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# Finiteness for the $k$ -factor model and chirality varieties

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## Abstract

This paper deals with two families of algebraic varieties arising from applications. First, the  $k$ -factor model in statistics, consisting of  $n \times n$  covariance matrices of  $n$  observed Gaussian random variables that are pairwise independent given  $k$  hidden Gaussian variables. Second, *chirality varieties* inspired by applications in chemistry. A point in such a chirality variety records chirality measurements of all  $k$ -subsets among an  $n$ -set of ligands. Both classes of varieties are given by a parameterisation, while for applications having polynomial equations would be desirable. For instance, such equations could be used to test whether a given point lies in the variety. We prove that in a precise sense, which is different for the two classes of varieties, these equations are finitely characterisable when  $k$  is fixed and  $n$  grows.

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## 1. Results

### 1.1. The $k$ -factor model

Factor analysis addresses the problem of testing whether  $n$  observed random variables are conditionally independent given  $k$  hidden variables, called the factors. In the case where the joint

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distribution of all  $n + k$  variables is multivariate Gaussian, the parameter space  $F_{n,k}$  for the  $k$ -factor model is the set of  $n \times n$ -covariance matrices of the form  $\Sigma + \Gamma$  where  $\Sigma$  is diagonal positive definite and  $\Gamma$  is positive semidefinite of rank at most  $k$ . An algebraic approach to factor analysis seeks to determine all polynomial relations among the matrix entries in  $F_{n,k}$ ; these relations are called model invariants [4].

Clearly, any principal  $m \times m$ -submatrix of a matrix in  $F_{n,k}$  lies in  $F_{m,k}$ . An important question of theoretical interest is whether, for fixed  $k$ , there exists an  $m$  such that for  $n \geq m$  the model  $F_{k,n}$  is completely characterised by the fact that each principal  $m \times m$ -submatrix lies in  $F_{k,m}$ . For  $k = 2$  this question was settled in the affirmative very recently;  $m = 6$  suffices [5]. We prove the corresponding statement for the Zariski closure of the model, i.e., for the set of all real (or complex)  $n \times n$ -matrices satisfying all model invariants. Apart from polynomial equalities the definition of the model  $F_{k,n}$  also involves inequalities, which our approach does not take into consideration.

Our theorem to this effect needs the following notation. Let  $K$  be a field; all varieties and schemes will be defined over  $K$ . If  $X$  is a scheme over  $K$  and  $S$  is a  $K$ -algebra (commutative with 1), then we write  $X(S)$  the set of  $S$ -rational points of  $X$ . Our schemes will be affine, but not necessarily of finite type. So  $X = \text{Spec } R$  for some  $K$ -algebra  $R$ , and  $S$ -rational points are the  $K$ -algebra homomorphisms  $R \rightarrow S$ .

For a natural number  $n$  we write  $[n]$  for the set  $\{1, \dots, n\}$  and  $M_n, SM_n$  for the affine spaces over  $K$  of  $n \times n$ -matrices and of symmetric  $n \times n$ -matrices, respectively. We also write  $OM_n$  for the affine space of *off-diagonal*  $n \times n$ -matrices. This is the space  $\mathbb{A}^{n^2-n}$  where we think of the coordinates as the off-diagonal entries  $y_{ij}, i \neq j$  of an  $n \times n$ -matrix, so that the notion of principal submatrix of an off-diagonal matrix has an obvious meaning. Similarly we write  $SOM_n$  for the space of symmetric off-diagonal  $n \times n$ -matrices. There are natural projections  $M_n \rightarrow OM_n$  and  $SM_n \rightarrow SOM_n$ . Given a second natural number  $k$  we write  $M_n^{\leq k} \subseteq M_n$  for the subvariety of matrices of rank at most  $k$  and  $SM_n^{\leq k} \subseteq SM_n$  for the subvariety of symmetric matrices of rank at most  $k$ . Our first finiteness result concerns the varieties  $OM_n^{\leq k}$  and  $SOM_n^{\leq k}$ , which are the Zariski closures of the images of  $M_n^{\leq k}$  and of  $SM_n^{\leq k}$  in  $OM_n$  and  $SOM_n$ , respectively. In concrete terms, the ideal of  $OM_n^{\leq k}$  is the intersection of the ideal of  $M_n^{\leq k}$  with the polynomial algebra in the off-diagonal matrix entries, and similarly for  $SOM_n^{\leq k}$ . It seems rather hard to determine these ideals explicitly; for fixed  $k$  and  $n$  they can in principle be computed using Gröbner basis techniques [4].

**Remark 1.1.** The images of  $M_n^{\leq k}$  and  $SM_n^{\leq k}$  in  $OM_n$  and  $SOM_n$  are in general not closed. Indeed, for  $a, b \in K \setminus \{0\}$  the symmetric off-diagonal matrix

$$\begin{bmatrix} * & a & 0 \\ a & * & b \\ 0 & b & * \end{bmatrix}$$

can clearly not be extended to a rank-one symmetric matrix, while for any  $\epsilon \neq 0$  the matrix

$$\begin{bmatrix} * & a & \epsilon \\ a & * & b \\ \epsilon & b & * \end{bmatrix}$$

can be extended to the rank-one matrix

$$\begin{bmatrix} a\epsilon/b & a & \epsilon \\ a & ab/\epsilon & b \\ \epsilon & b & b\epsilon/a \end{bmatrix}.$$

This shows that the image of  $SM_3^{\leq 1}$  in  $SOM_3$  is not closed, which is why we defined  $SOM_n^{\leq k}$  as the Zariski closure of the image of  $SM_n^{\leq k}$ .

**Example 1.2.** For  $k = 2$  and  $n = 5$  the variety  $SOM_5^{\leq 2}$  is a hypersurface in  $SOM_5$  with equation

$$\frac{1}{10} \sum_{\sigma \in \text{Sym}(5)} \text{sgn}(\sigma) \sigma(y_{12}y_{23}y_{34}y_{45}y_{51}) = 0,$$

where  $y_{ij} = y_{ji}$  is the  $(i, j)$ -matrix entry and  $\text{Sym}(5)$  acts by simultaneously permuting rows and columns. The factor  $1/10$  comes from the dihedral group stabilising the 5-cycle, so this equation has 12 terms. This equation is called the *pentad* in [4]. Experiments there show that for  $n$  up to 9 pentads and off-diagonal  $3 \times 3$ -minors generate the ideal of  $SOM_n^{\leq 2}$ .

For any subset  $I \subseteq [n]$  of size  $m$  and any matrix  $y$  in  $M_n$  we write  $y[I] \in M_m$  for the principal submatrix of  $y$  with rows and columns labelled by  $I$ ; this notation is also used for off-diagonal matrices. If  $y \in M_n^{\leq k}(K)$ , then also  $y[I] \in M_m^{\leq k}(K)$ . Conversely,  $y \in M_n(K)$  lies in  $M_n^{\leq k}(K)$  if and only if all its  $(k + 1) \times (k + 1)$ -minors vanish. This implies that if  $n \geq 2(k + 1)$ , then  $y \in M_n(K)$  lies in  $M_n^{\leq k}(K)$  if and only if  $y[I] \in M_{2(k+1)}^{\leq k}(K)$  for all  $I \subseteq [n]$  of size  $2(k + 1)$ . Moreover, this statement holds scheme-theoretically, as well: the ideal of  $M_n^{\leq k}$  is generated by the pullbacks of the ideal of  $M_{2(k+1)}^{\leq k}$  under the morphisms  $y \mapsto y[I]$ ; this is just a restatement of the well-known fact that the  $(k + 1) \times (k + 1)$ -minors generate the ideal of  $M_n^{\leq k}$  [2]. Note that we need  $2(k + 1)$  here, rather than for instance  $(k + 1)$ , because we are only taking *principal* submatrices.

**Theorem 1.3** (Set-theoretic finiteness for the  $k$ -factor model). *There exists a natural number  $N_0$ , depending only on  $k$ , such that for all  $n \geq N_0$  we have*

$$OM_n^{\leq k}(K) = \{y \in M_n(K) \mid y[I] \in OM_{N_0}^{\leq k}(K) \text{ for all } I \subseteq [n] \text{ of size } N_0\}.$$

Similarly, there exists a natural number  $N_1$ , depending only on  $k$ , such that for all  $n \geq N_1$  we have

$$SOM_n^{\leq k}(K) = \{y \in SM_n(K) \mid y[I] \in SOM_{N_1}^{\leq k}(K) \text{ for all } I \subseteq [n] \text{ of size } N_1\}.$$

This theorem settles the “radical” part of [4, Question 29]. To relate this theorem to the  $k$ -factor model take  $K = \mathbb{R}$ . Since the diagonal entries in  $F_{n,k}$  are “free parameters”, all model invariants are generated by those involving only the off-diagonal entries. Hence a matrix lies in the Zariski closure of  $F_{n,k}$  if and only if its image in  $SOM_n(\mathbb{R})$  lies in  $SOM_n^{\leq k}(\mathbb{R})$ . Note that the theorem does not claim the stronger finiteness property in [4, Question 29], that the entire

ideal of  $\text{SOM}_n^{\leq k}$  is generated by the pull-back of the ideal of  $\text{SOM}_{N_1}^{\leq k}$  under taking principal submatrices. Although we expect this to be true, our methods do not suffice to prove this result.

### 1.2. Chirality varieties

Our second finiteness result concerns another family of algebraic varieties, motivated by applications in chemistry [11]. In one such application, one imagines four distinct ligands in the vertices of a regular tetrahedron  $T$ , which bond to an atom in the centre of  $T$ , and one is interested in some measurable property of the resulting structure which is invariant under orientation preserving symmetries of  $T$  but not under reflection. An example of such a property is optical activity. One assumes that the ligands can be characterised by scalars  $x_1, \dots, x_4$  and that the property is captured by a scalar valued function  $F$  of those four variables. The smallest-degree case is that where  $F(x_1, \dots, x_4) := \prod_{i < j} (x_i - x_j)$ . This function is called a *chirality product*; it changes its sign under reflections. Next one assumes that one has  $n$  ligands, and wants to know the polynomial relations among the values  $F(x_{i_1}, \dots, x_{i_4})$  as  $\{i_1, \dots, i_4\}$  runs over all 4-subsets of  $[n]$ . Again one hopes that these relations are characterised by finitely many types; and this is exactly what we shall prove.

More precisely and generally, fix a natural number  $k$  and for all  $n \geq k$  consider the affine space  $S_n^{(k)}$  whose coordinates  $y_J$  are parameterised by all  $k$ -subsets of  $[n]$ . In the example above  $k$  equals 4. Consider the morphism  $\mathbb{A}^n \rightarrow S_n^{(k)}$  that sends  $x$  to the point  $y \in S_n^{(k)}$  whose coordinate  $y_J$  equals the Vandermonde determinant  $\prod_{i, j \in J: i < j} (x_i - x_j)$ . The Zariski closure of the image of this map is denoted  $V_n^{(k)}$ . We call  $V_n^{(k)}$  a *chirality variety*; its ideal is the set of all relations between the Vandermonde determinants, or outcomes of chirality measurements for all  $k$ -subsets of  $n$  ligands. For any subset  $I$  of  $[n]$  of size  $m \geq k$  we have a natural morphism  $\pi_I : S_n^{(k)} \rightarrow S_m^{(k)}$ ,  $y \mapsto y[I]$ , which forgets the coordinates corresponding to  $k$ -subsets  $J \not\subseteq [m]$ . By construction, this map sends  $V_n^{(k)}$  into  $V_m^{(k)}$ .

**Theorem 1.4** (*Scheme-theoretic finiteness for chirality varieties in characteristic zero*). *In the (chemically relevant) case where the characteristic of  $K$  is zero there exists a natural number  $N_2$ , depending only on  $k$ , such that for all  $n \geq N_2$  the scheme  $V_n^{(k)}$  is the scheme-theoretic intersection of the pre-images  $\pi_I^{-1} V_{N_2}^{(k)}$  over all  $N_2$ -subsets  $I$  of  $[n]$ .*

In more concrete terms, the ideal of  $V_n^{(k)}$  is generated by the pull-backs of  $V_{N_2}^{(k)}$  under all projections  $\pi_I$ . The condition on the characteristic of  $K$  comes from the fact that our proof uses the existence of a Reynolds operator.

### 1.3. Preview

The remainder of this paper is organised as follows. In Section 2 we cast our two main results in a common framework, in which a sequence of schemes with group actions is replaced by its limit, which is a scheme of infinite type. For the  $k$ -factor model this scheme is the scheme of all (off-diagonal, symmetric)  $\mathbb{N} \times \mathbb{N}$ -matrices of rank at most  $k$ , and for chirality varieties it is the image of the map  $\mathbb{A}^{\mathbb{N}} \rightarrow S_{\mathbb{N}}^{(k)}$ , the infinite analogue of the map defining  $V_n^{(k)}$ . The slightly technical details of this section can be skipped at first reading. Section 3 introduces and develops the new notion of  $G$ -Noetherianity for topological spaces. Lemma 3.5 is particularly important

for our proof of Theorem 1.3. Section 4 introduces and develops the analogous notion for rings (or schemes). Theorem 4.2 by Aschenbrenner, Hillar, and Sullivant is used extensively in all proofs, while Proposition 4.9 shows how to use the Reynolds operator to prove scheme-theoretic finiteness for certain schemes, including the chirality varieties in characteristic 0.

### 2. Finiteness problems for chains of schemes

Our results above fit into the following set-up, which is similar to that of [1, Section 4]. First, we are given an infinite sequence

$$A_1 \begin{matrix} \xleftarrow{\tau_{12}} \\ \xrightarrow{\pi_{21}} \end{matrix} A_2 \begin{matrix} \xleftarrow{\tau_{23}} \\ \xrightarrow{\pi_{32}} \end{matrix} A_3 \begin{matrix} \xleftarrow{\tau_{34}} \\ \xrightarrow{\pi_{43}} \end{matrix} \cdots$$

where each  $A_n = \text{Spec}(T_n)$  is an affine scheme over  $K$ ,  $\tau_{n,n+1}$  is a closed embedding, and  $\pi_{n+1,n}$  is a morphism satisfying  $\pi_{n+1,n}\tau_{n,n+1} = \text{id}_{A_n}$ . For  $m \leq n$  we define  $\tau_{m,n} := \tau_{n-1,n} \cdots \tau_{m,m+1}$  and  $\pi_{n,m} := \pi_{m+1,m} \cdots \pi_{n,n-1}$ . Second, we are given a sequence

$$Y_1 \begin{matrix} \xleftarrow{\tau_{12}} \\ \xrightarrow{\pi_{21}} \end{matrix} Y_2 \begin{matrix} \xleftarrow{\tau_{23}} \\ \xrightarrow{\pi_{32}} \end{matrix} Y_3 \begin{matrix} \xleftarrow{\tau_{34}} \\ \xrightarrow{\pi_{43}} \end{matrix} \cdots$$

where each  $Y_n$  is a closed subscheme of  $A_n$  with ideal  $I_n \subseteq A_n$ , and we require that  $\pi_{n+1,n}$  maps  $Y_{n+1}$  into  $Y_n$  and that  $\tau_{n,n+1}$  maps  $Y_n$  into  $Y_{n+1}$ . Third, we are given a sequence of groups

$$G_1 \subseteq G_2 \subseteq \cdots$$

with for each  $n$  an action of  $G_n$  on  $A_n$  by automorphisms stabilising  $Y_n$ , and such that  $\pi_{n+1,n}$  is  $G_n$ -equivariant. We do *not* require that  $\tau_{n,n+1}$  be  $G_n$ -equivariant. We make the following somewhat technical assumption: for any triple  $q \geq n \geq m$  and any  $g \in G_q$  there exist a  $p \leq m$ ,  $g' \in G_n$ , and  $g'' \in G_m$  such that

$$\pi_{q,m}g\tau_{n,q} = g''\tau_{p,m}\pi_{n,p}g' \tag{*}$$

Now let  $A_\infty := \lim_{\leftarrow n} A_n$  be the projective limit of the  $A_n$ , i.e., the affine scheme corresponding to the algebra  $T_\infty := \bigcup_n T_n$ , where for  $m \leq n$  the algebra  $T_m$  is identified with a subalgebra of  $T_n$  by means of  $\pi_{n,m}^*$ . Since the latter map is  $G_m$ -equivariant, the union  $G$  of all  $G_n$  acts naturally on  $T_\infty$  and hence on  $A_\infty$ . Define  $Y_\infty \subseteq A_\infty$  similarly; then  $G$  stabilises  $Y_\infty$ . We write  $\tau_{n,\infty}$  and  $\pi_{\infty,n}$  for the natural embedding  $A_n \rightarrow A_\infty$  and the natural projection  $A_\infty \rightarrow A_n$ , respectively.

**Lemma 2.1.** *Let  $m$  be a natural number. For any  $K$ -algebra  $S$  the following two statements are equivalent:*

- (1) *for all  $n \geq m$  the set  $Y_n(S)$  consists of all  $y \in A_n(S)$  for which  $\pi_{n,m}G_n y \subseteq Y_m(S)$ , and*
- (2) *the set  $Y_\infty(S)$  consists of all  $y \in A_\infty(S)$  such that  $\pi_{\infty,m}G y \subseteq Y_m(S)$ .*

**Proof.** For the implication (1)  $\Rightarrow$  (2) let  $y \in A_\infty(S)$  have the property that  $\pi_{\infty,m}G y \subseteq Y_m(S)$ . By definition, the element  $y$  lies in  $Y_\infty(S)$  if and only if  $\pi_{\infty,n}y$  lies in  $Y_n(S)$  for all  $n$ . For  $n \leq m$  this condition is fulfilled because  $\pi_{\infty,n}(y) = \pi_{m,n}\pi_{\infty,m}(y)$  and the right-hand side lies

in  $Y_n(S)$  by the property of  $y$  and the fact that  $\pi_{m,n}$  maps  $Y_m(S)$  into  $Y_n(S)$ . Next consider  $y' := \pi_{\infty,n}y \in A_n(S)$  for  $n \geq m$ . For  $g \in G_n$  we have  $\pi_{n,m}gy' = \pi_{\infty,m}gy$ , which lies in  $Y_m(S)$  by assumption. Hence by (1)  $y'$  lies in  $Y_n(S)$ , as needed.

For the implication (2)  $\Rightarrow$  (1) let  $y \in A_n(S)$  satisfy  $\pi_{n,m}G_n y \subseteq Y_m(S)$ . Let  $y' := \tau_{n,\infty}y$ ; we prove that  $y'$  lies in  $Y_\infty(S)$  by showing that  $\pi_{\infty,m}Gy' \subseteq Y_m(S)$ . Indeed, let  $g \in G$ , say  $g \in G_q$ . If  $q \leq n$  then  $g \in G_n$  and  $\pi_{\infty,m}gy' = \pi_{n,m}gy \in Y_m(S)$  by the condition on  $y$ . Hence suppose that  $q \geq n$  and consider the element  $\pi_{\infty,m}gy' = \pi_{q,m}g\tau_{n,q}y$ . Now we invoke (\*) above to find a  $p \leq m$  and a  $g' \in G_n$  and a  $g'' \in G_m$  such that the right-hand side equals  $g''\tau_{p,m}\pi_{n,p}g'y$ , which by the property of  $y$  lies in  $Y_m(S)$ . By (2) we conclude that  $y'$  does indeed lie in  $Y_\infty(S)$ , hence  $y$  lies in  $Y_n(S)$ .  $\square$

The condition that (1) be true for all  $K$ -algebras  $S$  is equivalent to the statement that for  $n \geq m$  the ideal  $I_n \subseteq T_n$  of  $Y_n$  be the smallest  $G_n$ -stable ideal containing  $I_m$ . Similarly, the condition that (2) be true for all  $K$ -algebras  $S$  is equivalent to the statement that the ideal  $I_\infty$  be the smallest  $G$ -stable ideal of  $T_\infty$  containing  $I_m$ . Consequently, these two scheme-theoretic statements are also equivalent. We now fit our main results into this set-up.

### 2.1. The $k$ -factor model

Here  $A_n$  equals  $OM_n$  or  $SOM_n$ ,  $Y_n$  equals  $OM_n^{\leq k}$  or  $SOM_n^{\leq k}$ , and  $G_n = \text{Sym}(n)$  acts by simultaneous row and column permutations. The map  $\pi_{n+1,n}$  sends an (off-diagonal)  $(n + 1) \times (n + 1)$ -matrix to its principal  $n \times n$ -submatrix in the upper left corner, and  $\tau_{n,n+1}$  augments an (off-diagonal)  $n \times n$ -matrix with zero  $(n + 1)$ -st row and column. That  $\pi_{n+1,n}$  and  $\tau_{n,n+1}$  map  $Y_{n+1}$  into  $Y_n$  and vice versa follows from the fact that they map  $M_{n+1}^{\leq k}$  into  $M_n^{\leq k}$  and vice versa. Finally, condition (\*) is fulfilled since any  $m \times m$ -principal submatrix of a  $q \times q$ -matrix obtained from an  $n \times n$ -matrix  $y$  by augmenting with zeros and applying an element of  $\text{Sym}(q)$  can also be obtained by applying an element of  $\text{Sym}(n)$  to  $y$ , taking a suitable principal  $p \times p$ -submatrix, augmenting with zeroes, and reordering rows and columns with a permutation from  $\text{Sym}(m)$ .

Our proof of Theorem 1.3, set-theoretic finiteness of the  $k$ -factor model, will focus on the  $K$ -rational points of the schemes  $A_n = OM_n$  and  $Y_n = OM_n^{\leq k}$ . More specifically, we shall prove that there exist finitely many elements  $f_1, \dots, f_l \in T_\infty$  such that  $y \in A_\infty(K)$  lies in  $Y_\infty(K)$  if and only if  $f_i(gy) = 0$  for  $i = 1, \dots, l$  and  $g \in G$ . By Lemma 2.1 with  $S = K$  this implies Theorem 1.3.

### 2.2. Chirality varieties

Here  $A_n$  equals  $S_n^{(k)}$ ,  $Y_n$  equals  $V_n^{(k)}$ , and  $G_n = \text{Sym}(n)$  acts on the coordinates as follows:  $gy_J$  equals  $(-1)^a y_{gJ}$  where  $a$  is the number of pairs  $i < j$  in  $J$  such that  $gi > gj$ ; we call  $a$  the number of inversions of  $g$  on  $J$ . Note that this action makes the parameterisation  $y_J = \prod_{i,j \in J: i < j} (x_i - x_j)$  of  $V_n^{(k)}$   $\text{Sym}(n)$ -equivariant. The map  $\pi_{n+1,n}$  projects onto the coordinates  $y_J$  with  $J$  a subset of  $[n]$ . That  $\pi_{n+1,n}$  maps  $Y_{n+1}$  into  $Y_n$  is clear from the parameterisation. The map  $\tau_{n,q}$  for  $q \geq n$  is defined by its dual as follows: for  $J$  a  $k$ -subset of  $[q]$  we set

$$\tau_{n,q}^* : y_J \mapsto \begin{cases} y_J & \text{if } J \subseteq [n - 1], \\ 0 & \text{if } |J \setminus [n - 1]| \geq 2, \text{ and} \\ y_{J - \{j\} + \{n\}} & \text{if } J \setminus [n - 1] = \{j\}. \end{cases}$$

This reflects the effect of taking  $x_j = x_n$  for all  $j > n$  in the parameterisation, which shows that  $\tau_{n,q}$  does indeed map  $Y_n$  into  $Y_q$ . Note that  $\tau_{n,q} = \tau_{q-1,q} \cdots \tau_{n,n+1}$ , as in our set-up. Now we need to verify condition (\*), whose dual statement reads

$$\tau_{n,q}^* g \pi_{q,m}^* = g' \pi_{n,p}^* \tau_{p,m}^* g''$$

for suitable  $p \leq m, g' \in \text{Sym}(n), g'' \in \text{Sym}(m)$ . Let  $L$  be the set of elements in  $[m]$  that are mapped into  $[n - 1]$  by  $g$ . For any  $k$ -subset  $J$  of  $[m]$  the map  $\tau_{n,q}^* g \pi_{q,m}^*$  sends

$$y_J \mapsto \begin{cases} (-1)^a y_{gJ} & \text{if } J \subseteq L, \\ 0 & \text{if } |J \setminus L| \geq 2, \text{ and} \\ (-1)^a y_{gJ - \{gj\} + \{n\}} & \text{if } J \setminus L = \{j\}, \end{cases}$$

where  $a$  is the number of inversions of  $g$  on  $J$ . We distinguish two cases. First suppose that  $L$  is all of  $[m]$ . Then the last two cases do not occur, and we may take  $g'' := 1 \in \text{Sym}(m)$ ,  $p := m$ , and any  $g' \in \text{Sym}(n)$  which agrees with  $g$  on  $[m]$ . Second, suppose that  $L \subsetneq [m]$ . Set  $p := |L| + 1 \leq m$  and choose  $g'' \in \text{Sym}(m)$  such that  $g''$  maps  $L$  bijectively into  $[p - 1]$  and such that for all  $i, j$  in  $L$  we have  $g''i < g''j$  if and only if  $gi < gj$ . This ensures that the number of inversions of  $g''$  on any subset of  $[m]$  containing at most one element outside of  $L$  is the same as the number of inversions of  $g$  on that subset. Then  $\pi_{n,p}^* \tau_{p,m}^* g''$  maps

$$y_J \mapsto \begin{cases} (-1)^a y_{g''J} & \text{if } J \subseteq L, \\ 0 & \text{if } |J \setminus L| \geq 2, \text{ and} \\ (-1)^a y_{g''J - \{g''j\} + \{p\}} & \text{if } J \setminus L = \{j\}, \end{cases}$$

where  $a$  is both the number of inversions of  $g$  on  $J$  and that of  $g''$  on  $J$ . Hence if we compose this with an element  $g' \in \text{Sym}(n)$  which is increasing on  $[p]$  and satisfies  $g'g'' = g$  on  $L$  and  $g'p = n$ , then we are done.

For Theorem 1.4, scheme-theoretic finiteness of chirality varieties in characteristic 0, we shall prove that there exist finitely many elements  $f_1, \dots, f_l$  in the ideal  $I_\infty$  of  $Y_\infty = V_\infty^{(k)}$  whose  $G$ -orbits generate the ideal of  $Y_\infty$ . By Lemma 2.1 and the remark following it this implies Theorem 1.4.

### 3. Topological $G$ -Noetherianity

In this section we develop a purely topological notion that can be used to prove set-theoretic finiteness results.

**Definition 3.1.** Let  $G$  be a group acting by homeomorphisms on a topological space  $X$ ; we shall call  $X$  a  $G$ -space. Then  $X$  is called  $G$ -Noetherian if every chain  $X_1 \supseteq X_2 \supseteq \dots$  of closed  $G$ -stable subsets of  $X$  stabilises in the sense that there exists an  $m$  such that  $X_n = X_m$  for all  $n \geq m$ .

**Lemma 3.2.** Let  $X$  be a  $G$ -space and  $Y$  a  $G$ -stable closed subset of  $X$ . If  $X$  is  $G$ -Noetherian, then so is  $Y$ .

**Proof.** Every descending chain of  $G$ -stable closed subsets in  $Y$  is also such a chain in  $X$ , hence stabilises.  $\square$

**Lemma 3.3.** *Let  $X$  and  $Y$  be  $G$ -spaces. If  $Y$  is  $G$ -Noetherian and if there exists a surjective  $G$ -equivariant continuous map  $Y \rightarrow X$ , then  $X$  is  $G$ -Noetherian.*

**Proof.** The pre-image of a descending chain of  $G$ -stable closed subsets of  $X$  is such a chain in  $Y$ , hence stabilises. By surjectivity the chain in  $X$  also stabilises.  $\square$

**Lemma 3.4.** *Let  $X$  and  $Y$  be  $G$ -spaces. Then their disjoint union, equipped with the disjoint-union topology, is  $G$ -Noetherian if and only if both  $X$  and  $Y$  are.*

**Proof.** A closed  $G$ -stable subset of  $X \cup Y$  is of the form  $C \cup D$  with  $C$  and  $D$  closed and  $G$ -stable in  $X$  and  $Y$ , respectively. A descending chain of such sets stabilises if and only if the induced chains in  $X$  and  $Y$  stabilise.  $\square$

The above lemmas are exact analogues of statements on ordinary Noetherianity of topological spaces. Now, however, we introduce a construction where the  $G$ -structure plays an essential role. Suppose that  $Y$  is an  $H$ -space, where  $H$  is a subgroup of  $G$ . Construct the space  $G \times_H Y := G \times Y / \sim$  where  $\sim$  is the smallest equivalence relation with  $(gh, y) \sim (g, hy)$  for all  $g \in G$ ,  $h \in H$ ,  $y \in Y$ . The space  $G \times Y$  carries the product topology with  $G$  discrete, so that every closed subset is of the form  $\bigcup_{g \in G} \{g\} \times Y_g$  with each  $Y_g$  closed in  $Y$ . The space  $G \times_H Y$  carries the quotient topology of the product topology and a topological  $G$ -action by left multiplication.

**Lemma 3.5.** *If  $Y$  is  $H$ -Noetherian, then  $G \times_H Y$  is  $G$ -Noetherian.*

**Proof.** Let  $Z_1 \supseteq Z_2 \supseteq \dots$  be a chain of  $G$ -stable closed subsets of  $G \times_H Y$ . The pre-image of  $Z_i$  in  $G \times Y$  is of the form  $\bigcup_{g \in G} \{g\} \times Y_{i,g}$  with  $Y_{i,g}$  closed in  $Y$ . As  $Z_i$  is  $G$ -stable, we have  $Y_{i,g} = Y_{i,e}$  for all  $g \in G$ , and as  $Z_i$  is a union of equivalence classes  $Y_{i,e}$  is  $H$ -stable. The chain  $Y_{1,e} \supseteq Y_{2,e} \supseteq \dots$  stabilises as  $Y$  is  $H$ -Noetherian, hence so does the chain  $Z_1 \supseteq Z_2 \supseteq \dots$ .  $\square$

In our application to the  $k$ -factor model, the topological spaces will be sets of rational points of affine schemes over  $K$ , equipped with the Zariski topology where closed sets are given by the vanishing of elements in the corresponding  $K$ -algebra. The following lemma describes what  $G$ -Noetherianity means in this case.

**Lemma 3.6.** *Suppose that  $A = \text{Spec } T$  is an affine scheme over  $K$ , where  $T$  is a  $K$ -algebra, and that  $G$  acts by automorphisms on  $A$ , hence on  $T$ . Then the following two statements are equivalent:*

- (1)  $A(K)$ , equipped with the Zariski topology, is  $G$ -Noetherian; and
- (2) for every  $G$ -stable ideal  $I$  of  $T$  there exist finitely many elements  $f_1, \dots, f_l$  such that  $y \in A(K)$  lies in the closed set defined by  $I$  if and only if  $f_i(gy) = 0$  for all  $i = 1, \dots, l$  and for all  $g \in G$ .

**Proof.** Suppose first that  $A(K)$  is  $G$ -Noetherian and let  $I$  be a  $G$ -stable ideal in  $T$ . Construct a chain  $X_0 \supseteq X_1 \supseteq \dots$  of  $G$ -stable closed subsets of  $A(K)$  as follows:  $X_0 := A(K)$ , and for

$i \geq 1$  either choose  $f_i \in I$  which does not vanish identically on  $X_{i-1}$  and set  $X_i := \{y \in X_{i-1} \mid f_i(gy) = 0 \text{ for all } g \in G\}$  or, if such an  $f_i$  does not exist, then  $X_i = X_{i-1}$ . By  $G$ -Noetherianity this  $G$ -stable chain stabilises at some  $X_l$ , and then  $f_1, \dots, f_l$  have the required property.

For the converse suppose that every  $G$ -stable ideal  $I$  has the stated property, and consider a chain  $X_1 \supseteq X_2 \supseteq \dots$  of  $G$ -stable closed subsets. Let  $I_n$  be the ideal in  $T$  vanishing on  $X_n$ . The union of all  $I_n$  is a  $G$ -stable ideal in  $T$ , hence let  $f_1, \dots, f_l$  be as in the assumption. Let  $n$  be such that  $f_1, \dots, f_l \in I_n$ . Then we claim that  $X_m = X_n$  for  $m \geq n$ . Indeed, if not, then let  $y \in X_n \setminus X_m$ . This means that some element of  $I_m \subseteq I$  does not vanish on  $y$ , which contradicts the fact that  $f_i(gy) = 0$  for all  $i$  and  $g$ .  $\square$

#### 4. Scheme-theoretic $G$ -Noetherianity

In this section we introduce and develop the notion of  $G$ -Noetherianity for rings. Dually, we will also adopt this terminology for the corresponding affine schemes.

**Definition 4.1.** Let  $G$  be a group acting by automorphisms on a ring  $R$ ; we shall simply call  $R$  a  $G$ -ring. Then  $R$  is called  $G$ -Noetherian if every chain  $I_1 \subseteq I_2 \subseteq \dots$  of  $G$ -stable ideals stabilises.

For all main theorems we need the following fundamental result.

**Theorem 4.2.** (See [1,7].) *The ring  $K[x_{ij} \mid i = 1, \dots, l, j = 0, 1, 2, \dots]$ , on which the group  $\text{Sym}(\mathbb{N})$  of bijections from  $\mathbb{N}$  to itself acts by  $\sigma x_{ij} = x_{i\sigma j}$ , is  $\text{Sym}(\mathbb{N})$ -Noetherian.*

This theorem was first proved in [1] for the case where  $l = 1$ , and then generalised in [7]. Its proof boils down to showing that a certain order on monomials is a well-quasi-order. The fact that  $\text{Sym}(\mathbb{N})$ -stable monomial ideals are finitely generated up to the action of  $\text{Sym}(\mathbb{N})$  boils down to the statement that Young diagrams are well-quasi-ordered by inclusion. This, in turn, is a special case of the theorem in [10] that antichains of monomial ideals are finite.

**Remark 4.3.** In view of Section 2 it is more natural to replace  $\text{Sym}(\mathbb{N})$  by the direct limit  $G$  of all  $\text{Sym}(n)$ , where  $\text{Sym}(n)$  is considered as the stabiliser in  $\text{Sym}(n + 1)$  of  $n + 1$ . As both groups have the same orbits on the ring above, that ring is also  $G$ -Noetherian.

**Lemma 4.4.** *If  $R$  is a  $G$ -Noetherian ring, then so is  $R[X]$ , where  $X$  is a variable and  $G$  acts only on the coefficients of the polynomials in  $R[X]$ .*

**Proof.** One can copy the proof of Hilbert’s basis theorem from [9] word-by-word.  $\square$

**Lemma 4.5.** *Let  $R$  be a  $K$ -algebra, and let  $A = \text{Spec } R$  be the corresponding affine scheme over  $K$ . Suppose that a group  $G$  acts on  $R$  by  $K$ -algebra automorphisms. Then if  $R$  is  $G$ -Noetherian, then the set  $A(K)$  with the Zariski topology is a  $G$ -Noetherian topological space.*

**Proof.** Consider a chain  $X_1 \supseteq X_2 \supseteq \dots$  of  $G$ -stable closed subsets of  $Y(K)$ . Let  $I_n$  be the vanishing ideal in  $R$  of  $X_n$ . The  $I_n$  form an ascending chain of  $G$ -stable ideals, which stabilises as  $R$  is  $G$ -Noetherian. Hence, since  $X_n$  is the zero set of  $I_n$ , the chain  $X_1 \supseteq X_2 \supseteq \dots$  also stabilises.  $\square$

We collect some further elementary properties of scheme-theoretic  $G$ -Noetherianity. First, an analogue of Lemma 3.2.

**Lemma 4.6.** *Let  $R$  and  $S$  be  $G$ -rings. If there exists a  $G$ -equivariant epimorphism  $R \rightarrow S$  and  $R$  is  $G$ -Noetherian, then  $S$  is  $G$ -Noetherian.*

**Proof.** Every chain  $I_1 \subseteq I_2 \subseteq \dots$  of  $G$ -stable ideals in  $S$  lifts to a chain in  $R$ . As the latter stabilises by  $G$ -Noetherianity of  $R$ , so does the chain in  $S$  by surjectivity of the morphism  $R \rightarrow S$ .  $\square$

An analogue of Lemma 3.4 is this.

**Lemma 4.7.** *Let  $R$  and  $S$  be  $G$ -rings. Then  $R \oplus S$  is  $G$ -Noetherian (respectively, radically  $G$ -Noetherian) if and only if both  $R$  and  $S$  are  $G$ -Noetherian (respectively, radically  $G$ -Noetherian).*

**Proof.** A  $G$ -stable ideal of  $R \oplus S$  is of the form  $I \oplus J$  with  $I$  a  $G$ -stable ideal in  $R$  and  $J$  a  $G$ -stable ideal in  $S$ . A chain of such ideals stabilises if and only if the two component chains stabilise.  $\square$

Here is one possible analogue of Lemma 3.3.

**Lemma 4.8.** *Let  $R$  and  $S$  be  $G$ -rings. If there exists a  $G$ -equivariant homomorphism  $\phi : R \rightarrow S$  such that  $\phi^{-1}(\phi(I)S) = I$  for all  $G$ -stable ideals  $I$  of  $R$ , and if  $S$  is  $G$ -Noetherian, then so is  $R$ .*

**Proof.** For a chain  $I_1 \subseteq I_2 \subseteq \dots$  of  $G$ -stable ideals in  $R$  the ideals  $J_i := \phi(I_i)S$  form a chain of  $G$ -stable ideals in  $S$ . As  $S$  is  $G$ -Noetherian, we have  $J_{i+1} = J_i$  for all sufficiently large  $i$ . But then also  $I_{i+1} = \phi^{-1}(J_{i+1}) = \phi^{-1}(J_i) = I_i$ , as required.  $\square$

For scheme-theoretic finiteness of chirality varieties in characteristic 0 we need another construction of  $G$ -Noetherian algebras. Consider a  $K$ -algebra  $R$  acted upon by two groups  $G$  and  $H$ , where the actions have the following four properties:  $G$  and  $H$  act by  $K$ -algebra automorphisms; the actions of  $G$  and  $H$  on  $R$  commute; every element of  $R$  is contained in a finite-dimensional  $H$ -module; and every finite-dimensional  $H$ -submodule of  $R$  splits as a direct sum of irreducible  $H$ -modules. By the second property, the  $K$ -algebra  $R^H$  of  $H$ -invariants is  $G$ -stable.

**Proposition 4.9.** *If  $R$  is  $G$ -Noetherian, then so is  $R^H$ .*

The proof of this proposition uses the Reynolds operator  $\rho : R \rightarrow R^H$ , defined as follows. For  $f \in R$  let  $U$  be a finite-dimensional  $H$ -submodule of  $R$  containing  $f$ . Split  $U = U_0 \oplus U_1$  where  $U_0$  is the sum of all trivial  $H$ -modules in  $U$  and  $U_1$  is the sum of all non-trivial irreducible  $H$ -modules in  $U$ . Split  $f = f_0 + f_1$  accordingly. Then  $\rho(f) := f_0$ . A standard verification shows that this map is well-defined and an  $R^H$ -module homomorphism  $R \rightarrow R^H$ . See, for instance, [3,6,8].

**Proof of Proposition 4.9.** By Lemma 4.8 it suffices to show that  $RI \cap R^H = I$  for all ideals  $I$  of  $R^H$ . This follows from a standard argument involving the Reynolds operator: Write  $f \in RI \cap R^H$  as  $\sum_i r_i f_i$  with  $r_i \in R$  and  $f_i \in I$ . As  $f$  is  $H$ -invariant we have

$$f = \rho(f) = \sum_i \rho(r_i) f_i \in I,$$

where the last step uses that  $\rho$  is an  $R^H$ -module homomorphism.  $\square$

**5. Proofs of the main theorems**

We retain the setting and notation of Section 2. If we can prove that the ambient topological space  $A_\infty(K)$  is  $G$ -Noetherian, then by Lemma 3.6 there exist finitely many elements  $f_1, \dots, f_l$  of the ideal  $I_\infty$  of  $Y_\infty$  such that  $y \in A_\infty(K)$  lies in  $Y_\infty(K)$  if and only if  $f_i(y) = 0$  for all  $i$  and all  $g \in G$ . Choosing  $m$  such that  $f_1, \dots, f_l \in I_m$  we then have, for  $n \geq m$ ,

$$Y_n(K) = \{y \in A_n(K) \mid f_i(gy) = 0 \text{ for } i = 1, \dots, l \text{ and all } g \in G_n\}$$

by Lemma 2.1, which proves the desired set-theoretic result. A similar reasoning, assuming that  $A_\infty$  is scheme-theoretically  $G$ -Noetherian, would yield that for some  $m$  and all  $n \geq m$  the ideal of  $Y_n$  is generated by the  $G_n$ -translates of the ideal of  $Y_m$ . Unfortunately, neither the topological space  $\text{OM}_\infty(K)$  nor the scheme  $S_\infty^{(k)}$  is  $G$ -Noetherian, as the following example shows.

**Example 5.1.** Consider the monomials

$$f_2 := y_{12}y_{21}, \quad f_3 := y_{12}y_{23}y_{31}, \quad f_4 := y_{12}y_{23}y_{34}y_{41}, \quad \dots$$

in the coordinate ring of  $\text{OM}_\infty$ , as well as the points  $p_2, p_3, \dots \in \text{OM}_\infty(K)$  where  $p_i$  is an off-diagonal  $\mathbb{N} \times \mathbb{N}$ -matrix with 1’s on the positions corresponding to the variables appearing in  $f_i$  and zeroes elsewhere. Then we have  $f_i(Gp_j) = 0$  for all  $i \neq j$  and  $f_i(p_i) = 1$ . Hence the sequence  $X_1 \supseteq X_2 \supseteq \dots$  of  $G$ -stable closed sets defined by

$$X_i := \{p \in \text{OM}_\infty(K) \mid f_j(Gp) = \{0\} \text{ for all } j \leq i\}$$

does not stabilise, since  $p_{i+1} \in X_i \setminus X_{i+1}$ . It is easy to find a similar example showing that  $S_\infty^{(k)}$  is not  $G$ -Noetherian; see [1, Proposition 5.2].

Our strategy in both cases is to replace  $A_\infty$  by a closed  $G$ -stable subscheme  $\widetilde{A}_\infty$ , which contains  $Y_\infty$  and such that  $\widetilde{A}_\infty$  is  $G$ -Noetherian (for chirality varieties) or at least  $\widetilde{A}_\infty(K)$  is  $G$ -Noetherian (for the  $k$ -factor model).

*5.1. The  $k$ -factor model*

In this section  $\widetilde{A}_\infty$  equals the subscheme  $\widetilde{\text{OM}}_\infty^{\leq k}$  of  $\text{OM}_\infty$  whose ideal is generated by all off-diagonal  $(k + 1) \times (k + 1)$ -minors of the off-diagonal matrix  $(y_{ij})_{i \neq j}$ .

**Theorem 5.2.** *The topological  $\text{Sym}(\mathbb{N})$ -space  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$  is  $\text{Sym}(\mathbb{N})$ -Noetherian.*

**Proof.** We proceed by induction on  $k$ . For  $k = 0$  the statement is trivial, since  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$  consists of a single point. Suppose that the statement is true for  $k - 1$ . We shall construct a continuous

and  $\text{Sym}(\mathbb{N})$ -equivariant map  $\phi$  from a  $\text{Sym}(\mathbb{N})$ -Noetherian space to  $\text{OM}_\infty(K)$  whose image contains  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$  as a closed subset. By Lemmas 3.3 and 3.2 we are then done. The required  $\text{Sym}(\mathbb{N})$ -Noetherian space is the disjoint union of  $\widetilde{\text{OM}}_\infty^{\leq k-1}(K)$ , on which  $\phi$  is the inclusion map, and a second space  $Z$ , which will cover all points of  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$  that are not in  $\widetilde{\text{OM}}_\infty^{\leq k-1}(K)$ .

For any (possibly infinite) matrix  $Q$  and subsets  $L, N$  of its row index set and column index set, respectively, we write  $Q[L, N]$  for the corresponding submatrix of  $Q$ . To motivate the construction of  $Z$ , set  $I := \{1, \dots, k\}$  and  $J := \{k + 1, \dots, 2k\}$  and consider a point  $Y$  in  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$  such that  $\det Y[I, J]$  is non-zero. We argue that the matrix  $Y[\mathbb{N} \setminus J, \mathbb{N} \setminus I]$  is an honest rank- $k$  matrix in the sense that there exist  $b_{ip}, c_{pj}, i \in \mathbb{N} \setminus J, j \in \mathbb{N} \setminus I, p = 1, \dots, k$  such that for all such  $i, j$  with  $i \neq j$  we have

$$y_{ij} = \sum_{p=1}^k b_{ip}c_{pj}.$$

Indeed, it is clear that we can choose the  $b_{ip}$  and  $c_{pj}$  such that this relation is satisfied for  $(i, j) \in I \times (\mathbb{N} \setminus I) \cup (\mathbb{N} \setminus J) \times J$ . Then for  $(i, j) \in (\mathbb{N} \setminus (I \cup J)) \times (\mathbb{N} \setminus (I \cup J))$  the relation will automatically be satisfied due to the vanishing of the determinant of  $Y[I \cup \{i\}, J \cup \{j\}]$  and the non-vanishing of the determinant of  $Y[I, J]$ . We shall think of the remaining entries of  $Y$ , i.e., those  $y_{ij}$  with  $i \in J$  or  $j \in I$ , as “free variables”. This leads us to consider the ring

$$R := K[(b_{ip})_{i \in \mathbb{N} \setminus J, 1 \leq p \leq k}, (c_{pj})_{j \in \mathbb{N} \setminus I, 1 \leq p \leq k}, (d_{ij})_{i \in \mathbb{N}, j \in I}, (e_{ij})_{i \in J, j \in \mathbb{N} \setminus I}].$$

On this ring the group  $H := \text{Sym}(\mathbb{N} \setminus (I \cup J))$ , considered as the pointwise stabiliser of  $I \cup J$  in  $\text{Sym}(\mathbb{N})$ , acts by permuting the row indices of  $b$  and  $d$  and the column indices of  $c$  and  $e$ . By Theorem 4.2 and Lemma 4.4 the ring  $R$  is  $H$ -Noetherian, since apart from  $4k$  copies of countably many variables on which the full symmetric group acts,  $R$  has only finitely many further variables. Hence by Lemma 4.5 the topological space  $X := (\text{Spec } R)(K)$  is also  $H$ -Noetherian. Consider the map  $\phi_X : X \rightarrow \text{OM}_\infty(K)$  sending  $(B, C, D, E)$  to the off-diagonal matrix

$$\begin{bmatrix} D[I, I] & (B.C)[I, J] & (B.C)[I, \mathbb{N} \setminus (I \cup J)] \\ D[J, I] & E[J, J] & E[J, \mathbb{N} \setminus (I \cup J)] \\ D[\mathbb{N} \setminus (I \cup J), I] & (B.C)[\mathbb{N} \setminus (I \cup J), J] & (B.C)[\mathbb{N} \setminus (I \cup J), \mathbb{N} \setminus (I \cup J)] \end{bmatrix},$$

where the blocks on the diagonal are projected into the relevant spaces of off-diagonal matrices. The map  $\phi_X$  is continuous and  $H$ -equivariant, and hence gives rise to a unique continuous and  $\text{Sym}(\mathbb{N})$ -equivariant map  $\phi_Z$  from  $\text{Sym}(\mathbb{N}) \times_H X$  into  $\text{OM}_\infty(K)$  which maps the equivalence class of  $(e, x)$  to  $\phi_X(x)$  for all  $x$ . By Lemma 3.5 the space  $Z$  is  $\text{Sym}(\mathbb{N})$ -Noetherian. As all off-diagonal  $k \times k$ -minors are in the same  $\text{Sym}(\mathbb{N})$ -orbit (up to a sign), the above discussion shows that the image of  $Z$  contains  $\widetilde{\text{OM}}_\infty^{\leq k}(K) \setminus \widetilde{\text{OM}}_\infty^{\leq k-1}(K)$ . Now the disjoint union of  $\widetilde{\text{OM}}_\infty^{\leq k-1}$  and  $Z$  is  $\text{Sym}(\mathbb{N})$ -Noetherian by Lemma 3.4, and the map  $\phi$  which is the inclusion on  $\widetilde{\text{OM}}_\infty^{\leq k-1}$  and  $\phi_Z$  on  $Z$  has  $\text{im}(\phi) \supset \widetilde{\text{OM}}_\infty^{\leq k}$ . This proves the theorem.  $\square$

**Proof of set-theoretic finiteness for the  $k$ -factor model.** We spell out the proof of the first statement, which characterises  $\text{OM}_n^{\leq k}$  for large  $n$  by the condition that all principal  $N_0 \times N_0$ -submatrices lie in  $\text{OM}_{N_0}^{\leq k}$ . First we argue that there exist finitely many elements  $f_1, \dots, f_l$  in

the ideal of  $\text{OM}_\infty^{\leq k}$  such that  $y \in \text{OM}_\infty(K)$  lies in  $\text{OM}_\infty^{\leq k}(K)$  if and only if  $f_i(gy) = 0$  for  $i = 1, \dots, l$  and all  $g \in \text{Sym}(\mathbb{N})$ . Take  $f_1$  equal to any off-diagonal  $(k + 1) \times (k + 1)$ -determinant; these form a single  $\text{Sym}(\mathbb{N})$ -orbit up to a sign. Requiring that  $f_1(gy) = 0$  for all  $g$  forces  $y$  to lie in  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$ . As the latter space is  $\text{Sym}(\mathbb{N})$ -Noetherian, the closed  $\text{Sym}(\mathbb{N})$ -stable subspace  $\text{OM}_\infty^{\leq k}(K)$  is cut out by finitely many further equations  $f_2, \dots, f_l$ ; see Lemma 3.6. Now take  $N_0$  large enough such that  $f_1, \dots, f_l$  lie in the coordinate ring of  $\text{OM}_{N_0}$ . Then we have  $y \in \text{OM}_\infty^{\leq k}(K)$  if and only if  $\pi_{\infty, N_0}(gy) \in \text{OM}_{N_0}^{\leq k}(K)$  for all  $g \in \text{Sym}(\mathbb{N})$ . By Lemma 2.1 this implies that for all  $n \geq N_0$  an element  $y \in \text{OM}_n(K)$  lies in  $\text{OM}_n^{\leq k}(K)$  if and only if  $\pi_{n, N_0}gy$  lies in  $\text{OM}_{N_0}^{\leq k}$  for all  $g \in \text{Sym}(n)$ . This proves the first statement of the theorem.

The second statement, which concerns the Zariski closure of the  $k$ -factor model  $F_{n,k}$ , is proved in a similar fashion:  $\text{SOM}_\infty^{\leq k}(K)$  is a closed  $\text{Sym}(\mathbb{N})$ -stable subspace of  $\widetilde{\text{OM}}_\infty^{\leq k}(K)$ , hence characterised by finitely many equations.  $\square$

### 5.2. Chirality varieties

For chirality varieties we take  $\widetilde{A}_\infty$  to be the subscheme  $\widetilde{V}_\infty^{(k)}$  of  $S_\infty^{(k)}$  defined by all Plücker relations among the  $y_J$  with  $|J| = k$ . From the parameterisation

$$y_J = \prod_{i, j \in J, i < j} (x_i - x_j) = \det(x_j^i)_{j \in J, i=0, \dots, k-1}$$

it is clear that  $V_\infty^{(k)}$  is a subscheme of  $\widetilde{V}_\infty^{(k)}$ . Theorem 1.4 will follow from the following theorem.

**Theorem 5.3.** *Assume that the characteristic of  $K$  is zero. Then  $\widetilde{V}_\infty^{(k)}$  is scheme-theoretically  $\text{Sym}(\mathbb{N})$ -Noetherian.*

**Proof.** Consider the scheme  $X = M_{k, \mathbb{N}}$  of  $k \times \mathbb{N}$ -matrices with coordinate ring  $R = K[x_{ij} \mid i \in [k], j \in \mathbb{N}]$ . Let  $G = \text{Sym}(\mathbb{N})$  act on  $X$  by permuting the columns, and let  $H = \text{SL}_k(K)$  act on  $X$  by multiplication from the left. Now the conditions of Proposition 4.9 are satisfied: complete reducibility for  $H$  in characteristic 0 is classical, and  $G$ -Noetherianity of  $R$  is Theorem 4.2. Hence  $R^H$  is  $G$ -Noetherian. Now we claim that the homomorphism sending  $y_J$  to  $\det(x[[k], J])$  is a  $\text{Sym}(\mathbb{N})$ -equivariant isomorphism from the coordinate ring of  $\widetilde{V}_\infty^{(k)}$  to  $R^H$ . This claim follows from two well-known facts: First, the kernel of this homomorphism is generated by the Plücker relations, which generate the defining ideal of  $\widetilde{V}_\infty^{(k)}$ . Second, by the First Fundamental Theorem for  $\text{SL}_k$  the ring of  $\text{SL}_k$ -invariants on any space  $M_{k,n}$  of finite matrices is generated by the determinants  $\det(x[[k], J])$  with  $J \subseteq [n]$  of size  $k$  see [3,6,8,12]; this readily implies that  $R^H$  is generated by these determinants as  $J$  runs through all  $k$ -sets in  $\mathbb{N}$ . Hence  $V_\infty^{(k)} = \text{Spec } R^H$  is  $\text{Sym}(\mathbb{N})$ -Noetherian, as claimed.  $\square$

**Proof of scheme-theoretic finiteness of chirality varieties in characteristic zero.** The scheme  $\widetilde{V}_\infty^{(k)}$  is cut out scheme-theoretically from  $S_\infty^{(k)}$  by the Plücker relations, which form a single  $\text{Sym}(\mathbb{N})$ -orbit. By Theorem 5.3 the scheme  $\widetilde{V}_\infty^{(k)}$  is  $\text{Sym}(\mathbb{N})$ -Noetherian, hence its subscheme  $V_\infty^{(k)}$  is cut out scheme-theoretically from  $\widetilde{V}_\infty^{(k)}$  by finitely many  $\text{Sym}(\mathbb{N})$ -orbits of equations. Hence the ideal of  $V_\infty^{(k)}$  in the coordinate ring of  $S_\infty^{(k)}$  is generated by finitely many  $\text{Sym}(\mathbb{N})$ -orbits of equations, and Lemma 2.1, together with the remark following it, concludes the proof.  $\square$

## 6. Remarks

We conclude the paper with a few remarks.

- (1) If one drops the characteristic-0 assumption in Theorem 1.4 one can still prove a set-theoretic finiteness result:  $\tilde{V}_\infty^{(k)}(K)$  is a  $\text{Sym}(\mathbb{N})$ -Noetherian topological space.
- (2) In Theorem 1.4 one may replace the Vandermonde determinant by any other determinant of a square matrix of which the entry at position  $(i, j)$  equals  $p_i(x_j)$  for some fixed polynomials  $p_1, \dots, p_k$ . Indeed, the resulting scheme is still a closed subscheme of  $\tilde{V}_\infty^{(k)}$ , and the argument in Section 2 putting the chirality varieties into the framework of Lemma 2.1 applies unaltered.
- (3) So far we have not succeeded to prove scheme-theoretic finiteness for the  $k$ -factor model. Even in the case where  $k = 1$ , in which all off-diagonal  $2 \times 2$ -determinants of an infinite off-diagonal matrix are known to generate the ideal, it is not obvious that the quotient by these determinants is  $\text{Sym}(\mathbb{N})$ -Noetherian.

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