ELSEVIER

Discrete Applied Mathematics 85 (1998) 87–97

DISCRETE APPLIED MATHEMATICS

Bisecting de Bruijn and Kautz graphs ☆

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Received 12 December 1995; received in revised form 10 September 1997; accepted 26 January 1998

Abstract

De Bruijn and Kautz graphs have been intensively studied as perspective interconnection networks of massively parallel computers. One of the crucial parameters of an interconnection network is its bisection width. It has an influence on both communication properties of the network and the algorithmic design. We prove optimal bounds on the edge and vertex bisection widths of the k-ary n-dimensional de Bruijn digraph. This generalizes known results for $k = 2$ and improves the upper bound for the vertex bisection width. We extend the method to prove optimal upper and lower bounds on the edge and vertex bisection widths of Kautz graphs.

1. Introduction

The de Bruijn and Kautz digraphs were originally studied as asymptotically largest digraphs w.r.t. degree and diameter [4] and were proposed as promising topologies for massively parallel computer architectures [3]. Several graph–theoretic properties and algorithmic design problems have been widely studied for these graphs [1, 2, 13, 17].

In this paper, we study vertex and edge bisection width of the k-ary n-dimensional de Bruijn and Kautz digraphs, respectively. The *edge bisection width* of a graph $G = (V, E)$, denoted $b_E(G)$, is the smallest number of edges removal of which divides G into two parts of equal size (up to 1 vertex). Similarly, the *vertex bisection width*, denoted $b_V(G)$, is the smallest number of vertices removal of which divides G into two

☆ A preliminary version of this paper appeared at The 2nd Colloquium on Structural Information and Communication Complexity (SIROCCO'95), Olympia, Greece, 12–14, June 1995.

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[1] Partially supported by the Swiss National Science Foundation grant No. 20-40354-94 and by the grant No. 95/5305/277 of the Slovak Grant Agency.

0166-218X/98/$19.00
PII S0166-218X(98)00031-6

parts having at most $\lceil |V|/2 \rceil$ vertices each. The edge bisection width is a fundamental concept in the theory of interconnection networks [13]. In many problems, where a large portion of data must be moved from one half of a parallel computer to the second half (sorting, routing, gossiping), the computation and communication time depends heavily on the bisection width [10, 11]. Monien et al. [15] defined the communication capacity of a network in terms of the edge bisection width. Another connection between the bisection width and the communication complexity appears in deriving lower bounds on the *area* and *area* $\times$ *time*^{2} complexity of VLSI circuits [20, 22]. Bisection widths also play an important role in the algorithmic design [14] and in computing lower bounds on crossing numbers [12].

Thompson [20] proved a lower bound on the bisection width of the binary shuffle–exchange graph to be $\Omega(2^n/n)$. Hoey and Leiserson [8] found an optimal upper bound. Using a strong similarity between the shuffle-exchange and de Bruijn digraph, Leighton [13] derived the same optimal bound on the bisection width of the binary de Bruijn digraph. The same result was independently proved by Samatham and Pradhan [17]. Bounds on bisection widths of small de Bruijn digraphs are given in [15]. Recently, Feldmann et al. [6] have proved the best upper bound so far for the edge bisection width of the binary de Bruijn graph: $2\ln(2)2^n/n$, which differs from the lower bound by a multiplicative factor of 1.4 only.

As the binary de Bruijn digraph is of bounded degree, the vertex and edge bisection widths are of the same order. For the k-ary n-dimensional de Bruijn digraph, Pellegrini [16] proved an upper bound on the vertex bisection width to be $O(k^n/\sqrt{n})$.

The main results of this paper are as follows. We show that the vertex and edge bisection widths of the k-ary n-dimensional de Bruijn digraph are $\Theta(k^n/n)$ and $\Theta(k^{n+1}/n)$, respectively. This generalizes known results for $k = 2$ from [13, 17] and improves the upper bound on the vertex bisection width of Pellegrini [16] by a factor of $\sqrt{n}$. Using structural similarities between de Bruijn and Kautz digraphs, we extend the method to show that similar optimal bounds hold also for the Kautz digraph. It is noteworthy that our constant factors are very small.

2. Preliminaries

Given a graph G, let $V(G)$ and $E(G)$ denote its set of vertices and edges, respectively, and $\Delta(G)$ the maximum degree of G. A similar notation will be used for digraphs.

Let $\mathscr{Z}$ denote the set of all integers and let $\mathscr{Z}_k = \{0, 1, \dots, k-1\}$. The k-ary n-dimensional de Bruijn digraph $B(k,n)$ consists of k^n vertices. Vertices are labeled with strings $u_{n-1}u_{n-2}\dots u_1u_0$ over the alphabet $\mathscr{Z}_k$. Vertex x is adjacent to vertex y iff the last $n-1$ letters of x are the same as the first $n-1$ letters of y. The k-ary n-dimensional Kautz digraph $K(k,n)$ consists of $k^n + k^{n-1}$ vertices labeled with strings $u_{n-1}\dots u_0$ over the alphabet $\mathscr{Z}_{k+1}$ so that $\forall\, i = 1, \dots, n-1$, $u_i \neq u_{i-1}$. Similar to the de Bruijn digraph, adjacency between vertices is based on left shifting:

$u_{n-1}\dots u_0$ is adjacent to $u_{n-2}\dots u_0\alpha$ for all $\alpha \in \mathscr{Z}_{k+1} - \{u_0\}$. An edge joining $u = u_{n-1}\dots u_0$ to $v = u_{n-2}\dots u_0\alpha$ will be denoted by $\langle u, v\rangle = \langle u_{n-1}\dots u_0\alpha\rangle$. More generally, a path starting in $u_{n-1}\dots u_0$, passing through $u_{n-2}\dots u_0 v_{n-1}$, $u_{n-3}\dots u_0 v_{n-1} v_{n-2}$, ..., and ending in $w_{n-1}\dots w_0$ will be denoted by $\langle u_{n-1}\dots u_0 v_{n-1} v_{n-2}\dots w_{n-1}\dots w_0\rangle$.

Since the orientation of arcs has no influence on bisection widths, we consider de Bruijn and Kautz graphs obtained from the digraphs by omitting orientations and we also use the same notation as for digraphs. Note that the resulting graphs have double edges, the de Bruijn graph has loops, and $\Delta(B(k,n)) = \Delta(K(k,n)) = 2k$.

The operation of cyclic shifting, rotation, induces an equivalence on the vertex set of the de Bruijn graph. The equivalence classes are called *de Bruijn necklaces*. For a given vertex $u = u_{n-1}\dots u_0 \in V(B(k,n))$, the *length* j of the necklace containing u is the number of rotations of u needed to get u again. Clearly, $j|n$. Necklaces of length n are called *full* necklaces of $B(k,n)$ and shorter necklaces are called *degenerate* necklaces.

It has been shown in [21] that Kautz vertices can also be partitioned into necklaces. The lengths of necklaces of $K(k,n)$ are divisors of either n or $n-1$. There is a one-to-one correspondence between necklaces of $B(k,n) \cup B(k,n-1)$ and $K(k,n)$ of length $\geqslant 3$. Necklaces of length n and $n-1$ are called the *full Kautz* necklaces of the *first* and *second* type, respectively. Necklaces of length $j < n$, $j|n$ and $j < n-1$, $j|(n-1)$ are called the *degenerate Kautz* necklaces of the *first* and *second* type, respectively. Vertex $u_{n-1}\dots u_0$ belongs to a Kautz necklace of the first type if $u_{n-1} \neq u_0$ and to a necklace of the second type otherwise.

Lemma 2.1 (Tvrdík, 1994). *Let $m_1(k,n)(m_2(k,n))$ denote the number of vertices of necklaces of the first (second) type in $K(k,n)$. Then $m_1(k,n) = k^n \pm k$ and $m_2(k,n) = k^{n-1} \mp k$, where the upper sign is taken iff n is even.*

Let $\mathscr{C}$ denote the complex plane. For $z \in \mathscr{C}$, let $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$ denote the real and imaginary part of z, respectively. If $\xi = \exp(2\pi\sqrt{-1})/n$ is the nth primitive root of unity, then $\mathscr{Z}(\xi)$ denotes the *cyclotomic ring* over $\mathscr{Z}$, i.e., set $\{\sum_{i=0}^{n-1} u_i\xi^i : u_i \in \mathscr{Z}\}$. Let $\phi(n)$ denote the Euler's *totient* function, i.e., the number of all positive integers smaller than n and relatively prime to n. A fundamental result about $\mathscr{Z}(\xi)$ says that $\{1, \xi, \xi^2, \dots, \xi^{\phi(n)-1}\}$ is a basis of $\mathscr{Z}(\xi)$, see [18]. It means that every element of $\mathscr{Z}(\xi)$ can be expressed as a linear combination of $1, \xi, \dots, \xi^{\phi(n)-1}$ over $\mathscr{Z}$ and if $\sum_{i=0}^{\phi(n)-1} u_i\xi^i = 0, u_i \in \mathscr{Z}$, then $u_i = 0$ for all $i = 0, 1, \dots, \phi(n)-1$.

A *routing* R of a graph $G = (V,E)$ is a set of $|V|(|V|-1)$ paths connecting each ordered pair of distinct vertices of G. For any $u, v \in V(G)$, the path from u to v in R will be denoted by $R(u,v)$. The *load* of a vertex $v \in V$ in routing R is the number of paths, denoted $\zeta(G,R,v)$, going through v where v is not an end vertex. The load of an edge $e \in E$ in R is the number of paths, denoted $\pi(G,R,e)$, going through e. The *vertex forwarding index* of G is defined as

$$\zeta(G) = \min_{R} \max_{v\in V}\{\zeta(G,R,v)\}.$$

The *edge forwarding index* of G is defined as

$$\pi(G) = \min_{R} \max_{e \in E} \{\pi(G,R,e)\}.$$

Since we consider de Bruijn and Kautz graphs derived from digraphs just by omitting the orientation of the arcs, the forwarding indexes are different compared to those derived in [7].

Lemma 2.2. *For* $k,n \geqslant 2$

$$\pi(B(k,n)) \leqslant nk^{n-1} \tag{1}$$

$$\pi(K(k,n)) \leqslant nk^{n-1} + (n-1)k^{n-2} \tag{2}$$

Proof. To prove (1), consider the routing R where $R(u,v) = \langle u_{n-1} \dots u_0 v_{n-1} \dots v_0 \rangle$ for any $u,v \in V(B(k,n))$. Every pair of vertices is connected by a path of length exactly n, we ignore shorter paths implied by possible periodicity of string patterns. For estimating the upper bound on forwarding indexes, this is acceptable since the truly shortest path routing gives forwarding index of the same order, or even exactly the same values if n is prime. Consider an edge $e = \langle x_n x_{n-1} \dots x_1 x_0 \rangle$. Then e is an edge of $R(u,v)$ iff for some $0 \leqslant i \leqslant n-1$, $u_{n-1} \dots u_0 = \alpha_1 \dots \alpha_i x_n \dots x_{i+1}$ and $v_{n-1} \dots v_0 = x_i \dots x_0 \beta_1 \dots \beta_{n-i-1}$, where $\alpha_1 \dots \alpha_i$ and $\beta_1 \dots \beta_{n-i-1}$ are arbitrary k-ary strings of length i and $n-i-1$, respectively. It follows that the number of paths $R(u,v)$ containing e is at most $\sum_{i=0}^{n-1} k^i k^{n-i-1} = nk^{n-1}$.

In case of Kautz graphs, we will use a similar routing. Due to the constraints on Kautz string patterns, we have to distinguish two cases. Consider two vertices $u_{n-1} \dots u_0$ and $v_{n-1} \dots v_0$ of $K(k,n)$. Then $R(u,v) = \langle u_{n-1} \dots u_0 v_{n-1} \dots v_0 \rangle$ if $u_0 \neq v_{n-1}$ and $R(u,v) = \langle u_{n-1} \dots u_0 v_{n-2} \dots v_0 \rangle$ otherwise. Every vertex and every edge of $K(k,n)$ will be loaded both by paths of length n and by paths of length $n-1$. Using a similar argument as above, we get (2). □

3. De Bruijn graph

The main result follows from the following four lemmas:

Lemma 3.1. *For* $k \geqslant 2, n \geqslant 3$,

$$b_V(B(k,n)) \leqslant \frac{2k^n}{n} \left(1 + \frac{n-2}{2k^{1/2\sqrt{n}}}\right).$$

Proof. We extend the complex plane diagram method, first used by Hoey and Leiserson to prove an upper bound on the edge bisection width of the binary shuffle-exchange graph [8]. The original method relies on a suitable mapping of the vertices of the shuffle-exchange graph into the complex plane and on joining the images of adjacent vertices by line segments. An edge bisection is obtained by deleting all edges incident with the real line.

We use the same function for mapping the vertices of a de Bruijn graph into the complex plane. However, the construction of the bisection and the analysis of the upper bound essentially differs from [8] as it will be seen later. Define a mapping

$$f : V(B(k,n)) \to \mathscr{C},$$

so that

$$f(u_{n-1}\dots u_0) = u_{n-1}\xi^{n-1} + u_{n-2}\xi^{n-2} + \cdots + u_1\xi + u_0.$$

Observe that if $\langle u,v\rangle \in E(B(k,n))$, then

$$\text{either } f(u) = \xi f(v) + r_1 \text{ or } f(v) = \xi f(u) + r_2,$$

where $r_1, r_2 \in \{-k+1,\dots,-1,0,1,\dots,k-1\}$. If we join images of adjacent vertices by straight-line segments, we get a drawing of $B(k,n)$ in the complex plane. From now on, we will not distinguish between vertices and edges of $B(k,n)$ and their images in $\mathscr{C}$, respectively.

The mapping f is not one-to-one. The main problem when using this method is to estimate the number of vertices mapped into the origin. In the binary case [8, 13], the vertices mapped into the origin are precisely the vertices of degenerate necklaces. This does not hold in the general case. For example, vertex 020111 of $B(3,6)$ is mapped into the origin, but it belongs clearly to a full necklace.

Define

$$M_{11} = \{u \in V(B(k,n)) \mid f(u) \neq 0, \mathrm{Im}\{f(u)\}\mathrm{Im}\{\xi f(u)\} < 0\},$$

$$M_{12} = \{u \in V(B(k,n)) \mid f(u) \neq 0, \mathrm{Im}\{f(u)\} = 0\},$$

$$M_2 = \{u \in V(B(k,n)) \mid \mathrm{Re}\{f(u)\} = \mathrm{Im}\{f(u)\} = 0\}.$$

Let $M_1 = M_{11} \cup M_{12}$ and $M = M_1 \cup M_2$. We claim that M is a vertex bisection of $B(k,n)$ dividing $V(B(k,n))$ into sets X and Y defined as follows

$$X = \{u \in V(B(k,n)) \mid \mathrm{Im}\{f(u)\} > 0\} - M_{11},$$

$$Y = \{u \in V(B(k,n)) \mid \mathrm{Im}\{f(u)\} < 0\} - M_{11}.$$

First, we show that there is no edge joining X and Y. Suppose that there is an edge $\langle u,v\rangle \in E(B(k,n))$ such that $u \in X$ and $v \in Y$. Then

$$\mathrm{Im}\{f(u)\} > 0, \mathrm{Im}\{f(v)\} < 0, \mathrm{Im}\{\xi f(u)\} \geqslant 0, \mathrm{Im}\{\xi f(v)\} \leqslant 0. \tag{3}$$

Assume w.l.o.g. that $f(v) = \xi f(u) + r$, where $r \in \{-k+1,\dots,-1,0,1,\dots,k-1\}$. This together with (3) implies that $0 > \mathrm{Im}\{f(v)\} = \mathrm{Im}\{\xi f(u)\} \geqslant 0$, a contradiction.

Second, we prove that $|X|, |Y| \leqslant \lceil V(B(k,n))/2 \rceil$. There is a 1–1 correspondence between vertices of $B(k,n)$ with positive and negative imaginary parts. Vertex $u = u_{n-1} \dots u_0$ corresponds to vertex $\bar{u} = \bar{u}_{n-1}\bar{u}_{n-2} \dots \bar{u}_0$, where $\bar{u}_i = k-1-u_i$, since

$$f(u) + f(\bar{u}) = (k-1)\xi^{n-1} + \cdots + (k-1)\xi + (k-1) = 0.$$

It follows from the definition of X and Y that

$$|X| = |Y| \leqslant |V(B(k,n)) - M|/2 < |V(B(k,n))|/2.$$

Third, we estimate the cardinality of the set M. Let us start with M_2. We claim that

$$|M_2| \leqslant k^{n-1/2\sqrt{n}}.$$

Observe that $|M_2|$ is the number of strings $u_{n-1} \dots u_0$ satisfying

$$u_{n-1}\xi^{n-1} + \cdots + u_1\xi + u_0 = 0, \tag{4}$$

where $0 \leqslant u_i \leqslant k-1$. As $\{1, \xi, \dots, \xi^{\phi(n)-1}\}$ is a basis of $\mathscr{Z}(\xi)$, for all $j = \phi(n), \dots, n-1$, there exist $a_{ij} \in \mathscr{Z}$ such that

$$\xi^j = \sum_{i=0}^{\phi(n)-1} a_{ij}\xi^i. \tag{5}$$

By substituting (5) into (4), we get

$$\sum_{j=0}^{\phi(n)-1} u_j\xi^j + \sum_{j=\phi(n)}^{n-1} u_j \sum_{i=0}^{\phi(n)-1} a_{ij}\xi^i = 0.$$

By rearranging the terms we have

$$\sum_{j=0}^{\phi(n)-1} (u_j + \sum_{i=\phi(n)}^{n-1} u_i a_{ji})\xi^j = 0.$$

The linear independence of $\{1, \xi, \dots, \xi^{\phi(n)-1}\}$ implies a system of $\phi(n)$ linear equations with unknowns $u_0, u_1, \dots, u_{n-1}$

$$u_j + \sum_{i=\phi(n)}^{n-1} u_i a_{ji} = 0, \tag{6}$$

where $j = 0, 1, \dots, \phi(n)-1$. Thus, $|M_2|$ is the number of solutions of (6) satisfying $0 \leqslant u_i \leqslant k-1, i = 0, 1, \dots, n-1$. By choosing $u_{\phi(n)}, \dots, u_{n-1}$ arbitrarily from $\mathscr{Z}_k$ and substituting them into (6), we get a system of $\phi(n)$ linear equations with unknowns $u_0, \dots, u_{\phi(n)-1}$. Clearly, this system has at most one solution satisfying $0 \leqslant u_i \leqslant k-1, i = 0, 1, \dots, \phi(n)-1$. Hence,

$$|M_2| \leqslant k^{n-\phi(n)}.$$

Using the lower bound on $\phi(n)$ from [19, p. 230],

$$\phi(n) \geqslant 1/2\sqrt{n}, \tag{7}$$

for $n \geqslant 1$, we get the claimed bound.

Observe that

$$\begin{aligned}\xi f(u_{n-1} \dots u_0) &= u_{n-1}\xi^n + u_{n-2}\xi^{n-1} + \cdots + u_0\xi \\ &= u_{n-2}\xi^{n-1} + u_{n-3}\xi^{n-2} + \cdots + u_0\xi + u_{n-1} \\ &= f(u_{n-2} \dots u_0 u_{n-1}).\end{aligned}$$

If $f(u) = 0$, then the whole necklace (either full or degenerate) containing u is mapped into the origin. If $f(u) \neq 0$, then the necklace containing u is full and consists of vertices $\xi^j f(u)$, for $j = 0, 1, \dots, n-1$, centered symmetrically around the origin. Each of these necklaces contributes by exactly 2 vertices to M_1. Hence,

$$\begin{aligned}|M| &= |M_1| + |M_2| = 2\left(\frac{k^n - |M_2|}{n}\right) + |M_2| = \frac{2k^n}{n} + \frac{n-2}{n}|M_2| \\ &\leqslant \frac{2k^n}{n}\left(1 + \frac{n-2}{2k^{\frac{1}{2}}\sqrt{n}}\right). \qquad \square\end{aligned}$$

Lemma 3.2. *For $k \geqslant 2, n \geqslant 3$*

$$b_{\mathrm{e}}(B(k,n)) \leqslant \frac{2k^{n+1}}{n}\left(1 + \frac{n-2}{2k^{1/2}\sqrt{n}}\right).$$

Proof. Consider the vertex bisection M from the proof of Lemma 3.1. Take arbitrarily a subset M' of M having $\lceil |M|/2 \rceil$ vertices. Adding M' either to X or to Y induces 2 edge bisections. It is easy to see that the smaller one has at most $k|M|$ edges, which implies the result. $\square$

Lemma 3.3. *For $k \geqslant 2, n \geqslant 14$*

$$b_{\mathrm{V}}(B(k,n)) \geqslant \frac{k^n}{2n}\left(1 - \frac{169}{n^2}\right).$$

Proof. Consider a vertex bisection C of $B(k,n)$, i.e., there exists a partition of V into three sets A, B, C such that $|C| = b_{\mathrm{V}}(B(k,n))$ and no edge joins A with B. Assume $|A| \leqslant |B|$. Since $|C| \geqslant 1$, then $|A| \leqslant \lceil |V|/2 \rceil - 1$ and $|B| \leqslant \lfloor |V|/2 \rfloor$. For any routing R, there are at least $2|A||B|$ paths of R going through vertices of C. This implies

$$\zeta(B(k,n)) \geqslant \frac{2|A||B|}{|C|} = \frac{2|A||B|}{b_{\mathrm{V}}(B(k,n))}.$$

Chung et al [5] proved that $\zeta(B(k,n)) \leqslant (n-1)k^n$. Hence,

$$b_{\mathrm{V}}(B(k,n)) \geqslant \frac{2|A||B|}{(n-1)k^n} \geqslant \frac{2|A|(k^n - |A| - b_{\mathrm{V}}(B(k,n)))}{(n-1)k^n}$$

which implies

$$b_V(B(k,n)) \geqslant \frac{2|A|(k^n - |A|)}{(n-1)k^n + 2|A|} \geqslant \frac{2|A|(k^n - |A|)}{nk^n}.$$

Observing that for $n \geqslant 14$

$$\begin{aligned} |A| &= k^n - |B| - b_V(B(k,n)) \geqslant \frac{k^n - 1}{2} - \frac{2k^n}{n}\left(1 + \frac{n-2}{2k^{1/2}\sqrt{n}}\right) \\ &= \frac{k^n}{2}\left(1 - \frac{1}{k^n} - \frac{4}{n} - \frac{2(n-2)}{nk^{1/2}\sqrt{n}}\right) \geqslant \frac{k^n}{2}\left(1 - \frac{13}{n}\right), \end{aligned}$$

we get

$$b_V(B(k,n)) \geqslant \frac{2(k^n/2)(1 - 13/n)(k^n/2)(1 + 13/n)}{nk^n} \geqslant \frac{k^n}{2n}\left(1 - \frac{169}{n^2}\right). \qquad \square$$

Lemma 3.4. *For* $k \geqslant 2, n \geqslant 2$

$$b_e(B(k,n)) \geqslant \frac{k^{n+1}}{2n}\left(1 - \frac{1}{k^{2n}}\right).$$

Proof. We apply the Leighton's lower bound formula [13] for the bisection width. For any graph $G = (V, E)$

$$b_e(G) \geqslant \frac{2\lfloor |V|/2 \rfloor \lceil |V|/2 \rceil}{\pi(G)}. \tag{8}$$

From Lemma 2.2, $\pi(B(k,n)) \leqslant nk^{n-1}$. By substituting this into (8), we get

$$b_e(B(k,n)) \geqslant \frac{[2(k^n - 1)/2][(k^n + 1)/2]}{nk^{n-1}} = \frac{k^{n+1}}{2n}\left(1 - \frac{1}{k^{2n}}\right). \qquad \square$$

Note that if k is even, then the second-order term disappears. By combining the above lemmas, we get directly the main theorem.

Theorem 3.1. *For* $k \geqslant 2, n \geqslant 3$

$$\max\left\{\frac{k^n}{4n}\left(1 - \frac{1}{k^{2n}}\right), \frac{k^n}{2n}\left(1 - \frac{169}{n^2}\right)\right\} \leqslant b_V(B(k,n)) \leqslant \frac{2k^n}{n}\left(1 + \frac{n-2}{2k^{1/2}\sqrt{n}}\right)$$

$$\frac{k^{n+1}}{2n}\left(1 - \frac{1}{k^{2n}}\right) \leqslant b_e(B(k,n)) \leqslant \frac{2k^{n+1}}{n}\left(1 + \frac{n-2}{2k^{1/2}\sqrt{n}}\right).$$

4. Kautz graph

In this section, we extend the previous arguments to prove optimal bounds on bisection widths of Kautz graphs.

Lemma 4.1. *For* $k \geqslant 2, n \geqslant 3$

$$b_V(K(k,n)) \leqslant \frac{2k^n}{n}\left(1+\frac{n}{2k}+\frac{n+2}{2k^{n-1}}+\frac{(n-2)(k+1)}{2k^{1/2\sqrt{n}+1}}\right).$$

Proof. From the definitions of the de Bruijn and the Kautz graph, it is easy to see that $K(k,n) \subset B(k+1,n)$. Consider the vertex bisection set $M = M_1 \cup M_2$ of $B(k+1,n)$ from Lemma 3.1. We claim that the set $M^* = M \cap V(K(k,n))$ is a vertex bisection of $K(k,n)$. Using a similar argument as in the proof of Lemma 3.1, we can show that there is a one-to-one correspondence between vertices u and $\bar{u}$ of $K(k,n)$ such that u and $\bar{u}$ are symmetrical w.r.t. the origin of the complex plane. Let $M_1^* = M_1 \cap V(K(k,n))$ and $M_2^* = M_2 \cap V(K(k,n))$. First, we estimate the cardinality of M_2^*. Clearly, $|M_2^*|$ equals to the number of integer solutions of (6) satisfying $0 \leqslant u_i \leqslant k, i = 0,1,\ldots,n-1$, and $u_i \neq u_{i+1}, i = 0,1,\ldots,n-2$. Similarly as in the proof of Lemma 3.1, we can choose arbitrarily $u_{n-1},\ldots,u_{\phi(n)}$ from $\mathscr{Z}_{k+1}$ so that $u_i \neq u_{i-1}, i = n-1,\ldots,\phi(n)+1$, and substitute them into (6). This can be done in $(k+1)k^{n-2-\phi(n)+1} = k^{n-\phi(n)} + k^{n-1-\phi(n)}$ ways and for each of them we get a system of $\phi(n)$ linear equations with at most one integer solution for unknowns $u_{\phi(n)-1},\ldots,u_0$ from Z_{k+1}. Hence,

$$|M_2^*| \leqslant k^{n-\phi(n)} + k^{n-1-\phi(n)}.$$

Since M_2^* consists of vertices of necklaces of the first ($u_{n-1} \neq u_0$) and second ($u_{n-1} = u_0$) type, we can decompose $M_2^* = M_{21}^* \cup M_{22}^*$, where M_{21}^* (M_{22}^*) is the set of solutions of (6) of the first (second, resp.) type. Full necklaces of the first type which are not mapped into the origin contribute by exactly two vertices to M_1^*. Since the full necklaces of the second type that are not mapped into the origin can contribute by more than two vertices to M_1^*, we have by Lemma 2.1

$$|M_1^*| \leqslant \frac{2(k^n \pm k - |M_{21}^*|)}{n} + (k^{n-1} \mp k - |M_{22}^*|).$$

Since $|M_2^*| = |M_{21}^*| + |M_{22}^*|$, we have $|M_{21}^*| \leqslant k^{n-\phi(n)} + k^{n-\phi(n)-1}$ and

$$\begin{aligned} |M^*| &= |M_1^*| + |M_2^*| \leqslant \frac{2(k^n+k)}{n} + k^{n-1} + k + \frac{n-2}{n}|M_{21}^*| \\ &\leqslant \frac{2k^n}{n}\left(1+\frac{n}{2k}+\frac{n+2}{2k^{n-1}}+\frac{(n-2)(k+1)}{2k^{\phi(n)+1}}\right). \end{aligned}$$

Using (7), we get the result. □

Lemma 4.2. *For* $k \geqslant 2, n \geqslant 3$

$$b_e(K(k,n)) \leqslant \frac{2k^{n+1}}{n}\left(1+\frac{n}{2k}+\frac{n+2}{2k^{n-1}}+\frac{(n-2)(k+1)}{2k^{1/2\sqrt{n}+1}}\right).$$

Proof. We apply the argument as in the proof of Lemma 3.2. □

Lemma 4.3. *For* $k \geqslant 2, n \geqslant 2$

$$b_{\mathrm{E}}(K(k,n)) \geqslant \frac{(k+1)k^n}{2n}.$$

Proof. The proof is the same as for Lemma 3.4 using $\pi(K(k,n)) \leqslant nk^{n-1} + (n-1)k^{n-2}$ from Lemma 2.2. □

By combining the above lemmas, we get

Theorem 4.1. *For* $k \geqslant 2, n \geqslant 3$

$$\frac{(k+1)k^{n-1}}{4n} \leqslant b_{\mathrm{V}}(K(k,n)) \leqslant \frac{2k^n}{n}\left(1 + \frac{n}{2k} + \frac{n+2}{2k^{n-1}} + \frac{(n-2)(k+1)}{2k^{1/2\sqrt{n}+1}}\right),$$

$$\frac{(k+1)k^n}{2n} \leqslant b_{\mathrm{E}}(K(k,n)) \leqslant \frac{2k^{n+1}}{n}\left(1 + \frac{n}{2k} + \frac{n+2}{2k^{n-1}} + \frac{(n-2)(k+1)}{2k^{1/2\sqrt{n}+1}}\right).$$

Observe that if $n = \mathrm{O}(k)$, then the upper bounds are optimal.

5. Final remarks

Due to the method used, our upper bounds hold for $n \geqslant 3$ only. In case $n = 2$ asymptotically optimal upper bounds for the edge bisection follow from a result of Hromkovič and Monien [9]. They proved that in any n vertex d-regular graph there exists a bisection of size $nd/4 + \mathrm{o}(n)$. For the vertex bisection width it is sufficient to remove one half of vertices. It remains an open problem to improve our bounds. The experimental results on the edge bisection widths of binary de Bruijn graphs, computed for $n \leqslant 10$ by Monien et al. [15], suggest that the upper bound could be improved by a factor of 2.5. A possible approach is to extend the method used by Feldmann et al. [6] for the case $k = 2$. Another open problem is to prove optimal bounds for the Kautz graph if $n = \omega(k)$ by improving the upper bound on the number of intersections of the Kautz necklaces of the second type with the real axis.

Acknowledgements

We thank S. Jakubec and M. Paštéka for valuable number-theoretic discussions on cyclotomic rings and anonymous referee for careful proofreading and correcting mistakes in the first draft.

References

[1] T. Andreae, M. Nölle, G. Schreiber, Cartesian products of graphs as spanning subgraphs of de Bruijn graphs, in: Proceedings of the 20th International Workshop on Graph-Theoretic Concepts in Computer Science, Lecture Notes in Computer Science, vol. 903, Springer, Berlin, 1995, pp. 140–150.

[2] D. Barth, F. Pellegrini, A. Raspaud, J. Roman, On bandwidth, cutwidth and quotient graphs, RAIRO - Informatique, Théor. Appl. 29 (1996) 487–508.
[3] J.-C. Bermond, C. Peyrat, De Bruijn and Kautz networks: a competitor for the hypercube?, in: Proceedings of the Hypercube and Distributed Computing, Elsevier, Amsterdam, 1989, pp. 279–274.
[4] N.G. de Bruijn, A combinatorial problem, Neder. Akad. Wetensch. Proc. Ser. A 49 (1946) 758–764.
[5] F.R.K. Chung, E.G. Coffman, M.I. Reiman, B.E. Simon, The forwarding index of communication networks, IEEE Trans. Inform. Theory 33 (1987) 224–232.
[6] R. Feldmann, B. Monien, P. Mysliwietz, S. Tschöke, a better upper bound on the bisection width of de Bruijn networks, in: Proceedings of the 14th International Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Computer Science, vol. 1200, Springer, Berlin, 1997 pp. 511–522.
[7] M.-C. Heydemann, J.C. Meyer, D. Sotteau. On forwarding indices of networks, Discrete Appl. Math. 23 (1989) 103–123.
[8] D. Hoey, Ch.E. Leiserson, A layout for the shuffle-exchange graph, in: Proceedings of the International Conference on Parallel Processing, IEEE Computer Society Press, Los Alamitos, CA, 1980, pp. 329–336.
[9] J. Hromkovič, B. Monien, The bisection problem for graphs of degree 4, in: Proceedings of the 16th International Symposium on Foundations of Computer Science, Lecture Notes in Computer Science, vol. 520 Springer, Berlin, 1991, pp. 211–220.
[10] R. Klasing, The relationship between gossiping in vertex-disjoint paths mode and bisection width, in: Proceedings of the 19th International Symposium on Mathematical Foundations of Computer Science, Lecture Notes in Computer Science vol. 841, Springer, Berlin, 1994, pp. 473–483.
[11] M. Kunde, Concentrated data stream on grids: Sorting and routing near to the bisection bound, in: Proceedings of the 32nd Annual IEEE Symposium on Foundation of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, 1991 pp. 141–150.
[12] F.T. Leighton, Complexity Issues in VLSI, MIT Press, Cambridge, 1983.
[13] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays, Trees, and Hypercubes, Morgan Kaufmann Publishers, San Mateo, 1992.
[14] R.J. Lipton, R.E. Tarjan, Application of planar separator theorem, SIAM J. Comput. 9 (1980) 615–627.
[15] B. Monien, R. Diekmann, R. Lülling, Communication throughput of interconnection networks, in: Proc. 19th International Symposium on Mathematical Foundations of Computer Science, Lecture Notes in Computer Science vol. 841, Springer, Berlin, 1994 pp. 72–86.
[16] F. Pellegrini, Bounds for the bandwidth of the d-ary de Bruijn graph, Parallel Process. Lett. 3 (1993) 431–443.
[17] M.R. Samatham, D.K. Pradhan, The de Bruijn multiprocessor network: a versatile parallel processing and sorting network for VLSI, IEEE Trans. Comput. 38 (1989) 567–581.
[18] I.R. Shafarevich, Z. Borevich, Theory of Numbers, Nauka, Moscow, 1967.
[19] W. Sierpiński, Elementary Theory of Numbers, Państwowe Wydawnictwo Naukowe, Warszawa, 1964.
[20] C.D. Thompson, Area-time complexity for VLSI, in: Proceedings of the 11th Annual Symposium on Theory of Computing, ACM Press, New York, 1979 pp. 81–88.
[21] P. Tvrdík, Necklaces, scalability of Kautz digraphs, in: Proceedings of the Sixth IEEE Symposium on Parallel and Distributed Processing, IEEE Computer Society Press, Los Alamitos, CA 1994, pp. 409–415.
[22] J.D. Ullman, Computational Aspects of VLSI, Computer Science Press, Rockville, 1984.