : U \to \mathrm{GL}(n, \mathbb{C})$ for some n. Applying the above to π, we conclude that ι has to be injective. We can therefore assume that U is a subgroup of $U_{\mathbb{C}}$. Since U is compact, it follows that U is closed in $U_{\mathbb{C}}$.

Lemma 2.4. *Let $\mu \in \Lambda_K^+(U)$. Then the spherical function ψ_μ extends to a holomorphic function on $U_{\mathbb{C}}$. The extension is given by*

$$\psi_\mu(g) = \big((\pi_\mu)_{\mathbb{C}}(g)w_\mu, w_\mu\big).$$

Let G to be the analytic subgroup of $U_{\mathbb{C}}$ with the Lie algebra $\mathfrak{g} := \mathfrak{k} \oplus i\mathfrak{q}$. Then G is closed in $U_{\mathbb{C}}$ and $K \subset G$. We set $\mathfrak{p} := i\mathfrak{q}$ and notice that $\mathfrak{a} = i\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{p}$. Denote by $\tau_{\mathbb{C}}$ the holomorphic extension of τ to $U_{\mathbb{C}}$ and set $\theta = \tau_{\mathbb{C}}|_G$. Then θ is a Cartan involution on G. We have $K = G^\theta$. The symmetric space G/K is called the *noncompact dual* of U/K. We set $A = \exp(\mathfrak{a})$ and $A_1 = \exp(\mathfrak{a}_1)$. Finally we set $T_{\mathbb{C}} = \exp(\mathfrak{z}_{\mathbb{C}})$ and $T_{\mathbb{R}} = T_{\mathbb{C}} \cap G = \exp(i\mathfrak{z}) \subseteq A$.

Let us recall the standard notations and definition for the Iwasawa decomposition of G. Let

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha,$$

where, as usual, $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a}\colon\ [H, X] = \alpha(H)X\}$. Then the Iwasawa map

$$K \times A \times N \ni (k, a, n) \mapsto kan \in G \tag{2.6}$$

is a diffeomorphism. For $x \in G$ define $(k(x), a(x), n(x)) \in K \times A \times N$ by the inverse of the map in (2.6). We normalize the Haar measure dn on N so that $\int_N a(\theta(n))^{-2\rho}\, dn = 1$. Here ρ is as in (2.1). Moreover, we normalize the Haar measure da on A (and similarly dt on $T_{\mathbb{R}}$) so that the *Fourier transform* on $C_c^\infty(A)$, defined by

$$\hat{f}(\lambda) = \mathcal{F}_A(f)(\lambda) := \int_A f(a)a^{-\lambda}\,da, \quad f \in C_c^\infty(A),\ \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

has inverse

$$f(a) = \int_{i\mathfrak{a}^*} \hat{f}(\lambda)a^{\lambda}\,d\lambda.$$

Finally we normalize the Haar measure dg on G so that the equality

$$\int_G f(g)\,dg = \int_K \int_A \int_N f(kan)a^{2\rho}\,dn\,da\,dk$$

holds for all $f \in C_c(G)$.

For $\lambda \in \mathfrak{b}_{\mathbb{C}}^* = \mathfrak{a}_{\mathbb{C}}^*$ let

$$\text{(2.7)} \qquad \varphi_\lambda(g) = \int_K a\big(g^{-1}k\big)^{-\lambda-\rho}\,dk = \int_K a(gk)^{\lambda-\rho}\,dk$$

be the corresponding spherical function on G. Let G' be the analytic subgroup of G corresponding to the Lie algebra $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$. Notice that if $g = th$ with $t \in T_{\mathbb{R}}$ and $h \in G'$, then

$$\varphi_\lambda(g) = t^{\lambda_1}\varphi_{\lambda_2}(h)$$

where λ_1 is the restriction of λ to $\mathfrak{z}_{\mathbb{C}}$, λ_2 is the restriction of λ to $\mathfrak{a}_{1\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \cap [\mathfrak{u}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$, and φ_{λ_2} is the spherical function on the semisimple Lie group G' with spectral parameter λ_2. Recall that $\varphi_\lambda = \varphi_\mu$ if and only if there exists $w \in W$ such that $\lambda = w\mu$. We also recall the following well known fact.

Lemma 2.5. *Let $\mu \in \Lambda_K^+(U)$ and let ψ_μ denote the holomorphic extension to $U_{\mathbb{C}}$ of the spherical function ψ_μ on U. Then*

$$\psi_\mu|_G = \varphi_{\mu+\rho}.$$

Proof. (Cf., e.g., [14, pp. 540–541].) Fix a highest weight vector u for π_μ such that $w_\mu = \int_K \pi_\mu(k)u\,dk$. In particular $(u, w_\mu) = 1$ and for $b \in B$ we have

$$\text{(2.8)} \qquad \psi_\mu(b) = \big(\pi_\mu(b)w_\mu, w_\mu\big) = \int_K \big(\pi_\mu(bk)u, w_\mu\big)\,dk.$$

As K is compact, it follows that (2.8) remains valid for the holomorphic extension of ψ_μ. In particular, it is valid for $b \in A$. Thus, as $(u, w_\mu) = 1$,

$$\big(\pi_\mu(bk)u, w_\mu\big) = a(bk)^{\mu}(u, w_\mu) = a(bk)^{(\mu+\rho)-\rho}$$

and hence

$$\psi_\mu(b) = \int_K a(bk)^{(\mu+\rho)-\rho}\, dk = \varphi_{\mu+\rho}(b). \qquad \square$$

2.3. The dimension function $d(\mu)$ and the c-function

Set $\overline{N} := \theta(N)$ and normalize the Haar measure $d\bar{n}$ on $\overline{N}$ so that $\int_{\overline{N}} a(\bar{n})^{-2\rho}\, d\bar{n} = 1$. If $\mathrm{Re}(\lambda_\alpha) > 0$ for all $\alpha \in \Delta^+$, then the Harish-Chandra **c**-function for G/K is given on $\mathfrak{a}^*_{\mathbb{C}}$ by

$$\mathbf{c}(\lambda) = \int_{\bar{N}} a(\bar{n})^{-\lambda-\rho}\, d\bar{n}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}. \tag{2.9}$$

Observe that $\mathbf{c}(\rho) = 1$. By the Gindikin–Karpelevic formula we have

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha\in\Delta^+} \mathbf{c}_\alpha(\lambda_\alpha), \tag{2.10}$$

where $\mathbf{c}_\alpha(\lambda_\alpha)$ corresponds to a rank-one **c**-function, i.e.

$$\mathbf{c}_\alpha(\lambda_\alpha) = \frac{2^{-\lambda_\alpha}\Gamma(\lambda_\alpha)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \lambda_\alpha))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \lambda_\alpha))}, \tag{2.11}$$

and the constant c_0 is determined by $\mathbf{c}(\rho) = 1$. In particular, this formula gives the meromorphic extension of **c** to all of $\mathfrak{a}^*_{\mathbb{C}}$. If $m_{2\alpha} = 0$ for all α then (2.11) simplifies to

$$\mathbf{c}(\lambda) = c_1 \prod_{\alpha\in\Delta^+} \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha + m_\alpha/2)}.$$

If m_α is even for all $\alpha \in \Delta$, then $m_{2\alpha} = 0$ and, by the classification or by [22, Appendix C], there exists $m \in \mathbb{N}$ such that $m_\alpha = 2m$ for all $\alpha \in \Delta$.

The relation $\Gamma(z+1) = z\Gamma(z)$ implies then the following lemma.

Lemma 2.6. *Suppose that m_α is even for all $\alpha \in \Delta$. Let $2m \in 2\mathbb{N}$ be the resulting common value of m_α for all $\alpha \in \Delta$. Then*

$$\frac{1}{\mathbf{c}(\lambda)} = C \prod_{\alpha\in\Delta^+} \prod_{k=0}^{m-1} (\lambda_\alpha + k)$$

where the constant C is given by

$$C = \prod_{\alpha\in\Delta^+} \prod_{k=0}^{m-1} \frac{1}{\rho_\alpha + k}.$$

The dimension $d(\mu)$ of the spherical representation π_μ can be expressed as a limit of ratios of **c**-functions by means of Vretare's formula, cf. [30] or [17, Theorem 9.10, p. 337]. In the even multiplicity case this formula simplifies because the limit involved can be computed as the quotient of the limits of the **c**-functions appearing in the numerator and in the denominator of the formula.

Lemma 2.7. *Assume that for all $\alpha \in \Delta$ the multiplicities m_α are even, and let $2m$ be their resulting common value. Then the following properties hold:*

(1) $\rho_\alpha = m$ *for every simple root* α;
(2) $\rho_\alpha \in \mathbb{Z}$ *for every* $\alpha \in \Delta$;
(3) $\rho_\alpha \geqslant m$ *for every* $\alpha \in \Delta^+$;
(4) *For all* $\mu \in \Lambda_K^+(U)$ *we have*

$$d(\mu) = \frac{\mathbf{c}(-\rho)}{\mathbf{c}(\mu+\rho)\mathbf{c}(-(\mu+\rho))}.$$

Proof. If α is a simple root in a reduced root system Δ, then $\rho_\alpha = m$. Indeed, $\rho - m\alpha$ is fixed by the reflection s_α. This proves (1). All the remaining statements follow easily from the first and from Vretare's formula. □

Theorem 2.8. *Assume that m_α is even for each $\alpha \in \Delta$, and let $2m$ be their resulting common value. Then the dimension function d extends as to a polynomial function on $\mathfrak{a}_\mathbb{C}^*$ given by*

$$d(\lambda) = \prod_{\alpha\in\Delta^+} \prod_{k=0}^{m-1} \frac{k^2 - (\lambda+\rho)_\alpha^2}{k^2 - \rho_\alpha^2}.$$

Proof. This follows from Lemmas 2.6 and 2.7. □

2.4. The differential shift operator

In this section we suppose that all multiplicities m_α are even. Then the function $\delta(b)$ of (1.1) extends as a W-invariant holomorphic function on $B_\mathbb{C} = AB$. (See Lemma 1.2 in [22].) The following theorem, which is a slight extension of Theorem 5.1(c) of [22], provides an explicit formula on A for the spherical functions on a symmetric space G/K. This theorem is our starting point for investigating the holomorphic extension of the spherical functions on G/K to its complexification $U_\mathbb{C}/K_C$. By restriction, we shall then deduce explicit formulas for the spherical functions on the compact dual symmetric space U/K. Because of our context, we shall only consider the case for which m_α is even for each $\alpha \in \Delta$. Recall that since we also assume that $\mathfrak{u}'$ is simple, this means that the m_α have a common value $2m \in 2\mathbb{N}$.

Theorem 2.9. *Assume that all multiplicities m_α are even. Then there exists a W-invariant differential operator D on A with analytic coefficients, such that for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^* = \mathfrak{a}_{\mathbb{C}}^*$ and all $a \in A$ we have*

$$(2.12) \qquad \delta(a)\varphi_\lambda(a) = \frac{1}{d(\lambda-\rho)} D\Big(\sum_{w\in W} a^{w\lambda}\Big).$$

The right-hand side of (2.12) *is holomorphic in* λ.

Proof. If G (and hence U) is semisimple, then this follows from Lemma 2.7, Theorem 2.8, and [22, Theorem 5.1(c)]. For the general case, let D' be the differential operator from [22] on A_1. By assumption, we have $A \simeq T_{\mathbb{R}} \times A_1$. For $t \in T_{\mathbb{R}}$ and $a \in A_1$ we define the operator D by $D(f)(ta) := D'_a f(ta)/\mathfrak{c}(-\rho)$, where the subscript a indicates differentiation with respect to the variable a. □

We remark that the operator D' occurring in the proof of Theorem 2.9 is of the form $\delta(a)\widetilde{D}$, where $\widetilde{D}$ is Opdam's shift operator of shift $2m$. Multiplication by $\delta(a)$ cancels the singularities of the coefficients of $\widetilde{D}$. By construction, D' can be considered as differential operator on B_1 with holomorphic coefficients on $(B_1)_{\mathbb{C}} = A_1 B_1$. Consequently the operator D itself can be considered as differential operator on B with holomorphic coefficients on $B_{\mathbb{C}} = AB$.

The following corollary, which allows us to holomorphically extend the right-hand side of (2.12), will also play a crucial role in the proof of the local Paley–Wiener theorem.

Corollary 2.10. *Suppose that $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ is such that $\lambda_\alpha \in \pm\{0, 1, \ldots, m-1\}$ for some $\alpha \in \Delta^+$. Then $D(\sum_{w\in W} b^{w\lambda}) = 0$ for all $b \in B_{\mathbb{C}} := AB$.*

Proof. The last statement in Theorem 2.9 ensures that $D(\sum_{w\in W} b^{w\lambda}) = 0$ for all $b \in A$. Since $D(\sum_{w\in W} b^{w\lambda})$ is holomorphic (possibly multivalued) on $B_{\mathbb{C}}$ and it vanishes on A, it must be identically zero on $B_{\mathbb{C}}$. □

Because of Corollary 2.10, the only obstruction to the holomorphic extension to $B_{\mathbb{C}}$ of the right-hand side of (2.12) is the fact that the functions $b^{w\lambda}$ might be multivalued. This obstruction is solved by choosing a domain where the exponential function is a diffeomorphism.

Let $0 \in \mathcal{V} \subset \mathfrak{b}_{\mathbb{C}}$ and $e \in \mathcal{U} \subset B_{\mathbb{C}}$ be open, connected and such that $\exp: \mathcal{V} \to \mathcal{U}$ is an analytic diffeomorphism. We will assume furthermore that $\mathcal{V} \cap \mathfrak{b}$ is open and connected, and that $\mathcal{V}$ (and hence also $\mathcal{U}$) is W-invariant and contains $\mathfrak{a}$. Then, by Theorem 2.9, and Corollary 2.10, φ_λ has an analytic extension to $\mathcal{U}$.

Theorem 2.11. *The function*

$$\Lambda_K^+(U) \times \mathcal{U} \cap B \ni (\mu, b) \mapsto \psi_\mu(b) \in \mathbb{C}$$

has a holomorphic extension to $\mathfrak{b}_{\mathbb{C}}^* \times \mathcal{U}$ given by:

$$
\begin{aligned}
\psi_\lambda(b) &= \varphi_{\lambda+\rho}(b)\\
&= \frac{1}{d(\lambda)\delta(b)} D\Big(\sum_{w\in W} b^{w(\lambda+\rho)}\Big)\\
&= \frac{1}{d(\lambda)\delta(b)} \sum_{w\in W} Db^{w(\lambda+\rho)}\\
&= \delta(b)^{-1}\Big(\prod_{\alpha\in\Delta^+}\prod_{k=0}^{m-1}\frac{k^2-\rho_\alpha^2}{k^2-(\lambda+\rho)_\alpha^2}\Big) D\Big(\sum_{w\in W} b^{w(\lambda+\rho)}\Big).
\end{aligned}
$$

Furthermore, the analytic continuation satisfies

$$\psi_\lambda = \psi_{w(\lambda+\rho)-\rho}$$

for all $w \in W$ and $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$.

Proof. The first claim follows from Lemma 2.5, Theorem 2.9 and Corollary 2.10, as the right-hand side of (2.12) extends to a analytic function on $\mathfrak{b}_{\mathbb{C}}^* \times \mathcal{U}$. The statement $\psi_\lambda = \psi_{w(\lambda+\rho)-\rho}$ follows from the Weyl group invariance of $\lambda \mapsto \varphi_\lambda(b)$. □

As a corollary, we obtain the following explicit formulas for the spherical functions on Riemannian symmetric spaces of the compact type with even multiplicities.

Corollary 2.12. *Let $\mu \in \Lambda_K^+(U)$. Suppose that all multiplicities m_α are even. Then the following holds on B:*

$$
\begin{aligned}
\delta(b)\psi_\mu(b) &= \frac{1}{d(\mu)} D\Big(\sum_{w\in W} b^{w(\mu+\rho)}\Big)\\
&= \frac{1}{d(\mu)} \sum_{w\in W} Db^{w(\mu+\rho)}.
\end{aligned}
\tag{2.13}
$$

Proof. This follows from the fact that $w(\mu+\rho) \in \Lambda_K^+(U)$ and that for all $\nu \in \Lambda_K^+(U)$ the function b^ν is single valued and holomorphic on $B_{\mathbb{C}}$. □

2.5. The classification

We finish this section by giving the classification – up to coverings – of the symmetric spaces U/K with even multiplicities with the property that U is semisimple and U/K irreducible. Here K stands for an arbitrary connected, compact and simple Lie group. We list also the noncompact Riemannian dual G/K as well as $r := \operatorname{rank}(U/K) = \dim(\mathfrak{b})$, the multiplicities m_α, and the dimension d of U/K. In all these cases the multiplicities m_α are constant.

The first line of Table 1 corresponds to the complex case, in which the Lie algebra $\mathfrak{g}$ admits a complex structure.

Table 1

U/K	G/K	m_α	rank r	dimension d
$K\times K/K\simeq K$	$K_{\mathbb{C}}/K$	2	r	d
$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$	4	$n-1$	$(n-1)(2n+1)$
$\mathrm{E}_{6(-78)}/\mathrm{F}_4$	$\mathrm{E}_{6(-26)}/\mathrm{F}_4$	8	2	26
$\frac{\mathrm{SO}(2(n+1))}{\mathrm{SO}_0(2n+1)}\simeq S^{2n+1}$	$\frac{\mathrm{SO}_0(2n+1,1)}{\mathrm{SO}(2n+1)}$	$2n$	1	$2n+1$

3. THE LOCAL PALEY–WIENER THEOREM FOR COMPACT SYMMETRIC SPACES WITH EVEN MULTIPLICITY

In this section we introduce the spherical Fourier transform of K-invariant functions on the compact symmetric spaces U/K. We then assume that all multiplicities are even and use the results from the last section, in particular Theorem 2.11, to show that the Fourier transform, which in the beginning is only defined on a discrete set, extends holomorphically to $\mathfrak{b}_{\mathbb{C}}^*$ as long as the K-invariant function has sufficiently small support. We then define the Paley–Wiener space on $\mathfrak{b}_{\mathbb{C}}^*$ and prove the local Paley–Wiener theorem. This theorem generalizes the results obtained by Gonzalez in [10] for the case $U=K\times K$, where K is a connected, compact, simple Lie group. Notice that in this case $U/K\cong K$.

Let $\|\cdot\|$ be the norm on $\mathfrak{u}$ with respect to the U-invariant inner product constructed in Section 1, and let d be the associated Riemannian distance function on U/K. For $R>0$ let $B_R:=\{X\in\mathfrak{q}\mid \|X\|\leqslant R\}$ and $D_R:=\{x\in U/K\mid d(x,x_0)\leqslant R\}$ denote the corresponding balls of radius R with center 0 and x_0, respectively. We suppose that R is chosen so that the map $\mathrm{Exp}: X\to \exp X\cdot x_0$ is a diffeomorphism of B_R onto D_R. Finally we define

$$(3.1)\qquad C_R^\infty(U/K)^K:=\left\{f\in C^\infty(U/K)^K\mid \mathrm{Supp}(f)\subseteq D_R\right\}.$$

Here and in the following the superscript K denotes K-invariance.

Note that $\pi: B\to B\cdot x_0$ is a finite covering. We will identify B with the image $B\cdot x_0$. This is allowed, as we will only be considering K-invariant functions. Then, for every $f\in C_R^\infty(U/K)^K$ we have $\mathrm{Supp} f\subseteq D_R$ if and only if $\mathrm{Supp}(f|_B)\subseteq D_R$.

3.1. The spherical Fourier transform on U/K

For $f\in L^2(U/K)^K$ define $\hat f:\Lambda_K^+(U)\to\mathbb{C}$ by

$$(3.2)\qquad \begin{aligned}\mathcal{F}(f)(\mu)=\hat f(\mu)&:=(f,\psi_\mu)\\ &=\int_U f(u)\psi_\mu\left(u^{-1}\right)du\\ &=c\int_B f(b)\psi_\mu\left(b^{-1}\right)\delta(b)\,db.\end{aligned}$$

Here we have used the equality $\overline{\psi_\mu(u)} = \psi_\mu(u^{-1})$ from Lemma 2.3, and c is a suitable positive constant depending on the fixed normalization of measures. We call $\hat{f}$ the *spherical Fourier transform* of f and the map $\mathcal{F}$ the *spherical Fourier transform*. It is well known that $f \mapsto \{\sqrt{d(\mu)}\hat{f}(\mu)\}$ is a unitary isomorphism of $L^2(U/K)^K$ onto $\ell^2(\Lambda_K^+(U))$. The inversion formula is stated in the next theorem. In the following we shall often consider the K-bi-invariant functions ψ_λ on U as K-invariant functions on U/K.

Theorem 3.1. *Let $f \in L^2(U/K)^K$. Then*

$$f = \sum_{\mu \in \Lambda_K^+} d(\mu)\hat{f}(\mu)\psi_\mu,$$

where the sum is taken in $L^2(U/K)^K$. If $f \in C^\infty(U/K)^K$ then the above sum converges in the C^∞-topology.

Let E be a differential operator on B. Then the *formal adjoint* operator E^* is defined by the relation

$$\int_B f(b)Eg(b)\,db = \int_B E^*f(b)g(b)\,db$$

for all $f, g \in C^\infty(B)$. In this section D denotes the differential operator from Theorem 2.11. Hence

$$\delta(b)\psi_\mu(b) = d(\mu)^{-1} \sum_{w \in W} Db^{w(\mu+\rho)}. \tag{3.3}$$

If $\lambda = \lambda_R + i\lambda_I \in \mathfrak{b}_\mathbb{C}^*$ with $\lambda_R, \lambda_I \in \mathfrak{b}^*$, then we set

$$\operatorname{Re}\lambda := \lambda_R \quad \text{and} \quad \operatorname{Im}\lambda := \lambda_I. \tag{3.4}$$

Definition 3.2. A holomorphic function $f : \mathfrak{b}_\mathbb{C}^* \to \mathbb{C}$ is said to be of *exponential type* $R > 0$ if for each $N \in \mathbb{N}_0$ there exists a constant $C_N > 0$ such that

$$|f(\lambda)| \leqslant C_N\big(1 + \|\lambda\|\big)^{-N} e^{R\|\operatorname{Re}\lambda\|}$$

for all $\lambda \in \mathfrak{b}_\mathbb{C}^*$.

Lemma 3.3. *Let $p : \mathfrak{b}_\mathbb{C}^* \to \mathbb{C}$ be a polynomial function. Assume that $F : \mathfrak{b}_\mathbb{C}^* \to \mathbb{C}$ is an entire function so that $\lambda \mapsto p(\lambda)F(\lambda)$ is of exponential type R. Then F itself is of exponential type R.*

Proof. Let $n = \dim_\mathbb{C} \mathfrak{b}_\mathbb{C}^*$. Choose a basis $\lambda_1, \dots, \lambda_n$ of $\mathfrak{b}_\mathbb{C}^*$ and identify $\mathfrak{b}_\mathbb{C}^*$ with $\mathbb{C}^n$ by $\sum z_j\lambda_j \mapsto (iz_1, \dots, iz_n)$. The fact that F is of exponential type follows then from [17, Chapter III, Theorem 5.13]. The type R, which is not explicitly computed in this reference, can be easily deduced from formula (62) in that proof. Indeed one obtains the estimate

$$|F(z)| \leqslant C \sup_{\|w\|\leqslant 1} |p(z+w)F(z+w)|,$$

which shows that F and pF have the same exponential type. □

Theorem 3.4. *Suppose that all multiplicities are even. Let $\mu \in \Lambda_K^+(U)$, $w \in W$ and $f \in C^\infty(U/K)^K$. Then*

$$\hat{f}(\mu) = \frac{c}{d(\mu)}|W| \int_B [D^* f(b)] b^{-w(\mu+\rho)}\, db,$$

where $|W|$ denotes the cardinality of W.

Assume that $R > 0$ is chosen so that Exp *is a diffeomorphism of B_R onto D_R, and let $f \in C_R^\infty(U/K)^K$. Then $\Lambda_K^+(U) \ni \mu \mapsto \hat{f}(\mu) \in \mathbb{C}$ extends to a holomorphic function on $\mathfrak{b}_\mathbb{C}^*$ of exponential type R such that for all $\lambda \in \mathfrak{b}_\mathbb{C}^*$ and all $w \in W$ we have:*

$$\hat{f}(\lambda) = \hat{f}\big(w(\lambda+\rho)-\rho\big).$$

Proof. Observe that $\delta(b^{-1}) = \delta(b)$ because the multiplicities are even. The formula for $\hat{f}$ follows then directly from Corollary 2.12, formula (3.2) and the W-invariance of D and f. For the second part, we note that, by Theorem 2.11, for every fixed $b \in D_R \cap B$ the map

$$\mathfrak{b}_\mathbb{C}^* \ni \lambda \mapsto \delta(b^{-1})\psi_\lambda(b^{-1}) \in \mathbb{C}$$

is holomorphic. Hence, as we are integrating over a compact set, the map

$$\mathfrak{b}_\mathbb{C}^* \ni \lambda \mapsto c \int_{D_R} f(b)\psi_\lambda(b^{-1})\delta(b)\, db \in \mathbb{C}$$

is holomorphic. By Theorem 2.11, we have for the holomorphic extension:

$$d(\lambda)\hat{f}(\lambda) = c|W| \int_{D_R} [D^* f](b) b^{-(\lambda+\rho)}\, db. \tag{3.5}$$

The statement on the exponential type R of $\hat{f}$ follows now using Lemma 3.3 because the right-hand side of (3.5) is a Fourier transform for the torus B. □

3.2. The local Paley–Wiener theorem

In this subsection we prove the non-trivial part of the local Paley–Wiener theorem for compact symmetric spaces with even multiplicities. But first let us introduce some notation. Let $\eta : \widetilde{U}' \to U'$ be the universal covering of U', and set $\widetilde{U} =$

$T \times \widetilde{U}'$. Let $\exp_{\widetilde{U}} : \mathfrak{u} \to \widetilde{U}$ be the exponential map, and let $\eta_1 := \mathrm{id} \times \eta : \widetilde{U} \to U$. Then $\exp_U = \eta_1 \circ \exp_{\widetilde{U}}$. Set $\widetilde{K} := \exp_{\widetilde{U}} \mathfrak{k}$. Then $\eta_1(\widetilde{K}) = K$. Since $K \cap T = \{e\}$, we have $\widetilde{K} \subset \widetilde{U}'$. The symmetric space $\widetilde{U}'/\widetilde{K}$ is the universal covering manifold of U'/K. Let $\widetilde{d}$ denote the Riemannian metric on $\widetilde{U}/\widetilde{K}$ induced by the fixed U-invariant inner product on $\mathfrak{u}$. The induced map $\eta_1 : \widetilde{U}/\widetilde{K} \to U/K$ is a local isometry. Let $\widetilde{x}_0 = \{\widetilde{K}\}$ be the base point in $\widetilde{U}/\widetilde{K}$. Setting $\widetilde{D}_R := \{\widetilde{x} \in \widetilde{U}/\widetilde{K} \mid \widetilde{d}(\widetilde{x}, \widetilde{x}_0) \leqslant R\}$, we have $\eta_1(\widetilde{D}_R) = D_R$. Finally, let $\mathrm{Exp}_{\widetilde{U}} : \mathfrak{q} \to \widetilde{U}/\widetilde{K}$ be defined by $\mathrm{Exp}_{\widetilde{U}}(X) := \exp_{\widetilde{U}} X \cdot \widetilde{x}_0$. Then $\mathrm{Exp}_U = \eta_1 \circ \mathrm{Exp}_{\widetilde{U}}$.

Definition 3.5. We say that $R > 0$ is *small* if the following two conditions are satisfied:

(1) $\mathrm{Exp}_{\widetilde{U}} : B_R \to \widetilde{D}_R$ is a diffeomorphism;
(2) $\eta_1^{-1}(D_R)$ is a disjoint union of copies diffeomorphic to D_R under η_1.

Note that we are using the base point x_0 to define small. By translation, any other point could be used. The following statements would then still be valid. Notice also that, if R is small, then η_1 gives a diffeomorphism of $\widetilde{D}_R$ onto D_R. Moreover, in this case, the restriction of Exp is a diffeomorphism of B_R onto D_R.

We underline that we are employing the following notion of diffeomorphism for closed subsets: If C and C' are closed subsets of manifolds M and M', respectively, then a map $\vartheta : C \to C'$ is said to be a diffeomorphism if there exists open sets U in M and U' in M' with $C \subset U$ and $C' \subset U'$, so that $\vartheta : U \to U'$ is a diffeomorphism. According to Theorem 2.1, we have

$$(3.6)\qquad \Lambda_K^+(U) \subseteq \Lambda_{\widetilde{K}}^+(\widetilde{U}) = \Lambda^+ := \left\{\mu \in i\Gamma_0^* \oplus \mathfrak{a}^* \,\middle|\, \forall \alpha \in \Delta^+ : \mu_\alpha := \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}_0\right\}.$$

We set

$$(3.7)\qquad \Lambda := \left\{\mu \in i\Gamma_0^* \oplus \mathfrak{a}^* \mid \forall \alpha \in \Delta : \mu_\alpha \in \mathbb{Z}\right\}.$$

The decomposition

$$\left(\mathcal{L}, L^2(U/K)\right) = \bigoplus_{\mu \in \Lambda_K^+(U)} (\pi_\mu, V_{\pi_\mu}),$$

where $\mathcal{L}$ stands for the left-regular representation of U in $L^2(U/K)$ and V_{π_μ} denotes the Hilbert space of π_μ, implies the following lemma.

Lemma 3.6. *A function $f \in C(\widetilde{U}/\widetilde{K})$ is of the form $g \circ \eta_1$ for some $g \in C(U/K)$ if and only if $\hat{f}(\mu) = 0$ for all $\mu \in \Lambda^+ \setminus \Lambda_K^+(U)$. If f has support in $\widetilde{D}_R$, then g has support in D_R.*

Definition 3.7 (*Local Paley–Wiener space*). We shall denote by $\mathrm{PW}_R(\mathfrak{b}^*)$ the space of holomorphic functions of exponential type R satisfying $F(w(\lambda+\rho)-\rho) = F(\lambda)$ for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and all $w \in W$. Furthermore, we set

$$(3.8) \qquad \mathrm{PW}_R(\mathfrak{b}^*, U) := \{F \in \mathrm{PW}_R(\mathfrak{b}^*) \mid \forall \mu \in \Lambda^+ \setminus \Lambda_K^+(U)\colon\ F(\mu) = 0\}.$$

We call $\mathrm{PW}_R(\mathfrak{b}^*, U)$ the *local Paley–Wiener space* on $\mathfrak{b}_{\mathbb{C}}^*$ relative to U.

Theorem 3.8 (Local Paley–Wiener theorem). *Suppose that all multiplicities are even. Let $R > 0$ be small (according to Definition 3.5). Then the Fourier transform $\mathcal{F}$ is a bijection of $C_R^\infty(U/K)^K$ onto $\mathrm{PW}_R(\mathfrak{b}^*, U)$.*

We have already seen that the Fourier transform maps $C_R^\infty(U/K)^K$ injectively into $\mathrm{PW}_R(\mathfrak{b}^*)$, so we only have to show the surjectivity. Given $F \in \mathrm{PW}_R(\mathfrak{b}^*)$, then, by the inversion formula in Theorem 3.1, we have to define

$$(3.9) \qquad f = \sum_{\mu \in \Lambda_K^+(U)} d(\mu) F(\mu) \psi_\mu .$$

We must show:

(1) f is smooth and K-invariant;
(2) $\hat{f} = F$;
(3) $\mathrm{Supp}(f) \subseteq D_R$.

We start with some necessary preliminaries. Let $X \in \mathfrak{u}$ and $\mu \in \Lambda_K^+(U)$. Denote by $\pi_\mu^\infty(X)$ the bounded linear map defined on V_μ by

$$\pi_\mu^\infty(X)v := \lim_{t\to 0} \frac{\pi_\mu(\exp tX)v - v}{t}, \qquad v \in V_\mu .$$

We can extend π_μ^∞ to all of $\mathfrak{u}_{\mathbb{C}}$ by complex linearity.

Lemma 3.9. *Let $X \in \mathfrak{u}$ and $\mu \in \Lambda_K^+(U)$. Then*

$$\|\pi_\mu^\infty(X)\| \leqslant \|\mu\| \|X\|.$$

Proof. Notice that, if $X \in \mathfrak{u}$, then there exists $k \in U$ such that $\mathrm{Ad}(k)X \in \mathfrak{c}$, where $\mathfrak{c}$ is the Cartan subalgebra from Section 2. Furthermore,

$$\|\pi_\mu^\infty(\mathrm{Ad}(k)X)\| = \|\pi_\mu(k)\pi_\mu^\infty(X)\pi_\mu(k^{-1})\| = \|\pi_\mu^\infty(X)\|.$$

We can therefore assume that $X \in \mathfrak{c}$.

Denote the set of roots of $\mathfrak{c}_{\mathbb{C}}$ in $\mathfrak{u}_{\mathbb{C}}$ by $\Delta(\mathfrak{c})$, the set of positive roots from Section 2 by $\Delta^+(\mathfrak{c})$ and the corresponding set of simple roots by $\Sigma(\mathfrak{c})$. Finally, let $W(\mathfrak{c})$ denote the corresponding Weyl group. As π_μ extends to a representation of $U_{\mathbb{C}}$ it is easily seen that $\|\pi_\mu^\infty(w(X))\| = \|\pi_\mu^\infty(X)\|$ for all $w \in W(\mathfrak{c})$ and $X \in \mathfrak{c}$. Let

$$i\mathfrak{c}^+ = \{X \in i\mathfrak{c} \mid \forall \alpha \in \Delta^+(\mathfrak{c})\colon \alpha(X) > 0\}.$$

Then, if $X \in i\mathfrak{c}$, there exists $w \in W$ such that $w(X) \in \overline{i\mathfrak{c}^+}$. Let w_0 be the longest element in $W(\mathfrak{c})$. Then there exists a orthogonal basis v_λ consisting of weight vectors for $\mathfrak{b}$, i.e. for all $X \in \mathfrak{b}$ we have

$$\pi_\mu^\infty(X)v_\lambda = \lambda(X)v_\lambda.$$

Furthermore, each weight is of the form

$$\lambda = \mu - \sum_{\alpha \in \Sigma(\mathfrak{c})} n_\alpha \alpha = w_0\mu + \sum_{\alpha \in \Sigma(\mathfrak{c})} k_\alpha \alpha$$

for some $n_\alpha, k_\alpha \in \mathbb{N}_0$. If $X \in i\mathfrak{c}^+$ we get:

$$\mu(X) \geqslant \mu(X) - \sum n_\alpha \alpha(X) = (w_0\mu)(X) + \sum k_\alpha \alpha(X) \geqslant (w_0\mu)(X).$$

As $\|\mu\| = \|w_0\mu\|$ it follows that

$$|\lambda(X)| \leqslant \|\mu\| \|X\|$$

and hence for $X \in \mathfrak{c}$:

$$|\lambda(X)| = |\lambda(iX)| \leqslant \|\mu\| \, \|X\|.$$

The claim in the lemma therefore follows. □

Lemma 3.10. *Let $F \in \mathrm{PW}_R(\mathfrak{b}^*, U)$ and define f by* (3.9). *Then $f \in C^\infty(U/K)^K$, and $\hat{f}(\mu) = F(\mu)$ for all $\mu \in \Lambda_K^+(U)$.*

Proof. Let $\mu \in \Lambda_K^+(U) \subseteq i\mathfrak{b}^*$. By Theorem 2.8 we have that $d(\lambda)$ is a polynomial of degree $L = 2m|\Delta^+|$. Furthermore,

$$|\psi_\mu(x)| = |(\pi_\mu(x)w_\mu, w_\mu)| \leqslant 1.$$

It follows that for each $N \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$|d(\mu)F(\mu)\psi_\mu(b)| \leqslant C_N(1 + \|\mu\|)^{-N}.$$

By choosing N large enough, it follows that the series (3.9) converges uniformly. Hence f is continuous. As each ψ_μ is K-bi-invariant, we deduce that f is K-invariant. Choose a basis $X_1, \dots, X_k$ of $\mathfrak{u}$ with $\|X_j\| = 1$ for all j, and let

$$g_\mu(t_1, \dots, t_k) := \psi_\mu(\exp(t_1X_1) \cdots \exp(t_kX_k)).$$

For a fixed j, let $g_1 = \exp(t_1 X_1) \cdots \exp(t_{j-1} X_{j-1})$ and $g_2 = \exp(t_{j+1} X_{j+1}) \cdots \exp(t_k X_k)$. Then

$$\begin{aligned}\left|\frac{\partial}{\partial t_j} g_\mu(t_1, \ldots, t_k)\right| &= \left|\left(\pi_\mu(g_1)\pi_\mu^\infty(X_j)\pi_\mu(g_2)w_\mu, w_\mu\right)\right| \\ &\leqslant \left\|\pi_\mu^\infty(X_j)\right\| \\ &\leqslant \|\mu\|\end{aligned}$$

by Lemma 3.9. Iteration shows that for any multi-index $\alpha \in \mathbb{N}_0^k$, we have

$$\left|D^\alpha g_\mu(t_1, \ldots, t_k)\right| \leqslant \|\mu\|^{|\alpha|}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_k$. It follows, as above, that the series $\sum_\mu d(\mu)F(\mu)D^\alpha g_\mu \times (\exp(t_1 X_k) \cdots \exp(t_k X_k))$ converges uniformly. Hence f is smooth. In particular, we have $f \in L^2(U/K)^K$, and therefore

$$\sum_{\mu\in\Lambda_K^+(U)} d(\mu)\hat{f}(\mu)\psi_\mu = f = \sum_{\mu\in\Lambda_K^+(U)} d(\mu)F(\mu)\psi_\mu$$

in $L^2(U/K)$. Taking the inner product with ψ_μ, we see that $\hat{f}(\mu) = F(\mu)$ for all $\mu \in \Lambda_K^+(U)$. □

We will now show that $\operatorname{Supp}(f) \subseteq D_R$. For this, it is enough to show that $\operatorname{Supp}(\delta f) \subseteq D_R$.

Lemma 3.11. *Let R be small according to Definition* 3.5. *Let $F \in \mathrm{PW}_R(\mathfrak{b}^*, U)$, and define f by* (3.9). *Then for $b \in B$:*

$$\delta(b)f(b) = D\Bigg(\sum_{\mu\in\Lambda_K^+(U),\, w\in W} F(\mu)b^{w(\mu+\rho)}\Bigg).$$

Proof. If $b \in B$, then by the proof of Lemma 3.10, we have

$$\sum_{\mu\in\Lambda_K^+(U)} F(\mu)\sum_{w\in W} Db^{w(\mu+\rho)} = D\Bigg(\sum_{\mu\in\Lambda_K^+(U)} F(\mu)\sum_{w\in W} b^{w(\mu+\rho)}\Bigg).$$

Hence, for all $b \in B$ with $\delta(b) \neq 0$, we get

$$\begin{aligned}\delta(b)f(b) &= \delta(b)\sum_{\mu\in\Lambda_K^+(U)} d(\mu)F(\mu)\psi_\mu(b) \\ &= \sum_{\mu\in\Lambda_K^+(U)} F(\mu)\sum_{w\in W} Db^{w(\mu+\rho)} \\ &= D\Bigg(\sum_{\mu\in\Lambda_K^+(U)} F(\mu)\sum_{w\in W} b^{w(\mu+\rho)}\Bigg).\end{aligned}$$

As both sides are continuous in b, it follows that this holds on $\overline{\{b \in B \mid \delta(b) \neq 0\}} = B$. $\square$

Before finishing the proof of the main theorem, we need the following well-known lemma. Recall that, for $\mu \in i\mathfrak{b}^*$, we have introduced the notation $\chi_\mu(b) := b^\mu$, provided b^μ is defined for all $b \in B$.

Lemma 3.12. *Let* $\Gamma_1 := \{X \in \mathfrak{b}_1 \mid \exp_{\widetilde{U'}}(X) \in \widetilde{K}\}$. *Then*

$$\Gamma_1 = \{X \in \mathfrak{b}_1 \mid \forall \mu \in \Lambda\colon \mu(X) \in 2\pi i \mathbb{Z}\}.$$

Furthermore, if $\Gamma = \Gamma_0 \oplus \Gamma_1$, *then* $i\Gamma^* = \Lambda$ *and the map*

$$\Lambda \ni \mu \mapsto \chi_\mu \in \widehat{\mathfrak{b}/\Gamma}$$

is a bijection.

Proof. See the proof of Lemma 4.1 in [14, p. 535]. $\square$

Lemma 3.13. *Let* $F \in \mathrm{PW}_R(\mathfrak{b}^*)$ *and* $b \in B$. *Then*

$$D\Big(\sum_{\mu \in \Lambda^+,\ w \in W} F(\mu) b^{w(\mu+\rho)}\Big) = D\Big(\sum_{\mu \in \Lambda} F(\mu - \rho) b^\mu\Big).$$

Proof. Set $G(\mu) := F(\mu - \rho)$. Then G is of exponential type R. It follows, as in the proof of Lemma 3.10, that $\sum_{\mu \in \Lambda} G(\mu) b^\mu$ defines a smooth function on B. From $F(w(\lambda + \rho) - \rho) = F(\lambda)$ we obtain that $G(w(\lambda + \rho)) = G(\lambda + \rho)$ for all $\lambda \in \mathfrak{b}^*_{\mathbb{C}}$ and $w \in W$. Finally, part (2) of Lemma 2.7 implies that $\mu \mapsto \mu + \rho$ is a bijection on Λ. As Λ^+ is a fundamental domain for the action of W on Λ, we have:

$$\begin{aligned}\sum_{\mu \in \Lambda} G(\mu) b^\mu &= \sum_{\mu \in \Lambda} G(\mu + \rho) b^{\mu+\rho} \\ &= \sum_{\mu + \rho \in \Lambda^+} F(\mu) \frac{1}{|W^{\mu+\rho}|} \sum_{w \in W} b^{w(\mu+\rho)}\end{aligned}$$

where $W^{\mu+\rho} := \{w \in W \mid w(\mu + \rho) = \mu + \rho\}$.

For the final step, assume first that $\mu + \rho \in \Lambda^+$, but $\mu \notin \Lambda^+$. Then there is a simple root $\beta \in \Delta^+$ such that $\langle \mu, \beta \rangle < 0$. As $\mu \in \Lambda$, it follows that $\mu_\beta \in \mathbb{Z}$. In particular, $\mu_\beta \leqslant -1$. Since $\langle \mu + \rho, \beta \rangle \geqslant 0$ we have, using part (1) of Lemma 2.7,

$$0 \leqslant \frac{\langle \mu + \rho, \beta \rangle}{\langle \beta, \beta \rangle} = (\mu + \rho)_\beta \leqslant -1 + m.$$

By Corollary 2.10 it follows that

$$D\Big(\sum_{w \in W} b^{w(\mu+\rho)}\Big) = 0.$$

Finally, if $\mu + \rho \in \Lambda^+_K$ and $\mu \in \Lambda^+_K$, then $W^{\mu+\rho} = \{e\}$. The claim thus follows. $\square$

Lemma 3.14. *Suppose that $R > 0$ is small in the sense of Definition* 3.5. *Let $F \in \mathrm{PW}_R(\mathfrak{b}^*)$. Define $h : \mathfrak{b} \to \mathbb{C}$ by*

$$h(X) = \sum_{\mu \in \Lambda} F(\mu - \rho) e^{\mu(X)}.$$

Then h is a Γ-periodic smooth function on $\mathfrak{b}$ and $\mathrm{Supp}(h) \subseteq B_R + \Gamma$.

Proof. It follows from Lemma 3.12 and from the proof of Lemma 3.10 that h is smooth and Γ-periodic. It therefore defines a smooth function on the abelian group $\mathfrak{b}/\Gamma$. Hence

$$F(\mu - \rho) = \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} \int_{\mathfrak{b}/\Gamma} h(X) e^{-\mu(X)}\, dX = \widehat{h}(\mu).$$

By the classical Paley–Wiener theorem there is a $g \in C_R^\infty(\mathfrak{b})$ such that $\widehat{g}(\lambda) = F(\lambda - \rho)$. Here the Fourier transform of g is defined by

$$\widehat{g}(\lambda) = \frac{1}{(2\pi)^n} \int_{\mathfrak{b}} g(X) e^{-\lambda(X)}\, dX, \quad \lambda \in i\mathfrak{b}^*$$

where $n = \dim \mathfrak{b}$, as before. We claim that there exists a constant $\gamma \neq 0$ such that

$$\sum_{Y \in \Gamma} g(X + Y) = \gamma h(X).$$

Indeed, let

$$G(X) = \sum_{Y \in \Gamma} g(X + Y).$$

Then G is Γ-periodic and

$$\begin{aligned}
\widehat{G}(\mu) &= \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} \int_{\mathfrak{b}/\Gamma} G(X) e^{-\mu(X)}\, dX \\
&= \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} \int_{\mathfrak{b}} g(X) e^{-\mu(X)}\, dX \\
&= \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} (2\pi)^n \widehat{g}(\mu) \\
&= \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} (2\pi)^n F(\mu - \rho) \\
&= \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} (2\pi)^n \widehat{h}(\mu).
\end{aligned}$$

But this implies that

$$G = \mathrm{vol}(\mathfrak{b}/\Gamma)^{-1} (2\pi)^n h.$$

Observe that $(B_R+\gamma_1)\cap(B_R+\gamma_2)=\emptyset$ if $\gamma_1,\gamma_2\in\Gamma$ and $\gamma_1\neq\gamma_2$. Hence $\mathrm{Supp}(g)\subseteq B_R$ implies that $\mathrm{Supp}(G)\subseteq B_R+\Gamma$. The same must therefore hold for h. $\square$

We now finish the proof of the local Paley–Wiener theorem by proving the following lemma:

Lemma 3.15. *Assume that $R>0$ is small in the sense of Definition 3.5 and that $F\in \mathrm{PW}_R(\mathfrak{b}^*,U)$. Then there exists a $f\in C_R^\infty(U/K)^K$ such that $\hat{f}(\mu)=F(\mu)$ for all $\mu\in\Lambda_K^+$. Hence the spherical Fourier transform $\mathcal{F}:C_R^\infty(U/K)^K\to \mathrm{PW}_R(\mathfrak{b}^*,U)$ is surjective.*

Proof. By Lemma 3.6 we can assume that $U=\widetilde{U}$ and $K=\widetilde{K}$. Hence $\Lambda_K^+=\Lambda^+$. Define a smooth K-invariant function f on U/K by

$$f(x)=\sum_{\mu\in\Lambda^+} d(\mu)F(\mu)\psi_\mu(x).$$

Then $\hat{f}(\mu)=F(\mu)$ for all $\mu\in\Lambda^+$, cf. Lemma 3.10. As already observed, it suffices to prove that $\mathrm{Supp}(f|_B)\subseteq D_R$, which is equivalent to the condition $\mathrm{Supp}(\delta f|_B)\subseteq D_R$. By Lemmas 3.11 and 3.13 we have for $X\in\mathfrak{b}$:

$$(3.10)\qquad (\delta f)(\exp X)=D\Big(\sum_{\mu\in\Lambda^+}F(\mu)\sum_{w\in W}e^{w(\mu+\rho)(X)}\Big)$$
$$=D\Big(\sum_{\mu\in\Lambda}F(\mu-\rho)e^{\mu(X)}\Big)=Dh(X)$$

where h is the function defined in Lemma 3.14. Here Dh is defined locally by $Dh:=D(h\circ\exp^{-1})$. According to Lemma 3.14, we know that h (and hence Dh) has support in $B_R+\Gamma$. Thus δf has support in D_R. $\square$

We conclude this section by proving two integral formulas for the smooth functions on U/K with "small support". The first formula, which can be deduced from the proof of the local Paley–Wiener theorem, will play a decisive role in proving the validity of Huygens' principle on U/K.

Corollary 3.16. *Suppose that $R>0$ is small in the sense of Definition 3.5 and that $f\in C_R^\infty(U/K)^K$. Then the following integral formulas hold on B:*

$$(3.11)\qquad \delta(b)f(b)=D\Big(\int_{i\mathfrak{b}^*}\hat{f}(\lambda-\rho)b^\lambda\,d\lambda\Big),$$

where D is the differential operator of Theorem 2.11, and

$$f(b)=\frac{1}{|W|}\int_{i\mathfrak{b}^*}\hat{f}(\lambda-\rho)\varphi_\lambda(b)\,d(\lambda)\,d\lambda.$$

Proof. Suppose $b = \exp X \in D_R \cap B$. Then by (3.10), we have $\delta(b) f(b) = Dh(X)$. The function h is determined by the proof of Lemma 3.14 with $F = \hat{f}$. Keeping the notation of that proof, we obtain $h(X) = g(X)$ because $X \in B_R$, and $\widehat{g}(\lambda) = \hat{f}(\lambda - \rho)$ for all $\lambda \in \mathfrak{b}^*_{\mathbb{C}}$. Hence

$$h(X) = g(X) = \int_{i\mathfrak{b}^*} \widehat{g}(\lambda) e^{-\lambda(X)}\, d\lambda = \int_{i\mathfrak{b}^*} \hat{f}(\lambda - \rho) e^{-\lambda(X)}\, d\lambda$$
$$= \int_{i\mathfrak{b}^*} \hat{f}(\lambda - \rho) b^{-\lambda}\, d\lambda.$$

This proves (3.11) because both $f|_B$ and $\int_{i\mathfrak{b}^*} \hat{f}(\lambda - \rho) b^{-\lambda}\, d\lambda$ are supported in $\exp B_R$. The last formula follows then immediately from Theorem 2.11 and the W-invariance of $\lambda \mapsto \hat{f}(\lambda - \rho)$.

Remark 3.17. So far we have only discussed the Paley–Wiener theorem for compact symmetric spaces with even multiplicities. The reason is that our main tool, Theorem 2.9, which gives the holomorphic extension of the spherical functions in terms of the usual exponential functions on $\mathfrak{a}$ and a differential operator D with analytic coefficients, only holds for even multiplicities. The differential operator D and its formal adjoint are used to reduce the proofs and formulas to similar statements for Euclidean spaces. The same method will be used in the next section, and in fact those results are only valid for even multiplicities. But, in response to a request by one of the referees, we would like to comment briefly on the general case here; see also [1] for the Paley–Wiener theorem on the sphere S^p, and [7] for the Paley–Wiener theorem on the complex Grassmann manifolds $\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$, using different methods.

In the general case of compact symmetric spaces, one first needs to show that the Fourier transform maps the compactly supported smooth functions into the Paley–Wiener space, i.e., the generalization of our Theorem 3.4. For that, the first task is to prove the analytic extension as in Theorem 2.11. Because of Lemma 2.5, this essentially reduces to the analytic extension of the spherical functions on the noncompact dual G/K to the complex domain $K_{\mathbb{C}} A \exp(2i\Omega) K_{\mathbb{C}}$, where $\Omega = \{X \in \mathfrak{a} \mid \forall \alpha \in \Delta\colon |\alpha(X)| < \pi/2\}$ (see [18] for more information on the domain $K_{\mathbb{C}} A \exp(2i\Omega) K_{\mathbb{C}}$). Several approaches are possible in accomplishing this. A proof using the Heckman–Opdam system of differential equations was communicated by J. Faraut at the conference "Harmonic Analysis on Complex Homogeneous Domains and Lie Groups", Rome, May 17–19, 2001. We reproduce the ideas of this proof just below. Note that this proof holds for all positive multiplicity functions. For the special case of Riemannian symmetric spaces, another proof was given in [20, Proposition 1.3].

The second task is then to derive the Paley–Wiener estimates stated in Theorem 3.4. The first statement in this direction was given by E. Opdam in [23, Proposition 6.1], for all positive multiplicity functions. For the special case of Riemannian symmetric spaces, a second proof was given by B. Krötz in [18]. Both statements give estimates for the spherical functions only on the smaller domain $K_{\mathbb{C}} A \exp(i\Omega) K_{\mathbb{C}}$. Thus taking this approach would require a smaller support than assumed in this article, cf. [5]. For this smaller domain, one would obtain a generalization of our Theorem 3.4. But the surjectivity, which as so often is the difficult part, is still open.

We finish this remark by a short explanation of Faraut's proof of the holomorphic extension of the spherical functions. On $A_{\mathbb{C}}$ the spherical functions are solutions of the Heckman–Opdam system, which is a holonomic system of differential equations with meromorphic coefficients and regular singularities. Let δ be as in (1.1) and set $\Gamma := \{h \in A_{\mathbb{C}} \mid \delta(h) = 0\}$. Since the coefficients are regular in $\exp(2i\Omega) \setminus \Gamma$, all solutions of the Heckman–Opdam system must be regular on this set. Moreover, the spherical functions are also holomorphic on a neighborhood of A. So they must extend to be holomorphic on $\exp(2i\Omega)$.

The result that the spherical functions are regular in a neighborhood of A was obtained by Heckman and Opdam; cf. the article by Heckman in [11], or [23].

4. THE LOCAL HUYGENS' PRINCIPLE FOR COMPACT SYMMETRIC SPACES WITH EVEN MULTIPLICITIES

Let L_X denote the Laplace–Beltrami operator on a Riemannian symmetric space X of the noncompact or compact type. The *modified wave equation* on X is the partial differential equation

$$(L_X \pm \|\rho\|^2)u = u_{tt}, \tag{4.1}$$

where $u = u(x,t)$ is a function of $(x,t) \in X \times I$ and $I \subseteq \mathbb{R}$ is an interval containing 0. The sign in front of $\|\rho\|^2 := \langle \rho, \rho \rangle$ has to be chosen $+$ if X is of the noncompact type, and $-$ if X is of the compact type.

Let $f \in C_c^\infty(X)$ be fixed. Huygens' principle concerns specific support properties for the smooth solution u of (4.1) which satisfies the Cauchy conditions

$$\begin{aligned} &u(x,0) = 0, \\ &u_t(x,0) = f(x). \end{aligned} \tag{4.2}$$

Recall that solving a Cauchy problem with initial conditions $u(x,0) = g(x)$, $u_t(x,0) = f(x)$, where $f, g \in C_c^\infty(X)$ are arbitrary, can always be reduced to solving a Cauchy problem with initial conditions of the form (4.2), and that support properties like Huygens' principle reduce at the same time. Indeed, if u_i for $i = 1,2$ is the solution to $Lu_i = (u_i)_{tt}$ (for L the operator on the left in (4.1)) with Cauchy data $(0, f_i)$, then $u := u_2 + (u_1)_t$ is a solution with Cauchy data

$$\begin{aligned} &u(x,0) = f_1(x), \\ &u_t(x,0) = f_2(x) + (u_1)_{tt}(x,0) = f_2(x) + (Lu_1)(x,0) = f_2(x). \end{aligned}$$

Moreover, if $u(x,t)$ is the solution corresponding to (4.2), then $u(x,-t)$ corresponds to the initial conditions $u(x,0) = 0$, $u_t(x,0) = -f(x)$. This allows us to restrict our analysis to values $t \geqslant 0$. Finally, the general case can be reduced to the K-invariant one. We shall therefore assume in the following that $f \in C_c^\infty(X)^K$. In this case, the solution u will be a K-invariant function of the variable $x \in X$.

The property that the support of the solution u is compact is stated by the *principle of finite propagation speed.* This principle holds, more generally, for solutions of wave equations on arbitrary Riemannian manifolds. (See, e.g., [9, Chapter 5].) In our setting, it corresponds to the following lemma. We recall the notation $D_R := \{x \in X \mid d(x,x_0) \leqslant R\}$ for the closed ball of center x_0 and radius R.

Lemma 4.1 (Finite propagation speed). *Let $u \in C^\infty(X \times I)$ be solution to the Cauchy problem* (4.2). *Let $\varepsilon > 0$ and let $t \in I \cap (0,+\infty)$. Suppose that the Cauchy datum f in* (4.2) *satisfies* $\mathrm{Supp}(f) \subseteq D_\varepsilon$. *Then* $\mathrm{Supp}(u(\cdot,t)) \subset D_{\varepsilon+t}$.

For positive values of $t \in I$, the solution $u(x,t)$ to the Cauchy problem is therefore supported inside the positive cone

$$C_\varepsilon := \big\{(x,t) \in X \times [0,\infty) \mid d(x,x_0) \leqslant \varepsilon + t\big\}. \tag{4.3}$$

Let ε, t, f and u be as in Lemma 4.1. We say that the *(strong) Huygens' principle* holds provided $\mathrm{Supp}(u(\cdot,t)) \subseteq \{x \in X \mid t-\varepsilon \leqslant d(x,x_0) \leqslant t+\varepsilon\}$.

Thus, when the strong Huygens' principle holds, the solution of $u(x,t)$ to the Cauchy problem is supported for positive $t \in I$ inside the conical shell

$$S_\varepsilon := \big\{(x,t) \in X \times [0,\infty) \mid t-\varepsilon \leqslant d(x,x_0) \leqslant t+\varepsilon\big\}. \tag{4.4}$$

Huygens' principle holds true (at least for small values of t) for the wave equation on odd dimensional Riemannian symmetric spaces X of the noncompact or compact type for which all root multiplicities are even. (See the references in the introduction.) Observe that $\dim(U/K)$ is odd if and only if $\dim(\mathfrak{b}) = \mathrm{rank}(U/K)$ is odd. Indeed $\dim_{\mathbb{R}} U/K = \dim \mathfrak{u}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}} = \dim \mathfrak{n}_{\mathbb{C}} + \dim \mathfrak{b}$ – see the tables in Section 2.5 for the list of possible values.

The main result of this section is the following local version of Huygens' principle on symmetric spaces of the compact type.

Theorem 4.2. *Let U/K be a Riemannian symmetric space of the compact type with all multiplicities even. Let $R > 0$ be small according to Definition* 3.5. *Let $0 < \varepsilon < R$, and let $f \in C_\varepsilon^\infty(U/K)^K$. Assume that U/K is odd dimensional* (*that is,* $\mathrm{rank}(U/K) = \dim \mathfrak{b}$ *is odd*).

Suppose that $u(x,t)$ is a smooth solution of Cauchy's problem

$$\begin{aligned} &\big(L_{U/K} - \|\rho\|^2\big)u = u_{tt},\\ &u(x,0) = 0,\\ &u_t(x,0) = f(x). \end{aligned} \tag{4.5}$$

Then the following properties are satisfied:

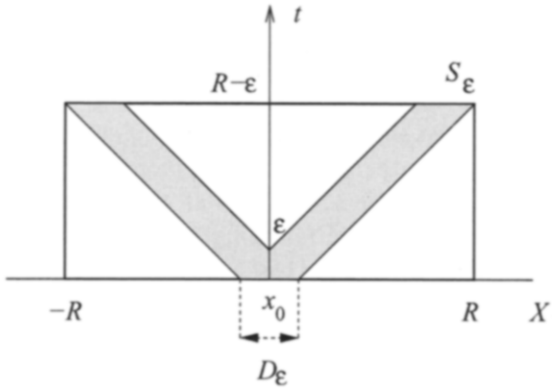

Figure 1.

(a) (Local exponential Huygens' principle) *There is a constant $C > 0$ so that for all $(x, t) \in U/K \times [0, R - \varepsilon]$ and all $\gamma \in [0, \infty)$ we have*

$$(4.6) \qquad \left|\delta(x)u(x,t)\right| \leqslant Ce^{-\gamma(t-d(x,x_0)-\varepsilon)}.$$

Here δ denotes the K-invariant extension to U/K of the W-invariant function δ defined in (1.1).

(b) (Local strong Huygens' principle)

$$\operatorname{Supp}(u) \cap \big(U/K \times [0, R-\varepsilon]\big) = \operatorname{Supp}(u) \cap \big(D_R \times [0, R-\varepsilon]\big) \subseteq S_\varepsilon,$$

where S_ε denotes the ε-shell (4.4).

(c) *Suppose* $\dim(U/K) \geqslant 3$. *Let D be the differential operator of Theorem* 2.11. *Then for all $b = \exp X \in B$ and $t \in [0, R - \varepsilon]$ the smooth solution $u(b,t)$ to* (4.14) *is given by formula*

$$(4.7) \qquad \delta(b)u(b,t) = \frac{\Omega_n/2}{[(n-3)/2]!\,\Omega_{n-1}} D\left(\frac{\partial}{\partial(t^2)}\right)^{(n-3)/2} \big(t^{n-2}\big(M^t g\big)(X)\big).$$

Here $g \in C_\varepsilon^\infty(\mathfrak{b})^W$ is the inverse Euclidean Fourier transform of $\hat{f}(\lambda - \rho)$ and

$$(4.8) \qquad \big(M^r g\big)(X) := \frac{1}{\Omega_{n-1}(r)} \int_{S_r(X)} g(s)\, d\sigma(s)$$

is the mean value of a function $g : \mathfrak{b} \to \mathbb{C}$ on the Euclidean sphere $S_r(X) := \{X \in \mathfrak{b} \mid \|X\| = r\}$ in $\mathfrak{b} \cong \mathbb{R}^n$ with respect to the $\mathrm{O}(n)$*-invariant surface measure $d\sigma$. Moreover, $\Omega_{n-1}(r)$ denotes the surface area of $S_r(x)$, and $\Omega_{n-1} := \Omega_{n-1}(1)$.*

Parts (a) and (c) of Theorem 4.2 seem to be new in the context of symmetric spaces of the compact type. As we shall see in the following, they both imply the local strong Huygens' principle of part (b). Another independent proof of Theorem 4.2 will be given in Corollary 4.4.

The remainder of this section is devoted to the proofs of the three parts of Theorem 4.2. To underline the various necessary steps, we have subdivided them into different lemmas and corollaries. Before entering the details of the proofs, we remark that, since the solution $u(x,t)$ is smooth and K-invariant in the x-variable, it suffices to examine its restriction to $B \times [0, R-\varepsilon]$. This will be common to all three methods which we are going to describe.

Recall that for all $\mu \in \Lambda_K(U)^+$ we have

$$L_{U/K}\psi_\mu = -\langle \mu + 2\rho, \mu\rangle \psi_\mu. \tag{4.9}$$

By Lemma 4.1, for fixed $t > 0$ the solution $u(\cdot, t)$ to (4.5) is supported inside $D_{t+\varepsilon}$. This allows us to interchange integration and differentiation with respect the variable $x \in U/K$. Hence, taking the spherical Fourier transform of (4.5) for fixed t, we obtain:

$$\begin{aligned} -\|\mu+\rho\|^2\, \widehat{u}(\mu,t) &= \widehat{u}_{tt}(\mu,t), \\ \widehat{u}(\mu,0) &= 0, \\ \widehat{u}_t(\mu,0) &= \hat{f}(\mu). \end{aligned} \tag{4.10}$$

Lemma 4.3. *Let U/K be a Riemannian symmetric space of the compact type with all even multiplicities. Let $R > 0$ be small according to Definition* 3.5, *and let* $0 < \varepsilon < R$. *Let $u(x,t)$ be a smooth solution of Cauchy's problem* (4.5) *with Cauchy datum $f \in C_\varepsilon^\infty(U/K)^K$. Then for $\lambda \in i\mathfrak{b}^*$*

$$\widehat{u}(\lambda-\rho,t) = \hat{f}(\lambda-\rho)\frac{\sin(\|\lambda\|t)}{\|\lambda\|}. \tag{4.11}$$

Consequently, for all $(b,t) \in B \times [0, R-\varepsilon]$ we have

$$\delta(b)u(b,t) = D\Big(\int_{i\mathfrak{b}^*} \hat{f}(\lambda-\rho)\frac{\sin(\|\lambda\|t)}{\|\lambda\|} b^\lambda \, d\lambda\Big). \tag{4.12}$$

Proof. Suppose that $t \in (0, R-\varepsilon)$. Then $t+\varepsilon < R$ is small, and the local Paley–Wiener Theorem 3.8 ensures that $\mu \mapsto \widehat{u}(\mu,t)$ extends uniquely to $\lambda \mapsto \widehat{u}(\lambda,t) \in \mathrm{PW}_{t+\varepsilon}(\mathfrak{b}^*, U)$. Likewise, $\widehat{u}_{tt}(\mu,t)$ and $-\langle \mu+\rho, \mu+\rho\rangle \widehat{u}(\mu,t)$ admit unique holomorphic extensions in $\mathrm{PW}_{t+\varepsilon}(\mathfrak{b}^*, U)$, respectively to $\widehat{u}_{tt}(\lambda,t)$ and $-\langle \lambda+\rho, \lambda+\rho\rangle \widehat{u}(\lambda,t)$. Finally, $\widehat{u}(\mu,0)$, $\widehat{u}_t(\mu,0)$ and $\hat{f}(\mu)$ extend uniquely to $\mathrm{PW}_\varepsilon(\mathfrak{b}^*, U)$. By uniqueness, we conclude that the equations in (4.10) hold for the holomorphic extensions. Setting $\omega(\lambda,t) := \widehat{u}(\lambda-\rho,t)$, we are therefore reduced to the Cauchy problem

$$\begin{aligned} \omega_{tt}(\lambda,t) &= -\|\lambda\|^2 \omega(\lambda,t), \\ \omega(\lambda,0) &= 0, \\ \omega_t(\lambda,0) &= \hat{f}(\lambda-\rho), \end{aligned}$$

from which (4.11) follows.

By Lemma 4.1, $u(\cdot,t) \in C^{\infty}_{t+\varepsilon}(U/K)^K$. Since $t+\varepsilon$ is small according to Definition 3.5, formula (4.12) is then a consequence of (3.11) and (4.11). □

Define $v : \mathfrak{b} \times (-R+\varepsilon, R-\varepsilon) \to \mathbb{C}$ by

$$(4.13) \qquad v(X,t) := \int_{i\mathfrak{b}^*} \hat{f}(\lambda-\rho)\frac{\sin(\|\lambda\|t)}{\|\lambda\|}e^{\lambda(X)}\,d\lambda.$$

Then $v(X,t)$ is the solution of the Cauchy problem for the wave equation on $\mathfrak{b} \cong \mathbb{R}^n$:

$$(4.14) \qquad \begin{aligned} L_{\mathfrak{b}} v(X,t) &= v_{tt}(X,t) \\ v(X,0) &= 0 \\ v_t(X,0) &= g(X), \end{aligned}$$

where $g \in C^{\infty}_{\varepsilon}(\mathfrak{b})^W$ is the inverse Euclidean Fourier transform of $\hat{f}(\lambda-\rho)$.

Since $\exp : B_R \to D_R$ is a diffeomorphism and since the operator D preserves supports, we have proved the following corollary, yielding the first proof of the local strong Huygens' principle of Theorem 4.2.

Corollary 4.4. *Suppose U/K is a symmetric space of the compact type with even multiplicities. Let R be small according to Definition* 3.5, *and let $0<\varepsilon<R$. Then the strong Huygens' principle holds for* (4.5) *on $U/K \times [0, R-\varepsilon]$ provided it holds for* (4.14) *on $\mathfrak{b} \times [0, R-\varepsilon]$. Hence the local strong Huygens' principle holds if* $\dim(U/K)$ *is odd (i.e. if* $\operatorname{rank}(U/K) = \dim \mathfrak{b}$ *is odd).*

To prove the local exponential Huygens' principle, we apply the procedure of [6] to the integral appearing at the right-hand side of (4.12). Our computations are nonetheless easier than those in that article. Since we only consider the even multiplicity situation, we can employ our differential operator D. This allows us to work in a Euclidean setting by replacing the spherical functions appearing in the integral formulas studied in [6] with exponential functions.

Let S denote the unit sphere in $i\mathfrak{b}^*$, and, as before, let $n = \dim \mathfrak{b} = \operatorname{rank}(U/K)$. With respect to polar coordinates $(\omega, p) \in S \times [0, +\infty)$ in $i\mathfrak{b}^*$, we have $d\lambda = p^{n-1}\,d\omega\,dp$.

Setting

$$(4.15) \qquad \Psi_\varepsilon(p,X) := p^{n-1}\int_S \hat{f}(p\omega-\rho)e^{p\omega(X)}\,d\omega,$$

we obtain for $b = \exp X \in B$

$$(4.16) \qquad \begin{aligned} \delta(b)u(b,t) &= D\bigg(\int_{i\mathfrak{b}^*} \hat{f}(\lambda-\rho)\frac{\sin(\|\lambda\|t)}{\|\lambda\|}e^{\lambda(X)}\,d\lambda\bigg) \\ &= D\bigg(\int_0^{\infty} \frac{\Psi_\varepsilon(p,X)}{p}\sin(pt)\,dp\bigg). \end{aligned}$$

Lemma 4.5. *Suppose $n := \dim \mathfrak{b}$ is odd. Then the following properties hold.*

(a) *The function $\Psi_\varepsilon(p, X)$ extends to a holomorphic function on $\mathbb{C} \times \mathfrak{b}_{\mathbb{C}}$. It is even in $p \in \mathbb{C}$ and W-invariant in $X \in \mathfrak{b}$. Moreover, for every $N \in \mathbb{N}$ there is a constant $K_N > 0$ such that*

$$(4.17) \qquad \left|\Psi_\varepsilon(p, X)\right| \leqslant K_N |p|^{n-1}\big(1 + |p|\big)^{-N} e^{|\operatorname{Im} p|(\varepsilon + \|X\|)}$$

for all $p \in \mathbb{C}$ and $X \in \mathfrak{b}$.

(b) *Suppose, furthermore, that $n \neq 1$. Let D be the differential operator of Theorem 2.11. Then the function $p^{-1} D\Psi_\varepsilon(p, X)$ is holomorphic on $\mathbb{C} \times \mathfrak{b}_{\mathbb{C}}$. It is odd in $p \in \mathbb{C}$ and W-invariant in $X \in \mathfrak{b}$. Moreover, for every $N \in \mathbb{N}$ and every compact $Q \subset \mathfrak{b}$ there is a constant $K_{N,Q} > 0$ such that*

$$(4.18) \qquad \left|\frac{D\Psi_\varepsilon(p, X)}{p}\right| \leqslant K_{N,Q} |p|^{n-2}\big(1 + |p|\big)^{-N} e^{|\operatorname{Im} p|(\varepsilon + \|X\|)}$$

for all $p \in \mathbb{C}$ and $X \in Q$.

Proof. As the integrand in (4.15) is holomorphic in $(p, X) \in \mathbb{C} \times \mathfrak{b}_{\mathbb{C}}$ and continuous in $\omega \in S$, it follows from Morera's theorem that Ψ_ε is holomorphic in $\mathbb{C} \times \mathfrak{b}_{\mathbb{C}}$. The fact that Ψ_ε is even in p and W-invariant in X is a consequence of the $\mathrm{O}(\mathfrak{b}^*)$-invariance of $d\omega$, the W-invariance of $\lambda \mapsto \hat{f}(\lambda - \rho)$, and the fact that n is odd.

Recall the notation (3.4) for the real and imaginary parts in $\mathfrak{b}^*_{\mathbb{C}}$. If $(p, \omega) \in \mathbb{C} \times S$ and $X \in \mathfrak{b}$, then

$$\big\|\operatorname{Re}(p\omega - \rho)\big\| = \big\|\operatorname{Re}(p\omega)\big\| = |\operatorname{Im} p|\|\omega\| = |\operatorname{Im} p|$$

and

$$\operatorname{Re}\big(p\omega(X)\big) = -\operatorname{Im} p \operatorname{Im} \omega(X) \leqslant |\operatorname{Im} p|\|X\|.$$

Since $\hat{f} \in \mathrm{PW}_\varepsilon(\mathfrak{b}^*)$, we therefore obtain for all $p \in \mathbb{C}$, $\omega \in S$ and $X \in \mathfrak{b}$:

$$\begin{aligned} \big|\hat{f}(p\omega - \rho) e^{p\omega(X)}\big| &\leqslant C_N\big(1 + \|p\omega - \rho\|\big)^{-N} e^{\varepsilon\|\operatorname{Re}(p\omega - \rho)\|} e^{\operatorname{Re}(p\omega(X))} \\ &\leqslant C_N\big(1 + |p|\big)^{-N} e^{\varepsilon|\operatorname{Im} p|} e^{|\operatorname{Im} p|\|X\|}, \end{aligned}$$

from which the estimate (4.17) immediately follows. Formula (4.15) shows then that $p^{-1}\Psi_\varepsilon(p, X)$ remains holomorphic provided $n > 1$. It is odd in $p \in \mathbb{C}$ and W-invariant in $X \in \mathfrak{b}$. The same property holds therefore also for $p^{-1} D\Psi_\varepsilon(p, X)$ because D is W-invariant and has holomorphic coefficients. Suppose $n \geqslant 3$. Differentiation under integral sign gives

$$\frac{D\Psi_\varepsilon(p, X)}{p} = p^{n-2} \int_S \hat{f}(p\omega - \rho) D e^{p\omega(X)}\, d\omega.$$

Notice that $De^{p\omega(X)} = f(X, \omega, p)e^{p\omega(X)}$ where $f(X, \omega, p)$ is holomorphic in $X \in \mathfrak{b}_{\mathbb{C}}^*$, polynomial in $\omega \in S$ and polynomial in $p \in \mathbb{C}$. For every compact subset Q of $\mathfrak{b}$ there is a constant C_K so that

$$\left|f(X, \omega, p)\right| \leqslant C_K\left(1 + |p|\right)^s$$

where $s = \deg D$ is the polynomial degree of f in the variable p. This, with the same argument used for (4.17), proves the estimate (4.18). □

We now use Lemma 4.5 to prove exponential estimates for the solution $u(x,t)$. Since $\Psi_\varepsilon(p, X)/p$ is odd, we obtain from (4.16) that for all $b = \exp X \in B$, $t \in [0, R - \varepsilon]$ and $\gamma > 0$ we have

$$(4.19) \qquad \delta(b)u(b,t) = \frac{1}{2}D\left(\int_{-\infty}^{\infty} \frac{\Psi_\varepsilon(p, X)}{p} e^{ipt}\, dp\right)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{D\Psi_\varepsilon(p, X)}{p} e^{ipt}\, dp$$
$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{D\Psi_\varepsilon(p + i\gamma, X)}{p + i\gamma} e^{ipt}\, dp\right)e^{-\gamma t}.$$

In the above computations, the differentiation under integral sign and the shift in the path of integration are justified by the estimates of Lemma 4.5.

Lemma 4.6. *Under the assumptions of Theorem* 4.2, *the local exponential Huygens' principle of Theorem* 4.2(b) *holds when* $\operatorname{rank}(U/K) > 1$.

Proof. Equation (4.19) together with estimate (4.18) give for all $b = \exp X \in B$ and $\gamma \in [0, \infty)$

$$\left|\delta(b)u(b,t)\right| \leqslant \frac{1}{2}\left(\int_{-\infty}^{\infty} \left|\frac{D\Psi_\varepsilon(p + i\gamma, X)}{p + i\gamma}\right| \left|e^{ipt}\right| dp\right)e^{-\gamma t}$$
$$\leqslant C_N\left(\int_{-\infty}^{\infty} \left(1 + |p|\right)^{-N} dp\right)e^{-\gamma(t - \|X\| - \varepsilon)}.$$

Since $\|X\| = d(x, x_0)$ when $x \in K\exp(X)K$, we conclude that for all $(b,t) \in B \times [0, R - \varepsilon]$ and all $t \in [0, \infty)$, we have

$$\left|\delta(b)u(b,t)\right| \leqslant Ce^{-\gamma(t - d(x, x_0) - \varepsilon)},$$

where C is a positive constant. The inequality then extends by K-invariance to U/K. □

As in [6], the exponential estimates need to be worked out directly when $\mathrm{rank}(U/K) = 1$. In this case the definition of $\Psi_\varepsilon(p, X)$ simplifies since $S = \{\pm i\}$. We shall identify $\mathfrak{b}_{\mathbb{C}}^*$ with $\mathbb{C}$ by $\lambda \equiv \langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle$. As $\lambda \mapsto \hat{f}(\lambda - \rho)$ is even, it follows that

$$\begin{aligned}\Psi_\varepsilon(p, X) &= \hat{f}(ip - \rho)e^{ipX} + \hat{f}(-ip - \rho)e^{-ipX} \\ &= 2\hat{f}(ip - \rho)\cos(pX).\end{aligned}$$

In the rank one case we can write the operator D in the form $D = D'(d/dX)$, where the D' is an odd differential operator with holomorphic coefficients. (See Corollary 4.16 of [22].) Hence

$$\begin{aligned}\frac{D\Psi_\varepsilon(p, X)}{p} &= \frac{D'(d/dX)\Psi_\varepsilon(p, X)}{p} \\ &= \frac{2\hat{f}(ip - \rho)D'(d/dX)\cos(pX)}{p} \\ &= -2\hat{f}(ip - \rho)D'\sin(pX).\end{aligned}$$

Formula (4.16) then yields, for $b = \exp X \in B$ and $t \in [0, R - \varepsilon]$,

$$\begin{aligned}\text{(4.20)} \qquad \delta(b)u(b, t) &= D\left(\int_0^\infty \frac{\Psi_\varepsilon(p, X)}{p} \sin(pt)\, dp\right) \\ &= \frac{1}{2i} D\left(\int_{-\infty}^\infty \frac{\Psi_\varepsilon(p, X)}{p} e^{ipt}\, dp\right) \\ &= \frac{1}{2i} \int_{-\infty}^\infty \frac{D\Psi_\varepsilon(p, X)}{p} e^{ipt}\, dp \\ &= i \int_{-\infty}^\infty \hat{f}(ip - \rho)D'\sin(pX)\, e^{ipt}\, dp.\end{aligned}$$

Since $\hat{f} \in \mathrm{PW}_\varepsilon(\mathfrak{b}^*)$, for all $N \in \mathbb{N}$ there are positive constants C_N and C'_N so that

$$\begin{aligned}\left|\hat{f}(ip - \rho)\sin(pX)\right| &\leqslant C_N\big(1 + \|ip - \rho\|\big)^{-N} e^{\varepsilon\|\mathrm{Re}(ip-\rho)\|} e^{\mathrm{Re}(ip\omega(X))} \\ &\leqslant C'_N\big(1 + |p|\big)^{-N} e^{\varepsilon|\mathrm{Im}\, p|} e^{|\mathrm{Im}\, p|\|X\|}\end{aligned}$$

for all $p \in \mathbb{C}$ and $X \in \mathfrak{b}$. As in Lemma 4.5(b) we conclude that $\hat{f}(ip - \rho)D'\sin(pX)$ is a holomorphic function of $(p, X) \in \mathbb{C} \times \mathfrak{b}$, and for every $N \in \mathbb{N}$ and every compact $Q \subset \mathfrak{b}$ there is a constant $K_{N,Q} > 0$ such that

$$\text{(4.21)} \qquad \left|\hat{f}(ip - \rho)D'\sin(pX)\right| \leqslant K_{N,Q}\big(1 + |p|\big)^{-N} e^{|\mathrm{Im}\, p|(\varepsilon + \|X\|)}$$

for all $p \in \mathbb{C}$ and $X \in Q$. This allows us to shift the contour of integration in (4.20) and get for all $p \in \mathbb{C}$ and $b = \exp X \in B$:

$$(4.22) \qquad \delta(b)u(b,t) = \left(\int_{-\infty}^{\infty} \hat{f}(ip - \gamma + \rho) D' \sin\big((p + i\gamma)X\big) e^{ipt}\, dp \right) e^{-\gamma t}.$$

The same argument used in Lemma 4.6, together with (4.21) and (4.22), yields the following lemma.

Lemma 4.7. *Keep the assumptions of Theorem* 4.2. *Then the local exponential Huygens' principle of Theorem* 4.2(a) *holds when* $\operatorname{rank}(U/K) = 1$.

The local exponential Huygens' principle provides a second proof of the local strong Huygens' principle.

Corollary 4.8. *Keep the assumptions of Theorem* 4.2. *If* $\dim(U/K)$ *is odd, then the local strong Huygens' principle of Theorem* 4.2(b) *holds for the modified wave equation on* U/K.

Proof. The finite propagation speed ensures that

$$\operatorname{Supp}(u) \cap \big(U/K \times [0, R - \varepsilon]\big) = \operatorname{Supp}(u) \cap \big(D_R \times [0, R - \varepsilon]\big) \subseteq C_\varepsilon,$$

where C_ε denotes the positive ε-cone (4.3). As $\gamma \to \infty$, we obtain from (4.6) that $\delta(x)u(x,t) = 0$ for all x with $t - d(x, x_0) - \varepsilon > 0$. Thus $\operatorname{Supp}(u) \cap (U/K \times [0, R - \varepsilon])$ is contained in the ε-shell S_ε of (4.4). □

We now turn to the proof of the explicit formulas for the smooth solution of the Cauchy problem (4.2) for the modified wave equation on U/K. These formulas are a consequence of (4.12) and of the explicit formulas known for the solution to the Cauchy problem (4.14) for the Euclidean wave equation.

For $r > 0$ we denote by $S_r(X) := \{X \in \mathfrak{b} \mid \|X\| = r\}$ the Euclidean sphere in $\mathfrak{b} \cong \mathbb{R}^n$ of center X and radius r. Again, we let $\Omega_{n-1}(r)$ denote the surface area of $S_r(x)$, and write simply Ω_{n-1} for $\Omega_{n-1}(1)$. Recall the definition (4.8) of the mean value $(M^r g)(X)$ of a function $g : \mathfrak{b} \to \mathbb{C}$ on $S_r(X)$.

Lemma 4.9. *Suppose* $\dim \mathfrak{b} = n$ *is at least* 2.

If n is odd, then the solution to (4.14) *is given by*

$$(4.23) \qquad v(X,t) = \frac{\Omega_n/2}{[(n-3)/2]!\,\Omega_{n-1}} \left(\frac{\partial}{\partial(t^2)} \right)^{(n-3)/2} \big(t^{n-2}(M^t g)(X)\big).$$

If n is even, then the solution to (4.14) *is given by*

$$(4.24) \qquad v(X,t) = \frac{1/2}{[(n-2)/2]!} \int_0^t r\big(t^2 - r^2\big) \left(\frac{\partial}{\partial r^2} \right)^{(n-2)/2} \big(r^{n-2}(M^r g)(X)\big)\, dr.$$

Proof. See, e.g., [17, p. 481]. □

Corollary 4.10. *Let U/K be a symmetric space of the compact type with even multiplicities. Suppose* $\operatorname{rank}(U/K) \geqslant 2$. *Let D be the differential operator of Theorem* 2.11, *and keep the notation of Lemma* 4.9. *As in* (4.14), *let $g \in C_\varepsilon^\infty(\mathfrak{b})^W$ be the inverse Euclidean Fourier transform of $\hat{f}(\lambda - \rho)$. If $n = \operatorname{rank}(U/K) = \dim \mathfrak{b}$ is odd, then the smooth solution $u(b,t)$ to* (4.14) *is given, for all $b = \exp X \in B$ and $t \in [0, R-\varepsilon]$, by the formula*

$$\delta(b)u(b,t) = \frac{\Omega_n/2}{[(n-3)/2]!\,\Omega_{n-1}} D\left(\frac{\partial}{\partial(t^2)}\right)^{(n-3)/2} \left(t^{n-2}\big(M^t g\big)(X)\right).$$

If n is even, then the solution to (4.14) *is given for all $b = \exp X \in B$ and $t \in [0, R-\varepsilon]$ by the formula*

$$(4.25)\qquad \delta(b)u(b,t) = \frac{1/2}{[(n-2)/2]!} D \int_0^t r\left(t^2 - r^2\right) \times \left(\frac{\partial}{\partial r^2}\right)^{(n-2)/2} \left(r^{n-2}\big(M^r g\big)(X)\right) dr.$$

Proof. This is immediate from (4.12), (4.13), and Lemma 4.9. □

Corollary 4.10 proves, in particular, Theorem 4.2(c). It also yields a third proof of the local strong Huygens' principle. Indeed, (4.7) shows that $u(b,t)$ is determined by the values of the Cauchy datum f in a thin shell around $S_t(X)$, where $b = \exp X$.

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