An L^p-L^q Transference Theorem

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A transference theorem is, roughly, a result which deduces the L^p-L^q boundedness of an operator on a certain space from the same sort of boundedness of a related operator on another space. For example, Theorem 1.6 of [CW] shows that the radial kernel $k(|y|)$ induces a bounded convolution operator on $L^p(\mathbb{R}^n)$ if $|y|\,k(|y|)$ does so on $L^p(\mathbb{R}^{n-1})$. For some other transference theorems, see [R] and the references cited there. The purpose of this note is to present a transference theorem which proves the L^p-L^q boundedness of certain Fourier multiplier operators on $\mathbb{R}^n$. The novelty here lies in the fact that p and q differ—in most transference theorems the domain and range spaces coincide. Before stating our result we fix some notation:

◇ $\mathcal{F}f$ is the Fourier transform of a function f on $\mathbb{R}$, while $\hat{g}$ will stand for the Fourier transform of a function g on $\mathbb{R}^n$;

◇ $\mathcal{S}_0$ is the subspace of the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ consisting of the functions $f \in \mathcal{S}$ with the property that $\hat{f}$ vanishes on some neighborhood of the origin—$\mathcal{S}_0$ is dense in $L^p(\mathbb{R}^n)$ if $p < \infty$;

◇ $\|\cdot\|_p$ will always denote an L^p norm with respect to the Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$, the appropriate choice being clear from the context;

◇ $\|\cdot\|_{p-q}$ will denote the norm of an operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$;

◇ S^{n-1} is the unit sphere in $\mathbb{R}^n$, ω is one of its elements, and the L^p norm $\|\cdot\|_{p,\omega}$ will be taken with respect to the Lebesgue measure on S^{n-1};

◇ for $\alpha \in \mathbb{R}$, the fractional integration operator I_n^α on $\mathbb{R}^n$ is defined by $\widehat{I_n^\alpha f}(\xi) = |\xi|^{-\alpha} \hat{f}(\xi)$, $\xi \in \mathbb{R}^n$;

◇ and finally $C = C(\cdot)$ will represent a constant, depending on certain parameters, which may vary from line to line within a proof.

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Here is our transference theorem:

THEOREM. *Fix* $n \geqslant 2$, $p \in (1, 2)$, *and* $q \in (2, \infty)$. *Let* $1/r = 1/p - 1/q$. *Suppose that* $\{A_\omega\}_{\omega \in S^{n-1}}$ *is a weakly continuous family of convolution operators on* $\mathbb{R}$ *such that*

$$\| \, \| I_1^{(n-1)(1/q-1/p)} A_\omega \|_{p-q} \|_{r,\omega} < \infty.$$

Then the convolution operator A *on* $\mathbb{R}^n$ *with symbol*

$$\hat{A}(sw) = \hat{A}_\omega(s), \qquad \omega \in S^{n-1} \quad \text{and} \quad s > 0$$

is bounded from $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

The proof of this result is based on certain mapping properties of the Radon transform which were established by Strichartz. Thus we begin by recalling a definition: the Radon transform Rf of $f \in \mathscr{S}$ is the function on $\mathbb{R} \times S^{n-1}$ defined by

$$Rf(s, \omega) = \int_{x \cdot \omega = s} f(x)\, dx,$$

where the integral is with respect to the Lebesgue measure on the hyperplane defined by $x \cdot \omega = s$. Following [F] we also define the modified Radon transform $\tilde{R}$ by

$$\tilde{R}f(s, \omega) = C(n)\, I_1^{1-n} Rf(\cdot, \omega)(s).$$

(The value of $C(n)$, irrelevant to our purposes, is given on p. 236 of [F].) The equality

$$\hat{f}(s\omega) = \mathscr{F} R(\cdot, \omega)(s), \qquad \omega \in S^{n-1}, \quad s \in \mathbb{R}, \tag{1}$$

can be easily verified.

The operator T taking functions h on $\mathbb{R} \times S^{n-1}$ to functions on $\mathbb{R}^n$ is defined by

$$Th(x) = \int_{S^{n-1}} h(x \cdot \omega, \omega)\, d\omega.$$

An important relation between T and the Radon transform is the equality

$$\int_{\mathbb{R}^n} Th(x) f(x)\, dx = \int_{-\infty}^{\infty} \int_{S^{n-1}} Rf(u, \omega)\, h(u, \omega)\, |u|^{n-1}\, du, \tag{2}$$

a well-known consequence of the formula

$$\int_{S^{n-1}} g\left(\frac{\omega_2}{\omega_1}, ..., \frac{\omega_n}{\omega_1}\right) \frac{d\omega}{|\omega_1|^n} = 2 \int_{\mathbb{R}^{n-1}} g(y_2, ..., y_n)\, dy.$$

The proof of Theorem 1 depends heavily on Theorem 2.2 in [S], which we now restate in the case $k=n-1$: fix $p\in(1,2)$ and let p' be the dual exponent. There exists $C=C(p,n)$ such that

$$\left(\int_{S^{n-1}} \|I_1^{(n-1)/p}\tilde{R}f(\cdot,\omega)\|_p^{p'}\,d\omega\right)^{1/p'} \leqslant C\,\|f\|_p \tag{3}$$

for f in $\mathscr{S}_0$. Dualizing (3) with the aid of (2) gives the following: fix $q\in(2,\infty)$. There is $C=C(q,n)$ such that

$$\|Th\|_q \leqslant C\left(\int_{S^{n-1}} \|I_1^{(n-1)/q}h(\cdot,\omega)\|_q^{q'}\,d\omega\right)^{1/q'} \tag{4}$$

for suitable functions h on $\mathbb{R}\times S^{n-1}$.

With p, q, and r as in the hypotheses, fix $f\in\mathscr{S}_0$. Define

$$h(s,\omega)=A_\omega\tilde{R}f(\cdot,\omega)(s), \qquad \omega\in S^{n-1}, \quad s\in\mathbb{R}.$$

Then

$$\begin{aligned}
&\left(\int_{S^{n-1}} \|I_1^{(n-1)/q}h(\cdot,\omega)\|_q^{q'}\,d\omega\right)^{1/q'}\\
&\quad\leqslant\left(\int_{S^{n-1}} (\|I_1^{(n-1)/p}\tilde{R}f(\cdot,\omega)\|_p\,\|I^{(n-1)(1/q-1/p)}A_\omega\|_{p-q})^{q'}\,d\omega\right)^{1/q'}\\
&\quad\leqslant\left(\int_{S^{n-1}} \|I_1^{(n-1)/p}\tilde{R}f(\cdot,\omega)\|_p^{p'}\,d\omega\right)^{1/p'}\\
&\qquad\times\left(\int_{S^{n-1}} \|I^{(n-1)(1/q-1/p)}A_\omega\|_{p-q}^{r}\,d\omega\right)^{1/r}\\
&\quad\leqslant C\,\|f\|_p.
\end{aligned}$$

Here the next to last inequality follows from $1/p'+1/r=1/q'$ while the last is a consequence of (3) and the hypotheses. If we define $Bf=Th$, then (4) yields

$$\|Bf\|_q\leqslant C\,\|f\|_p. \tag{5}$$

Now if g is nice function on $\mathbb{R}$ and $\omega\in S_{n-1}$, let $U_\omega g$ be the function on $\mathbb{R}^n$ defined by

$$U_\omega g(x)=g(x\cdot\omega), \qquad x\in\mathbb{R}^n.$$

For $\psi \in \mathscr{S}$ one can verify the following formula for the (distributional) Fourier transform of $U_\omega g$:

$$\langle \widehat{U_\omega g}, \psi \rangle = \int_{-\infty}^{\infty} \mathscr{F}g(s)\, \psi(s\omega)\, ds.$$

Recalling that

$$Th = \int_{S^{n-1}} U_\omega h(\cdot, \omega)\, d\omega$$

we thus have

$$\begin{aligned}\langle \widehat{Bf}, \psi \rangle &= \langle \widehat{Th}, \psi \rangle \\ &= \int_{S^{n-1}} \langle U_\omega h(\cdot, \omega)^\wedge, \psi \rangle\, d\omega \\ &= \int_{S^{n-1}} \int_{-\infty}^{\infty} \mathscr{F}h(\cdot, \omega)(s)\, \psi(s\omega)\, d\omega, \qquad \psi \in \mathscr{S}.\end{aligned}$$

Since

$$h(\cdot, \omega) = A_\omega \tilde{R}f(\cdot, \omega) = C(n)\, A_\omega I_1^{1-n} Rf(\cdot, \omega),$$

it follows from (1) that

$$\langle \widehat{Bf}, \psi \rangle = C(n) \int_{S^{n-1}} \int_{-\infty}^{\infty} \hat{f}(s\omega)\, \hat{A}_\omega(s)\, \psi(s\omega)\, |s|^{n-1}\, ds\, d\omega. \qquad (6)$$

It is a consequence of (5) and (6) that B is a bounded convolution operator on $\mathbb{R}^n$ with symbol

$$\hat{B}(s\omega) = C(n)(\hat{A}_\omega(s) + \hat{A}_{-\omega}(-s)), \qquad \omega \in S^{n-1} \quad \text{and} \quad s > 0.$$

Let H be the multiple of the Hilbert transform on $\mathbb{R}$ defined by

$$\hat{H}(s) = \operatorname{sgn}(s), \qquad s \in \mathbb{R}.$$

Define $A'_\omega = HA_\omega$ and let B' be the operator obtained as above by replacing $\{A_\omega\}_{\omega \in S^{n-1}}$ with $\{A'_\omega\}_{\omega \in S^{n-1}}$. Then B' is also a bounded convolution operator on $\mathbb{R}^n$, and

$$\widehat{B'}(s\omega) = C(n)(\hat{A}_\omega(s) - \hat{A}_{-\omega}(-s)), \qquad \omega \in S^{n-1}, \quad s > 0.$$

Since A is a multiple of $\frac{1}{2}(B + B')$, the theorem follows.

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